SAMPLE PATH LARGE DEVIATIONS IN FINER TOPOLOGIES

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Abstract. In this paper we present sufficient conditions for sample path large deviation principles to be extended to finer topologies. We consider extensions of the uniform topology by Orlicz functionals and we consider Lipschitz spaces: the former are concerned with cumulative path behaviour while the latter are more sensitive to extremes in local variation. We also consider sample paths indexed by the half line, where the usual projective limit topologies are not strong enough for many applications.

1. Introduction

Let Z be a topological space. A good rate function is a lower semi-continuous function $I : Z \to \mathbb{R}_+$, with compact level sets $\{z : I(z) \leq \alpha\}$. A sequence of random variables $(Z_n)_{n\in\mathbb{N}}$, taking values in Z, is said to satisfy the large deviation principle (LDP) with rate function I if for all closed $B \subset Z$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(Z_n \in B) \leq - \inf_{z\in B} I(z),$$

and for all open $G \subset Z$,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(Z_n \in G) \geq - \inf_{z\in B} I(z).$$

Let $(X_k)_{k\in\mathbb{N}}$ be a sequence of random variables with values in $\mathbb{R}^d$. For each $0 \leq t \leq 1$ set

$$S_n(t) = \frac{1}{n} \sum_{k=1}^{[nt]} X_k.$$

Denote by $L_\infty[0,1]$ the space of (equivalence classes modulo equality *a.e.* of) bounded measurable functions on $[0,1]$, equipped with the uniform topology, by $\mathcal{C}[0,1]$ the subspace of continuous functions, and by $\mathcal{A}[0,1]$ the subspace of absolutely continuous functions ϕ on $[0,1]$ with $\phi(0) = 0$. Denote by $L^d_\infty[0,1]$ the space of bounded measurable functions on $[0,1]$ with values in $\mathbb{R}^d$, equipped with the uniform topology.

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It is a classical result, originally due to Varadhan [13] and Mogulskii [9], that, if the $(X_k)_{k\in\mathbb{N}}$ are *i.i.d.* and

$$\Lambda(\theta) := \log \mathbb{E}\,\exp(\langle\theta, X_1\rangle) < \infty \tag{1.1}$$

for all $\theta \in \mathbb{R}^d$, then the sequence $\{S_n(\cdot), n \in \mathbb{N}\}$ satisfies the LDP in $L^d_\infty[0,1]$ with good rate function given by

$$I(\phi) = \begin{cases} \int_0^1 \Lambda^*(\dot{\phi})ds & \phi \in \mathcal{A}^d[0,1], \\ \infty & \text{otherwise,} \end{cases} \tag{1.2}$$

where Λ^* is the convex dual of Λ:

$$\Lambda^*(x) = \sup_{\theta\in\mathbb{R}^d} [\langle x, \theta\rangle - \Lambda(\theta)],$$

and where $\dot{\phi}$ denotes the derivative of ϕ, which exists almost everywhere for $\phi \in \mathcal{A}^d[0,1]$. The assumption that the moment generating function is finite has since been relaxed by Pinelis [11]; more recently, Dembo and Zajic [3] have relaxed the assumption of independence, replacing it with a variety of mixing and tightness hypotheses.

We find it convenient to consider, instead of S_n, its polygonal approximation:

$$\tilde{S}_n(t) = S_n(t) + \left(t - \frac{[nt]}{n}\right) X_{[nt]+1}. \tag{1.3}$$

Note that $\tilde{S}_n(\cdot)$ is continuous, and carries the same information as $S_n(\cdot)$. The LDP for $\tilde{S}_n(\cdot)$ can typically be shown to hold under milder hypotheses, and in the Polish space $\mathcal{C}^d[0,1]$ (equipped with the uniform topology).

In this paper we present sufficient conditions for such an LDP to be extended to finer topologies. In Section 3 we consider extensions of the uniform topology by Orlicz functionals and in Section 4 we consider Lipschitz spaces; the former are concerned with cumulative path behaviour while the latter are more sensitive to extremes in local variation. We also consider sample paths indexed by the half line, where the usual projective limit topologies are not strong enough for many applications, particularly in queueing theory. This extends work of O'Connell [10] and work of Dobrushin and Pechersky [5].

2. Some facts about Orlicz spaces

What follows is mostly a summary of the relevant details from [1, Chapter 8]. A function $C : \mathbb{R}_+ \to \mathbb{R}_+$ is called an *N-function* if it is convex and satisfies

$$\lim_{x\to 0^+} C(x)/x = 0, \qquad \lim_{x\to\infty} C(x)/x = \infty.$$

Examples of N-functions are

$$C(x) = x^p, \quad (1 < p < \infty),$$

$$C(x) = e^x - x - 1,$$

$$C(x) = (1+x)\log(1+x) - x,$$

or

$$C_a(x) = C(x-a)1_{x\geq a}$$

for any N-function C. We will write C^* for the convex dual of C.
The *Orlicz class* K_C is the set of all (equivalence classes modulo equality *a.e.* of) measurable functions $f : [0,1] \to \mathbb{R}$ satisfying

$$\int_0^1 C(|f(s)|)ds < \infty;$$

the *Orlicz space* L_C is defined to be the linear hull of K_C, and is a Banach space when equipped with the *Luxemburg norm*

$$\|f\|_C = \inf\left\{k > 0 : \int_0^1 C\left(|f(s)|/k\right) ds \leq 1\right\}.$$

We have the following generalisation of Hölder's inequality:

$$\left|\int_0^1 f(s)g(s)ds\right| \leq 2\|f\|_C\|g\|_{C^*}. \tag{2.1}$$

An N-function C is said to satisfy a Δ_2-*condition near infinity* if there exists a positive constant $k > 0$ such that for all x sufficiently large,

$$C(2x) \leq kC(x).$$

This is a useful condition, because $L_C = K_C$ if, and only if, C satisfies a Δ_2-condition near infinity [1, Lemma 8.8]; furthermore, L_C is reflexive, with dual L_{C^*}, if, and only if, both C and C^* satisfy a Δ_2-condition near infinity [1, Theorem 8.19]. Note that if L_C is reflexive, then it is weakly complete and the unit ball is weakly compact.
We will now record a lemma that will be useful for interpreting the results of the next section. A proof is given in Section 6.

