AUSLANDER-REITEN COMPONENTS FOR LIE ALGEBRAS OF REDUCTIVE GROUPS

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0. Introduction

In this paper we shall study nonperiodic connected components of the stable Auslander-Reiten quiver of a reduced enveloping algebra $u(L, \chi)$ associated to a restricted Lie algebra $(L, [p])$. According to earlier results (cf. [13, 15]) the tree classes of these AR-components are either Euclidean diagrams, or the infinite Dynkin diagrams A_∞, D_∞, A_∞^∞. In the context of finite groups K. Erdmann has recently shown that the latter two trees cannot occur for wild blocks (cf. [12]). As demonstrated in [13, 18], a similar result holds for the restricted enveloping algebra $u(L) := u(L, 0)$ in case the underlying Lie algebra L is nilpotent and of characteristic $p \geq 3$. Here neither tame blocks nor components of tree class A_∞^∞ or D_∞ exist. For $p = 2$ the restricted enveloping algebra of the 2-unipotent Heisenberg algebra possesses components of tree class A_∞^∞. Presently, no Lie algebra admitting components of tree class D_∞ is known, and recent work [19] suggests that such components will be rather exceptional.

In default of a general block theory for enveloping algebras one is led to either employ geometric techniques (cf. [19]), or to focus on those cases, where the block structure of $u(L, \chi)$ is governed by well-understood "linkage principles" (cf. [13, 17, 18]). The former are most effective for Lie algebras of algebraic groups, while the latter for instance play a rôle in the AR-theory of supersolvable Lie algebras.

Our paper can roughly be divided into two parts. The purpose of sections 1 through 3 is to furnish basic properties of components of infinite Dynkin type. By using a modification of a recognition criterion due to K. Erdmann [12], we provide in section 2 general results pertaining to the aforementioned components. Components of tree class A_∞ are discussed in section 3. In particular, we show that such a component contains at most one module of length 2.

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The second part combines these results with geometric techniques in case $L = Lie(G)$ is the Lie algebra of an algebraic group G. In this setting the adjoint representation $Ad : G \longrightarrow GL(L)$ induces for every linear form $\chi \in L^*$ an action of its stabilizer G_χ on the set of stable Auslander-Reiten components of $u(L, \chi)$. Following some technical preparations, we turn in section 5 to the study of AR-components of restricted enveloping algebras for Lie algebras of reductive groups. We focus on invariant components, i.e., those that are fixed by the above action. In view of [20, (2.2)] these are precisely those components whose vertices are Ad-stable in the sense that twisting of the module by the adjoint representation yields an isomorphic module. Components of Euclidean tree class, and those containing indecomposable constituents of rational G-modules belong to this class. Our main results, Theorems 5.2 and 5.5, show that invariant components are of type $\mathbb{Z}[A_\infty]$ or $\mathbb{Z}[\tilde{A}_{12}]$, with the latter occurring only if $SL(2)$ or $PSL(2)$ is a direct factor of G. In particular, $\mathbb{Z}[\tilde{A}_{12}]$ is the only component of Euclidean tree class, a result that contrasts with the absence of such components for group algebras of finite groups of characteristic $p \geq 3$ (cf. [40]). The same methods afford the determination of those blocks of the restricted enveloping algebras of the Lie algebras of reductive groups that possess a simple module of complexity 2 (cf. (5.2)). These turn out to be full matrix algebras over blocks of $u(s\ell(2))$, thereby providing in this context a refinement of Voigt's work (cf. [51]) on tame enveloping algebras. In view of the Morita equivalence given in [26], our results continue to hold for reduced enveloping algebras whose defining linear forms belong to the Zariski dense subset of semisimple elements. However, for the so-called nilpotent linear forms different phenomena are known to occur (cf. [17, 44]).

In the final section we illustrate how recent results by Suslin, Friedlander and Bendel [49] may be employed to study blocks of higher Frobenius kernels of reductive groups. Although some important features of the theory no longer hold in this context, one can still determine the Morita equivalence classes of the tame blocks (cf. (7.1)).

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1. Preliminaries

The purpose of this preparatory section is to modify various ideas of [12] to obtain information concerning modules belonging to components of tree classes A_∞^∞, D_∞ and $\tilde{D}_n$.

Let Λ be a Frobenius algebra, defined over a field F, with Nakayama automorphism $\mu : \Lambda \longrightarrow \Lambda$. Given a Λ-module M and an automorphism $\gamma \in Aut(\Lambda)$, we let $M^{(\gamma)}$ be the module with underlying vector space M and action

$$a \cdot m := \gamma^{-1}(a)\, m \quad \forall\ a \in \Lambda,\ m \in M.$$

For $i \in \mathbb{Z}$, we put $M^{(i)} := M^{(\mu^i)}$. Recall that the stable Auslander-Reiten quiver $\Gamma_s(\Lambda)$ has the isomorphism types $[M]$ of nonprojective indecomposable Λ-modules as its vertices. By general theory (cf. [3, p.138]) the Auslander-Reiten translation τ of $\Gamma_s(\Lambda)$ is given by

$$\tau([M]) = \Omega^2_\Lambda([M^{(1)}]) \quad \forall\ [M] \in \Gamma_s(\Lambda).$$

Here Ω_Λ denotes the Heller operator. Barring possible ambiguities we shall write $\Omega := \Omega_\Lambda$. The reader is referred to [2] concerning basic properties of AR-quivers. An injective homomorphism $g : M \longrightarrow N$ between two indecomposable Λ-modules is called *left properly irreducible* if it is irreducible, but not left almost split. Dually, a surjective homomorphism $g : M \longrightarrow N$ is referred to as *right properly irreducible* if it is irreducible, but not right almost split. We say that a vertex $[M] \in \Theta$ is *located at an end of a component* Θ if $[M]$ has exactly one predecessor in Θ. To every principal indecomposable Λ-module P there corresponds a *standard almost split sequence*

$$(0) \longrightarrow Rad(P) \longrightarrow H(P) \oplus P \longrightarrow P/Soc(P) \longrightarrow (0),$$

where $H(P) := Rad(P)/Soc(P)$ is the *heart* of P. The standard sequence is the only almost split sequence involving P.

The following result slightly modifies a very useful criterion by K. Erdmann (cf. [12, (1.5)]). We formulate it for Frobenius algebras with Nakayama automorphism of finite order. All finite-dimensional Hopf algebras are known to enjoy this property (see [23]).

Proposition 1.1 *Suppose that $ord(\mu) = r < \infty$, and let $g : B \longrightarrow C$ be a right properly irreducible homomorphism between two nonprojective indecomposable modules such that $[C]$ belongs to a component of $\Gamma_s(\Lambda)$ of tree class A^∞_∞, D_∞ or $\tilde{D}_n$. If V is a Λ-module such that there exists a projective module P with $\Omega^2(V) \oplus P \cong V$, then one of the following statements holds:*

(1) There is an embedding $ker\, g \hookrightarrow V^{(i_0)}$ for some $i_0 \in \{0, \ldots, r-1\}$.

(2) Every map $\phi : ker\, g \longrightarrow \Omega^{-1}(V)^{(i)}$ which does not factor through a projective module is a monomorphism $0 \leq i \leq r-1$.

Proof. Put $A := ker\, g$, and consider the short exact sequence

$$(0) \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow (0)$$

of indecomposable Λ-modules. We set $W := \bigoplus_{i=0}^{r-1} V^{(i)}$ and note that there exists a projective module Q such that $\tau(W) \oplus Q \cong W$. According to [14, (3.2)] the function

$$d_W : \Theta \longrightarrow \mathbb{N}_0 \quad ; \quad d_W([X]) = dim_F Ext^1_\Lambda(X, W)$$

is additive. Moreover, we have $d_W \circ \tau = d_W$. If $[C]$ was located at an end of Θ, then the middle term of the almost split sequence terminating in C would have a projective summand P. Thus $C \cong P/Soc(P)$ and $B \cong H(P)$, contradicting the surjectivity of g. Hence $[C]$ is not located at an end of Θ. Suppose $[B]$ to be located at an end. Then Θ has tree class D_∞ or $\tilde{D}_n$, and $[C]$ is the only successor of $[B]$. Since g is surjective, we conclude that the middle term of the almost split sequence originating in $[B]$ has a projective summand P. Consequently, $B \cong Rad(P)$ and the set of successors of $[B]$ in $\Gamma_s(\Lambda)$ is given by the indecomposable constituents of $H(P)$. In particular, $H(P)$ is indecomposable and the original sequence coincides with

$$(0) \longrightarrow Soc(P) \longrightarrow Rad(P) \xrightarrow{g} H(P) \longrightarrow (0).$$

As a result, (2) holds in this case. We may therefore assume that $[B]$ is not located at an end of Θ. It thus follows from [2, (VII.3.4)] and [3, (4.6.3)] that

$$d_W([B]) = d_W([C]).$$

Suppose that (1) fails. Given $i \in \{0, \dots, r-1\}$, we obtain from [2, (V.5.7)] that the map

$$Hom_\Lambda(B, V^{(i)}) \longrightarrow Hom_\Lambda(A, V^{(i)})$$

is surjective. There thus results an exact sequence

$$(0) \longrightarrow Ext^1_\Lambda(C, V^{(i)}) \xrightarrow{g^*_{V^{(i)}}} Ext^1_\Lambda(B, V^{(i)}) \xrightarrow{f^*_{V^{(i)}}} Ext^1_\Lambda(A, V^{(i)}).$$

Additivity of Ext^1_Λ gives rise to an exact sequence

$$(0) \longrightarrow Ext^1_\Lambda(C, W) \xrightarrow{g^*_W} Ext^1_\Lambda(B, W) \xrightarrow{f^*_W} Ext^1_\Lambda(A, W).$$

Since $d_W([B]) = d_W([C])$, the map g^*_W is an isomorphism, so that $f^*_W = 0$. Consequently, $f^*_{V^{(i)}}$ is the zero map $0 \le i \le r-1$.

Let $\phi : A \longrightarrow \Omega^{-1}(V)^{(i)}$ be a map that does not factor through a projective module. Then the class $[\phi] \in Ext^1_\Lambda(A, V^{(i)})$ is non-zero. If ϕ is not a monomorphism, then [2, (V.5.7)] provides a homomorphism $h : B \longrightarrow \Omega^{-1}(V^{(i)})$ with $\phi = h \circ f$. Consequently,

$$[\phi] = [h \circ f] = f^*_{V^{(i)}}([h]) = 0,$$

a contradiction. $\square$

For future reference we also record the dual version of (1.1):

Proposition 1.2 *Suppose that $ord(\mu) = r < \infty$, and let $g : C \longrightarrow B$ be a left properly irreducible homomorphism between two nonprojective indecomposable modules such that $[C]$ belongs to a component of $\Gamma_s(\Lambda)$ of tree class A_∞^∞, D_∞ or $\tilde{D}_n$. If V is a Λ-module such that $\Omega^{-2}(V) \oplus P \cong V$ for some projective module P, then one of the following statements holds:*

(1) There is a surjection $V^{(i_0)} \longrightarrow coker\, g$ for some $i_0 \in \{0, \dots, r-1\}$.

(2) Every map $\phi : \Omega(V)^{(i)} \longrightarrow coker\, g$ which does not factor through a projective module is an epimorphism $0 \le i \le r-1$. $\square$

Throughout the remainder of the paper $(L, [p])$ is assumed to be a finite dimensional restricted Lie algebra, defined over an algebraically closed field F of characteristic $p > 0$. We fix a linear form $\chi \in L^*$ and consider the stable Auslander-Reiten quiver $\Gamma_s(L, \chi)$ of the reduced enveloping algebra $u(L, \chi)$. By definition, the algebra $u(L, \chi)$ is obtained from the ordinary enveloping algebra $\mathcal{U}(L)$ by factoring out the ideal generated by

$$\{x^p - x^{[p]} - \chi(x)^p 1 \ ; \ x \in L\}.$$

We recall that the Nakayama automorphism μ of $u(L,\chi)$ is determined by

$$\mu(x) = x - tr(ad\,x)1 \quad \forall\ x \in L.$$

In particular, μ has finite order $r \in \{1, p\}$ (cf. [21, p.158]).

Given a finite dimensional $u(L,\chi)$-module M, the affine variety

$$\mathcal{V}_L(M) := \{x \in L \ ; \ x^{[p]} = 0 \ \ and \ \ M|_{u(Fx,\chi|_{Fx})} \ is\ not\ projective\ \} \cup \{0\}$$

is the *cohomological support variety* of M (cf. [26, p. 1083]). The dimension of $\mathcal{V}_L(M)$ is known to coincide with the *complexity* $c_{u(L,\chi)}(M)$. By definition, the latter is the growth of a minimal projective resolution of M. Thus,

$$c_{u(L,\chi)}(M) = min\{c \in \mathbb{N}_0 \ ; \ \exists\lambda > 0 \ with\ dim_F \Omega^n(M) \leq \lambda\, n^{c-1} \quad \forall\ n \geq 1\}.$$

The reader is referred to [25, 26] regarding basic properties of support varieties.

