DERIVED DYNKIN EXTENSIONS

PETER DRÄXLER AND AMREY KRAUSE

1. Introduction and main result

1.1. Let k be an algebraically closed field. For basic notation and results we refer to [GR] and [Ri]. All algebras which we will consider are finite-dimensional basic algebras over k. Sometimes we will consider such an algebra as a finite spectroid. For an algebra A we denote by mod$-A$ the category of finite-dimensional right A-modules and by ind$-A$ a spectroid of this aggregate. If A is considered as spectroid, then mod$-A$ is identified with the category of contravariant functors from A to the category of finite-dimensional k-spaces.

It is well-known that for any extended Dynkin diagram $\widetilde{\Delta}$ the path algebra kQ is hereditary and of tame representation type where Q is a quiver whose underlying graph is $\widetilde{\Delta}$. Recall that the extended Dynkin diagrams are $\widetilde{\mathbb{A}}_n$ $(n \geq 1)$, $\widetilde{\mathbb{D}}_n$ $(n \geq 4)$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ and $\widetilde{\mathbb{E}}_8$. Obviously, to each extended Dynkin diagram $\widetilde{\Delta}$ there corresponds a unique Dynkin diagram Δ in the list $\mathbb{A}_n$ $(n \geq 1)$, $\mathbb{D}_n$ $(n \geq 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$ obtained by cancelling an extremal vertex.

The Auslander-Reiten quiver of the algebra $H = kQ$ consists of a preprojective component $\mathcal{P}$, a preinjective component $\mathcal{I}$ and a family $\mathcal{T} = (\mathcal{T}_\varrho)_{\varrho \in \mathbb{P}(k)}$ of stable tubes. For each of these components we denote by the same symbol the induced full subcategory of ind$-H$ resp. mod$-H$. In particular, each of the tubes $\mathcal{T}_\varrho$ is an abelian subcategory of mod$-H$ with simple modules $E_1, \dots, E_{r(\varrho)}$. The positive integer $r(\varrho)$ is called the rank of the tube $\mathcal{T}_\varrho$. The map $r : \mathbb{P}(k) \to \mathbb{N}$ is said to be the tubular rank function of H. From the tubular rank function we form a graph by considering the collection $(\mathbb{A}_{r(\varrho)})_{\varrho \in \mathbb{P}(k)}$, choosing an extremal vertex in all these linear graphs and glueing them together in the chosen vertices. This star is said to be the tubular type of H. Since most of the tubes $\mathcal{T}_\varrho$ are homogeneous i.e. $r(\varrho) = 1$, the tubular type is actually a finite graph. It turns out that the tubular type of a tame hereditary algebra $H = kQ$ such that Q is a quiver with underlying extended Dynkin diagram $\widetilde{\Delta}$ is the associated Dynkin diagram Δ.

1.2. For a finite-dimensional algebra A we denote by $D^b(A)$ the derived category of bounded complexes in mod$-A$ and by ind$(D^b(A))$ a spectroid of this aggregate. The algebras A such that there is a triangle equivalence $F : D^b(A) \to D^b(H)$ where H is a tame hereditary algebra of the form $H = kQ$ are called piecewise tame hereditary of type Q.

Since H is hereditary the category ind$(D^b(H))$ is the disjoint union of all the shifts ind$-H[\nu]$ $(\nu \in \mathbb{Z})$ where mod$-H$ is included into $D^b(H)$ in the canonical way as complexes concentrated in degree 0. More precisely, the Auslander-Reiten quiver of $D^b(H)$ consists of all the shifts $\mathcal{T}[\nu]$ of the tubular family $(\mathcal{T}_\varrho)_{\varrho \in \mathbb{P}(k)}$ and directed components $\mathcal{C}[\nu]$ obtained by glueing together $\mathcal{I}[\nu - 1]$ and $\mathcal{P}[\nu]$ at the injective resp. projective vertices.

The most simple examples of the piecewise tame hereditary algebras are the tame concealed algebras where the triangle equivalence F is induced by a preprojective tilting module. In particular, the tame concealed algebras were classified in [HV].

1.3. Recall that for a given finite-dimensional algebra B and a module R in $B-$mod the one-point extension algebra $B[R]$ is by definition the matrix algebra $\begin{pmatrix} k & R \\ 0 & B \end{pmatrix}$ which is endowed with the usual addition and matrix multiplication. When considering $B[R]$ as a finite spectroid the one-point extension proecess means the creating of a new object which we usually call ω. In this interpretation the morphism space from an object x of B to ω is $R(x)$, there are no morphisms from ω to x and the endomorphism space of ω is k.

We will also need the dual construction. The one-point coextension algebra $[R]B$ is defined as $\begin{pmatrix} B & \mathrm{D}(R) \\ 0 & k \end{pmatrix}$ where $D := \mathrm{Hom}_k(-, k)$ is the usual duality mod$-B \to B-$mod from the finite-dimensional right to the finite-dimensional left modules.

If now B is a piecewise tame hereditary algebra of type Q and $F : D^b(B) \to D^b(kQ)$ is a triangle equivalence, then we consider a B-module R such that $F(R)$ is a simple object in a shifted tube $\mathcal{T}_\sigma[\nu]$ for some $\sigma \in \mathbb{P}(k)$ and some $\nu \in \mathbb{Z}$. Given the tubular rank function r of kQ we define the *derived tubular extension rank function* of the one-point extension $B[R]$ and also the one-point coextension $[R]B$ as the map r' such that $r'(\varrho) = r(\varrho)$ for all $\varrho \neq \sigma$ and $r'(\sigma) = r(\sigma) + 1$. The star associated with r' as above will be called the derived tubular extension type of $B[R]$ and also of $[R]B$. We say that A is a *derived Dynkin extension* of B by R if A can be identified with $B[R]$ or $[R]B$ and the derived tubular extension type of A is again a Dynkin diagram.

