On components of perfect binary codes *

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Abstract

Special components, called i-components, of perfect binary codes are investigated. The existence of maximal cardinality nonisomorphic i-components of different perfect codes of length n for all $n = 2^k - 1, k > 3$, is proved.

1 Introduction

The principal point of our investigation is the i-component structure of single-error correcting perfect binary codes (briefly perfect codes). Investigations of i-components may be helpful for the decision of the complicated problems of the construction and enumeration of perfect codes. We prove that there exist maximal cardinality nonisomorphic i-components of different perfect codes of length n for all $n = 2^k - 1, k > 3$. The construction is given by using simultaneously the switching [1] and the concatenation [2] constructions of perfect codes.

Let C be a perfect code and M be a subset of C. Exchanging the bit in the i'th coordinate of all vectors of M with the opposite bit we obtain a new set, denoted by $M \oplus i$. We identify with i the vector having an one in the i'th coordinate and zeroes elsewhere. If $C' = (C \setminus M) \cup (M \oplus i)$ is a perfect code, we call the set M an *i-component* of the code C and say that C' is obtained from C by a *translation* of an i-component M. An i-component is *minimal* if it cannot be subdivided into smaller i-components. We omit the word minimal because we consider below only minimal i-components.

It is known [3, 4] that upper and lower bounds on the number m of i-components of an arbitrary perfect code of length n, $n = 2^k - 1$, are given by

$$2 \leq m \leq 2^{\frac{n+1}{2}}/(n+1). \quad (1)$$

Both of these bounds can be achieved, see [1, 3, 4]. The cardinality of the minimal i-components can vary from $2^{(n-1)/2}$ to $2^{n-1}/(n+1)$. According to [3] there exist nonextremal cardinality i-components of perfect codes of length n for all admissible $n > 7$. A perfect code of length $n, n > 7$, with i-components of different structures and cardinalities was presented in [5].

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2 Definitions

A *binary code* C of length n is a subset of the vector space E^n of dimension n over $GF(2)$. The elements of C are called *codewords* or *vectors*. Two codes $C, C' \subset E^n$ are said to be *isomorphic* if there exists a permutation π such that $C' = \pi(C) = \{\pi(x) : x \in C\}$. Codes $C, C' \subset E^n$ are *equivalent* if there exists a vector $b \in E^n$ and a permutation π such that $C' = b \oplus \pi(C) = \{b \oplus \pi(x) : x \in C\}$. *The Hamming distance* $d(x, y)$ between vectors $x, y \in C$ is the number of coordinates in which x and y differ. *The Hamming weight* of $x \in C$ is given by $wt(x) = d(x, \mathbf{0})$, where $\mathbf{0}$ is the all-zero vector. A code distance is given by $d = min\, d(x, y)$ for any different codewords $x, y \in C$. A code $C \subset E^n$ of length n is *perfect* with distance 3 if every $x \in E^n$ is within distance 1 from exactly one codeword of C. The linear perfect codes called *Hamming codes* are unique up to equivalence. There are many constructions of nonlinear perfect codes. Short summary of them is given, for example, in [6].

3 Construction

We further identify a vector $x = (x_1, \ldots, x_n) \in E^n$ with its block presentation $(i_1, \ldots, i_k)$ of all those positions of coordinates of vector x, which equaled to 1.

Consider the Hamming code H^n of length n given by Vasil'ev construction

$$H^n = \{(\alpha, \alpha \oplus \beta, |\alpha|) : \alpha \in E^{(n-1)/2}, \beta \in H^{(n-1)/2}\}, \tag{2}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_{(n-1)/2} \pmod 2$ for $\alpha = (\alpha_1, \ldots, \alpha_{(n-1)/2})$ and $H^{(n-1)/2}$ be a Hamming code of length $(n-1)/2 = 2^k - 1, k \geq 2$.

According to Glagolev's lemma, see in [7], for any binary linear code of length n and distance d there exists a binary linear code of length n and distance d with a basis among codewords of weight d. As mentioned above Hamming codes are unique up to equivalence, therefore there exists a basis of the Hamming code among codewords of weight 3.