Suppose that $K \subset L$ is a p-subalgebra that operates nilpotently on L via the adjoint representation. Given an indecomposable $u(L,\chi)$-module A, a classical result due to Zassenhaus (cf. [48, (I.4.4)]) asserts that there exists a map $\eta : K \longrightarrow F$ such that $\eta(x)$ is the only eigenvalue of x on A. This function is customarily referred to as the *eigenvalue function of A relative to K*. Given a p-subalgebra $K \subset L$ and an algebra homomorphism $\lambda : u(K,\chi|_K) \longrightarrow F$, we denote by F_λ the corresponding one-dimensional $u(K,\chi|_K)$-module. Note that the eigenvalue function η relative to an abelian p-subalgebra $K \subset L$ extends in a unique fashion to a homomorphism $\eta : u(K,\chi|_K) \longrightarrow F$ of F-algebras.

Recall that a p-subalgebra $T \subset L$ is called a *torus* if the p-map is injective on T. Tori are known to be abelian. Given a p-subalgebra $K \subset L$ and a $u(L,\chi)$-module M, we will occasionally write $M|_{u(K,\chi|_K)}$ for the restriction of M to $u(K,\chi|_K)$.

We will require the following subsidiary result:

Lemma 1.3 *Let $T \subset C(L)$ be a central torus. Let A be a nonprojective indecomposable $u(L,\chi)$-module, $x \in \mathcal{V}_L(A) \setminus \{0\}$. If η is the eigenvalue function of A relative to the p-subalgebra $K_x := T \oplus Fx$, then the module $M_x(T,\eta) := u(L,\chi) \otimes_{u(K_x,\chi|_{K_x})} F_\eta$ has the following properties:*

(1) $\mathcal{V}_L(M_x(T,\eta)) = Fx$.

(2) $Ext^1_{u(L,\chi)}(M_x(T,\eta), A) \neq (0) \neq Ext^1_{u(L,\chi)}(A, M_x(T,\eta))$.

(3) $M_x(T,\eta)^{(1)} \cong M_x(T,\eta)$.

Proof. (1), (2). Since the module $M_x(T,\eta)$ is $(u(L,\chi) : u(Fx,\chi|_{Fx}))$-projective, an application of [16, (3.4)] yields $\mathcal{V}_L(M_x(T,\eta)) = \mathcal{V}_{Fx}(M_x(T,\eta)) \subset Fx$.

According to [15, (4.3)] the block $\mathcal{B}_\eta \subset u(K_x,\chi|_{K_x})$ belonging to F_η is a Nakayama algebra. Furthermore, [16, (2.6)] guarantees that $\mathcal{B}_\eta$ is primary and that its principal indecomposable module has dimension p. Since F_η is the only $u(K_x,\chi|_{K_x})$-composition factor of A, this readily implies that

$$A|_{u(K,\chi|_K)} \cong \bigoplus_{i=1}^{p} n_i X_i(\eta),$$

where $X_i(\eta)$ is the unique indecomposable $\mathcal{B}_\eta$-module of dimension i and with top F_η. Consequently,

$$\Omega_{u(K_x,\chi|_{K_x})}(X_i(\eta)) \cong X_{p-i}(\eta) \cong \Omega^{-1}_{u(K_x,\chi|_{K_x})}(X_i(\eta)) \qquad 1 \leq i \leq p-1,$$

so that, observing the fact that A is not $u(K_x, \chi|_{K_x})$-projective, we obtain

$$dim_F Ext^1_{u(L,\chi)}(M_x(T,\eta), A) = dim_F Ext^1_{u(K_x,\chi|_{K_x})}(F_\eta, A) = \sum_{i=1}^{p-1} n_i \neq 0.$$

This verifies the first part of (2) and also shows that $M_x(T,\eta)$ is not a projective $u(L,\chi)$-module. Thus, $\mathcal{V}_L(M_x(T,\eta)) \neq \{0\}$, so that (1) holds.

According to [21, p. 158f] $(u(L,\chi) : u(K_x, \chi|_{K_x}))$ is a Frobenius extension of first kind. Thus, by general properties of Frobenius Extensions (cf. [39, p.96f]),

$$M_x(T,\eta) \cong Hom_{u(K_x,\chi|_{K_x})}(u(L,\chi), F_\eta),$$

and Frobenius reciprocity yields

$$Ext^1_{u(L,\chi)}(A, M_x(T,\eta)) \cong Ext^1_{u(K_x,\chi|_{K_x})}(A, F_\eta).$$

Consequently, the foregoing arguments ensure that the latter space has dimension $\sum_{i=1}^{p-1} n_i$.

(3). As $\mu|_{u(K_x,\chi|_{K_x})} = id_{u(K_x,\chi|_{K_x})}$ the map

$$\psi : M_x(T,\eta) \longrightarrow M_x(T,\eta) \;\; ; \;\; \psi(a \otimes 1) = \mu(a) \otimes 1$$

is readily seen to induce an isomorphism $M_x(T,\eta) \cong M_x(T,\eta)^{(1)}$. $\square$

2. Components of Tree Classes $\mathbf{A}_\infty^\infty$, $\mathbf{D}_\infty$ and $\tilde{\mathbf{D}}_\mathbf{n}$

In this section we employ the criteria of §1 in order to provide subsidiary results concerning kernels and cokernels of properly irreducible homomorphisms of components of tree classes A_∞^∞, D_∞ and $\tilde{D}_n$. Recall that a vertex $[M]$ of $\Gamma_s(L,\chi)$ is said to be located at an end of a component Θ if $[M]$ has exactly one predecessor in $\Gamma_s(L,\chi)$. The following result was inspired by [13, Thm.3].

Proposition 2.1 *Let $g : B \longrightarrow C$ be a right properly irreducible homomorphism between two nonprojective indecomposable modules such that $[C]$ belongs to a component of $\Gamma_s(L,\chi)$ of tree class A_∞^∞, D_∞ or $\tilde{D}_n$. Then either ker g is located at an end of an infinite periodic component, or there exists a principal indecomposable $u(L,\chi)$-module P such that $B \cong Rad(P)$ is located at an end of Θ and $H(P) \cong C$ is indecomposable.*

Proof. We consider the exact sequence

$$(0) \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow (0).$$

Since this sequence does not split, A is not projective and owing to [26, (6.2)] there exists $x \in \mathcal{V}_L(A) \setminus \{0\}$. Let

$$P_1 \longrightarrow P_0 \longrightarrow M_x \longrightarrow (0), \ and \ \ (0) \longrightarrow M_x \longrightarrow E_0 \longrightarrow E_1$$

be the initial terms of a minimal projective and minimal injective resolution of the module $M_x := u(L,\chi) \otimes_{u(Fx,\chi|_{Fx})} F_\chi$, respectively. We put

$$\ell := max\{dim_F P_i \ , \ dim_F E_i \ \ ; \ \ 0 \leq i \leq 1\}$$

and proceed as in $[13, Thm.3]$.

According to (1.3) M_x is a periodic module, and [15, (2.5)] implies that there are projective $u(L,\chi)$-modules P and Q such that

$$\Omega^2(M_x) \oplus P \cong M_x \cong \Omega^{-2}(M_x) \oplus Q.$$

Let $k \geq 0$. Since Ω^k defines a stable equivalence, it follows from [2, (X.1.3)] that $\Omega^k(g)$ is irreducible and thereby either injective or surjective. If $\Omega^k(g) : \Omega^k(B) \longrightarrow \Omega^k(C)$ is surjective, then there exists an exact sequence

$$(0) \longrightarrow \Omega^k(A) \stackrel{\Omega^k(f)}{\longrightarrow} \Omega^k(B) \stackrel{\Omega^k(g)}{\longrightarrow} \Omega^k(C) \longrightarrow (0).$$

If this sequence is almost split, then [2, (X.1.6)] provides a principal indecomposable module P and maps $u : A \longrightarrow P$ and $v : P \longrightarrow C$ such that

$$(0) \longrightarrow A \stackrel{\binom{f}{u}}{\longrightarrow} B \oplus P \stackrel{(g,v)}{\longrightarrow} C \longrightarrow (0)$$

is almost split. In particular, $C \cong P/Soc(P)$ and B is a direct summand of $H(P)$, contradicting the surjectivity of g. Consequently, $\Omega^k(g)$ is right properly irreducible.

Directly from the definition we obtain $\mathcal{V}_L(\Omega^k(A)) = \mathcal{V}_L(A)$, so that (1.3) yields

$$Ext^1_{u(L,\chi)}(\Omega^k(A), M_x) \neq (0).$$

We may now apply (1.1) to see that

$$dim_F \Omega^k(A) \leq \ell.$$

If $\Omega^k(g) : \Omega^k(B) \longrightarrow \Omega^k(C)$ is injective, then general theory provides an exact sequence

$$(0) \longrightarrow \Omega^k(B) \stackrel{\Omega^k(g)}{\longrightarrow} \Omega^k(C) \longrightarrow \Omega^{k-1}(A) \longrightarrow (0).$$

Suppose first that there exists $k > 0$ such that $\Omega^k(g)$ is left almost split. According to [2, (X.1.3)] there exists a projective module P and a map $h : B \longrightarrow P$ such that

$$(0) \longrightarrow B \stackrel{\binom{g}{h}}{\longrightarrow} C \oplus P \longrightarrow \tau^{-1}(B) \longrightarrow (0)$$

is the almost split sequence originating in B. Since g is surjective, it follows that $P \neq (0)$ is a principal indecomposable module and that $B \cong Rad(P)$, $C \cong H(P)$. In particular, B is located at an end of Θ.

Alternatively, $\Omega^k(g)$ is left properly irreducible whenever $\Omega^k(g)$ is injective. Owing to (1.3) we have $Ext^1_{u(L,\chi)}(M_x, \Omega^{k-1}(A)) \neq (0)$. Thus, (1.2) now shows that there either is a surjection $M_x \longrightarrow \Omega^{k-1}(A)$, so that $dim_F \Omega^k(A) \leq dim_F P_0$, or a surjection $\Omega(M_x) \longrightarrow \Omega^{k-1}(A)$, whence $dim_F \Omega^k(A) \leq dim_F P_1$. As a result, the sequence $(dim_F \Omega^n(A))_{n \geq 0}$ is bounded in this case and $c_{u(L,\chi)}(A) = 1$. According to [4, (3.4)] this entails the periodicity of A. Suppose A to be simple. Then [15, (3.2)] implies that the block $\mathcal{B}$ containing A is a Nakayama algebra, a contradiction. By the main theorem of [36] it now follows that A has exactly one successor and is thereby located at an end of a periodic component. If the tree class of the component Ψ containing $[A]$ is a Dynkin diagram, then [2, (VI.1.4)] implies that $\mathcal{B}$ is of finite representation type, a contradiction. Consequently, Ψ has tree class A_∞. $\square$

The dual result reads as follows:

Proposition 2.2 *Let $g : C \longrightarrow B$ be a left properly irreducible homomorphism between two indecomposable modules such that $[C]$ belongs to a component of tree class A_∞^∞, D_∞ or $\tilde{D}_n$. Then either coker g is located at an end of an infinite periodic component or there exists a principal indecomposable $u(L,\chi)$-module P such that $B \cong P/Soc(P)$ is located at an end of Θ and $H(P) \cong C$ is indecomposable.* $\square$

Remark. The foregoing results actually characterize components of tree class A_∞^∞ and D_∞ in the sense they are precisely the nonperiodic, regular components admitting a properly irreducible homomorphism with periodic kernel or cokernel.

We continue by studying periodic kernels and cokernels of irreducible homomorphisms of components of tree classes D_∞ or A_∞^∞.

Lemma 2.3 *Let $\Theta \subset \Gamma_s(L,\chi)$ be a component of tree class A_∞^∞, D_∞, and let $g : B \longrightarrow C$ be a right properly irreducible map with periodic kernel A. Then one of the following statements holds:*

(1) There exists an embedding $A \hookrightarrow \Omega(A)^{(i_0)}$ for some $i_0 \in \{0, \dots, r-1\}$.

(2) $dim_F Ext^2_{u(L,\chi)}(A, A) = 1$.

Proof. We assume property (1) to fail and verify (2). According to [15, (2.2)] the group $Ext^2_{u(L,\chi)}(A, A)$ does not vanish. Let J_A be the radical of the local algebra $End_{u(L,\chi)}(A)$. From [15, (2.5)] we obtain $\Omega^2(A) \cong A$, so that (1.1) applies to $V := \Omega^1(A) \cong \Omega^{-1}(A)$. Accordingly, every non-zero element $\psi \in Ext^2_{u(L,\chi)}(A, A)$ is defined by a monomorphism $\phi : A \longrightarrow A$. Hence the natural map

$$Ext^2_{u(L,\chi)}(A, A) \longrightarrow End_{u(L,\chi)}(A)/J_A$$

is injective, so that $dim_F Ext^2_{u(L,\chi)}(A, A) = 1$. $\square$

We record the dual result:

Lemma 2.4 *Let $\Theta \subset \Gamma_s(L,\chi)$ be a component of tree class A_∞^∞, D_∞, and let $g : C \longrightarrow B$ be a left properly irreducible map with periodic cokernel A. Then one of the following statements holds:*

(1) There exists a surjection $\Omega(A)^{(i_0)} \longrightarrow A$ for some $i_0 \in \{0,\ldots,r-1\}$.