1.4. In the next section we will recall a result of Barot and Lenzing (see [BL]) which allows to extend triangle equivalences between derived categories of algebras to triangle equivalences of derived categories of certain one-point extensions. This will show that a derived Dynkin extension of a piecewise tame hereditary algebra is again piecewise tame hereditary. This allows to iterate the process of forming derived Dynkin extensions. More precisely, given a finite-dimensional algebra A and a piecewise tame hereditary algebra B, we say that A is an *iterated derived Dynkin extension* of B if there is a non-negative integer n, a sequence $B = B_0, \ldots, B_n = A$ of algebras and a sequence $R_0, \ldots, R_{n-1}$ of modules R_i in mod$-B_i$ such that for all $i = 1, \ldots, n$ the algebra B_i is a derived Dynkin extension of B_{i-1} by R_{i-1}. In particular, the derived tubular extension type of A is well-defined and a Dynkin diagram.

With this notation we can formulate the main theorem of this paper:

Theorem. *Let A be a finite-dimensional k-algebra and Δ be a Dynkin diagram. Then A is a representation infinite piecewise tame hereditary algebra of tubular type Δ if and only if there is a tame concealed algebra B such that A is an iterated derived Dynkin extension of B of derived tubular extension type Δ.*

We will prove this result at the end of section 2. In section 3 we will use it to reprove a characterisation due to Assem and Skowroński (see [AS]) saying that the representation infinite piecewise tame hereditary algebras are precisely the domestic tubular branch enlargements of the tame concealed algebras.

The investigations of this paper were motivated by [DS] were tame multiple one-point extensions by modules R over piecewise tame hereditary algebras which are sent by F to indecomposable objects of length 2 in non-homogeneous tubes in $D^b(H)$ are considered. The obvious question what happens when modules R are sent to objects of length 1 i.e. simple objects is answered in this paper.

2. One-point extension and derived equivalence

2.1. The first three parts of this section are essentially taken from [BL]. We arrange it in a way suitable for our purposes.

For an algebra A and an object $M^\bullet$ of $D^b(A)$ the left perpendicular category ${}^\perp(M^\bullet)$ is the full subcategory of $D^b(A)$ given by all complexes $X^\bullet$ such that $\mathrm{Hom}_{D^b(A)}(X^\bullet, M^\bullet[\nu]) = 0$ for all $\nu \in \mathbb{Z}$. The right perpendicular category $(M^\bullet)^\perp$ is defined in the dual fashion.

If B is an algebra of finite global dimension, then the one-point extension $A = B[R]$ by a module R in mod$-B$ is of finite global dimension as well. Therefore $D^b(A)$ can be identified with the homotopy category $\mathcal{K}^b(\mathrm{proj}{-}A)$ of the category $\mathcal{C}^b(A{-}\mathrm{proj})$ of bounded complexes of projective A-modules. Since the canonical inclusion functor mod$-B \to$ mod$-A$ is fully faithful exact and preserves projectives, we can consider $D^b(B)$ as a full and in fact thick subcategory of $D^b(A)$. Denoting by P_ω and S_ω the indecomposable projective and the simple module associated with the extension vertex ω of $B[R]$, we can describe this subcategory as a perpendicular category.

Lemma. *Let B be an algebra of finite global dimension and $A = B[R]$ a one-point extension of B by a B-module R. Then the following equalities hold:*

$$D^b(B) ={}^\perp S_\omega = P_\omega^\perp$$

Proof. By [Ha] we know that $\tau P_\omega = S_\omega[-1]$ and $\mathrm{D}\,\mathrm{Hom}_{D^b(A)}(-, \tau Y^\bullet) \cong \mathrm{Hom}_{D^b(A)}(Y^\bullet, -)$ for all $Y^\bullet$ in $D^b(A)$. This shows that ${}^\perp S_\omega = P_\omega^\perp$. We consider the minimal projective resolution $\cdots \to P'_{-1} \to R \to 0$ of R in mod$-B$ and obtain a minimal projective resolution $\cdots \to P'_{-1} \to P_\omega \to S_\omega \to 0$ of S_ω in mod$-A$ which we identify with S_ω in $D^b(A)$.

To establish $D^b(B) \subseteq{}^\perp S_\omega$ we show that any map $\phi^\bullet : P^\bullet \to S_\omega$ in $\mathcal{C}^b(\mathrm{proj}{-}A)$ is homotopic to zero where $P^\bullet$ is an arbitrary object in $D^b(B)$ given by its minimal projective resolution. We put $\alpha_i = 0$ for $i \geq 0$ and construct by induction maps $\alpha_{-i} : P_{-i+1} \to P'_{-i}$ for all $i \in \mathbb{N}$ to obtain a homotopy α between $\phi^\bullet$ and 0. Since P_0 is a B-module, we have $\mathrm{Im}\,\phi_0 \subseteq \mathrm{Rad}\,P_\omega = R$. Because the map $p'_{-1} : P'_{-1} \to \mathrm{Im}\,p'_{-1} = R$ is surjective and P_0 is projective, we can lift ϕ_0 to a morphism $\alpha_{-1} : P_0 \to P'_{-1}$ such that $\phi_0 = p'_{-1}\alpha_{-1}$. For the induction step we know that $\mathrm{Im}(\phi_{-i} - \alpha_{-i}p_{-i}) \subseteq \mathrm{Ker}\,p'_{-i} = \mathrm{Im}\,p'_{-i-1}$ as $p'_{-i}(\phi_{-i} - \alpha_{-i}p_{-i}) = p'_{-i}\phi_{-i} - (p'_{-i}\alpha_{-i} - \alpha_{-i+1}p_{-i+1})p_{-i} = p'_{-i}\phi_{-i} - \phi_{-i+1}p_{-i} = 0$. The projectivity of P_{-i} yields the existence of the next map $\alpha_{-i-1} : P_{-i} \to P'_{-i-1}$ satisfying $\phi_{-i} = p'_{-i-1}\alpha_{-i-1} + \alpha_{-i}p_{-i}$.