Lemma 2.2. *Suppose that C^* satisfies a Δ_2-condition near infinity. A sequence f_n is weakly convergent to a function f in L_C if, and only if,*

(i) *$\|f_n\|_C$ is bounded, and*
(ii) *for each $t \in [0,1]$, $\int_0^t f_n(s)ds \to \int_0^t f(s)ds$.*

Note that all of the above remarks apply to Orlicz spaces of functions on any finite interval $[0,t]$, which we denote by $L_C[0,t]$.

3. The LDP in a topology extended by Orlicz functionals

Consider the following Conditions:

(H1) For each $\theta \in \mathbb{R}^d$, the limit

$$\Lambda(\theta) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\, \exp(n\langle\theta, S_n(1)\rangle) \tag{3.1}$$

exists as an extended real number. The sequence $\tilde{S}_n(\cdot)$ satisfies the LDP in $\mathcal{C}^d[0,1]$ with good rate function given by

$$I(\phi) = \begin{cases} \int_0^1 \Lambda^*(\dot{\phi}) ds & \phi \in \mathcal{A}^d[0,1], \\ \infty & \text{otherwise}, \end{cases}$$

where Λ^* is the convex dual of Λ.

For each $j = 1, \dots, d$ and $\lambda \in \mathbb{R}$, set

$$\Lambda_j(\lambda) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E} \exp(n\,\lambda\, S_n^j(1)).$$

Note that the existence of Λ_j follows from the existence of Λ, and can be obtained by evaluating Λ at multiples of the j^{th} standard basis vector.

(H2) (i) For each $j = 1, \dots, d$, there exists an N-function B_j with

$$\limsup_{|x|\to\infty} \frac{B_j(|x|)}{\Lambda_j^*(x)} < \infty,$$

and both B_j and its convex dual satisfy a Δ_2-condition near infinity.

(ii) For some $\delta > 0$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E} \exp\left(\delta \sum_{k=1}^n B_j(|X_k^j|)\right) < \infty, \tag{3.2}$$

for each $j = 1, \dots, d$.

Remark 3.3. The hypothesis (H1) is fulfilled, if the sequence $(X_k)_{k\in\mathbb{N}}$ is *i.i.d.*, see for example [4, Section 5.1]. But (H1) is also fulfilled, if the assumptions (A-1) and (A-2) in [3] are fulfilled: by Theorem 2 in [3], the partial sums processes $\{S_n(\cdot), n \in \mathbb{N}\}$ satisfies the LDP under (A-1) and (A-2) in $D^d[0,1]$, the space of càdlàg functions on $[0,1]$ with values in $\mathbb{R}^d$, endowed with the uniform topology. By [3, Lemma 5], $\{S_n(\cdot), n \in \mathbb{N}\}$ and $\{\tilde{S}_n(\cdot), n \in \mathbb{N}\}$ are exponentially equivalent in $D^d[0,1]$, and thus the LDP for $\{\tilde{S}_n(\cdot), n \in \mathbb{N}\}$ in $\mathcal{C}^d[0,1]$ follows by [4, Lemma 4.1.5]. Moreover (A-1) implies (3.1) (see [3, Remark, page 201]). Thus some independent but not identically sequences (see [3, Corollary 1]), some Markov chains (see [3, Theorem 3]) and some stationary sequences with mixing conditions (see [3, Theorem 4 and 5]) come into consideration.

Remark 3.4. The hypothesis (H2) is guaranteed in many situations. For example, if the $(X_k)_{k\in\mathbb{N}}$ are bounded, it holds. If the $(X_k)_{k\in\mathbb{N}}$ are *i.i.d.*, then (i) implies (ii) (this follows from [4, Lemma 5.1.14]). The hypothesis (i) is certainly guaranteed if the effective domain of Λ^* is finite. The hypothesis (ii), although it is essentially a mixing condition, appears to have no direct connection with the mixing conditions introduced in [3] and in [2].

Let (B_j) be the N-functions of (H2), and write B for the $\mathbb{R}^d$-valued function with coordinates $(B_j)_{j=1}^d$. Recall that if ϕ is absolutely continuous, then $\dot\phi$ exists almost everywhere. Define a space

$$\mathcal{X}_B = \{\phi \in \mathcal{A}^d[0,1] : \; \dot\phi^j \in L_{B_j}, \; j = 1, \dots, d\}.$$

We will consider a variety of topologies on $\mathcal{X}_B$: the uniform topology, which we will denote by τ_u; the topology inherited from the product space $\prod_j L_{B_j}$, which we will denote by τ_B; the corresponding weak topology, τ_B^w; and the smallest topology containing both τ_u and τ_B^w, which we will denote by σ_B. The topology σ_B is the smallest extension of the uniform topology for which the functions

$$\phi \mapsto \int_0^1 \dot\phi^j(s) g(s) ds$$

are continuous for all $g \in L_{B_j}$, and for each $j = 1, \dots, d$. By Lemma 2.2, a sequence ϕ_n converges in this topology if, and only if, it converges uniformly and the sequences $\|\dot\phi_n^j\|_{B_j}$ are bounded.

Theorem 3.5. *If (H1) and (H2) are satisfied then $\{\tilde S_n(\cdot), n \in \mathbb{N}\}$ satisfies the LDP in $(\mathcal{X}_B, \sigma_B)$, with good rate function given by*

$$I(\phi) = \int_0^1 \Lambda^*(\dot\phi) ds.$$

Proof. It follows from (H1) and [4, Lemma 4.1.5] that $\{\tilde S_n(\cdot), n \in \mathbb{N}\}$ satisfies the LDP in the space $\mathcal{X}_B$ when equipped with the relative uniform topology: the effective domain of the rate function $I(\cdot)$, denoted by $\mathcal{D}_I$, is a subspace of $\mathcal{A}^d[0,1]$. To get the LDP on the subspace $\mathcal{X}_B$ we have to check that $\mathcal{D}_I \subset \mathcal{X}_B$. This follows from (H2) and the fact that $L_B = K_B$. To strengthen this to the space $(\mathcal{X}_B, \sigma_B)$ we appeal to the inverse contraction principle [4, Theorem 4.2.4, Corollary 4.2.6], by which it suffices to prove that the sequence $\{\tilde S_n(\cdot), n \in \mathbb{N}\}$ is exponentially tight in this space $(\mathcal{X}_B, \sigma_B)$.