(2) $dim_F Ext^2_{u(L,\chi)}(A,A) = 1$. □

The proof of the following result is essentially contained in [13, (3.3)]. We reproduce it for the sake of completeness. A homomorphism is called *properly irreducible* if it is left properly irreducible or right properly irreducible.

Lemma 2.5 *Let $\Theta \subset \Gamma_s(L,\chi)$ be a component of tree class A_∞^∞, D_∞. Then there exist vertices $[B]$, $[C] \in \Theta \cup \Omega^{-1}(\Theta)$ and a properly irreducible map $\psi : B \longrightarrow C$ such that*

(a) ψ and $\Omega^{-1}(\psi)$ are injective, or

(b) ψ and $\Omega^{-1}(\psi)$ are surjective.

Proof. We first consider the case where Θ has tree class A_∞^∞. Let $[C] \in \Theta$ be a vertex of minimal dimension. Then every irreducible map terminating in C is surjective, while every irreducible map originating in C is injective. In particular, the middle term of the almost split sequence originating in C has no projective summand.

Let $f : B \longrightarrow C$ be an irreducible map. Then $\tau^{-1}(f)$ is "parallel" to a map originating in C. Thus $\tau^{-1}(f)$, and thereby $\Omega^{-2}(f)$, is injective. Since Ω^{-1} is a stable equivalence, $\Omega^{-1}(f)$ is irreducible and thereby either surjective or injective. We may now set $\psi := f$ if $\Omega^{-1}(f)$ is surjective and $\psi := \Omega^{-1}(f)$ in case $\Omega^{-1}(f)$ is injective.

Now suppose that $\Theta \cong \mathbb{Z}[D_\infty]$. Since Θ contains at most finitely many projective meshes, there exits $n_0 \geq 2$ such that no module of quasilength $\geq n_0$ belongs to a projective mesh. Suppose the Lemma fails. Pick an irreducible map $f : A \longrightarrow B$ with quasilength $q\ell(A) \geq n_0$ and $q\ell(B) = q\ell(A)+1$, i.e., an arrow pointing towards infinity. Since the Lemma fails, either $\tau^n(f)$ is surjective for every $n \in \mathbb{Z}$ or $\tau^n(f)$ is injective for every $n \in \mathbb{Z}$. By the same token, passage to $\Omega(\Theta)$ allows us to assume that f is surjective. It now follows from the mesh relations that all arrows pointing towards infinity involving modules of quasilength $\geq n_0$ are surjective, a contradiction. □

Proposition 2.6 *Suppose that $\Theta \subset \Gamma_s(L,\chi)$ is a component of tree class A_∞^∞, D_∞, and let A be the kernel or cokernel of a properly irreducible map satisfying (a) or (b) of (2.5). Then A is periodic and has one of the following properties:*

(1) There exists an isomorphism $\Omega(A) \cong A^{(k)}$ for some $k \in \{0,\ldots,p-1\}$, or

(2) $dim_F Ext^2_{u(L,\chi)}(A,A) = 1$.

Proof. Let $\psi : B \longrightarrow C$ be the properly irreducible map given in (2.5). Without loss of generality we may assume that ψ and $\Omega^{-1}(\psi)$ are surjective. According to (2.1) the module $A := ker\,\psi$ is periodic. Suppose that (2) does not hold. Then (2) also fails for $\Omega(A) \cong \Omega^{-1}(A) \cong ker\,\Omega^{-1}(\psi)$ and (2.3) provides embeddings

$$A \hookrightarrow \Omega(A)^{(i_0)} \;\; ; \;\; \Omega(A) \hookrightarrow A^{(j_0)}.$$

As a result, both maps are isomorphisms and we have $\Omega(A) \cong A^{(k)}$ for some k. □

3. Regular Components of Tree Class $\mathbf{A}_\infty$

In this section we show that components of tree class A_∞ containing a module of length 2 are regular. Thanks to [15, (5.4)] infinite periodic components have this property, so it will suffice to consider nonperiodic components. Given $u(L,\chi)$-modules M and N, we let $\underline{Hom}_{u(L,\chi)}(M,N)$ denote the space of morphisms in the stable module category of $u(L,\chi)$. We also write $\ell(M)$ for the length of M.

Lemma 3.1 *Let $\Theta \subset \Gamma_s(L,\chi)$ be a nonperiodic component, P a principal indecomposable $u(L,\chi)$-module. Then the following statements hold:*

(1) Each τ-orbit of Θ contains at most one module of length ≤ 2.

(2) If $[Rad(P)] \in \Theta$, then the corresponding τ-orbit does not contain modules of length ≤ 2.

Proof. (1). Otherwise, there are indecomposable modules X, Y with $[X] \in \Theta$ and such that $\ell(X), \ell(Y) \leq 2$ and $\Omega^{2n}(X) \cong Y$. By virtue of [17, (3.1)] this implies that X is periodic, a contradiction.

(2). Assume for a contradiction that there exists a $u(L,\chi)$-module M of length $\ell(M) \leq 2$ such that $[M] \in \Theta$ belongs to the τ-orbit of $[Rad(P)]$. Then there exists $n \in \mathbb{Z}$ with

$$\tau^n(M) \cong Rad(P).$$

The assumption $n = 0, 1$ implies $\ell(P) \leq 3$, which contradicts (1). Now suppose that $n \geq 2$. It follows that $\Omega^{2(n-1)}(M) \cong (P/Soc(P))^{(1-n)}$ whence

$$Ext^{2(n-1)}_{u(L,\chi)}(M,M) \cong \underline{Hom}_{u(L,\chi)}(\Omega^{2(n-1)}(M), M) \cong \underline{Hom}_{u(L,\chi)}((P/Soc(P))^{(1-n)}, M).$$

We set $Q := P^{(1-n)}$ for notational convenience. Since $\ell(Q) \geq 3$, we obtain

$$Hom_{u(L,\chi)}(Q,M) \cong Hom_{u(L,\chi)}(Q/Soc(Q), M),$$

with the left-hand space having dimension ≤ 2. If its dimension equals 2, then Q is the projective cover of M and there exists a surjective map $\pi : Q \longrightarrow M$.

Let $\gamma : Q/Soc(Q) \longrightarrow M$ be a map such that $im\,\gamma \subset Rad(M)$. By composing γ with the canonical projection $Q \longrightarrow Q/Soc(Q)$ we obtain a map $\omega : Q \longrightarrow M$ with $im\,\omega \subset Rad(M)$. Since Q is projective, there exists a map $\alpha : Q \longrightarrow Q$ such that $\pi \circ \alpha = \omega$. Consequently,

$$\pi(\alpha(Q)) = \omega(Q) \subset Rad(M),$$

and we have $\alpha(Q) \neq Q$, so that α is not surjective. Hence α is not injective and thereby factors through the projection $Q \longrightarrow Q/Soc(Q)$. Consequently, γ factors through the projective module Q.

The discussion above implies that $dim_F \underline{Hom}_{u(L,\chi)}((P/Soc(P))^{(1-n)}, M) \leq 1$, and we may now apply [15, (2.1)] to conclude that M is periodic, a contradiction.

It remains to consider the case where $n < 0$. In that situation we have

$$Ext^{-2n}_{u(L,\chi)}(M,M) \cong \underline{Hom}_{u(L,\chi)}(M, \Omega^{2n}(M)) \cong \underline{Hom}_{u(L,\chi)}(M, Rad(P)^{(-n)}).$$

By dualizing the arguments of the first part, we see that the former space has dimension ≤ 1, giving again a contradiction. $\square$

The vertices of $\mathbb{Z}[A_\infty]$ of quasilength 1 are usually referred to as *quasisimple*. Thus, the quasisimple modules are those located at an end of $\mathbb{Z}[A_\infty]$. According to [19, (4.1)] every simple module belonging to a component of type $\mathbb{Z}[A_\infty]$ is quasisimple.

Proposition 3.2 *Let $\Theta \cong \mathbb{Z}[A_\infty]$ be a component of $\Gamma_s(L,\chi)$. If Θ contains a vertex $[M]$ of length 2, then the following statements hold:*
(1) $[M]$ is quasisimple.
(2) Θ is regular.
(3) $[M]$ is the only vertex of Θ of length 2.

Proof. (1). Given $[X] \in \Theta$, let $\varphi : X \longrightarrow M$ be an irreducible map. If φ is injective, then X is simple and [19, (4.1)] implies that $[X]$ is quasisimple. There results an almost split sequence

$$(0) \longrightarrow X \longrightarrow M \oplus P \longrightarrow \tau^{-1}(X) \longrightarrow (0)$$

with P projective indecomposable or $P = (0)$. If $P \neq (0)$, then $X \cong Rad(P)$, so that P is uniserial of length 2. It readily follows that X is periodic, a contradiction. Thus, $P = (0)$ and $\tau^{-1}(X)$ is simple, which contradicts (3.1). We conclude that φ is surjective. Dually, every irreducible morphism $\psi : M \longrightarrow X$ is injective.

Let $q\ell(M)$ denote the quasilength of M. If $q\ell(M) \geq 2$, then the mesh relations in Θ together with [19, (4.1)] imply that there exists a principal indecomposable $u(L,\chi)$-module P such that $[H(P)] = [M]$. Thus, P is uniserial and of length 4. In view of [15, (3.1)] we may therefore write $P = U(S, X, Y, S^{(-1)})$, where the entries denote the Loewy factors of P. From $Rad(P) \cong \tau(P/Soc(P)) \cong \Omega^2(P/Soc(P))^{(1)}$ we obtain

$$Ext^2_{u(L,\chi)}(P/Soc(P), P/Soc(P)) \cong \underline{Hom}_{u(L,\chi)}(Rad(P)^{(-1)}, P/Soc(P)).$$

If $Rad(P)^{(-1)}$ is isomorphic to $P/Soc(P)$, then $P/Soc(P)$ is periodic, a contradiction. Consequently, the space

$$Hom_{u(L,\chi)}(Rad(P)^{(-1)}, P/Soc(P)) \cong Hom_{u(L,\chi)}(M^{(-1)}, M)$$

has dimension ≤ 2. Since $P/Soc(P)$ is not periodic, it follows from [15, (2.1)] that

$$\begin{aligned} dim_F \underline{Hom}_{u(L,\chi)}(Rad(P)^{(-1)}, P/Soc(P)) &= dim_F Hom_{u(L,\chi)}(Rad(P)^{(-1)}, P/Soc(P)) \\ &= dim_F Hom_{u(L,\chi)}(M^{(-1)}, M) = 2. \end{aligned}$$

Consequently, $M \cong M^{(1)}$ and $X \cong Y$. The former property implies in particular that $[N] \mapsto [N^{(1)}]$ restricts to an automorphism of Θ of finite order. Since this automorphism commutes with Ω, it also defines an automorphism of $\Omega^{-1}(\Theta)$ of finite order. As $\Omega^{-1}(\Theta) \cong \Theta \cong \mathbb{Z}[A_\infty]$, it follows that

$$N \cong N^{(1)} \qquad \forall\ [N] \in \Theta \cup \Omega^{-1}(\Theta).$$

In particular, $S \cong S^{(-1)}$ so that $P = U(S, X, X, S)$.

Letting $\iota : Rad(P) \hookrightarrow P$, and $\pi : P \longrightarrow P/Soc(P)$ be the natural maps, we readily see that $\pi \circ \iota \neq 0$. This, however, contradicts the above equality of dimensions. Consequently, the vertex $[M]$ is quasisimple.

(2). If Θ is not regular, then [19, (4.1)] implies that there exists a principal indecomposable module P such that $[Rad(P)]$ is quasisimple. Thus, $[M]$ and $[Rad(P)]$ belong to the same τ-orbit. This, however, contradicts (3.1).

(3). This follows directly from (1) and (3.1). $\square$

4. A Basic Criterion for Sums of p-Ideals

In preparation for our investigation of Lie algebras of reductive groups, we collect in this section subsidiary results on Lie algebras that are direct sums of simple Lie algebras. The reader is referred to [46] concerning the definition and basic properties of classical Lie algebras. In virtue of [46, (II.6.1)], every classical Lie algebra is a direct sum of classical simple Lie algebras. Since we shall only be concerned with restricted enveloping algebras, we write $\Gamma_s(L)$ for the stable Auslander-Reiten quiver of $u(L)$. The following example will be used throughout the remainder of this paper.