For showing $B^b(B) \supseteq{}^\perp S_\omega$ we choose an indecomposable object $X^\bullet = \ldots \cdots \to P_{-1} \xrightarrow{p_{-1}} P_0 \xrightarrow{p_0} P_1 \xrightarrow{p_1} \cdots$ in $\mathcal{C}^b(\mathrm{proj}{-}A)$ representing a non-zero indecomposable object in ${}^\perp S_\omega$. The indecomposability yields that $\mathrm{Im}\,P_i \subseteq \mathrm{Rad}\,P_{i+1}$ for all $i \in \mathbb{Z}$. We wish to show that all the P_i are B-modules. To get a contradiction after possible shifting we assume that $P_0 = P \oplus P_\omega$ and will construct a non-zero map $\phi^\bullet : X^\bullet \to S_\omega$ in $D^b(A)$. We put $\phi_0 = (0, \mathrm{id}_{P_\omega}) : P \oplus P_\omega \to P_\omega$ and will construct by induction for all $i \in \mathbb{N}$ maps

$\phi_{-i} : P_{-i} \to P'_{-i}$ making the following diagram commutative.

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P_{-2} & \xrightarrow{p_{-2}} & P_{-1} & \xrightarrow{p_{-1}} & P \oplus P_\omega & \xrightarrow{p_0} & P_1 & \longrightarrow & \cdots \\
 & & {\scriptstyle\phi_{-2}}\big\downarrow & & {\scriptstyle\phi_{-1}}\big\downarrow & & {\scriptstyle\phi_0}\big\downarrow & & {\scriptstyle 0}\big\downarrow & & \\
\cdots & \longrightarrow & P'_{-2} & \xrightarrow{p'_{-2}} & P'_{-1} & \xrightarrow{p'_{-1}} & P_\omega & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

Let $p : P_\omega \to S_\omega$ be a projective cover. Then the kernel of p is R. Because of $\operatorname{Im} p_{-1} \subseteq \operatorname{Rad} P_0$ we obtain that $p\phi_0 p_{-1} = 0$ and the image of $\phi_0 p_{-1}$ has to be contained in R which is also the image of p'_{-1}. The projectivity of P_{-1} yields the existence of a map ϕ_{-1} satisfying $p'_{-1}\phi_{-1} = \phi_0 p_{-1}$. The induction step is left to the reader.

It remains to show that $\phi^\bullet$ is not homotopic to zero. Otherwise there would exist maps $\alpha_0 : P_0 \to P'_{-1}$ and $\alpha_1 : P_1 \to P_\omega$ such that $\phi_0 = p'_{-1}\alpha_0 + \alpha_1 p_0$ implying the contradiction $\operatorname{Im} \phi_0 = \operatorname{Im}(p'_{-1}\alpha_0 + \alpha_1 p_0) \subseteq \operatorname{Rad} P_\omega$. □

2.2. Observe that in the next lemma the 'middle term' of the Auslander-Reiten triangle is actually the module R.

Lemma. *If $A = B[R]$ is a one-point extension of a spectroid B of finite global dimension, then there is an Auslander-Reiten triangle in $D^b(A)$ of the following shape:*

$$S_\omega[-1] \longrightarrow R \longrightarrow P_\omega \longrightarrow S_\omega$$

Proof. By [Ha] there exists an Auslander-Reiten triangle

$$S_\omega[-1] \xrightarrow{f} E \xrightarrow{g} P_\omega \xrightarrow{h} S_\omega$$

in $D^b(A)$ where h is the canonical projection. Let us show that $E \cong R$.

By applying $\operatorname{Hom}_{D^b(A)}(P_\omega, -)$ to the triangle we obtain a short exact sequence as follows.

$$\operatorname{Hom}_{D^b(A)}(P_\omega, \tau P_\omega) \longrightarrow \operatorname{Hom}_{D^b(A)}(P_\omega, E) \longrightarrow \operatorname{Hom}_{D^b(A)}(P_\omega, P_\omega) \xrightarrow{h'}$$
$$\xrightarrow{h'} \operatorname{Hom}_{D^b(A)}(P_\omega, \tau P_\omega[-1]) \longrightarrow \operatorname{Hom}_{D^b(A)}(P_\omega, E[1]) \longrightarrow \operatorname{Hom}_{D^b(A)}(P_\omega, P_\omega[1])$$

The extremal terms vanish because of the calculations $\operatorname{Hom}_{D^b(A)}(P_\omega, \tau P_\omega) \cong \operatorname{D}\operatorname{Hom}_{D^b(A)}(P_\omega[-1], P_\omega) \cong \operatorname{D}\operatorname{Ext}^1_A(P_\omega, P_\omega) = 0$ and $\operatorname{Hom}_{D^b(A)}(P_\omega, P_\omega[1]) \cong \operatorname{Ext}^1_A(P_\omega, P_\omega) = 0$. Since $\operatorname{Hom}_{D^b(A)}(P_\omega, \tau P_\omega[1]) \cong \operatorname{D}\operatorname{Hom}_A(P_\omega, P_\omega)$ and $\operatorname{Hom}_{D^b(A)}(P_\omega, P_\omega) \cong \operatorname{Hom}_A(P_\omega, P_\omega)$ are 1-dimensional k-spaces and $h' = \operatorname{Hom}_A(P_\omega, h)$ is non-zero, the spaces $\operatorname{Hom}_{D^b(A)}(P_\omega, E)$ and $\operatorname{Hom}_{D^b(A)}(P_\omega, E[1])$ have to vanish as well. Moreover, for all $n \in \mathbb{Z}$ the triangle yields an exact sequence:

$$\operatorname{Hom}_{D^b(A)}(P_\omega, \tau P_\omega[n]) \longrightarrow \operatorname{Hom}_{D^b(A)}(P_\omega, E[n]) \longrightarrow \operatorname{Hom}_{D^b(A)}(P_\omega, P_\omega[n])$$