We construct such a basis for the Hamming code by induction. If $k = 2$ the Hamming code H^3 is generated by codeword $(1, 2, 3)$. Let $B_n = \{(i, j, k)\}$ be a basis of the Hamming code H^n, $|V_n| = n - log(n+1)$. Let $V_{2n+1} = V_n \cup \{(i, n+i, 2n+1) : i = 1, \ldots, n\}$. According to (2) the vector $(i, n+i, 2n+1)$ belongs to H^{2n+1}, $i = 1, \ldots, n$, and $V_n \subset H^{2n+1}$. Therefore $|V_{2n+1}| = 2n - log(n+1)$ and the set V_{2n+1} is a basis of weight 3 codewords of the Hamming code H^{2n+1}. We have then,

Proposition 1. (The Hamming code basis proposition.) *There exists a basis of weight 3 codewords of the Hamming code of length n for any $n = 2^k - 1, k \geq 2$.*

Proposition 2. *There exist Hamming codes H_1^n and H_2^n of length n, $n = 2^k - 1, k \geq 3$, with basis intersected exactly by one codeword.*

Proof. The proof is by induction on $k, k = log(n+1), k \geq 2$.

Let $k = 3$, e.g. $n = 2^k - 1 = 7$. Basis of Hamming codes H_1^7 and H_2^7 are given by

$$V_1^7 = \{(123), (147), (257), (367)\}, \tag{3}$$

$$V_2^7 = \{(123), (157), (267), (347)\}, \tag{4}$$

respectively, where $V_1^7 \cap V_2^7 = \{(123)\}$.

Assume now that for any $k, k = log(\frac{n+1}{2})$, two Hamming codes $H_1^{(n-1)/2}$ and $H_2^{(n-1)/2}$ with basis $V_1^{(n-1)/2}$ and $V_2^{(n-1)/2}$ respectively, $V_1^{(n-1)/2} \cap V_2^{(n-1)/2} = \{(123)\}$, are constructed.

Using Proposition 1 construct basis V_1^n and V_2^n of Hamming codes H_1^n and H_2^n of length n.

Let the set

$$V_1^n = V_1^{(n-1)/2} \cup \{(i, i + (n-1)/2, n) : i = 1, \ldots, (n-1)/2\} \tag{5}$$

from Proposition 1 be the basis of the first Hamming code H_1^n.

Let the basis of another code H_2^n be

$$V_2^n = V_2^{(n-1)/2} \cup \{(i, \pi(i) + (n-1)/2, n) : i = 1, \ldots, (n-1)/2\}, \tag{6}$$

where π is a permutation on the set $\{1, \ldots, (n-1)/2\}$ such that $\pi(i) \neq i, i = 1, \ldots, (n-1)/2$, for example $\pi(i) = (12\ldots(n-1)/2)$.

It is easy to see that $V_1^n \cap V_2^n = \{(123)\}$ and codes H_1^n, H_2^n generated by V_1^n and V_2^n respectively are Hamming codes of length n. $\square$

Let $H_1^n \times H_2^n = \{(x, y) : x \in H_1^n, y \in H_2^n\}$ be a product of two Hamming codes H_1^n and H_2^n of length n. We call a codeword $(i, j, k, n+i, n+s, n+k) \in H_1^n \times H_2^n$, where s is either equal or not to j, *a special codeword.* A basis (a set), which consists of special codewords, is called *a special basis* (*a special set*). We will use for a brevity the notation $(i, j, k, \bar{i}, \bar{s}, \bar{k})$ for the vector $(i, j, k, n+i, n+s, n+k)$.

The code $H_1^n \times H_2^n$ can be represented as

$$H_1^n \times H_2^n = (H_1^n \times H_2^n)^0 \cup ((H_1^n \times H_2^n)^0 \oplus (i, j, k)),$$

where $(H_1^n \times H_2^n)^0$ is the set of all even weight codewords of the code $H_1^n \times H_2^n$ and $(i, j, k) \in H_1^n$. Therefore $|(H_1^n \times H_2^n)^0| = 2^{2n-2log(n+1)-1}$.

Proposition 3. *There exists a special set of cardinality not more than* $2n - 2log(n+1) - 1$ *of the code* $(H_1^n \times H_2^n)^0$ *for* $n = 2^k - 1, k \geq 3$.

Proof. We construct the special set $B_{2n} \subset (H_1^n \times H_2^n)^0$ of cardinality not more than $2n - 2log(n+1) - 1$ by induction on $k, k = log(n+1), k \geq 3$.