Example: We consider $L = s\ell(2)$ for $p \geq 3$. It directly follows from [13, 15] and Pollack's work [42, §1] that the nonregular components of $\Gamma_s(s\ell(2))$ are isomorphic to $\mathbb{Z}[\tilde{A}_{12}]$. Let $\mathcal{B} \subset u(s\ell(2))$ be a nonsimple block. Then all projective indecomposable $\mathcal{B}$-modules have length 4. Since $\mathcal{B}$ has two simple modules S and T, say, there are at most 2 nonregular components. On the other hand, we have $Ext^1_{\mathcal{B}}(S, T) \neq (0)$, so that there exists a length 2 module M with top S and socle T. It follows from the structure of the principal indecomposable modules that M is periodic. Hence M defines a subadditive function

$$d_M : \Gamma_s(s\ell(2)) \longrightarrow \mathbb{N}_0 \quad ; \quad d_M([X]) = dim_F Ext^1_{u(s\ell(2))}(\Omega^{-1}(M), X)$$

with $d_M \circ \tau = d_M$, $d_M([S]) = 1$, $d_M([T]) = 0$. Accordingly, $[S]$ and $[T]$ belong to different components, and $\Gamma_s(\mathcal{B})$ possesses exactly 2 nonregular components, Θ and $\Omega(\Theta)$.

Now let M be an indecomposable module with $[M] \notin \Theta \cup \Omega(\Theta)$. In virtue of [14, (3.2)] the function

$$d_M : \Theta \cup \Omega(\Theta) \longrightarrow \mathbb{N}_0 \quad ; \quad d_M([X]) = dim_F Ext^1_{u(s\ell(2))}(M, X)$$

is additive and [52, (2.4)] provides $b \in \mathbb{N}$ such that

$$dim_F Ext^n_{u(s\ell(2))}(M, X) = dim_F Ext^1_{u(s\ell(2))}(M, \Omega^{1-n}(X)) \leq b \quad \forall\ [X] \in \Theta \cup \Omega(\Theta)$$

for every $n \geq 1$. By setting $X = S, T$ and using the long-exact sequence in cohomology, we see that the sequence $(dim_F Ext^n_{u(s\ell(2))}(M, M))_{n \geq 0}$ is bounded. Consequently, M is a periodic module with $[M]$ belonging to a component of type $\mathbb{Z}[A_\infty]/(\tau)$. One can also show that $\Theta \cup \Omega(\Theta)$ is the set of vertices of $\Gamma_s(\mathcal{B})$ having odd length.

By results of Fischer [22] the nonsimple blocks of $u(s\ell(2))$ all have a basic algebra given by the following quiver

$$\begin{tikzcd} \bullet \arrow[r, shift left=3, "x_1"] \arrow[r, shift left=1, "y_1"] & \bullet \arrow[l, shift left=1, "x_2"] \arrow[l, shift left=3, "y_2"] \end{tikzcd}$$

and relations $x_1y_2 = y_1x_2, \quad y_2x_1 = x_2y_1, \quad x_1x_2 = 0, \quad y_1y_2 = 0, \quad x_2x_1 = 0, \quad y_2y_1 = 0.$

Lemma 4.1 *Let $p > 7$ and suppose that $L = K_1 \oplus K_2$ is the direct sum of two p-ideals such that K_1 is simple. Let S be a simple $u(L)$- module, $\mathcal{B} \subset u(L)$ the block of S. If $dim\mathcal{V}_L(S) = 2$ and $\mathcal{V}_{K_1}(S) \neq \{0\}$, then the following statements hold:*

(1) There exist simple modules S_1, S_2 for $u(K_1)$ and $u(K_2)$, respectively such that $S \cong S_1 \otimes_F S_2$. Moreover, S_2 is $u(K_2)$-projective and $dim\mathcal{V}_{K_1}(S_1) = 2$.

(2) $K_1 \cong s\ell(2)$.

(3) There exists a block $\mathcal{C} \subset u(s\ell(2))$ and $n > 0$ such that $\mathcal{B} \cong Mat_n(\mathcal{C})$. In particular, $\mathcal{B}$ is tame and the Auslander-Reiten quiver $\Gamma_s(\mathcal{B})$ is the disjoint union of 2 components of type $\mathbb{Z}[\tilde{A}_{12}]$ and infinitely many infinite 1-tubes.

Proof. (1),(2). Since $u(L)$ is isomorphic to $u(K_1) \otimes_F u(K_2)$, general theory (cf. [10, §10.E]) provides simple modules S_1, S_2 such that

$$S \cong S_1 \otimes_F S_2.$$

The Künneth formula shows that there is an isomorphism

$$Ext^{ev}_{u(L)}(S, S) \cong Ext^{ev}_{u(K_1)}(S_1, S_1) \otimes_F Ext^{ev}_{u(K_2)}(S_2, S_2)$$

of graded algebras. Consequently, the center $\mathcal{Z}(S)$ of the even Yoneda algebra $Ext^{ev}_{u(L)}(S, S)$ is isomorphic to $\mathcal{Z}(S_1) \otimes_F \mathcal{Z}(S_2)$. We may now apply [8, (2.3)], which, according to the proof of [25, (3.2)] also holds in our context, to see that

$$\begin{aligned} dim\mathcal{V}_L(S) &= c_{u(L)}(S) = Krulldim\,\mathcal{Z}(S) = Krulldim\,\mathcal{Z}(S_1) + Krulldim\,\mathcal{Z}(S_2) \\ &= c_{u(K_1)}(S_1) + c_{u(K_2)}(S_2) = dim\mathcal{V}_{K_1}(S_1) + dim\mathcal{V}_{K_2}(S_2). \end{aligned}$$

(One can actually show that $\mathcal{V}_L(S) \cong \mathcal{V}_{K_1}(S_1) \times \mathcal{V}_{K_2}(S_2)$.) Since K_1 operates trivially on S_2, we obtain, observing [24, (2.1)],

$$\mathcal{V}_{K_1}(S) = \mathcal{V}_{K_1}(S_1 \otimes_F S_2) = \mathcal{V}_{K_1}((dim_F S_2)S_1) = \mathcal{V}_{K_1}(S_1).$$

Thanks to the Block-Wilson classification theorem [5], the algebra K_1 is either classical or of Cartan type. In the latter case [19, (2.7)] shows that $dim\mathcal{V}_{K_1}(S_1) \geq 3$, a contradiction. In particular, K_1 is classical simple. The arguments of [19, (2.5)] now yield $K_1 \cong s\ell(2)$ and $dim\mathcal{V}_{K_1}(S_1) = 2$. Consequently, $dim\mathcal{V}_{K_2}(S_2) = 0$, so that S_2 is $u(K_2)$-projective.

(3). For $i \in \{1, 2\}$ let $\mathcal{C}_i \subset u(K_i)$ be the block associated to the simple module S_i. Then $\mathcal{C}_1 \otimes_F \mathcal{C}_2$ is an indecomposable module for the enveloping algebra

$$u(L) \otimes_F u(L)^{op} \cong (u(s\ell(2)) \otimes_F u(s\ell(2))^{op}) \otimes_F (u(K_2) \otimes_F u(K_2)^{op})$$

as well as a direct summand of $u(L)$. Since S_2 is simple and projective, Wedderburn's Theorem shows that $\mathcal{C}_2 \cong Mat_n(F)$ for some $n \geq 1$. Consequently,

$$\mathcal{B} \cong \mathcal{C}_1 \otimes_F \mathcal{C}_2 \cong \mathcal{C}_1 \otimes_F Mat_n(F) \cong Mat_n(\mathcal{C}_1).$$

In particular, $\mathcal{B}$ and $\mathcal{C}_1$ are Morita equivalent, and there is an isomorphism $\Gamma_s(\mathcal{B}) \cong \Gamma_s(\mathcal{C}_1)$. By our example above, the latter quiver has the asserted form and $\mathcal{C}_1$ is tame. $\square$

Remark: If K_1 is known to be classical, then the foregoing result is valid for $p > 3$.

Given a block $\mathcal{B} \subset u(L,\chi)$ and a number $n \in \mathbb{N}_0$, we will say that $\mathcal{B}$ *has complexity* n ($c_{\mathcal{B}} = n$) if n is the minimum of the complexities of all simple $\mathcal{B}$-modules. According to [15, (3.2)] the blocks of complexity ≤ 1 are those of finite representation type. If $\mathcal{B}$ is tame, then a consecutive application of Rickard's Theorem [45, Thm.2] and [15, (3.2)] shows that $c_{\mathcal{B}} = 2$.

A component of the stable Auslander-Reiten quiver is called *regular* if it is not attached to a principal indecomposable module. Accordingly, a nonregular component contains the radical of a principal indecomposable module.

Theorem 4.2 *Let $p > 7$ and suppose that $L = L_1 \oplus \cdots \oplus L_n$ is a direct sum of restricted simple Lie algebras. Then the following statements hold:*

(1) If Θ is nonregular, then $\Theta \cong \mathbb{Z}[\tilde{A}_{12}]$ or $\Theta \cong \mathbb{Z}[A_\infty]$. The latter type occurs if and only if $L \not\cong s\ell(2)$.

(2) If $\Theta \subset \Gamma_s(L)$ is a component of Euclidean tree class, then $\Theta \cong \mathbb{Z}[\tilde{A}_{12}]$, and L is classical with at least one summand isomorphic to $s\ell(2)$.

(3) If $\mathcal{B} \subset u(L)$ is a block of finite representation type, then L is classical and $\mathcal{B}$ is simple.

(4) If $c_{\mathcal{B}} = 2$, then L is classical and there exists a block $\mathcal{C} \subset u(s\ell(2))$ and $n \geq 1$ such that $\mathcal{B} \cong Mat_n(\mathcal{C})$. In particular, $\mathcal{B}$ is tame and $\Gamma_s(\mathcal{B})$ is the disjoint union of 2 components of type $\mathbb{Z}[\tilde{A}_{12}]$ and infinitely many infinite 1-tubes.

Proof. (1),(2). Let Θ be nonregular. According to [52, Thm.A] this is automatically the case if Θ has Euclidean tree class. By definition, Θ is attached to a principal indecomposable module P and the type of the simple module $S := Top(P)$ is contained in $\Omega^{-1}(\Theta) \cong \Theta$. Since $u(L)$ is isomorphic to $\bigotimes_{i=1}^n u(L_i)$, general theory (cf. [10, §10.E]) provides for every $i \in \{1,\ldots,n\}$ a simple $u(L_i)$-module S_i such that

$$S \cong S_1 \otimes_F S_2 \otimes_F \cdots \otimes_F S_n.$$

The arguments of the proof of (4.1) show that

$$dim\mathcal{V}_L(S) = \sum_{i=1}^n dim\mathcal{V}_{L_i}(S_i) = \sum_{i=1}^n dim\mathcal{V}_{L_i}(S).$$

Hence there exists i_0 such that $\mathcal{V}_{L_{i_0}}(S) \neq \{0\}$.

According to [19, (2.6)] the module S is not periodic. Thus [15, (5.7)] implies that the tree class of Θ is either Euclidean, or A_∞^∞, or D_∞, or A_∞. Suppose the tree class of Θ belongs to one of the first three types. In virtue of [19, (2.1)] we then have

$$dim\mathcal{V}_L(S) = 2.$$

As a result (4.1) applies if we define $K_1 := L_{i_0}$ and $K_2 := \bigoplus_{j\neq i_0} L_j$. Thus, $L_{i_0} \cong s\ell(2)$ and since Θ is not a 1-tube, it is isomorphic to $\mathbb{Z}[\tilde{A}_{12}]$. It now follows from the above formula that S_i is $u(L_i)$-projective whenever $i \neq i_0$. In view of [28, (2.4)] this can only happen if L_i is classical.

We conclude the proof of (1) by supposing all nonregular components to have tree class $\tilde{A}_{12}$. Then the component containing the trivial $u(L)$-module has a two-dimensional support variety and

$$2 = dim\mathcal{V}_L(F) = \sum_{i=1}^{n} dim\mathcal{V}_{L_i}(F).$$

Thus, $n = 1$ and $L_1 \cong s\ell(2)$ (cf. [19, (2.5),(2.7)]).

(3),(4). Let S be a simple $\mathcal{B}$-module and write

$$S \cong S_1 \otimes_F S_2 \otimes_F \cdots \otimes_F S_n,$$

where S_i is a simple $u(L_i)$-module $1 \leq i \leq n$. Suppose $\mathcal{B}$ to have finite representation type. Then $dim\mathcal{V}_L(S) \leq 1$, and the formula

$$dim\mathcal{V}_L(S) = \sum_{i=1}^{n} dim\mathcal{V}_{L_i}(S_i) = \sum_{i=1}^{n} dim\mathcal{V}_{L_i}(S)$$

in conjunction with the foregoing observations readily yields the projectivity of S. Consequently, $\mathcal{B}$ is simple. According to [28, (2.4)] the restricted universal enveloping algebras of the Cartan type Lie algebras do not possess simple projective modules. Consequently, each L_i is classical simple.

Finally, let S be a simple $\mathcal{B}$-module of complexity 2. Thanks to the formula above, there exists i_0 such that $\mathcal{V}_{L_{i_0}}(S) \neq (0)$. Consequently, (3) of (4.1) yields the desired result. $\square$

Remark. If L is classical, then the conclusions of (4.2) are valid for $p > 3$.