For $n \neq 0, 1$ the extremal terms vanish because of $\operatorname{Hom}_{D^b(A)}(P_\omega, \tau P_\omega[n]) \cong \operatorname{D}\operatorname{Hom}_{D^b(A)}(P_\omega[n-1], P_\omega)$. Hence we proved that E lies in $P_\omega^\perp$ which coincides with $D^b(B)$ by the preceding lemma.

If E was not a B-module, then there would exist an integer $\nu \neq 0$ and an indecomposable projective B-module P such that $\operatorname{Hom}_{D^b(B)}(P[\nu], E) \neq 0$. We produce a contradiction by showing that also $\operatorname{Hom}_{D^b(A)}(P[\nu], P_\omega) \neq 0$ follows. The above Auslander-Reiten triangle yields for all integer n the following exact sequence of functors on $D^b(A)$.

$$\operatorname{Hom}_{D^b(A)}(-, \tau P_\omega[n]) \longrightarrow \operatorname{Hom}_{D^b(A)}(-, E[n]) \longrightarrow$$
$$\longrightarrow \operatorname{Hom}_{D^b(A)}(-, P_\omega[n]) \longrightarrow \operatorname{Hom}_{D^b(A)}(-, \tau P_\omega[n+1])$$

The extremal terms of this sequence vanish on $D^b(B) = P_\omega^\perp$ yielding that the middle terms become isomorphic.

The map $g : E \to P_\omega$ is minimal right almost split in $D^b(A)$. But, as we now have verified that E is a module, g is minimal right almost split in mod$-A$ as well. This forces E to be isomorphic to R. □

2.3. Now we present the result of Barot and Lenzing on derived equivalences of one-point extensions (see [BL]).

Theorem. *Let B_1, B_2 be algebras of finite global dimension and R_1 in* mod$-B_1$*, R_2 in* mod$-B_2$ *such that there is a triangle equivalence $F : D^b(B_1) \to D^b(B_2)$ with $F(R_1) \cong R_2$. Then the one-point extensions $A_1 = B_1[R_1]$ and $A_2 = B_2[R_2]$ are derived equivalent as well.*

Proof. We put $G = F^{-1}$ and identify the B_i with the spectroids of projective modules in ind$-B_i$. By [Rc] it suffices to show that $G(B_2) \cup P_\omega$ is a tilting spectroid in $D^b(A_1)$ which is isomorphic to A_2.

As $G(D^b(B_2)) = D^b(B_1) = P_\omega^\perp$, we know that $\mathrm{Hom}_{D^b(A_1)}(P_\omega, G(P_x))$ holds for all P_x in B_2. Using an argument from the proof of the previous lemma, we get $\mathrm{Hom}_{D^b(A_1)}(G(P_x), P_\omega) \cong \mathrm{Hom}_{D^b(A_1)}(G(P_x), M_1) \cong \mathrm{Hom}_{D^b(A_2)}(P_x, M_2) \cong \mathrm{Hom}_{B_2}(P_x, M_2) \cong M_2(x)$. This shows that $G(B_2) \cup P_\omega \cong A_2$.

Let us show next that $G(B_2) \cup P_\omega$ has no 'selfextensions'. For all $n \neq 0$ we have $\mathrm{Hom}_{D^b(A_1)}(P_\omega, P_\omega[n]) \cong \mathrm{Ext}^n_{A_1}(P_\omega, P_\omega) = 0$. Also $\mathrm{Hom}_{D^b(A_1)}(P_\omega, G(B_2)[n]) = 0$ for all $n \in \mathbb{Z}$ because $G(B_2) \in P_\omega^\perp$. Finally, $\mathrm{Hom}_{D^b(A_1)}(G(B_2)[n], P_\omega) \cong \mathrm{Hom}_{D^b(A_2)}(B_2[n], M_2) \cong \mathrm{Ext}^n_{B_2}(B_2, M_2) = 0$ for all $n \neq 0$.

To finish the proof we have to show that $G(B_2) \cup P_\omega$ generates $D^b(A_1)$ as a triangulated category. But the minimal triangulated category containing all objects of this spectroid contains $D^b(B_1)$ and P_ω. Hence it contains all indecomposable projective A_1-modules. □

2.4. **Proof of the main theorem.** As announced in the introduction we first have to establish that the derived Dynkin extension process can be iterated.

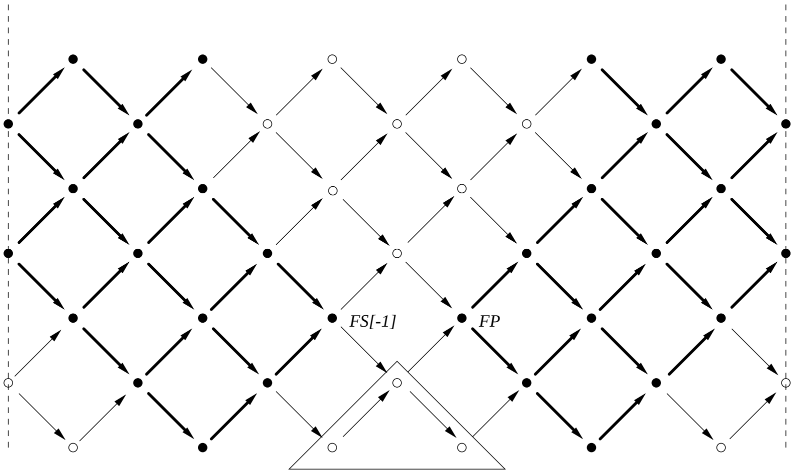
Proposition. *If B is a derived tame hereditary algebra of tubular type Δ and A is a derived Dynkin extension of B of derived tubular extension type Δ', then A is derived tame hereditary of tubular type Δ'.*