It is convenient to consider the natural imbedding of the weak topology τ_B^w in the space $\mathcal{C}^d[0,1]$. This is complete by the Δ_2-condition. Goodness of the rate function in (H1) implies exponential tightness in the space $\mathcal{C}^d[0,1]$, when equipped with the uniform topology, since this is a Polish space (see, for example, [4, Exercise 4.1.10]). Thus, by Lemma 6.1, we need only check that $\{\tilde S_n(\cdot), n \in \mathbb{N}\}$ is exponentially tight in $(\mathcal{X}_B^d, \tau_B^w)$.

For each $\alpha > 0$, set

$$K_\alpha = \{\phi \in \mathcal{X}_B : \; \max_j \|\dot\phi^j\|_{B_j} \le \alpha\}.$$

By the hypothesis (H2), the spaces L_{B_j} are reflexive and so K_α is compact in $(\mathcal{X}_B^d, \tau_B^w)$. For each $\alpha > 0$, set

$$C_\alpha = \left\{ \phi \in \mathcal{A}^d[0,1] : \; \max_j \int_0^1 B_j(|\dot\phi^j|) ds \le \alpha \right\}.$$

Lemma 3.6. *For α sufficiently large, $C_\alpha \subset K_\alpha$.*

Proof of Lemma 3.6. For α sufficiently large we have, by convexity of the B_j, that

$$\begin{aligned}\max_j \int_0^1 B_j(|\dot\phi^j|)ds \le \alpha &\Rightarrow \max_j \alpha \int_0^1 B_j(|\dot\phi^j|/\alpha)ds \le \alpha \\ &\Rightarrow \max_j \int_0^1 B_j(|\dot\phi^j|/\alpha)ds \le 1 \\ &\Rightarrow \max_j \|\dot\phi^j\|_{B_j} \le \alpha.\end{aligned}$$

□

The exponential tightness of $\tilde S_n(\cdot)$ in $(\mathcal{X}_B, \tau_B^w)$, and hence the statement of the theorem, is thus established by the following lemma.

Lemma 3.7. *If (H2) is satisfied, then*

$$\lim_{\alpha\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\big(\tilde S_n(\cdot) \in C_\alpha^c\big) = -\infty.$$

Proof of Lemma 3.7. By Chebyshev's inequality we have, for $\alpha, \delta > 0$,

$$\begin{aligned}\mathbb{P}(\tilde S_n(\cdot) \in C_\alpha^c) &= \mathbb{P}\left(\max_j \frac{1}{n}\sum_{k=1}^n B_j(|X_k^j|) > \alpha\right) \\ &\le \exp\big(-\delta\alpha n\big) \max_j \mathbb{E}\exp\left(\delta \sum_{k=1}^n B_j(|X_k^j|)\right).\end{aligned}$$

The result is obtained by choosing $\delta > 0$ as in (H2), considering the normalised logarithmic limit as $n \to \infty$, and then letting $\alpha \to \infty$. □

Thus the Theorem is proved. □

To get the LDP for $\{S_n(\cdot), n \in \mathbb{N}\}$ in $(\mathcal{X}_B, \tau_u)$ we have to assume the following exponential tail condition in addition to (H1):

(H3) For each $\gamma > 0$

$$a_n(\gamma) = \sup_{k,m\in\mathbb{N}, k\in[0,n\,m]} \frac{1}{m} \log \mathbb{E}\Big[\exp\Big(\gamma \Big\| \sum_{i=k+1}^{k+m} X_i \Big\|\Big)\Big] < \infty. \qquad (3.8)$$

This is a special case of Assumption (A-2) in [3].

Remark 3.9. As already been remarked in [3, Remark on page 201], the assumption (H3) becomes $a_0(\gamma) < \infty$ for every fixed $\gamma < \infty$, when $(X_i)_{i\in\mathbb{N}}$ is a stationary sequence. For an i.i.d. sequence $(X_i)_{i\in\mathbb{N}}$, this assumption holds as soon as $\|X\|$ has a finite logarithmic moment generating function, which is the standard condition for deriving the LDP of $\{S_n(1), n \in \mathbb{N}\}$.

Corollary 3.10. *If (H1) and (H3) are satisfied then $\{S_n(\cdot), n \in \mathbb{N}\}$ satisfies the LDP in $(\mathcal{X}_B, \tau_u)$, with good rate function given as in Theorem 3.5.*

Proof. It follows from (H1) that $\{\tilde{S}_n(\cdot), n \in \mathbb{N}\}$ satisfies the LDP in $\mathcal{X}_B$, equipped with the relative uniform topology. With Lemma 4.1.5(a) in [4] we obtain the LDP in $D^d[0,1]$, the space of all maps continuous from the right and having left limits, equipped with the uniform topology. Observing that $\sup_{t\in[0,1]} \|S_n(t) - \tilde{S}_n(t)\| = \max_{1\le k\le n} \|X_k\|/n$, it follows that for all $\delta > 0$ and every $\gamma > 0$

$$\mathbb{P}\big(\sup_{t\in[0,1]} \|S_n(t) - \tilde{S}_n(t)\| > \delta\big) \le n\, \exp(-\gamma\, \delta\, n) \max_{0\le k\le n} \mathbb{E}\big[\exp(\gamma \|X_k + 1\|)\big].$$