5. AR-Components of Reductive Lie Algebras

Throughout this section we shall study those components of the stable Auslander-Reiten quiver of the Lie algebra L associated to a connected reductive algebraic group G that are invariant relative to a natural action of G. We assume that $p > 3$.

Let $(L, [p])$ be a restricted Lie algebra with group $Aut_p(L)$ of automorphisms (of the restricted Lie algebra). Given a homomorphism $\pi : G \longrightarrow Aut_p(L)$ of linear algebraic groups, and a linear form $\chi \in L^*$ we consider the stabilizer

$$G_\chi := \{g \in G \; ; \; \chi \circ \pi(g^{-1}) = \chi\}$$

as well as its connected component G_χ°. For a $u(L,\chi)$-module M and $g \in G_\chi$, we write $M^{(g)} := M^{(\pi(g))}$. There results an operation

$$G_\chi \times \Gamma_s(L,\chi) \longrightarrow \Gamma_s(L,\chi) \;\; ; \;\; (g,[M]) \mapsto [M^{(g)}]$$

of G_χ on $\Gamma_s(L,\chi)$. As noted in [32, (2.2)], we have

$$\mathcal{V}_L(M^{(g)}) = \pi(g)(\mathcal{V}_L(M)) \quad \forall\, g \in G_\chi. \tag{1}$$

Let $g \in G_\chi$. The equivalence $M \mapsto M^{(g)}$ induces an automorphism $[M] \mapsto [M^{(g)}]$ of the quiver $\Gamma_s(L,\chi)$. Accordingly, G_χ operates on the set $\mathcal{C}_s(L,\chi)$ of components of $\Gamma_s(L,\chi)$, and for every component $\Theta \subset \Gamma_s(L,\chi)$ and $g \in G_\chi$ we either have

$$\Theta^{(g)} = \Theta$$

or

$$\Theta^{(g)} \cap \Theta = \emptyset.$$

Let G be a connected algebraic group with Lie algebra L, $\chi \in L^*$ a linear form. Considering the adjoint representation $Ad : G \longrightarrow Aut_p(L)$, we call a component $\Theta \subset \Gamma_s(L,\chi)$ *invariant* if $\Theta^{(g)} = \Theta \quad \forall\, g \in G_\chi^\circ$. A $u(L,\chi)$-module M is referred to as *Ad-stable* if $M \cong M^{(g)}$ for every $g \in G_\chi^\circ$. If M is a rational G-module, then the corresponding $u(L)$-module is Ad-stable. Moreover, since G_χ° is connected, every simple $u(L,\chi)$-module is Ad-stable. Evidently, every component containing an Ad-stable vertex is invariant.

We will call the Lie algebra of a (connected) reductive algebraic group *reductive.*

Proposition 5.1 *Let $L = Lie(G)$ be reductive, and suppose that M is an Ad-stable $u(L)$-module such that $dim\, \mathcal{V}_L(M) = 2$. Then the following statements hold:*

(1) There exist a normal subgroups $K, H \subset G$ such that

(a) $G = H \cdot K$,

(b) $L = Lie(H) \oplus Lie(K)$, and

(c) $Lie(K) \cong s\ell(2)$.

(2) If M is indecomposable, then $\mathcal{V}_L(M) = \mathcal{V}_{Lie(K)}(M)$ and M is a projective $u(Lie(H))$-module.

Proof. (1). According to general theory (cf. [30, §27], [47, (9.4)]) we have

$$G = G_1 \cdots G_r \cdot Z(G),$$

where G_i is a closed, connected, normal, almost simple subgroup of G and $Z(G)$ denotes the center of G. Since M is Ad-stable, we see from (1) that $\mathcal{V}_L(M)$ and its projective variety $Proj(\mathcal{V}_L(M))$ are stable under the adjoint representation. Owing to Borel's Fixed Point Theorem there exists $[x_0] \in Proj(\mathcal{V}_L(M))$ that is fixed by a given Borel subgroup $B \subset G$. Consequently, the isotropy group P_0 of $[x_0]$ is parabolic.

The assumption $P_0 = G$ implies that $Fx_0 \subset L$ is invariant under the adjoint representation. Hence Fx_0 is a p-unipotent ideal of L. As L is reductive (cf. [29, (11.8)]), it follows that $x_0 = 0$, a contradiction. Thus, in view of $dim\, Proj(\mathcal{V}_L(M)) = 1$, we have $dim\, P_0 = dim\, G - 1$.

Using the arguments of [6, Prop.13.13] we see that the action of G on G/P_0 induces a surjective homomorphism $\varphi : G \longrightarrow PGL_2$ of algebraic groups. It follows that there exists exactly one index $i_0 \in \{1, \ldots r\}$ such that $\varphi(G_{i_0}) = PGL_2$. Without loss of generality, we have $i_0 = 1$. Thus, setting $K := G_1$ and $H := G_2 \cdots G_r \cdot Z(G)$ we have $dim\, K = 3$ and $H \subset ker\, \varphi \subset P_0$. Since $(G_i, G_j) = \{1\}$ for $i \neq j$, the p-ideals $Lie(K) \cong s\ell(2)$ and $Lie(H)$ commute with each other. Hence $Lie(H) \subset C_L(Lie(K))$, the centralizer of $Lie(K)$ in L. Since $Lie(K)$ is complete (centerless with all derivations being inner), it readily follows that $L = Lie(K) \oplus C_L(Lie(K))$. Moreover, $H \cap K$ is finite, so that

$$dim_F L = dim\, G = dim\, H + dim\, K = dim_F Lie(H) + dim_F Lie(K).$$

This shows that

$$L = Lie(K) \oplus Lie(H).$$

(2). Let M be indecomposable. Owing to [24, (2.2)] the projective variety $Proj(\mathcal{V}_L(M))$ is connected. Since $dim\, \mathcal{V}_L(M) = 2$, we have

$$dim\, (G \cdot [x]) \leq 1 \quad \forall\ [x] \in Proj(\mathcal{V}_L(M)).$$

If $dim\, (G \cdot [x]) = 0$ for some x, then $G \cdot [x] = [x]$ and $Fx \neq (0)$ is a p-unipotent ideal of L, a contradiction. Hence all orbits have dimension 1, so that each of them is closed and thus an irreducible component of $Proj(\mathcal{V}_L(M))$. Since the orbits do not intersect and $Proj(\mathcal{V}_L(M))$ is connected, it follows that $Proj(\mathcal{V}_L(M))$ is irreducible. In particular, we have

$$Proj(\mathcal{V}_L(M)) = G \cdot [x_0] = K \cdot H \cdot [x_0] = K \cdot [x_0].$$

Since H and K commute, this implies that H operates trivially on $Proj(\mathcal{V}_L(M))$.

Let x be an element of $\mathcal{V}_L(M) \setminus \{0\}$. By the observation above there exists a character $\alpha_x : H \longrightarrow F^\times$ such that

$$Ad(h)(x) = \alpha_x(h)x \quad \forall\ h \in H.$$

Since $Z(G)$ operates trivially on L and $G_i = (G_i, G_i)\ \ 2 \leq i \leq r$, it readily follows that $\alpha_x(h) = 1 \quad \forall\ h \in H$. Consequently,

$$[Lie(H), x] = (0).$$

Writing $x = y + z$, where $y \in Lie(K)$ and $z \in Lie(H)$, we have $z^{[p]} = 0$ as well as

$$[L, z] = [Lie(H), z] = [Lie(H), x] = (0).$$

Hence $z \in C(L)$ defines a p-unipotent ideal $Fz \subset L$. Since L is reductive, we conclude that $z = 0$. Consequently, $x = y \in Lie(K)$.

According to [24, (2.1)] we now obtain

$$\mathcal{V}_L(M) = \mathcal{V}_L(M) \cap Lie(K) = \mathcal{V}_{Lie(K)}(M).$$

Thus, $\mathcal{V}_{Lie(H)}(M) = (0)$ and M is $u(Lie(H))$-projective. $\square$

Let $\Theta \subset \Gamma_s(L, \chi)$ be a component. It is easy to see (cf. [15, (5.2)]) that

$$\mathcal{V}_L(M) = \mathcal{V}_L(N) \quad \forall\, [M],\, [N] \in \Theta,$$

so that we can speak of the support variety $\mathcal{V}_L(\Theta)$ of Θ.

As noted before, tame blocks of $u(L, \chi)$ have complexity 2. While the converse does not hold in general, our next result shows its validity for blocks of reductive Lie algebras.

Theorem 5.2 *Let L be a reductive Lie algebra, $\mathcal{B} \subset u(L)$ a block. Then the following statements hold:*

(1) If $\Theta \subset \Gamma_s(\mathcal{B})$ is a nonregular component, then $\Theta \cong \mathbb{Z}[A_\infty]$ or $\Theta \cong \mathbb{Z}[\tilde{A}_{12}]$. In particular, $\Theta \cong \mathbb{Z}[\tilde{A}_{12}]$ is the only component of Euclidean tree class.

(2) If $\mathcal{B}$ has finite representation type, then $\mathcal{B}$ is simple.

(3) If $c_{\mathcal{B}} = 2$, then there exists a block $\mathcal{C} \subset u(s\ell(2))$ and $n \geq 1$ such that $\mathcal{B} \cong Mat_n(\mathcal{C})$. In particular, $\Gamma_s(\mathcal{B})$ is the disjoint union of 2 components of type $\mathbb{Z}[\tilde{A}_{12}]$ and infinitely many 1-tubes.

Proof. (1). Since Θ is not regular, the component $\Psi := \Omega(\Theta)$ is isomorphic to Θ and contains a simple, Ad-stable module S. If $dim\mathcal{V}_L(\Psi) \geq 3$, then [19, (2.1)] yields $\Theta \cong \Psi \cong \mathbb{Z}[A_\infty]$. According to [19, (3.1)] we have $dim\mathcal{V}_L(\Psi) \neq 1$, so that it remains to consider the case where $dim\mathcal{V}_L(S) = dim\mathcal{V}_L(\Psi) = 2$. Thanks to (5.1) Lemma 4.1 applies and the nonperiodic component Θ is isomorphic to $\mathbb{Z}[\tilde{A}_{12}]$.

(2). This follows directly from (1).

(3). The assertion follows from a consecutive application of (5.1) and (4.1). □

Remark. Under the hypotheses of (4.2) or (5.2) a block $\mathcal{B} \subset u(L)$ is tame if and only if $\Gamma_s(\mathcal{B})$ contains a Euclidean component. Moreover, all tame blocks have domestic representation type (cf. [14, (2.1)]).

Corollary 5.3 *Let L be reductive. If the principal block $\mathcal{B}_0 \subset u(L)$ has complexity 2, then there exists a torus $T \subset L$ such that $L \cong s\ell(2) \oplus T$.*

Proof. According to (5.2) the block $\mathcal{B}_0$ is tame, so that Rickard's Theorem [45, Thm.2] yields $dim\, \mathcal{V}_L(F) = 2$. Hence (5.1) provides a decomposition

$$L = K_1 \oplus K_2$$

of L by p-ideals such that $K_1 \cong s\ell(2)$ and $\mathcal{V}_{K_2}(F) = \{0\}$. Consequently, K_2 is a torus (cf. [48, (II.3.10)]). □

Remarks. (1). Corollary 5.3 should be compared with a result by D. Voigt [51, Thm.4], who showed that, for any restricted Lie algebra $(L, [p])$ with center $C(L)$, tameness of $u(L)$ implies $L/C(L) \cong s\ell(2)$.

(2). Suppose that G is solvable and let $\mathcal{B} \subset u(Lie(G))$ be a block. According to [16, (2.2)] $\mathcal{B}$ is isomorphic to the restricted enveloping algebra of a suitably chosen factor algebra of $Lie(G)$, which by virtue of [51, Thm.4] is not tame.

(3). Let G be reductive. Recall that a block $\mathcal{B} \subset u(L)$ is said to be *biserial* if the heart of each principal indecomposable $\mathcal{B}$-module is a direct sum of at most two uniserial modules. We note that, according to [9], biserial algebras are necessarily tame. Let $P \subset \mathcal{B}$ be a principal indecomposable $\mathcal{B}$-module. Since all known tame blocks of reduced enveloping algebras are biserial, it is perhaps of interest to observe the equivalence of the following statements:

(a) $H(P)$ is decomposable

(b) $\mathcal{B}$ is tame

(c) $\mathcal{B}$ is biserial.

Proof. (a) $\Rightarrow$ (b). Suppose $H(P)$ to be decomposable and let Θ be the nonregular component containing $[Rad(P)]$. Then $[Rad(P)]$ is not located at an end of Θ, and a consecutive application of [19, (4.1)] and (5.2) shows that $\Theta \cong \mathbb{Z}[\tilde{A}_{12}]$. Consequently, $dim\, \mathcal{V}_L(Soc(P)) = 2$ and (5.2) shows that the block $\mathcal{B}$ is tame.

(b) $\Rightarrow$ (c). According to [45, Thm.2], (3) of (5.2), and Pollack's work [42] every tame block of $u(L)$ is biserial.