Proof. The object $F(R)$ lies in a shifted tube $\mathcal{T}_\sigma[\nu]$ where $F : D^b(B) \to D^b(H)$ is a derived equivalence. Composing F with the shift by $-\nu$ we obtain a new derived equivalence $F' : \mathrm{D}^b(B) \to D^b(H)$ which maps R to a simple regular module R' in a tube $\mathcal{T}_\sigma$ in mod$-H$. By the above theorem A is derived equivalent to the one-point extension $H[R']$ which by [Ri] is tilted from a hereditary algebra H' of tubular type Δ'. By [Ha] the tilting gives rise to a derived equivalence $D^b(H[R']) \to D^b(H')$. □

With this proposition it is obvious that any iterated derived Dynkin extension is derived tame hereditary. To prove the converse we start with a representation infinite derived tame hereditary algebra of tubular type Δ which is induced by a tubular rank function r. By [AS, Lemma 4.3] there exists a tame concealed convex subspectroid C of A. We use induction on the difference $|A| - |C|$ of the number of objects in A and in C. If $|A| - |C| = 0$, the $A = C$ and nothing is to show. For $|A| - |C| > 0$ we obtain A from C by iterated one-point extension and coextension. Therefore there exists a convex connected subspectroid B of A containing C such that up to duality $A = B[R]$ for some B-module R.

The existence of a derived equivalence $D^b(A) \to D^b(H)$ for a piecewise tame hereditary algebra H yields an isometry of the Grothendieck groups $f : K_0(A) \to K_0(H)$. In particular, the Euler form χ_A has to be positive semidefinite of corank 1. On the other hand the Euler form of χ_C is positive semidefinite of corank 1 as well. Because of the convexity of C the minimal positive radical generator of μ_C of χ_C satisfies $\chi_A(\mu_C) = 0$ and because it has a coefficient 1 it has to be a radical generator of χ_A. Consequently, $f(\mu_C) = n\mu_H$ for some non-zero integer n where μ_H is the minimal positive radical generator of χ_H. Denoting by $\mathbf{dim}\, S_\omega$ the corresponding element of the Grothendieck group $K_0(A)$, we obtain that $n\langle \mu_H, \mathbf{dim}\, F(S_\omega)\rangle_H = \langle \mu_C, \mathbf{dim}\, S_\omega\rangle_A = 0$ using that S_ω is injective and $\mu_C(\omega) = 0$. This shows that $F(S_\omega)$ has to lie in some shift of the tubular family $\mathcal{T}$.

From $0 = \mathrm{Ext}^1_A(S_\omega, S_\omega)$ and $\mathrm{End}_A(S_\omega) \cong k$ we conclude that $F(S_\omega)$ lies in a non-homogeneous tube $\mathcal{T}_\sigma[\nu]$ and is of regular length smaller than $r(\sigma)$. We will show that $F(S_\omega)$ is a simple object i.e. of length $l = 1$ in $\mathcal{T}_\sigma[\nu]$. It is easy to see that $D^b(B)$ is the product of $D^b(H')$ for a tame hereditary algebra H' with $|A| - l$ objects and $D^b(H'')$ where $H'' = kQ''$ is a representation finite hereditary algebra such that the underlying graph of Q'' is $\mathbb{A}_{l-1}$. For a more rigorous proof for the structure of $D^b(H')$ we refer to [DS] where the case $l = 1$ is explained. The arguments generalise immediately. The structure of $D^b(H'')$ can be seen directly by looking at the tubes $\mathcal{T}_\sigma[\nu]$. We illustrate the situation in the following picture for the case $l = 3$. The bold parts do not belong to $D^b(B) = {}^\perp S_\omega$. The part belonging to $D^b(H'')$ is encircled.