In view of (H3), the exponential equivalence of $\{\tilde{S}_n(\cdot), n \in \mathbb{N}\}$ and $\{S_n(\cdot), n \in \mathbb{N}\}$ is obtained in the uniform topology by considering the logarithmic limit as $n \to \infty$ and then letting $\gamma \to \infty$. □

Remark 3.11. There is no hope to get a LDP for $\{S_n(\cdot), n \in \mathbb{N}\}$ in $(\mathcal{X}_B, \sigma_B)$, even when we change (H2) by the stronger assumption that (3.2) holds true for every $\delta > 0$. The picewise constant paths of $S_n(\cdot)$ come not into play. To see why, we only like to give some sketchy arguments: Notice that one can proof that if τ_1 and τ_2 are both complete metric topologies on a space $\mathcal{Y}$, and the sequences $\{Z_n, n \in \mathbb{N}\}$ and $\{\tilde{Z}_n, n \in \mathbb{N}\}$ are exponentially equivalent with respect to both topologies (see definition [4, Definition 4.2.10]), then they are exponentially equivalent with respect to $\tau_1 \vee \tau_2$ (the smallest topology containing τ_1 and τ_2). Thus one have only to check that $\{\tilde{S}_n(\cdot), n \in \mathbb{N}\}$ and $\{S_n(\cdot), n \in \mathbb{N}\}$ are exponentially equivalent in $(\mathcal{X}_B^d, \tau_B^w)$. One obtains for every $g \in L_{B_j}$:

$$\mathbb{P}\Big(\Big\| \int_0^1 \dot{S}_n^j(s)\, g(s)\, ds - \int_0^1 \dot{\tilde{S}}_n^j(s)\, g(s)\, ds \Big\| > \delta\Big) = \mathbb{P}\Big(\Big\| \frac{1}{n} \sum_{i=1}^n X_i \int_{\frac{i}{n}}^{\frac{i+1}{n}} g(s)\, ds \Big\| \ge \delta\Big).$$

But this probability is not exponentially small for each $\delta > 0$. (Consider $d = 1$, $B(t) = t^p/p$ with $1 < p < \infty$ and $g(t) = n$).

Remark 3.12. In [6], the LDP for partial sums U-processes was considered in a topology extended by Orlicz functionals.

4. The LDP in Lipschitz spaces

For $\alpha \ge 0$ we denote by $\mathrm{Lip}_\alpha[0,1]$ the space of α-Hölder continuous functions on $[0,1]$, equipped with the norm

$$\|\phi\|_\alpha = \sup_{s\ne t} \frac{|\phi(t) - \phi(s)|}{|t-s|^\alpha}. \tag{4.1}$$

Set $R_j(x) = \Lambda^*(x) \wedge \Lambda^*(-x)$. We will begin by recording some hypotheses. Note that (H5) follows automatically from (H1) if the $(X_k)_{k\in\mathbb{N}}$ are stationary.

(H1a) (H1) holds with Λ lower semi-continuous and differentiable at the origin.

(H4) For each j, there exists $0 \le \beta_j \le 1$ such that

$$\liminf_{r\to\infty} \inf_{0<\tau<1} \tau R_j(r\tau^{\beta_j - 1}) = +\infty.$$

(H5) For each $\epsilon > 0$ and $\theta \in \mathbb{R}^d$, there exists $C > 0$, such that

$$\mathbb{E}\big(\exp\big(\langle\theta, (S_n(t) - S_n(s))\rangle\big)\big) \le C \exp\big(n(t-s)[\Lambda(\theta)+\epsilon]\big),$$

for all $0 \le s < t \le 1$.

Theorem 4.2. *If (H1a), (H4) and (H5) are satisfied and $0 < \alpha_j < \beta_j$ for each j, then the sequence $\{\tilde{S}_n(\cdot), n \in \mathbb{N}\}$ satisfies the LDP in $\prod_j Lip_{\alpha_j}[0,1]$ with good rate function given by*

$$I(\phi) = \int_0^1 \Lambda^*(\dot{\phi}) ds.$$

Proof. First note that (H4) implies

$$\mathcal{D}_I \subset \prod_j \mathrm{Lip}_{\beta_j}[0,1] \subset \prod_j \mathrm{Lip}_{\alpha_j}[0,1].$$

Indeed, if $\|\phi^j\|_{\beta_j} \ge r$, for some j, then

$$\begin{aligned} I(\phi) = \int_0^1 \Lambda^*(\dot{\phi})ds &\ge \int_0^1 \Lambda_j^*(\dot{\phi}^j)ds \\ &\ge \inf_{0<\tau<1} \tau R_j(r\tau^{\beta_j - 1}). \end{aligned} \tag{4.3}$$

We also have, trivially, that

$$\mathbb{P}\bigg(\tilde{S}_n(\cdot) \in \prod_j \mathrm{Lip}_{\alpha_j}[0,1]\bigg) = 1,$$

for all n. Therefore, it suffices to prove that $\{\tilde{S}_n(\cdot), n \in \mathbb{N}\}$ is exponentially tight in the space $\prod_j \mathrm{Lip}_{\alpha_j}[0,1]$. To do this, we consider the sets

$$K(r) = \bigcap_j \{\phi : \|\phi^j\|_{\beta_j} \le r\}.$$

These are compact in $\prod_j \mathrm{Lip}_{\alpha_j}[0,1]$ (*c.f.* [8, Corollary 3.3]). By Chebychev's inequality and (H5) we have, for each $\epsilon > 0$ and for each pair $\theta_1, \theta_2 \in \mathbb{R}$,

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\|\tilde{S}_n^j(\cdot)\|_{\beta_j} \ge r) \\ &\le \limsup_{n\to\infty} \sup_{0<s<t<1} \frac{1}{n} \log \mathbb{P}(|S_n^j(t) - S_n^j(s)| > r(t-s)_j^{\beta}) \\ &\le \left[\sup_{0<s<t<1} [(t-s)\Lambda_j(\theta_1) - r(t-s)^{\beta_j}\theta_1] + \epsilon\right] \\ &\vee \left[\sup_{0<s<t<1} [(t-s)\Lambda_j(\theta_2) + r(t-s)^{\beta_j}\theta_2] + \epsilon\right]. \end{aligned} \tag{4.4}$$