Let $k = 3$, e.g. $n = 2^k - 1 = 7$. Using from Proposition 2 the basis (3) and (4) of the Hamming codes H_1^7 and H_2^7 respectively we construct the special set $B_{14} \subset (H_1^7 \times H_2^7)^0$:

$$B_{14} = \{(123\bar{1}\bar{2}\bar{3}), (145\bar{1}\bar{5}\bar{7})(257\bar{2}\bar{6}\bar{7}), (367\bar{3}\bar{4}\bar{7}), (147\bar{3}\bar{4}\bar{7}), (257\bar{1}\bar{5}\bar{7}), (367\bar{3}\bar{5}\bar{6})\}, \tag{7}$$

where $|B_{14}| = 7$ and the vector $(123) \oplus (157) \oplus (267) = (356)$ is from H_2^7.

Let for any $k, k = log(n+1)$, the special set B_{n-1}, $|B_{n-1}| = n - 2log(n+1)$, is constructed from basis $V_1^{(n-1)/2}$ and $V_2^{(n-1)/2}$ of the Hamming codes $H_1^{(n-1)/2}$ and $H_2^{(n-1)/2}$ respectively. Let $V_1^{(n-1)/2}$ and $V_2^{(n-1)/2}$ are sets from Proposition 2 and $B_{n-1} \subset (H_1^{(n-1)/2} \times H_2^{(n-1)/2})^0$, see (5) and (6).

Construct now a special set B_{2n}, $|B_{2n}| \leq 2n - 2log(n+1) - 1$ for $k+1 = log(n+1)+1$. Let $B_{n-1} \subset B_{2n}$. Using constructions of the codes $H_1^{(n-1)/2}$ and $H_2^{(n-1)/2}$ given by Proposition 2 we define the following special sets:

$$F_{2n}=\{(i,i+(n-1)/2,n,\overline{i},\overline{j+(n-1)/2},\overline{n}):(i,i+(n-1)/2,n)\in V_1^{(n-1)/2},$$
$$(i,j+(n-1)/2,n)\in V_2^{(n-1)/2}, i=1,\dots,(n-1)/2, j=\pi(i)\}, \tag{8}$$

where $\pi(i)$ is from Proposition 2 and

$$G_{2n}=\{(i,i+(n-1)/2,n,\overline{s},\overline{i+(n-1)/2},\overline{n}):(i,i+(n-1)/2,n)\in V_1^{(n-1)/2},$$
$$(s,i+(n-1)/2,n)\in V_2^{(n-1)/2}, i=1,\dots,(n-3)/2\}\cup$$
$$\{((n-1)/2,n-1,n,\overline{(n-1)/2},\overline{n-1},\overline{p}):$$
$$((n-1)/2,n-1,n)\in V_1^{(n-1)/2},((n-1)/2,n-1,p)\in V_2^{(n-1)/2}\}. \tag{9}$$

It is clear from (8) and (9) that $F_{2n}\subset(H_1^n\times H_2^n)^0$ and $G_{2n}\subset(H_1^n\times H_2^n)^0$. Including F_{2n} and G_{2n} in B_{2n} we obtain $B_{2n}\subset(H_1^n\times H_2^n)^0$ and

$$|B_{2n}|=|B_{n-1}\cup F_{2n}\cup G_{2n}|\le n-2log(n+1)+2(\frac{n-1}{2})=2n-2log(n+1)-1.$$

□

Denote by $r(A)$ a rank of a matrix A.

Proposition 4. *The special set $F_{2n}\cup G_{2n}$ is linear independent set of cardinality $n-1$ for $n=2^k-1, k\ge 3$.*

Proof. We write all vectors of $F_{2n}\cup G_{2n}$ as rows of the matrix M_{2n} of order $2n\times(n-1)$ in the same order as in (8),(9):

$$M_{2n}=\left(\begin{array}{c|c|c|c|c|c} \mathrm{E} & \mathrm{E} & \mathbf{1}^T & \mathrm{E} & \mathrm{C} & \mathbf{1}^T \\ \hline \mathrm{E} & \mathrm{E} & \mathbf{1}^T & \mathrm{A} & \mathrm{D} & \mathrm{a}^T \end{array}\right)$$

and find a rank $r(M_{2n})$ of M_{2n}. Here E is an identity matrix of the order $p, p=(n-1)/2$, the vector $\mathbf{1}^T$ is all-one vector-column of length p and $a^T=(1,\dots,1,0)^T$ is the vector-column having a zero in the last coordinate and ones elsewhere. According to (8) the matrix C is a cyclic shift of the identity matrix E by one column. The matrix D is coincide with E into the first $(p-1)$'th rows and has an one only in j'th and $(p-1)$'th coordinates in the last row, see (9). The structure of the matrix A is not important for clarifying of a rank of M_{2n}.