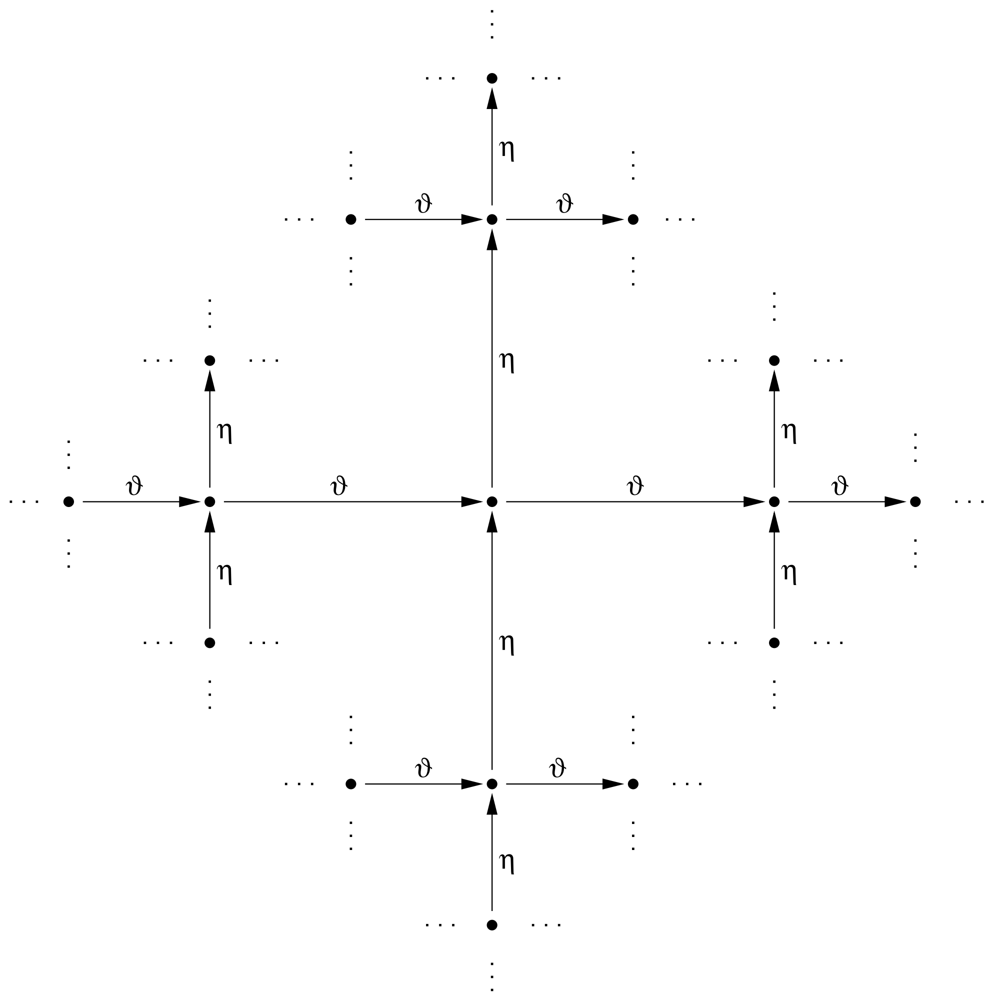
As B is connected also $D^b(B)$ has to be connected which forces l to be 1. We know that there is an Auslander-Reiten triangle

$$S_\omega[-1] \longrightarrow R \longrightarrow P_\omega \longrightarrow S_\omega$$

whose middle term R has to be indecomposable because $F(S_\omega)$ is simple regular. In particular, we see that $F(R)$ is simple regular in a tube of rank $r(\sigma) - 1$ in $D^b(H')$. It is obvious that H' is piecewise tame hereditary with tubular rank function r' where $r'(\varrho) = r(\varrho)$ for all $\varrho \neq \sigma$ and $r'(\sigma) = r(\sigma) - 1$ inducing a tubular Dynkin type with one arm shorter than that of A. By induction we get that B is an iterated Dynkin extension and therefore A as well.

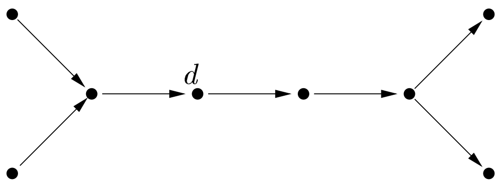
3. Domestic tubular branch enlargements

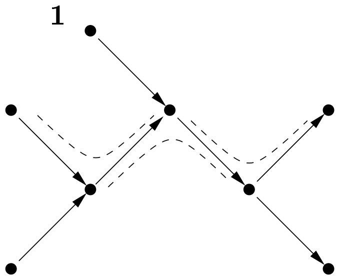
3.1. We recall that any algebra can be written as kQ/I where kQ is the path algebra of a quiver Q and I is an admissible ideal generated by a finite set of relations. This generalises to locally bounded spectroids. We need an easy example of this and follow [Ba] in the way of introducing it. The complete branch is an infinite connected quiver whose vertices are given as follows. We take two symbols η and ϑ and form the set of all words in this symbols and theirs inverses. On this set we consider the smallest equivalence relation such that $\eta\eta^{-1}$, $\eta^{-1}\eta$, $\vartheta\vartheta^{-1}$, $\vartheta^{-1}\vartheta$ are all equivalent to the trivial word $\mathbf{1}$. The set of equivalence classes is the set Q_0 of vertices of the complete branch. In the sequel we will not distinguish between words and their equivalence classes. Thus Q_0 consists of all products of the shape $w = \vartheta^{m_t}\eta^{n_t}\ldots\vartheta^{m_1}\eta^{n_1}$ where $t \in \mathbb{N}_0$, $n_i, m_i \in \mathbb{Z}$. For any vertex w there exist precisely two outgoing arrows namely $\eta_w : w \to \eta w$ and $\vartheta_w : w \to \vartheta w$. The relations generating our desired ideal are $\eta_{\vartheta w}\vartheta_w$ and $\vartheta_{\eta w}\eta_w$, w running through all $w \in Q_0$. Thus the complete branch may be viewed as follows endowed with all relations $\eta\vartheta = 0 = \vartheta\eta$.



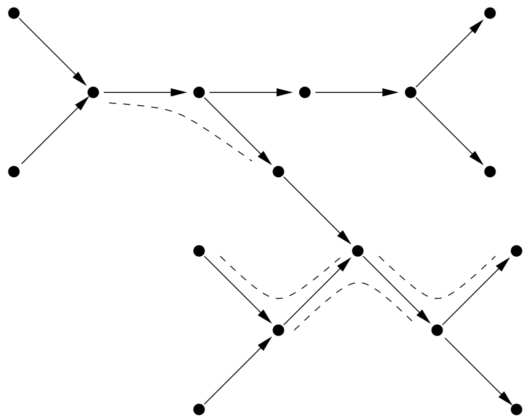
A finite connected subquiver Q' of Q containing $\mathbf{1}$ and endowed with the relations I' induced from the complete branch is called extension branch if it does not contain the vertex η^{-1}. Dually it is called coextension branch if is does not contain η. The length $|\Theta|$ of such a branch $\Theta = (Q', I')$ is by definition the number of vertices of Q'_0.