Since this holds for all ϵ we can let $\epsilon \to 0$ to get

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\|\tilde{S}_n^j(\cdot)\|_{\beta_j} \ge r) \\ &\le \sup_{0<\tau<1} [\tau\Lambda_j(\theta_1) - r\tau^{\beta_j}\theta_1] \vee \sup_{0<\tau<1} [\tau\Lambda_j(\theta_2) + r\tau^{\beta_j}\theta_2]. \end{aligned} \tag{4.5}$$

Next we take the infimum over $\theta_1 \geq 0$ and $\theta_2 \leq 0$. To justify the interchange of inf and sup, note that wherever $\theta_1 \geq 0$ and $\Lambda_j(\theta_1)$ is finite,

$$(\tau, \theta_1) \mapsto \tau\Lambda_j(\theta_1) - r\tau^{\beta_j}\theta_1$$

is a closed, proper, concave-convex function (see [12, Corollary 37.3.1]). Here we have used the fact that Λ is lower semi-continuous and proper, by (H1a) (see, for example, [4, Lemma 2.3.9]). A similar argument applies to the second term when $\theta_2 \leq 0$, so we have

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\|\tilde{S}_n^j(\cdot)\|_{\beta_j} \geq r) \\ &\leq \sup_{0<\tau<1} \inf_{\theta_1 \geq 0} [\tau\Lambda_j(\theta_1) - r\tau^{\beta_j}\theta_1] \vee \sup_{0<\tau<1} \inf_{\theta_2 \leq 0} [\tau\Lambda_j(\theta_2) + r\tau^{\beta_j}\theta_2]. \end{aligned} \tag{4.6}$$

Finally, we use that fact (see, for example, [4, Lemma 2.2.5]) that for $x > |\Lambda_j'(0)|$,

$$\Lambda_j^*(x) = \sup_{\theta \geq 0} [\theta x - \Lambda_j(\theta)]$$

and

$$\Lambda_j^*(-x) = \sup_{\theta \leq 0} [\theta x - \Lambda_j(\theta)],$$

to deduce that for r sufficiently large,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\|\tilde{S}_n^j(\cdot)\|_{\beta_j} \geq r) \leq -\inf_{0<\tau<1} \tau R_j(r\tau^{\beta_j - 1}).$$

Exponential tightness now follows from (H4). □

Note that if $\Lambda^*(x) = |x|^2$, then (H4) is satisfied with $\beta_j = 1/2$ for each j. This provides a generalisation of Schilder's theorem and was presented as an exercise in [4, Exercise 5.2.14]. If the effective domain of Λ^* is finite, then (H4) holds with $\beta_j = 1$ for each j.

5. The LDP for sample paths indexed by $\mathbb{R}_+$

Denote by $\mathcal{C}(\mathbb{R}_+)$ the space of continuous functions on $\mathbb{R}_+$ and by $\mathcal{A}(\mathbb{R}_+)$ the subspace of absolutely continuous functions starting at zero. Consider the sequence of partial sums processes $\tilde{S}_n$ defined by (1.3) on the entire half line. Using the Dawson-Gärtner theorem for projective limits, an immediate consequence of (H1) is that the sequence $\{\tilde{S}_n(\cdot), n \in \mathbb{N}\}$ satisfies the LDP in $\mathcal{C}^d(\mathbb{R}_+)$ when equipped with the topology of uniform convergence on compact intervals, with good rate function given by

$$I(\phi) = \begin{cases} \int_0^\infty \Lambda^*(\dot{\phi}) ds & \phi \in \mathcal{A}^d(\mathbb{R}_+) \\ \infty & \text{otherwise.} \end{cases} \tag{5.1}$$

However, as is pointed out by Dobrushin and Pechersky in [5], this topology is not strong enough for many applications. For typical quantities of interest in queueing theory (the steady-state queue-length at a deterministic buffer with inputs given by a real-valued stationary sequence) they introduce a finer topology (a gauge topology). O'Connell shows in [10] that the hypothesis (H1), plus differentiability of Λ at the origin, is sufficient to strengthen the LDP to the following topology: Consider the set of paths

$$\mathcal{Z} = \bigcap_j \left\{ \phi \in \mathcal{C}^d(\mathbb{R}_+) : \lim_{t\to\infty} \frac{\phi^j(t)}{1+t} \text{ exists } \right\},$$

and equip $\mathcal{Z}$ with the norm

$$\|\phi\|_u = \sup_j \sup_t \left| \frac{\phi^j(t)}{1+t} \right|.$$

Note that $\mathcal{Z}$ can be identified with the Polish space $\mathcal{C}^d(\mathbb{R}_+^*)$ of continuous functions on the extended (and compactified) real line, equipped with the supremum norm, via the bijective mapping $\phi(t) \mapsto \phi(t)/(1+t)$. In particular, $\mathcal{Z}$ is a Polish space. O'Connell shows that if (H1) holds, and Λ is differentiable at the origin, then $\{\tilde{S}_n(\cdot), n \in \mathbb{N}\}$ satisfies the LDP in $\mathcal{Z}$ with good rate function given by (5.1). This result provides a new tool for looking at large deviations for queueing systems in equilibrium.

In this section we consider an improvement of O'Connell's result in the i.i.d. case: With $\beta(t) := \sqrt{2(t \vee 1) \log\log(t \vee 3)}$, $t \in \mathbb{R}_+$, we consider the set of paths

$$\mathcal{Y} = \bigcap_j \left\{ \phi \in \mathcal{C}^d(\mathbb{R}_+) : \lim_{t\to\infty} \frac{\phi^j(t)}{\beta(t)} \text{ exists } \right\},$$

and equip $\mathcal{Y}$ with the norm

$$\|\phi\|_\beta = \sup_j \sup_t \left| \frac{\phi^j(t)}{\beta(t)} \right|.$$

Note that $\mathcal{Y}$ can be identified with the Polish space $\mathcal{C}^d(\mathbb{R}_+^*)$ of continuous functions in the same way as $\mathcal{Z}$.