(c) $\Rightarrow$ (a). This follows by applying [9] and (5.2) successively. $\square$

Corollary 5.4 *Let L be as in (4.2) or (5.2). If $\Theta \subset \Gamma_s(L)$ is a component, then $\Theta \cong \mathbb{Z}[A_\infty]/(\tau), \mathbb{Z}[A_\infty], \mathbb{Z}[A_\infty^\infty], \mathbb{Z}[D_\infty]$, or $\mathbb{Z}[\tilde{A}_{12}]$.*

Proof. If L is a direct sum of simple algebras, then $L = [L, L]$ and $u(L)$ is symmetric. Alternatively, we have $L = Lie(G)$, where G is reductive. From the presentation $G = (G, G) \cdot Z(G)$ we conclude that

$$det(Ad(g)) = 1 \qquad \forall\, g \in G$$

and differentiation yields

$$tr(ad\, x) = 0 \qquad \forall\, x \in L.$$

Consequently, $u(L)$ is symmetric in this case as well. Let $\bar{T}_\Theta$ be the tree class of Θ. Suppose that $\bar{T}_\Theta$ is a finite Dynkin diagram. According to [15, (5.7)] it follows that $\Theta \cong \mathbb{Z}[A_\ell]/(\tau)$ is finite, and [2, (VI.1.4)] shows that $u(L)$ possesses a nonsimple block of finite representation type. This, however, contradicts (4.2) or (5.2). In view of [15, (5.7)] it remains to rule out the case, where $\Theta \cong \mathbb{Z}[\tilde{A}_n]$. Owing to a result by Butler and Ringel [7, p.155] such a component is not regular, so the desired conclusion again follows from (4.2) or (5.2). $\square$

In the remainder of this section we will be concerned with the determination of the invariant components of $\Gamma_s(L)$. These arise in the following fashion. Let M be a rational G-module for a connected, reductive group G. The corresponding $u(L)$-module is Ad-stable and decomposes into its indecomposable constituents:

$$M = n_1 M_1 \oplus \cdots \oplus n_r M_r.$$

By the theorem of Krull-Remak-Schmidt an element $g \in G$ induces a permutation $\sigma_g \in S_r$ via

$$M_i^{(g)} := M_{\sigma_g(i)} \qquad 1 \leq i \leq r.$$

There results a homomorphism $\sigma : G \longrightarrow S_r$ of abstract groups. Consequently, $ker\,\sigma$ is a normal subgroup of finite index that contains the center $Z(G)$. It now follows from [30, (29.5)] that σ is trivial. Consequently, each M_i is contained in an invariant component of $\Gamma_s(L)$.

Theorem 5.5 *Let G be a connected, reductive algebraic group with Lie algebra L. If $\Theta \subset \Gamma_s(L)$ is an invariant component, then $\Theta \cong \mathbb{Z}[A_\infty]$ or $\Theta \cong \mathbb{Z}[\tilde{A}_{12}]$.*

Proof. Let $\Theta \subset \Gamma_s(L)$ be invariant. According to [20, (2.2)] every vertex $[M] \in \Theta$ is *Ad*-stable, and [19, (3.1)] implies that $dim\mathcal{V}_L(\Theta) > 1$. In view of (5.4) it therefore suffices to show that $\Theta \not\cong \mathbb{Z}[A_\infty^\infty],\ \mathbb{Z}[D_\infty]$.

We assume $\Theta \cong \mathbb{Z}[A_\infty^\infty],\ \mathbb{Z}[D_\infty]$ and derive a contradiction. Owing to [19, (2.1)] we have $dim\mathcal{V}_L(\Theta) = 2$ and Proposition 5.1 provides a decomposition

$$L = K \oplus H$$

of L into p-ideals such that $K \cong s\ell(2)$ and $\mathcal{V}_L(\Theta) = \mathcal{V}_K(\Theta) = \mathcal{V}_K(F)$.

The component Θ belongs to a block $\mathcal{B} \subset u(L)$. According to general theory there exist blocks $\mathcal{B}_H \subset u(H)$ and $\mathcal{B}_K \subset u(K)$ with $\mathcal{B} \cong \mathcal{B}_K \otimes_F \mathcal{B}_H$.

By our discussion of $u(s\ell(2))$, the block $\mathcal{B}_K$ possesses exactly two simple modules $S_0,\ S_1$, whose projective covers will be denoted $P_0,\ P_1$, respectively. Moreover, there are two non-periodic components $\Psi_0,\ \Psi_1 \subset \Gamma_s(\mathcal{B}_K)$, both of which are isomorphic to $\mathbb{Z}[\tilde{A}_{12}]$. We choose the indices so that $[Rad(P_i)] \in \Psi_i$. Thus, $[S_i] \in \Psi_{1-i}\ \ 0 \leq i \leq 1$.

Given a subalgebra $\Lambda \subset u(L)$ and a $u(L)$-module M, we will denote by $M|_\Lambda$ the restriction of M to Λ.

(a) *Every vertex $[M] \in \Theta$ is $u(H)$-projective, and $\mathcal{B}_H$ is not simple.*
Let $[M] \in \Theta$ be a vertex. Then we have, observing [24, (2.1)],

$$\mathcal{V}_H(M) = \mathcal{V}_L(M) \cap H = \mathcal{V}_L(\Theta) \cap H = \{0\},$$

so that $M|_{u(H)}$ is projective. If $\mathcal{B}_H$ is simple, then Wedderburn's Theorem provides $n > 0$ such such $\mathcal{B} \cong Mat_n(\mathcal{B}_K)$. This implies that $\Theta \cong \mathbb{Z}[\tilde{A}_{12}]$, a contradiction.

(b) *Let $[M] \in \Theta$. Then we have*

$$M|_{u(K)} \cong \bigoplus_{[N] \in \Psi_0 \cup \Psi_1} n_N(M) N \oplus P_M,$$

where P_M is a projective $\mathcal{B}_K$-module.
We decompose

$$M|_{u(K)} = n_1 N_1 \oplus \cdots \oplus n_q N_q \oplus P_M,$$

into a direct sum of $u(K)$-modules, where N_i is nonprojective indecomposable $1 \leq i \leq q$ and P_M is projective. Let e_K and e_H be the central idempotents defining $\mathcal{B}_K$ and $\mathcal{B}_H$, respectively. Then $e_K \otimes e_H$ is the central idempotent defining $\mathcal{B}$. Since $e_K \otimes e_H = (1 \otimes e_H)(e_K \otimes 1)$, it follows that $e_K \otimes 1$ operates invertibly on M. As a result, all constituents of M belong to $\mathcal{B}_K$.

According to (5.1) there exists a normal, connected, quasi-simple subgroup $\tilde{G} \subset G$ such that $K = Lie(\tilde{G})$. Since $M|_{u(K)}$ is Ad-stable relative to $\tilde{G}$ the arguments preceding (5.5) show that each N_i is Ad-stable relative to $\tilde{G}$. In particular, each variety $\mathcal{V}_K(N_i)$ is $\tilde{G}$-invariant and $dim\,\mathcal{V}_K(N_i) = 2$. As a result, $[N_i] \in \Psi_0 \cup \Psi_1 \quad 1 \leq i \leq q$.
The arguments of (b) also show the validity of

(c) *Let $[M] \in \Theta$. Then every indecomposable constituent of $M|_{u(H)}$ belongs to $\mathcal{B}_H$.*

We let $U(X,Y)$ denote a uniserial $\mathcal{B}_K$-module with top X and socle Y. From the structure of the P_i (cf. [42, §1]) we get

$$Ext^1_{u(K)}(S_i, S_{1-i}) \neq (0) \qquad 0 \leq i \leq 1,$$

as well as

$$\Omega_{u(K)}(U(S_i, S_{1-i})) \cong U(S_{1-i}, S_i).$$

Thus, by [17, (3.1)] every uniserial module of length 2 is periodic, and we have

$$dim_F Ext^1_{u(K)}(U(S_i, S_{1-i}), S_j) = \delta_{(1-i),j} \;\; ; \;\; dim_F Ext^1_{u(K)}(S_i, U(S_j, S_{1-j})) = \delta_{ij} \quad 0 \leq i,j \leq 1.$$

Consequently,

$$Ext^1_{u(K)}(U(S_i, S_{1-i}), X) \neq (0) \quad \forall\, [X] \in \Psi_i, \tag{2}$$

and

$$Ext^1_{u(K)}(X, U(S_i, S_{1-i})) \neq (0) \quad \forall\, [X] \in \Psi_{1-i}. \tag{3}$$

Let $\psi : B \longrightarrow C$ be a properly irreducible homomorphism in Θ. We set $A := ker\,\psi$ or $A := coker\,\psi$ according as ψ is surjective or injective.

(d) *The module $A|_{u(H)}$ is projective and belongs to $\mathcal{B}_H$.*
We assume that ψ is surjective. There results exact sequence

$$(0) \longrightarrow A|_{u(H)} \longrightarrow B|_{u(H)} \longrightarrow C|_{u(H)} \longrightarrow (0),$$

which, according to (a), is split exact. In particular, $A|_{u(H)}$ is, as a direct summand of the projective $\mathcal{B}_H$-module $B|_{u(H)}$, projective.

(e) *Suppose that ψ is surjective. There exists a principal indecomposable $\mathcal{B}_H$-module P and $i \in \{0,1\}$ such that $A \hookrightarrow U(S_i, S_{1-i}) \otimes_F P$.*
Let P be an arbitrary principal indecomposable $u(H)$-module. Given $[X] \in \Theta$, we have, setting $M_i := U(S_{1-i}, S_i)$

$$\begin{aligned} Ext^1_{u(L)}(X, M_i \otimes_F P) &\cong H^1(u(K), Hom_{u(H)}(X, M_i \otimes_F P)) \\ &\cong H^1(u(K), M_i \otimes_F Hom_{u(H)}(X, P)) \\ &\cong H^1(u(K), M_i \otimes_F Hom_{u(H)}(P^*, X^*)) \\ &\cong Ext^1_{u(K)}(M_i^*, Hom_{u(H)}(P^*, X^*)). \end{aligned}$$

Since $S_i \cong S_i^* \quad 0 \leq i \leq 1$, it follows that $M_i^* \cong U(S_i, S_{1-i})$ is uniserial with top S_i and socle S_{1-i}. By the same token, if N is an indecomposable $u(K)$-module belonging to $\mathcal{B}_K$, then N^* has the same property. Moreover,

$$[N^*] \in \Psi_i \quad \forall\, [N] \in \Psi_i \quad 0 \leq i \leq 1.$$

It now follows directly from (b) and (2) that there exists $i \in \{0,1\}$ and a principal indecomposable $u(H)$-module P such that

$$Ext^1_{u(L)}(X, M_i \otimes_F P) \neq (0).$$

Since $M_i \otimes_F P$ is periodic a consecutive application of [15, (2.5)] and (1.1) shows that $A \hookrightarrow U(S_{1-i}, S_i) \otimes_F P$ or $A \hookrightarrow U(S_i, S_{1-i}) \otimes_F P$. In view of (d), this implies $A|_{u(H)} \cong \ell P$, so that P belongs to $\mathcal{B}_H$.

We also record the dual statement:

(f) *Suppose that ψ is injective. There exists a principal indecomposable $\mathcal{B}_H$-module P and $i \in \{0,1\}$ such that there a surjection $U(S_i, S_{1-i}) \otimes_F P \longrightarrow A$.*

According to (2.5) we can pick a properly irreducible map $\psi : B \longrightarrow C$ such that $[B],\ [C] \in \Theta \cup \Omega^{-1}_{u(L)}(\Theta)$ and with the property that either $\psi,\ \Omega^{-1}_{u(L)}(\psi)$ are surjective or $\psi,\ \Omega^{-1}_{u(L)}(\psi)$ are injective. Assume ψ to be surjective and let $A := ker\psi$.

(g) *There exists a principal indecomposable $\mathcal{B}_H$-module P and $i \in \{0,1\}$ such that $A \cong U(S_i, S_{1-i}) \otimes_F P$.*

According to (e) there exists an embedding

$$A \hookrightarrow U(S_i, S_{1-i}) \otimes_F P.$$

By choice of ψ, $\Omega^{-1}_{u(L)}(A)$ is the kernel of the irreducible map $\Omega^{-1}_{u(L)}(\psi)$. Since this map belongs to the component $\Omega^{-1}_{u(L)}(\Theta) \cong \mathbb{Z}[A^\infty_\infty],\ \mathbb{Z}[D_\infty]$ another application of (e) provides an embedding

$$\Omega^{-1}_{u(L)}(A) \hookrightarrow U(S_j, S_{1-j}) \otimes_F Q.$$

Hence there are embeddings

$$A \hookrightarrow P_{1-i} \otimes_F P \ \ ; \ \ \Omega^{-1}_{u(L)}(A) \hookrightarrow P_{1-j} \otimes_F Q.$$

Owing to (d) the first isomorphism yields $A|_{u(H)} \cong \ell P$ and $\Omega^{-1}_{u(L)}(A)|_{u(H)} \cong kP$, while the second implies $\Omega^{-1}_{u(L)}(A)|_{u(H)} \cong sQ$. Hence $P \cong Q$, and we conclude

$$\ell(A) \leq 2\,\ell(P) \ \ ; \ \ \ell(\Omega^{-1}_{u(L)}(A)) \leq 2\,\ell(P),$$

while

$$\ell(A) + \ell(\Omega^{-1}_{u(L)}(A)) = 4\,\ell(P).$$

Consequently, $\ell(A) = 2\,\ell(P) = \ell(U(S_i, S_{1-i}) \otimes_F P)$, and $A \cong U(S_i, S_{1-i}) \otimes_F P$.