If now B is an algebra and R is a B-module, then the one-point extension $B[R]$ is given by some (Q', I') as quiver with relations where the extension vertex ω is a sink of Q'. If, moreover, an extension branch $\Theta = (Q'', I'')$ is given, then the simple branch enlargement $B[R, \Theta]$ of B by R and Θ is defined by the quiver with relations (Q, I) constructed in the following way. The quiver Q is obtained from Q' and Q'' by identifying ω and $\mathbf{1}$. The ideal I is generated by imposing I', I'' and all compositions $\vartheta_{\mathbf{1}}\gamma$ where γ runs through all arrows γ in Q' ending in $\omega = \mathbf{1}$. The simple branch coenlargement $[R, \Theta]B$ by a coextension branch Θ is defined in the dual way. As an example we consider $B = kQ'$ where Q' is a quiver whose underlying graph is $\widetilde{\mathbb{D}}_7$ and Θ is the following branch of length 8.





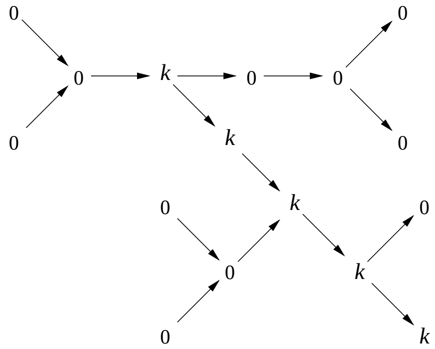
We put $R = S_d$ the simple module associated with the vertex d and display the simple branch enlargement $B[R, \Theta]$.

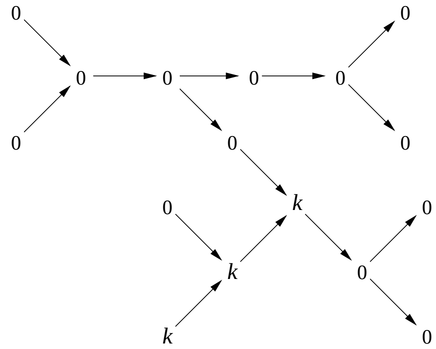


3.2. Let us consider a simple branch enlargement $A = B[R, \Theta]$. We are going to define certain A-modules. Let us say that a vertex w of the branch Θ is η-minimal if it is not the final point of an η-arrow. Clearly, if w is an η-minimal point of Θ, then there exists a unique vertex v of Θ and a non-negative integer n such that $v = \eta^n w$ and v is not the initial point of an η-arrow.

The chain $K(w)$ associated with an η-minimal w is the A-module sending all arrows $\eta_{\eta^{i-1}w} : \eta_{\eta^{i-1}w} \to \eta_{\eta^i w}$ to the identity map id_k of the 1-dimensional k-space k for all $i = 1, \dots, n$ whereas all other points and arrows of the extension branch Θ are sent to 0. The restriction of $K(w)$ to B is R if $w = \mathbf{1}$ and 0 otherwise. The module $K(w)$ is always indecomposable. The chain $K(w)$ is called big for $w = \mathbf{1}$ and small otherwise. The ϑ-minimal points and ϑ-chains are defined in an analogous, η- and ϑ-chains in the dual way.

We use the preceding example to show the big η-chain and a ϑ-chain.





3.3. Let us now start with a tame concealed algebra C and a sequence $R_1, \dots, R_t$ of pairwise non-isomorphic C-modules such that each R_i is simple in a tube $\mathcal{T}_{\sigma_i}$. For each $i = 1, \dots, t$ there is given an extension branch Θ_i (which may be of length 0). For all $j = 2, \dots, t$ we define the branch enlargement $C[R_i, \Theta_i]_{i=1}^{j} = (C[R_i, \Theta_i]_{i=1}^{j-1})[R_j, \Theta_j]$. The obtained algebra $A = C[R_i, \Theta_i]_{i=1}^{t}$ does not depend on the enumeration of the R_i and all algebras $C[R_i, \Theta_i]$ are convex subspectroids of A. In particular, all chains $K(w)$ for the various branches give rise to $C[R_i, \Theta_i]_{i=1}^{t}$-modules using extension by zero. Let us denote the extension point associated with R_i by ω_i if $|\Theta_i| > 0$. For $i = 1, \dots, t$ we put $R_i' = K(\mathbf{1}_i)$ if $|\Theta_i| > 0$ and $R_i' = R_i$ otherwise. Given a sequence $\Theta_1', \dots, \Theta_t'$ of coextension branches we put ${}_{i=1}^{t}[R_i, \Theta_i']C[R_i, \Theta_i]_{i=1}^{t} = {}_{i=1}^{t} [R_i', \Theta_i'](C[R_i, \Theta_i]_{i=1}^{t})$. Algebras of this kind are called *tubular branch enlargements*. We will always assume that for all $i = 1, \dots, t$ at least one of the branches Θ_i and Θ_i' has positive length. The tubular enlargement function r' associated with a tubular branch enlargement algebra is defined as $r'(\varrho) = r(\varrho) + \sum_{R_i \in \mathcal{T}_\varrho}(|\Theta_i| + |\Theta_i|)$. Hence, we can also associate a tubular enlargement type. The algebra ${}_{i=1}^{t}[R_i, \Theta_i']C[R_i, \Theta_i]_{i=1}^{t}$ is called a domestic tubular branch enlargement provided that its tubular enlargement type is Dynkin. The construction of branch enlargements immediately yields:

Lemma. *If $A = {}_{i=1}^{t}[R_i, \Theta_i']C[R_i, \Theta_i]_{i=1}^{t}$ is a domestic tubular branch enlargement of a tame concealed algebra C, then there is a sequence $C = A_0, A_1, \dots, A_m = A$ of convex connected subspectroids which are domestic tubular branch enlargements of C themselves such that for all $i = 1, \dots, m$ the algebra A_i is a one-point extension or a one-point coextension of A_{i-1} by a chain or one of the modules R_i which does not yet appear in any branch.*