Theorem 5.2. *In the i.i.d. case assume that* (1.1) *holds. Then* $\{\tilde{S}_n(\cdot), n \in \mathbb{N}\}$ *satisfies the LDP in* $\mathcal{Y}$ *with good rate function given by (5.1).*

Proof of Theorem 5.2. We first show that $\mathcal{D}_I \subset \mathcal{Y}$. By the convexity of Λ^*, and Jensen's inequality,

$$t\Lambda^*(\phi(t)/t) \leq I(\phi)$$

for each $t > 0$. For $x \in \mathbb{R}^d$ define the moderate rate function

$$\Lambda_m(x) := \sup_{\lambda \in \mathbb{R}^d} \left\{ \langle \lambda, x \rangle - \frac{\|x\|^2}{2\Lambda''(0)} \right\}.$$

Then for $u > 0$ small we obtain by Taylor-expansion

$$\Lambda^*(u\, x) = \frac{1}{2} u^2 \Lambda_m(x) + o(u^2),$$

and consequently $\Lambda^*\big(u\frac{\phi(t/u)}{t}\big) = \frac{1}{2}u^2\Lambda_m\big(\frac{\phi(t/u)}{t}\big) + o(u^2)$. Remark that there exists some $M < \infty$ with $\Lambda_m(x) \geq \frac{\|x\|^2}{M}$. Hence

$$\frac{t}{u}\Lambda^*\left(\frac{\phi(t/u)}{t/u}\right) \geq \frac{1}{2M}\frac{\|\phi(t/u)\|^2}{t/u} + \frac{t}{u}\,o(u^2).$$

Thus for every $\alpha > 0$ and every ϕ with $I(\phi) \leq \alpha$ we obtain

$$\|\phi(t/u)\|^2 \leq \Big((t/u)\,2\,M\,\alpha - (t/u)^2 o(u^2)\Big)^{1/2}.$$

This implies, that $\lim_{t\to\infty}\big\|\frac{\phi(t)}{\beta(t)}\big\|$ exists and therefore we get $\mathcal{D}_I \subset \mathcal{Y}$.
Next we want to check that $\mathbb{P}\big(\tilde{S}_n \in \mathcal{Y}\big) = 1$: From (1.3) we obtain

$$\tilde{S}_n(t) \leq \sqrt{n}S_n(t) + \frac{nt-[nt]}{\sqrt{n}}X_{[nt]+1}.$$

By the classical Hartman-Wintner LIL we get that $\sqrt{n}S_n(t) \in \mathcal{Y}$ almost surely and the same for $|\frac{nt-[nt]}{\sqrt{n}}X_{[nt]+1}|$.
Now by the the Dawson-Gärtner theorem for projective limits, and [4, Lemma 4.1.5], we have that $\{\tilde{S}_n(\cdot), n \in \mathbb{N}\}$ satisfies the LDP in $\mathcal{Y}$ when equipped with the topology of uniform convergence on compact intervals. To strengthen this to the topology induced by the norm $\|\cdot\|_\beta$, we appeal again to the inverse contraction principle, by which it suffices to prove exponential tightness in the space $(\mathcal{Y}, \|\cdot\|_\beta)$. For each t, denote by $\mathcal{C}^d[0,t]$ the projection of $\mathcal{C}^d(\mathbb{R}_+)$ onto the interval $[0,t]$, equipped with the uniform topology, and by $\phi[0,t]$, for $\phi \in \mathcal{C}^d(\mathbb{R}_+)$, the restriction of ϕ to the interval $[0,t]$. Goodness of the rate function I implies that the sequence $\{\tilde{S}_n[0,1], n \in \mathbb{N}\}$ is exponentially tight in the uniform topology on $\mathcal{A}^d[0,1]$. In other words, for each $\alpha > 0$, there exists a compact set K_α in $\mathcal{A}^d[0,1]$ such that

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\big(\tilde{S}_n[0,1] \notin K_\alpha\big) \leq -\alpha.$$

It follows that for each $t > 0$,

$$K_\alpha(t) := \{\phi \in \mathcal{C}^d[0,t] :\ \{s \mapsto \phi(st)\} \in K_\alpha\}$$

is compact in $\mathcal{C}^d[0,t]$, and for each $0 < \epsilon < \alpha$,

$$\begin{aligned}
\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}&\Big(\bigcup_{t>1}\{\tilde{S}_n[0,t] \notin K_\alpha(t)\}\Big)\\
&\leq \limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\bigcup_{t>1}\{\tilde{S}_{nt}[0,1] \notin K_\alpha(1)\}\Big)\\
&\leq \limsup_{n\to\infty}\frac{1}{n}\log\sum_{k>n}e^{-(\alpha-\epsilon)k}\\
&\leq -\alpha+\epsilon.
\end{aligned} \tag{5.3}$$

Since ϵ is arbitrary we have, for each $\alpha > 0$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Big(\bigcup_{t>1} \{\tilde{S}_n[0,t] \notin K_\alpha(t)\} \Big) \le -\alpha. \tag{5.4}$$

For $\alpha, t > 0$, define $c(\alpha) := \exp(\exp(\alpha))$ and

$$d_\alpha(t) = \begin{cases} \alpha^2 & t \le c(\alpha) \\ (\sqrt{\log\log t})^{-1} & t > c(\alpha) \end{cases}$$

and consider the sets

$$D_\alpha := \bigcap_{j=1}^{d} \left\{ \phi \in \mathcal{Y} : \ \left| \frac{\phi^j(t)}{\beta(t)} \right| \le d_\alpha(t), \text{ for all } t, \ \phi[0,t] \in K_\alpha(t) \text{ for all } t > 1 \right\}.$$

The exponential tightness of $\{\tilde{S}_n(\cdot), n \in \mathbb{N}\}$ in $(\mathcal{Y}, \|\cdot\|_\beta)$ will be established by the following two lemmas.