It is well known that linear transformations can not alter a rank of matrix. Consider some transformations of matrix M_{2n}.

First we summ the s'th row to the $(p+s)$'th row by the modulo 2, $s=1,\dots,p$. The resulting matrix

$$\left(\begin{array}{c|c|c|c|c|c} \mathrm{E} & \mathrm{E} & \mathbf{1}^T & \mathrm{E} & \mathrm{C} & \mathbf{1}^T \\ \hline \Theta & \Theta & \mathbf{0}^T & \mathrm{A}\oplus\mathrm{E} & \mathrm{C}\oplus\mathrm{D} & \mathrm{a}^T\oplus\mathbf{1}^T \end{array}\right)$$

is equivalent to M_{2n}.

After that using the matrix arranged from $(p+1)/2$'th to p'th columns one can easily get the matrix

$$M'_{2n}=\left(\begin{array}{c|c|c|c|c|c} \Theta & \mathrm{E} & \mathbf{0}^T & \Theta & \Theta & \mathbf{0}^T \\ \hline \Theta & \Theta & \mathbf{0}^T & \mathrm{A}\oplus\mathrm{E} & \mathrm{C}\oplus\mathrm{D} & \mathrm{a}^T\oplus\mathbf{1}^T \end{array}\right),$$

which is also equivalent to M_{2n}. Here Θ is all-zero matrix of the order p and $\mathbf{0}^T$ is all-vero vector-column of length p.

We now transform last $2p+1$ columns of M'_{2n} and show

$$r(A \oplus E|C \oplus D|a^T \oplus \mathbf{1}^T) = p. \tag{10}$$

It give us immediately

$$r(M_{2n}) = r(M'_{2n}) = r(E) + r(A \oplus E|C \oplus D|a^T \oplus \mathbf{1}^T) = p + p = n - 1.$$

By constructions of C and D the matrix $C \oplus D$ contains in the first $p-1$ rows cyclic shifts of the vector $(1, 1, 0, \ldots, 0)$ of length p. First add the last column of $(C \oplus D|a^T \oplus \mathbf{1}^T)$ to all those columns, which consist 1 in the last coordinate. The resulting matrix is equivalent to $(C \oplus D|a^T \oplus \mathbf{1}^T)$, and has the vector $(0, \ldots, 0, 1)$ as the last row. After that we add every j'th column to the $(j-1)$'th column, $j = p, p-1, \ldots, 2$, beginning from the p'th column. It is easy to see that the resulting matrix has a rank p and has in every row exactly one 1. Returning to the matrix $(A \oplus E|C \oplus D|a^T \oplus \mathbf{1}^T)$ one can transform the matrix $(A \oplus E)$ to the matrix Θ and obtain (10). Then $r(M_{2n}) = n - 1$ and $F_{2n} \cup G_{2n}$ is linear independent special set. $\square$

Proposition 5. *There exists a special basis of the code* $(H_1^n \times H_2^n)^0$ *for* $n = 2^k - 1, k \geq 3$.

Proof. Prove by induction on k, $k = log(n+1), k \geq 3$, that the special set B_{2n} constructed in Proposition 3 is linear independent.

It is easy to check that B_{14} is linear independent.

Let for any $k, k = log(n+1)$, the special set B_{n-1}, $|B_{n-1}| = n - 2log(n+1)$, is linear independent.