We are now in a position to derive the desired contradiction. Since

$$\Omega_{u(L)}(A) \cong \Omega_{u(K)}(U(S_i, S_{1-i})) \otimes_F P) \cong U(S_{1-i}, S_i) \otimes_F P,$$

it follows that $\Omega_{u(L)}(A) \not\cong A$. On the other hand, the Künneth formula yields

$$Ext^2_{u(L)}(A, A) \cong Ext^2_{u(K)}(U(S_i, S_{1-i}), U(S_i, S_{1-i})) \otimes_F Hom_{u(H)}(P, P).$$

As $\mathcal{B}_H$ is symmetric and not simple, we have $dim_F Hom_{u(H)}(P,P) \geq 2$. Consequently, $dim_F Ext^2_{u(L)}(A,A) \geq 2$. This, however, contradicts (2.6) and the proof is complete. $\square$

Remark. Let $L = H \oplus K$ be given as in (5.5). Let $\Theta \subset \Gamma_s(L)$ be a nonperiodic component containing a $(u(L) : u(K))$-projective vertex $[M]$. By definition, there exists an indecomposable $u(K)$-module N such that M is a direct summand of $u(L) \otimes_{u(K)} N$. If $u(H)$ has a decomposition $u(H) = \bigoplus_{i=1}^r n_i P_i$ into principal indecomposable modules, then we have

$$u(L) \otimes_{u(K)} N \cong u(H) \otimes_F N \cong \bigoplus_{i=1}^{r} n_i(P_i \otimes_F N).$$

Thus, $M \cong P_i \otimes_F N$ for some i. Let $\mathcal{B}_K \subset u(K)$ be the block of N. Since M is not periodic, N has full support and thus belongs to $\Psi_0 \cup \Psi_1$. This also shows that M is *Ad*-stable. By applying the Heller operator we see that there are only finitely many components containing a nonperiodic indecomposable $(u(L) : u(K))$-projective module. According to (5.5) we have $\Theta \cong \mathbb{Z}[A_\infty]$. Moreover, by applying the techniques of [27] one can show that every $(u(L) : u(K))$-projective vertex is quasisimple.

Suppose that $P := P_i$ is not simple. If Θ is not regular, then $\Omega(\Theta)$ contains a simple module $S \otimes_F T$. Since

$$\mathcal{V}_L(S \otimes_F T) = \mathcal{V}_H(S) \times \mathcal{V}_K(T) = \{0\} \times \mathcal{V}_K(N),$$

it follows that S is a projective $u(H)$-module. Thus, the block $\mathcal{B}_H$ of P is simple and $P \cong S$ is also simple, a contradiction.

Given $d > 0$, we let $\mathcal{C}_s(L,d)$ denote the set of those components of $\Gamma_s(L)$ that contain a vertex of dimension d. We also put $A_\infty(L,d) := \{\Theta \in \mathcal{C}_s(L,d) \ ; \ \Theta \cong \mathbb{Z}[A_\infty]\}$.

Corollary 5.6 *Let G be connected and reductive with Lie algebra L. Suppose there is $d > 0$ and a block $\mathcal{B} \subset u(L)$ such that $\mathcal{C}_s(L,d) \cap \Gamma_s(\mathcal{B})$ is finite. Then either $\mathcal{C}_s(L,d) \cap \Gamma_s(\mathcal{B}) \subset A_\infty(L,d)$ or $\Theta \cong \mathbb{Z}[\tilde{A}_{12}] \ \ \forall\ \Theta \in \mathcal{C}_s(L,d) \cap \Gamma_s(\mathcal{B})$. In the latter case $SL(2)$ or $PSL(2)$ is a direct factor of G.*

Proof. Thanks to [20, (2.2)] every component $\Theta \in \mathcal{C}_s(L,d) \cap \Gamma_s(\mathcal{B})$ is invariant, so that (5.5) yields $\Theta \cong \mathbb{Z}[A_\infty],\ \mathbb{Z}[\tilde{A}_{12}]$. If there exists $\Theta \in \mathcal{C}_s(L,d) \cap \Gamma_s(\mathcal{B})$ of the latter type, then $\mathcal{B}$ is Morita equivalent to a tame block of $u(s\ell(2))$, and all invariant components of $\Gamma_s(\mathcal{B})$ have tree class $\tilde{A}_{12}$ (cf. (5.2)). Moreover, (5.1) shows that $SL(2)$ or $PSL(2)$ is a direct factor of G. $\square$

Let $\mathcal{N}$ be the set of those components $\Theta \subset \Gamma_s(L)$, such that Θ or $\Omega(\Theta)$ is not regular. The second part of the foregoing remark establishes, for non-simple algebraic groups, the existence of invariant components not belonging to $\mathcal{N}$. In case $G = SL(2)$ the example of §4 shows that every invariant component belongs to $\mathcal{N}$. Our final result of this section illustrates that this case is exceptional.

Proposition 5.7 *Let G be an almost simple, connected, simply connected algebraic group of rank ≥ 3. Then $\Gamma_s(Lie(G))$ possesses a regular invariant component.*

Proof. Let $T \subset G$ be a maximal torus. Following the notation of [33] we denote the simple G-module of dominant highest weight λ by $L(\lambda)$.

Since G has rank $r \geq 3$ its Dynkin diagram $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ contains a subdiagram $I := \{\alpha_i, \alpha_j\}$ of type A_2. We let $\{\omega_1, \ldots, \omega_r\}$ denote the set of fundamental weights and consider the Weyl module $V(\lambda)$ of highest weight $\lambda := (p-2)\omega_i + \omega_j$. Since λ is "restricted", $L(\lambda)$ is a simple $u(Lie(G))$-module (cf. [33, (II.3.15)]). Direct computation shows that $\mu := \lambda - \alpha_i - \alpha_j$ is also restricted.

Let $P_I = L_I \cdot U_I$ be the Levi decomposition of the standard parabolic subgroup P_I associated to $I \subset \Delta$. Since G is simply connected, we have $L_I^{(1)} \cong SL(3)$. According to [33, II.2.11]

$$L(\lambda)^{U_I} = \bigoplus_{\nu \in \mathbb{Z}I} L(\lambda)_{\lambda-\nu}$$

is the simple L_I-module with highest weight λ. This module can be realized as the p-th homogeneous part of $F[X, Y, Z]/(X^p, Y^p, Z^p)$, whence

$$dim_F L(\lambda)_{\lambda-\nu} \leq 1 \quad \forall\ \nu \in \mathbb{Z}I.$$

On the other hand, Freudenthal's formula shows that

$$dim_F V(\lambda)_\mu = 2 \ \ ; \ \ dim_F V(\lambda)_{\lambda-\alpha_k} = 1 \quad k \in \{i, j\}.$$

Consequently, $Rad_G(V(\lambda))$ contains a primitive vector of weight μ, and $L(\mu)$ is a top composition factor of $Rad_G(V(\lambda))$. According to [33, (II.2.14)] we thus have

$$Ext^1_G(L(\lambda), L(\mu)) \cong Hom_G(Rad_G(V(\lambda), L(\mu)) \neq (0).$$

It now follows from [1, (5.2)] that the canonical restriction map

$$Ext^1_G(L(\lambda), L(\mu)) \longrightarrow Ext^1_{u(Lie(G))}(L(\lambda), L(\mu))$$

is injective. As a result, there exists an indecomposable, Ad-stable $u(Lie(G))$-module M of length 2. Let Θ be the component of $\Gamma_s(Lie(G))$ containing $[M]$. Then Θ is invariant and since G has rank ≥ 3, [19, (3.2)] shows that $\Theta \cong \mathbb{Z}[A_\infty]$.

We may now apply (3.2) to obtain the desired result. $\square$

Remark. In view of the results of [37, 53] the conclusion of (5.7) obtains also for groups of types C_2 and G_2 (and hence for all groups of rank ≥ 2) provided $p \geq 13$. Under this hypothesis the tables of [53, p.54] and [37, p.1913] provide restricted weights λ, μ such that $Ext^1_G(L(\lambda), L(\mu))$ has dimension 1.

6. Semisimple Linear Forms

In this section we employ a basic result due to Friedlander and Parshall [26, (3.2)] in order to extend the results of the previous section to semisimple linear forms.

Let $L = Lie(G)$ be the Lie algebra of a reductive algebraic group G whose derived group (G,G) is simply connected. We will also assume that $p = char(F)$ is good for L. For a Borel subalgebra $B \subset L$ and a maximal torus $T \subset B$, we consider the corresponding triangular decomposition

$$L = L^- \oplus L_0 \oplus L^+,$$

where $L_0 = T$ and L^+ is the p-unipotent radical of B. Recall that a linear form $\chi \in L^*$ is referred to as *semisimple* if there exists an element $g \in G$ such that $(g \cdot \chi)(L^- \oplus L^+) = (0)$. According to [35, (3.2)] the set of semisimple linear forms lies dense in L^*.

Let $\chi \in L^*$ be semisimple. Owing to ([35, (3.1)], [43, (3.1)]) the stabilizer $G_\chi \subset G$ is a reductive subgroup with Lie algebra

$$Lie(G_\chi) = L_\chi := \{x \in L^* \ ; \ x \cdot \chi = 0\}$$

and [26, (3.2)] shows that the algebras $u(L,\chi)$ and $u(L_\chi)$ are Morita equivalent (see also [34, (7.4)]). More precisely, there exists a parabolic subalgebra $L_\chi \subset P \subset L$ such that the induction functor $X \mapsto u(L,\chi) \otimes_{u(P,\chi|_P)} X$ is an equivalence between the module categories of $u(L_\chi, \chi|_{L_\chi})$ and $u(L,\chi)$. In addition, the former algebra is isomorphic to $u(L_\chi)$.

Theorem 6.1 *Let L be the Lie algebra of a reductive, connected algebraic group G such that (G,G) is simply connected and $p > 3$ is good for L. For a semisimple linear form $\chi \in L^*$ let $\mathcal{B} \subset u(L,\chi)$ be a block and $\Theta \subset \Gamma_s(L,\chi)$ a component.*

(1) If Θ is invariant, then $\Theta \cong \mathbb{Z}[A_\infty]$ or $\Theta \cong \mathbb{Z}[\tilde{A}_{12}]$.

(2) $\Theta \cong \mathbb{Z}[A_\infty]/(\tau)$, $\mathbb{Z}[A_\infty]$, $\mathbb{Z}[A_\infty^\infty]$, $\mathbb{Z}[D_\infty]$ or $\mathbb{Z}[\tilde{A}_{12}]$.

(3) If $\mathcal{B}$ is representation-finite, then $\mathcal{B}$ is simple.

(4) If $\mathcal{B}$ has complexity 2, then $\mathcal{B}$ is Morita equivalent to a tame block of $u(s\ell(2))$.

Proof. Statements (2), (3), and (4) readily follow from (5.2), (5.4) and the above observations. With regard to (1) it suffices to note that the above equivalence of categories induces an isomorphism $\Gamma_s(L,\chi) \cong \Gamma_s(L_\chi)$ that is G_χ-equivariant. Hence Theorem 5.5 yields the desired conclusion. $\square$

Remark. For arbitrary linear forms our methods break down. If $\chi \in L^*$ is regular nilpotent, then a result by Springer (cf. [31, (1.14)]) shows that the connected component of G_χ is abelian and unipotent. In that case [17, (4.7)] provides tame blocks of $u(L,\chi)$ that are isomorphic to full matrix algebras of the Kronecker algebra $F[X,Y]/(X^2,Y^2)$. Moreover, using [38] one can construct representation-finite, non-simple blocks for Lie algebras of arbitrarily large semisimple rank. By contrast, if all blocks of $u(L,\chi)$ are representation-finite, then [44, (5.3)] shows that the semisimple rank of L is bounded.

7. Blocks of Higher Frobenius Kernels

In this final section we provide a partial generalization of (5.2) to higher Frobenius kernels. As before we let G be a connected, reductive group, defined over an algebraically closed field

F of characteristic $p > 3$. The r^{th} Frobenius kernel of G will be denoted G_r. Following [33] we let $G_{a,r}$ be the r^{th} Frobenius kernel of the additive group G_a. We will also write $H(G_r)$ for the group algebra of the infinitesimal F-group G_r. General theory (cf. [50, p.40ff]) tells us that $H(G_1) \cong u(Lie(G))$.