Lemma 5.5. *For each $\alpha > 0$, D_α is compact in $(\mathcal{Y}, \|\cdot\|_\beta)$.*

Proof. Let ϕ_n be a sequence in D_α. By Tychonoff's theorem, the set $\cap_{t>1} K_\alpha(t)$ is compact in $\mathcal{Y}$ when equipped with the topology of uniform convergence on compact intervals, so there exists a subsequence $n(k)$ such that $\phi_{n(k)}$ converges to some $\phi \in \cap_{t>1} K_\alpha(t)$ in this topology. It follows that, for each $T > 0$, and for each j,

$$\lim_{k\to\infty} \sup_{t\le T} \left| \frac{\phi^j_{n(k)}(t)}{\beta(t)} - \frac{\phi^j(t)}{\beta(t)} \right| = 0.$$

Note that this implies, for each t and j,

$$\left| \frac{\phi^j(t)}{\beta(t)} \right| \le d_\alpha(t),$$

and so $\phi \in D_\alpha$. Now for each $\epsilon > 0$ (sufficiently small), we have for k sufficiently large,

$$\begin{aligned} \|\phi_{n(k)} - \phi\|_\beta \le & \sup_{t \le c(1/\epsilon^2)} \left| \frac{\phi^j_{n(k)}(t)}{\beta(t)} - \frac{\phi^j(t)}{\beta(t)} \right| \\ & + \sup_{t > c(1/\epsilon^2)} \left| \frac{\phi^j_{n(k)}(t)}{\beta(t)} - \frac{\phi^j(t)}{\beta(t)} \right| \le \epsilon + 2 d_\alpha(c(1/\epsilon^2)) = 3\epsilon. \end{aligned} \tag{5.6}$$

The set D_α is therefore sequentially compact, and hence compact, in $(\mathcal{Y}, \|\cdot\|_\beta)$. $\square$

Lemma 5.7. *Under the assumptions of Theorem 5.2 we get*

$$\lim_{\alpha\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\big(\tilde{S}_n \notin D_\alpha\big) = -\infty.$$

Proof. We have for some $\theta > 0$ and for each j:

$$\begin{aligned}
\mathbb{P}\Big(\bigcup_{t\le c(\alpha)}\{|\tilde S_n^j(t)| > c(\alpha)\beta(t)\}\Big) &\le \mathbb{P}\Big(\bigcup_{i=0}^{n-1}\bigcup_{k=0}^{[c(\alpha)]-1}\{|\tilde S_n^j(k+i/n)| > \beta(k)c(\alpha)\}\Big)\\
&\le n\sum_{k=0}^{[c(\alpha)]-1} C(\theta)e^{-\theta n\beta(k)c(\alpha)}\\
&\le nc(\alpha)e^{-\theta n\beta(3)c(\alpha)}.
\end{aligned} \tag{5.8}$$

Here we have used Chebyshev's inequality. It follows that

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\bigcup_{t\le c(\alpha)}\{|\tilde S_n^j(t)| > \beta(t)c(\alpha)\}\Big) \le -\theta\beta(3)c(\alpha). \tag{5.9}$$

We also have, for each j and some $\theta > 0$,

$$\begin{aligned}
\mathbb{P}\Big(\bigcup_{t> c(\alpha)}\{|\tilde S_n^j(t)| > \beta(t)d_\alpha(t)\}\Big) &\le \mathbb{P}\Big(\bigcup_{i=0}^{n-1}\bigcup_{k=[c(\alpha)]}^{\infty}\{|\tilde S_n^j(k+i/n)| > \beta(k)d_\alpha(k)\}\Big)\\
&\le n\sum_{k=[c(\alpha)]}^{\infty} C(\theta)e^{-\theta\, n\,\beta(k)d_\alpha(k)}\\
&\le nC(\theta)De^{-\theta n\sqrt{c(\alpha)-1}/2}.
\end{aligned} \tag{5.10}$$

Here we have used Chebyshev's inequality, and the inequality

$$\sum_{k\ge k_0} e^{-\rho\sqrt{k}} \le De^{-\rho\sqrt{k_0-1}/2}.$$

It follows that

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\bigcup_{t> c(\alpha)}\{|\tilde S_n^j(t)| > \beta(t)d_\alpha(t)\}\Big) \le -\theta\sqrt{c(\alpha)-1}/2. \tag{5.11}$$

The statement can now be obtained from (5.4), (5.9) and (5.11), via the principle of the largest term. $\square$

This conclude s the proof of the Theorem 5.2.
We will now conclude this section with the observation that the LDP holds true with respect to an even finer topology. Consider the space

$$\mathcal{Y}_B = \bigcap_j\{\phi\in\mathcal{Y} :\ \dot\phi^j[0,t]\in L_{B_j}[0,t] \text{ for all } t>0\},$$

where B_j are the N-functions of hypothesis (H2)(i). Denote by $\tau_B^w(t)$ the product of the weak topologies on $\prod_j L_{B_j}[0,t]$, and by $\tau_B^w(\infty)$ the projective limit, as $t\to\infty$, of the $\tau_B^w(t)$. The restriction of $\tau_B^w(\infty)$ to $\mathcal{Y}_B$ is the weakest topology for which the mappings $\phi\mapsto\int_0^t g(s)\dot\phi(s)ds$ are continuous, for all $t>0$ and $g\in\prod_j L_{B_j}[0,t]$.

Now write ω_B for the smallest topology on $\mathcal{Y}_B$ containing the topology of the norm $\|\cdot\|_\beta$ and the restriction of $\tau_B^w(\infty)$.