3.4. Let us discuss the role of the chains more precisely.

Lemma. *Suppose that B is a piecewise tame hereditary algebra of infinite representation type and $A = B[R]$ is a derived Dynkin extension. Then the modules in* $\mathrm{mod}-A$ *which are simple in some tube in the derived category $D^b(A)$ are precisely the modules in* $\mathrm{mod}-B$ *which are simple in some tube in $D^b(B)$ and different from R, together with the modules P_ω and S_ω.*

Proof. Our main theorem says that there is a derived equivalence $D^b(A) \to D^b(H)$ where H is tame hereditary. Invoking again the Auslander-Reiten-triangle

$$S_\omega[-1] \longrightarrow R \longrightarrow P_\omega \longrightarrow S_\omega$$

and remembering that $D^b(B) = P_\omega^\perp$, we see that P_ω and S_ω are the only A-modules which are not B-modules but are simple in some tube. Moreover, all simple modules in tubes in $D^b(B)$ remain simple with the exception of R which becomes of length 2. □

Proposition. *Suppose that $A =_{i=1}^t [R_i, \Theta_i']C[R_i, \Theta_i]_{i=1}^t$ is a domestic tubular branch enlargement of a tame concealed algebra C with tubular enlargement rank function r. Then the following assertions hold:*

(a) *A is an iterated derived Dynkin extension of C whose derived tubular extension type coincides with the tubular enlargement type of A.*
(b) *The modules in* mod$-A$ *which are simple in some tube in $D^b(A)$ are precisely the chains together with all the simple modules in tubes of* mod$-C$ *which do not appear among the $R_1, \dots, R_t$.*
(c) *All the chains associated with one of the modules R_i do lie in the same shift orbit of a tube $\mathcal{T}_\varrho$ in $D^b(A)$.*

Proof. We use induction on $|A| - |C|$. If this number is 0, then nothing has to be shown. If $|A| - |C| > 0$, then by a lemma above up to duality we can write A as $B[R]$ where B is a domestic tubular branch enlargement of C with one object less. By induction B is an iterated derived Dynkin extension of C satisfying the properties (a) to (c). By (b) the module R is simple regular in some shifted tube in $D^b(B)$. Therefore the derived tubular extension type of $B[R]$ is precisely the tubular enlargement type of A thus a Dynkin diagram. It remains to establish (b) and (c). Since P_ω and S_ω are chains by construction, (b) follows from the preceding lemma. Finally, (c) holds because P_ω and S_ω lie in the same shift orbit in $D^b(A)$ as R. □

The equivalence of the parts (a) and (c) in the theorem below is a theorem of Assem and Skowroński from [AS]. We will reprove this result by showing that (b) and (c) are equivalent and recalling that the equivalence of (a) and (b) is just our main theorem.

Theorem. *Let A be a finite-dimensional algebra and Δ a Dynkin diagram. Then the following assertions are equivalent:*

(a) *A is a representation infinite piecewise tame hereditary algebra of tubular type Δ.*
(b) *A is an iterated derived Dynkin extension of a tame concealed algebra of derived tubular extension type Δ.*
(c) *A is a domestic tubular branch enlargement of a tame concealed algebra of tubular enlargement type Δ.*

Proof. That (c) implies (b) follows from the Proposition. For the converse we know that A is an iterated derived tubular extension of some tame concealed algebra C and will use again induction on $|A| - |C|$ observing that the induction start is again trivial. So let us write A up to duality as $A = B[R]$ with B a derived tubular extension and R simple in some tube in $D^b(B)$. By induction B is a domestic tubular branch enlargement of a tame concealed algebra C. Remembering $D^b(B) = P_\omega^\perp$ we see that the B-module R is simple in a a tube of $D^b(B)$. By part (b) of the Proposition R has to be a chain or one of the simple modules in some tube of C. Consequently, A is a branch enlargement as well whose enlargement type coincides with its derived extension type. □

References

[AS] I. Assem, A. Skowroński, *Algebras with cycle-finite derived category,* Math. Annalen 180 (1988), 441-463.

[Ba] M. Barot, *Representation-finite derived tubular algebras,* Preprint.

[BL] M. Barot, H. Lenzing, *One-point extensions and derived equivalence,* Preprint.

[DR] V. Dlab, C.M. Ringel, *Indecomposable representations of graphs and algebras,* Mem. Amer. Math. Soc. 173 (1973).

[DS] P. Dräxler, A. Skowroński, *Tame biextensions by indecompsable modules of derived regular length 2,* to appear in Compositio Math.

[GR] P. Gabriel, A.V. Roiter, *Representations of finite-dimensional algebras,* Encyclopedia of the Mathematical Sciences, Vol. 73, Algebra VIII, A.I. Kostrikin and I.V. Shafarevich (Eds.), Berlin, Heidelberg, New York, 1992.

[Ha] D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras,* LMS Lecture Notes 119 (1988).

[HV] D. Happel, D. Vossieck, *Minimal algebras of infinite representation type with preprojective component,* Manuscr. Math. 42 (1983), 221-243.

[Qu] D. Quillen, *Higher algebraic K-theory I,* Springer LNM 341 (1973), 85-147.

[Rc] J. Rickard, *Morita theory for derived categories,* J. london Math. Soc. (2) 39 (1989), 436-456.

[Ri] C.M. Ringel, *Tame algebras and integral quadratic forms,* Springer LNM 1099 (1984).

Fakultät für Mathematik, Universität Bielefeld, POBox 100131, D-33501 Bielefeld, Germany