Consider for $k+1$ the block-matrix T_{2n} of vectors B_{2n} :

$$T_{2n} = \left(\frac{B_p^1 \mid \Theta_{r \times (p+1)} \mid B_p^2 \mid \Theta_{r \times (p+1)}}{M_{2n}} \right).$$

Here $r = |B_{n-1}| = n - 2log(n+1)$, $B_{n-1} = (B_p^1|B_p^2)$, where B_p^1 and B_p^2 contain codewords of H_p^1 and H_p^2 respectively, $p = (n-1)/2$. Make all changes described in Proposition 4 with the submatrix M_{2n}. It is clear that the linear transformations do not alter the first r rows of matrix T_{2n}. Therefore

$$r(T_{2n}) = r(B_p^1|B_p^2) + r(M_{2n}) = n - 2log(n+1) + n - 1 = 2n - 2log(n+1) - 1.$$

$\square$

For a perfect code C^n of length n we define the minimum distance graph $G(C^n, U)$ of C^n with the codewords in C^n as vertices and edges $(x, y) \in U$ if and only if the Hamming distance between x and y is equal to 3. Let $G(C^n, U_i)$ be the spanning subgraph of the graph $G(C^n, U)$ with U_i as the subset of all edges $(x, y) \in U$, where codewords x and y disagree in the i'th coordinate. The subgraph $G(C^n, U_i)$ is introduced in [3, 4], where it is established that the number of components m of the subgraph $G(C^n, U_i)$ satisfies the inequalities (1). It is not difficult to see that an i-component M_i of the perfect code C^n coincides with the set of vertices of one component of the subgraph $G(C^n, U_i)$. We also refer to such components as the i-components of the subgraph $G(C^n, U_i)$.

To make the paper self contained we give the following

Proposition 6. (See [3].) *The graph $G(C^n, U)$ of a perfect code C^n contains at least two i-components.*

Proof. W.l.o.g. let $\mathbf{0} \in C^n$. The set of codewords of weigth 3 of a perfect code C is a Steiner triple system $STS(C^n)$ of a code C^n, see [8]. $STS(C^n)$ contains $n(n-1)/6$ triples and exactly $(n-1)/2$ of them contain one in a fixed i'th coordinate, $i = 1, \ldots, n$. Then one can find for $n \geq 7$ a vector α of weight 3 with zero in the i'th coordinate. Consider now the i-component R_i of the graph $G(C^n, U_i)$, where $\mathbf{0} \in R_i$. It is evident that the vector α can not belong to the i-component R_i. $\square$

Consider now a concatenation construction of perfect codes given in [2]. Let $C_0, C_1, \ldots, C_n$ and $C'_0, C'_1, \ldots, C'_n$ be any partitions of the vector space E^n into perfect codes (constructions of nontrivial partitions of E^n can be found in [2]). Let φ be a permutation on the set $\{0, 1, \ldots, n\}$. Then the code

$$C^{2n+1} = \{(x, y, |y|) : x \in C_i, y \in C'_{\varphi(i)}, i = 0, 1, \ldots, n\} \tag{11}$$

is a perfect code of length $2n + 1$.

Let H_1^n and H_2^n be Hamming codes given above and φ be an identity permutation. Using for the concatenation construction (11) the partitions of E^n into cosets $H_1^n \oplus i$ and $H_2^n \oplus i$ of Hamming codes H_1^n and H_2^n respectively, $i = 0, 1, \ldots, n$, we obtain the perfect code, denote it by K^{2n+1}. In our case H_1^n is not equal to H_2^n then the code K^{2n+1} is not a linear code, see [9, 2].

Let K_0^{2n+1} and K_1^{2n+1} be the following subcodes of the code K^{2n+1} :

$$K_0^{2n+1} = \{(x, y, |y|) : x \in H_1^n \oplus i,\ y \in H_2^n \oplus i,\ i = 0, 1, \ldots, n;\ w(x, y) \equiv 0(mod2)\},$$

$$K_1^{2n+1} = K_0^{2n+1} \oplus (z, \mathbf{0}, 0),$$

where $z \in H_1^n, \mathbf{0} \in H_2^n, w(z) = 3$.

It is clear that $K^{2n+1} = K_0^{2n+1} \cup K_1^{2n+1}$ and

$$K_0^{2n+1} = \bigcup_{i=0}^{n} ((H_1^n \times H_2^n)^0 \oplus (i, \bar{i}, 2n + 1)).$$

Let K^{2n} be the code obtained from K^{2n+1} by shortening of the last coordinate. Then $K^{2n} = K_0^{2n} \cup K_1^{2n}$, where $K_0^{2n} = \bigcup_{i=0}^{n} ((H_1^n \times H_2^n)^0 \oplus (i, \bar{i}))$ and $K_1^{2n} = K_0^{2n} \oplus (z, \mathbf{0})$.