Theorem 7.1 *Let G be a connected reductive group, $\mathcal{B} \subset H(G_r)$ a block.*
(1) If $\mathcal{B}$ is representation-finite, then $\mathcal{B}$ is simple.
(2) If $\mathcal{B}$ is tame, then $\mathcal{B}$ is Morita equivalent to a tame block $\mathcal{C} \subset u(s\ell(2))$.

Proof. Let S be a simple $\mathcal{B}$-module. We denote by $\mathcal{V}_{G_r}(S)$ the F-rational points of its support variety, whose definition can be found in [49, §6]. We note that $\mathcal{V}_{G_r}(S)$ is a subvariety of $\mathcal{V}_{G_r}(F)$ and that G operates on the latter by composition with inner automorphisms. Thanks to [11, (I.§3 (6.8))] and [49, (6.8)] we have

$$c_{H(G_r)}(S) = dim\, \mathcal{V}_{G_r}(S).$$

Moreover, direct computation shows that

$$\mathcal{V}_{G_r}(S^{(g)}) = g(\mathcal{V}_{G_r}(S)) \qquad \forall\ g \in G.$$

Since S is simple and G is connected, we have $S \cong S^{(g)}\ \ \forall\ g \in G$, so that $\mathcal{V}_{G_r}(S)$ is invariant under conjugation.

We first show that $dim\, \mathcal{V}_{G_r}(S) \neq 1$. Otherwise, each irreducible component of the conical variety $\mathcal{V}_{G_r}(S)$ (cf. [49, (6.1)]) is a G-stable line, and G_r contains a normal subgroup isomorphic to $G_{a,s}$ for some $s \in \{1, \ldots, r\}$. Consequently, $Lie(G)$ contains a line Fx with $x^{[p]} = 0$ that is invariant under the adjoint representation. In view of [29, (11.8)] this contradicts the fact that G is reductive.

(1). If $\mathcal{B}$ has finite representation type, then $\mathcal{V}_{G_r}(S)$ has dimension ≤ 1. By our remark above we therefore have $dim\, \mathcal{V}_{G_r}(S) = 0$. Consequently, S is projective and $\mathcal{B}$ is simple.

(2). By Rickard's Theorem [45, Thm.2] and our observation above there exists a simple $\mathcal{B}$-module S such that $dim\, \mathcal{V}_{G_r}(S) = 2$. Since $\mathcal{V}_{G_r}(S)$ is conical, Borel's Fixed Point Theorem provides a point $[x_0] \in Proj(\mathcal{V}_{G_r}(S))$ whose stabilizer is a parabolic subgroup $P \subset G$. Thus, $dim\, G/P \leq 1$, and if $G = P$ then the above reasoning yields a contradiction. Alternatively, the arguments of (5.1 (1)) show that there exist closed normal subgroups $H, K \subset G$ such that

(a) $G = H \cdot K$, and $Lie(G) = Lie(H) \oplus Lie(K)$
(b) $K \cong SL(2)$ or $K \cong PSL(2)$, and $H \subset P$.

Since $SL(2)_r \cong PSL(2)_r$ for every $r \geq 1$, we obtain an isomorphism

$$G_r \cong SL(2)_r \times H_r.$$

Consequently, there are blocks $\mathcal{B}_1 \subset H(SL(2)_r)$ and $\mathcal{B}_2 \subset H(H_r)$ such that $\mathcal{B} \cong \mathcal{B}_1 \otimes_k \mathcal{B}_2$. By the same token, $S \cong S_1 \otimes_k S_2$, where S_i is a simple $\mathcal{B}_i$-module. From [49, (7.2)] we obtain

$$\mathcal{V}_{G_r}(S) = \mathcal{V}_{SL(2)_r}(S_1) \times \mathcal{V}_{H_r}(S_2).$$

If $dim\,\mathcal{V}_{SL(2)_r}(S_1) = 0$, then $\mathcal{V}_{G_r}(S) = \mathcal{V}_{H_r}(S_2)$ and G stabilizes the point $[x_0]$, a contradiction. Thus, $dim\,\mathcal{V}_{SL(2)_r}(S_1) = 2$, and $dim\,\mathcal{V}_{H_r}(S_2) = 0$. Consequently, S_2 is projective and $\mathcal{B}_2$ is a matrix ring over k. There results an isomorphism $\mathcal{B} \cong Mat_n(\mathcal{B}_1)$. In particular, $\mathcal{B}_1$ is a tame block of $H(SL(2)_r)$.

Thanks to [41, Satz 6] every tame block of $H(SL(2)_r)$ is Morita equivalent to a tame block of $H(SL(2)_1) \cong u(s\ell(2))$. Hence $\mathcal{B}$ also has this property. $\square$

Remark. The arguments of (7.1) also show that every block $\mathcal{B} \subset H(G_r)$ of complexity 2 is Morita equivalent to a block $\mathcal{C} \subset H(SL(2)_r)$ enjoying the same property. In contrast to the case $r = 1$, such a block need not be tame if $r \geq 2$: Consider the $SL(2)_r$-module $L(\lambda)$, associated to the weight $\lambda = p^r - p$. It readily follows from [49, (7.8)] that $\mathcal{V}_{SL(2)_r}(L(\lambda)) \cong \mathcal{V}_{s\ell(2)}(F)$ has dimension 2. However, [41, Satz 6] shows that the block associated to $L(\lambda)$ is wild.

References

[1] H. Andersen. *Extensions of Modules for Algebraic Groups.* Amer. J. Math. **106** (1984), 489-504

[2] M. Auslander, I. Reiten and S. Smalø. *Representation Theory of Artin Algebras.* Cambridge Studies in Advanced Math. **36.** Cambridge University Press, 1995

[3] D. Benson. *Representations and Cohomology I.* Cambridge Studies in Advanced Mathematics **30.** Cambridge University Press 1991.

[4] D. Benson and J. Carlson. *Complexity and Multiple Complexes.* Math. Z. **195** (1987), 221-238

[5] R. Block and R. Wilson. *Classification of Restricted Simple Lie Algebras.* J. Algebra **114** (1988), 115-259

[6] A. Borel. *Linear Algebraic Groups.* Graduate Texts in Mathematics **126.** Springer Verlag 1991

[7] M. Butler and C. Ringel. *Auslander-Reiten Sequences with few Middle Terms and Applications to String Algebras.* Comm. Algebra **15** (1987), 145-179

[8] J. Carlson. *Complexity and Krull Dimension.* Lecture Notes in Mathematics **903** (1981), 62-67

[9] W. Crawley-Boevey. *Tameness of Biserial Algebras.* Arch. Math. **65** (1995), 399-407

[10] C. Curtis and I. Reiner. *Methods of Representation Theory I.* John Wiley & Sons, New York 1990

[11] M. Demazure and P. Gabriel. *Groupes Algébriques I.* Masson/North Holland 1970

[12] K. Erdmann. *On Auslander-Reiten Components for Group Algebras.* J. Pure Appl. Algebra **104** (1995), 149-160

[13] K. Erdmann. *The Auslander-Reiten Quiver of Restricted Enveloping Algebras.* CMS Conf. Proc. **18** (1996), 201-214

[14] K. Erdmann and A. Skowroński. *On Auslander-Reiten Components of Blocks and Self-Injective Special Biserial Algebras.* Trans. Amer. Math. Soc. **330** (1992), 165-189

[15] R. Farnsteiner. *Periodicity and Representation Type of Modular Lie Algebras.* J. reine angew. Math. **464** (1995), 47-65

[16] R. Farnsteiner. *Representations of Blocks Associated to Induced Modules of Restricted Lie Algebras.* Math. Nachr. **179** (1996), 57-88

[17] R. Farnsteiner. *On Auslander-Reiten Quivers of Enveloping Algebras of Restricted Lie Algebras.* Math. Nachr. (to appear)

[18] R. Farnsteiner. *Representations of Nilpotent Lie Algebras.* Arch. Math. (to appear)

[19] R. Farnsteiner. *On Support Varieties of Auslander-Reiten Components.* Indag. Math. (to appear)

[20] R. Farnsteiner. *On the Distribution of AR-Components of Restricted Lie Algebras.* Contemp. Math. (to appear)

[21] R. Farnsteiner and H. Strade. *Shapiro's Lemma and its Consequences in the Cohomology Theory of Modular Lie Algebras.* Math. Z. **206** (1991), 153-168

[22] G. Fischer. *Darstellungstheorie des ersten Frobeniuskerns der SL_2.* Dissertation, Universität Bielefeld, 1982

[23] D. Fischman, S. Montgomery and H. Schneider. *Frobenius Extensions of Subalgebras of Hopf Algebras.* Trans. Amer. Math. Soc. **349** (1996), 4857-4895

[24] E. Friedlander and B. Parshall. *Support Varieties for Restricted Lie Algebras.* Invent. math. **86** (1986), 553-562

[25] E. Friedlander and B. Parshall. *Geometry of p-Unipotent Lie Algebras.* J. Algebra **109** (1987), 25-45

[26] E. Friedlander and B. Parshall. *Modular Representation Theory of Lie Algebras.* Amer. J. Math. **110** (1988), 1055-1098

[27] J. A. Green. *Functors on Categories of Finite Group Representations.* J. Pure Appl. Algebra **37** (1985), 265-298

[28] R. Holmes and D. Nakano. *Block Degeneracy and Cartan Invariants for Graded Lie Algebras of Cartan Type.* J. Algebra **161** (1993), 155-170

[29] J. Humphreys. *Algebraic Groups and Modular Lie Algebras.* Mem. Amer. Math. Soc. **71** (1967)

[30] J. Humphreys. *Linear Algebraic Groups.* Graduate Texts in Mathematics **21.** Springer-Verlag, 1981

[31] J. Humphreys. *Conjugacy Classes in Semisimple Algebraic Groups.* Mathematical Surveys and Monographs **43.** Amer. Math. Soc. 1995

[32] J. Jantzen. *Kohomologie von p-Lie-Algebren und nilpotente Elemente.* Abh. Math. Sem. Univ. Hamburg **56** (1986), 191-219

[33] J. Jantzen. *Representations of Algebraic Groups.* Pure and Applied Mathematics **131.** Academic Press, 1987

[34] J. Jantzen. *Representations of Lie Algebras in prime Characteristic.* University of Aarhus Prepint Series No. **1** (1998)

[35] V. Kac and B. Weisfeiler. *Coadjoint Action of a Semisimple Algebraic Group and the Center of the Enveloping Algebra in Characteristic p.* Indag. Math. **38** (1976), 136-151

[36] H. Krause. *The Kernel of an Irreducible Map.* Proc. Amer. Math. Soc. **121** (1994), 57-66

[37] J. Liu and J. Ye. *Extensions of Simple Modules for the Algebraic Group of Type G_2.* Comm. Algebra **21** (1993), 1909-1946

[38] I. Mirković and D. Rumynin. *Centers of Reduced Enveloping Algebras.* Math. Z. (to appear)

[39] T. Nakayama and T. Tsuzuku. *On Frobenius Extensions I.* Nagoya J. Math. **17** (1960), 89-110

[40] T. Okuyama. *On the Auslander-Reiten Quiver of a Finite Group.* J. Algebra **110** (1987), 425-430

[41] W. Pfautsch. *Die Köcher der Frobeniuskerne der SL_2.* Dissertation Universität Bielefeld 1983.

[42] R. Pollack. *Restricted Lie Algebras of Bounded Type.* Bull. Amer. Math. Soc. **74** (1968), 326-331

[43] A. Premet. *Irreducible Representations of Lie Algebras of Reductive Groups and the Kac-Weisfeiler Conjecture.* Invent. math **121** (1995), 79-117

[44] A. Premet. *Complexity of Lie Algebra Representations and Nilpotent Elements of the Stabilizers of Linear Forms.* Math. Z. (to appear)

[45] J. Rickard. *The Representation Type of Self-Injective Algebras.* Bull. London Math. Soc. **22** (1990), 540-546

[46] G. Seligman. *Modular Lie Algebras.* Ergebnisse der Mathematik **40.** Springer-Verlag, 1967

[47] T. Springer. *Linear Algebraic Groups.* Birkhäuser Verlag. Boston-Basel 1980

[48] H. Strade and R. Farnsteiner. *Modular Lie Algebras and their Representations.* Marcel Dekker Monographs **116** New York, 1988

[49] A. Suslin, E. Friedlander and C. Bendel. *Support Varieties for Infinitesimal Group Schemes.* J. Amer. Math. Soc. **10** (1997), 729-759

[50] D. Voigt. *Induzierte Darstellungen in der Theorie der endlichen, algebraischen Gruppen.* Lecture Notes in Mathematics **592**. Springer-Verlag 1977

[51] D. Voigt. *The Algebraic Infinitesimal Groups of Tame Representation Type.* C. R. Acad. Sci. Paris **311** (1990), 757-760

[52] P. Webb. *The Auslander-Reiten Quiver of a Finite Group.* Math. Z. **179** (1982), 97-121

[53] J. Ye. *Extensions of Simple Modules for the Group $Sp(4, K)$.* J. London Math. Soc. **41** (1990), 51-62

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