Theorem 5.12. *If in the i.i.d. case (H2)(i) is satisfied, then $\{\tilde{S}_n(\cdot), n \in \mathbb{N}\}$ satisfies the LDP in $(\mathcal{Y}_B, \omega_B)$ with good rate function given by (5.1).*

Proof. It suffices to prove that $\{\tilde{S}_n(\cdot), n \in \mathbb{N}\}$ is exponentially tight in $(\mathcal{Y}_B, \tau_B^w(\infty))$. This follows from Lemma 6.1, as in the proof of Theorem 3.5: $\tau_B^w(\infty)$ can be thought of as a projective limit of weak topologies, each of which is complete by (H2), and so the natural embedding of $\tau_B^w(\infty)$ on $\mathcal{Y}$ is also complete; by the previous theorem, we have exponential tightness in $\mathcal{Y}$, which is Polish.
The sets

$$K_\alpha = \bigcap_j \{\phi \in \mathcal{Y}_B : \sup_t \frac{1}{\beta(t)} \|\dot{\phi}^j[0,t]\|_{B_j} \leq \alpha\}$$

are compact in $(\mathcal{Y}_B, \tau_B^w(\infty))$, by Tychonoff's theorem. It follows (*c.f.* Lemma 3.6) that, for α sufficiently large, the sets

$$C_\alpha = \bigcap_{j=1}^d \{\phi \in \mathcal{Y}_B : \sup_t \frac{1}{\beta(t)} \int_0^t B_j(|\dot{\phi}^j(s)|) ds \leq \alpha\}$$

are precompact. Now, by (H2), we have for some $\theta > 0$ and for each j:

$$\begin{aligned}
&\mathbb{P}\Big(\bigcup_{t \leq \alpha} \Big\{ \Big| \sum_{k=1}^{[nt]} B_j(|X_k^j|) \Big| > n\,\alpha\beta(t) \Big\} \Big) \\
&\qquad \leq \mathbb{P}\Big(\bigcup_{i=0}^{n-1} \bigcup_{k=0}^{[\alpha)]-1} \Big\{ \Big| \sum_{k=1}^{[n(k+i/n)]} B_j(|X_k^j|) \Big| > \beta(k)\, n\, \alpha \Big\} \Big) \qquad (5.13) \\
&\qquad \leq n \sum_{k=0}^{[\alpha]-1} C(\theta) e^{-\theta n \beta(k) \alpha} \\
&\qquad \leq n\alpha e^{-\theta n \beta(3) \alpha}.
\end{aligned}$$

Analogously to the calculations in (5.13) and (5.10) we obtain

$$\mathbb{P}\Big(\bigcup_{t \geq \alpha} \Big\{ \Big| \sum_{k=1}^{[nt]} B_j(|X_k^j|) \Big| > n\,\alpha\beta(t) \Big\} \Big) \leq nC(\theta) D \exp(-\theta n \sqrt{\alpha - 1}/2) \qquad (5.14)$$

for some constants $C(\theta)$ and D. For α sufficiently large, the result follows as in the proof of Lemma 5.7. □

6. Abstract large deviations result, convergence in L_C

Lemma 6.1. *If τ_1 and τ_2 are both complete metric topologies on a space $\mathcal{Z}$, and the sequence Z_n is exponentially tight with respect to both topologies, then it is exponentially tight with respect to $\tau_1 \vee \tau_2$ (the smallest topology containing τ_1 and τ_2).*

Proof. Let d_1 and d_2 be a pair of complete metrics that respectively generate the topologies τ_1 and τ_2. By hypothesis, for every $\alpha > 0$, there exist τ_i-compact sets K^i_α for which

$$\limsup_{n\to\infty} \frac{1}{n} \log P(Z_n \notin K^i_\alpha) \leq -\alpha.$$

It follows from the principle of the largest term that

$$\limsup_{n\to\infty} \frac{1}{n} \log P(Z_n \notin K^1_\alpha \cap K^2_\alpha) \leq -\alpha.$$

Recall that a set in a complete metric topology is precompact if, and only if, it is totally bounded. The sets $K^1_\alpha \cap K^2_\alpha$ are thus totally bounded in both metrics, and hence in the metric $d_1 \vee d_2$, which generates the topology $\tau_1 \vee \tau_2$. The statement follows. □

Proof of Lemma 2.2. Suppose f_n converges weakly to f in L_C. In other words, for every $g \in L_{C^*}$,

$$\int_0^1 f_n(s)g(s)ds \to \int_0^1 f(s)g(s)ds. \tag{6.2}$$

Then (i) follows from general principles (see, for example, [7, Theorem 4.10.7]) and (ii) follows from (6.2). □

Now suppose that (i) and (ii) hold. To deduce that

$$\int_0^1 f_n(s)g(s)ds \to \int_0^1 f(s)g(s)ds,$$

for every $g \in L_{C^*}$, we need the following lemma.

Lemma 6.3. *If A is an N-function satisfying a Δ_2-condition near infinity, then any $h \in L_A$ can be written as an almost everywhere limit of step functions h_n with $\|h_n - h\|_A \to 0$.*

Proof. Without loss of generality we can assume that h is non-negative. (For general h simply split it into its positive and negative parts.) Then h can be written as a monotone increasing sequence of simple functions (see, for example, [7, Theorem 2.2.5]), which therefore converge in the mean and [1], since A satisfies a Δ_2-condition near infinity, in L_A. Furthermore, each simple function can be written as an almost everywhere limit of step functions, which also converge in the mean, by bounded convergence, and hence in L_A. □

Now (ii) implies that

$$\left| \int f_n g - \int f g \right| \to 0,$$

for all step functions g. For general $g \in L_{C^*}$, write g as a limit of step functions g_m, as in the above lemma, so that $\|g - g_m\|_{C^*} \to 0$. Now we have, applying (2.1) for each m,

$$\begin{aligned}
&\limsup_n \left| \int f_n g - \int fg \right| \\
&= \limsup_n \left| \int f_n g - \int f_n g_m + \int f_n g_m - \int f g_m + \int f g_m - \int fg \right| \\
&\le \limsup_n \left\{ 2\|f_n\|_C \|g - g_m\|_{C^*} + 2\|f\|_C \|g - g_m\|_{C^*} + \left| \int f_n g_m - \int f g_m \right| \right\} \\
&\le 2(\alpha + \|f\|_C)\|g - g_m\|_{C^*}
\end{aligned} \tag{6.4}$$

where $\alpha = \sup_n \|f_n\|_C$, and since this holds for all m we can let m tend to infinity for the result.

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