Consider furthermore a graph $G(K_0^{2n}, U')$ with codewords in K_0^{2n} as vertices and edges $(x, y) \in U'$ if and only if the Hamming distance between x and y is equal to 2.

Proposition 7. *The graph $G(K_0^{2n}, U')$ is adjaced graph for any $n = 2^k - 1, k \geq 3$.*

Proof. According to Proposition 5 the code $(H_1^n \times H_2^n)^0$ has a special basis. Then any two vectors in $(H_1^n \times H_2^n)^0$ are adjaced by a circuit of the special codewords like

$$\alpha = (i, j, k, \bar{i}, \bar{s}, \bar{k}),$$

where $(i, j, k) \in H_1^n$ and $(i, s, k) \in H_2^n$. By the construction of the code K_0^{2n} it is true that $\beta \oplus (t, \bar{t}) \in K_0^{2n}$ for any $\beta = (x, y) \in (H_1^n \times H_2^n)^0$, where $x \in H_1^n$, $y \in H_2^n$ and any $t \in \{1, \ldots, n\}$. It is true that $\beta \oplus \alpha \in (H_1^n \times H_2^n)^0$ by the construction of $(H_1^n \times H_2^n)^0$. Therefore $\beta \oplus (i, \bar{i})$, $\beta \oplus (i, j, \bar{i}, \bar{s}) = \beta \oplus \alpha \oplus (k, \bar{k}) \in K_0^{2n}$.

Then the subgraph $G(K_0^{2n}, U')$ is adjaced. $\square$

Theorem 1. *The perfect code K^{2n+1} is partitioned into two $(2n+1)$-components of maximal cardinality for $n = 2^k - 1, k \geq 3$.*

Proof. Consider the graph $G(K^{2n+1}, U)$ and show that the subgraph $G(K^{2n+1}, U_{2n+1})$ is partitioned into two $(2n+1)$-components $G(K_0^{2n+1}, U_{2n+1})$ and $G(K_1^{2n+1}, U_{2n+1})$. By the construction of K_0^{2n+1} and K_1^{2n+1} we have

$$|K_0^{2n+1}| = |K_1^{2n+1}| = \frac{1}{2}|K^{2n+1}| = \frac{2^{n-1}}{n+1}.$$

A connectivity of the graph $G(K_0^{2n+1}, U_{2n+1})$ is followed immediately from the connectivity of $G(K_0^{2n}, U')$ given by Proposition 7.

The set K_1^{2n+1} is a translation of the set K_0^{2n+1} then the graph $G(K_1^{2n+1}, U_{2n+1})$ is also adjaced. Therefore using the condition that the graph $G(C)$ of a perfect code C contains at least two i-components (see Proposition 6) one can conclude that the graph $G(K^{2n+1}, U_{2n+1})$ is partitioned exactly into two $(2n+1)$-components of maximal cardinality. $\square$

A map ψ between two perfect codes C and C' such that $d(x, y) = d(\psi(x), \psi(y))$ for every $x, y \in C$ is called an *isometric* map from C to C'.

A sequence $\{x^1, \ldots, x^l\}$ with $x^i \in E^n$ is a *cycle* in E^n if $d(x^i, x^j) \geq 1$ for $j = i+1$ or $i = l, j = 1$, $i = 1, \ldots, l$, and $d(x^i, x^j) \geq 2$ elsewhere.

Theorem 2. *There exist maximal cardinality nonisomorphic i-components of different perfect codes of length n for all $n = 2^k - 1, k > 3$.*

Proof. Compare the structure of the perfect code K^{2n+1} described above with the structure of the perfect code C^{2n+1} presented in [3], $n = 2^k - 1, k > 3$.

Every $(2n+1)$-component of the graph $G(K^{2n+1}, U_{2n+1})$ contains cycles of length 4, for example

$$\{0^{2n+1}, (1, \overline{1}, 2n+1), (2, \overline{2}, 2n+1), (1, 2, \overline{1}, \overline{2})\}.$$

At the same time an $(2n+1)$-component of the graph $G(C^{2n+1}, U_{2n+1})$ does not contain cycles of length 4. Then the $(2n+1)$-components of codes K^{2n+1} and C^{2n+1} are not isometric and therefore they are not isomorphic. $\square$

It is not difficult to see that the technique given above allows one to construct sets of perfect codes of length n with nonextremal cardinality i-components for all $n = 2^k - 1, k > 3$.

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