The numerical computation and geometrical analysis of heteroclinic tangencies

Jan–Martin Kleinkauf *

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Abstract

Quadratic tangential homoclinics, also called homoclinic bifurcations, are known to cause highly complex behavior of time–discrete dynamical systems. In this paper we develop a method to approximate these orbits. More generally, we consider the case of tangential heteroclinics. These orbits are approximated by finite orbit–segments solving a boundary value problem and satisfying a regularity constraint. A detailed error analysis for this method is given which includes the case of local branches of heteroclinic orbits. We apply our method to the computation of quadratic tangentials. They are approximated as turning points of the boundary value operator. Several examples illustrate this approach. Furthermore, equivalences are derived between geometric properties of the intersection of the stable and unstable manifolds at a heteroclinic point on one hand, and analytic properties of the variational equation along the heteroclinic orbit on the other hand.

Key words. dynamical systems – homoclinic bifurcations – heteroclinic orbits for maps – numerical methods

1 Introduction

One famous result on chaotic dynamics of time–discrete systems is the homoclinic theorem of Smale and Birkhoff, see [23], [22]. It states that in some neighborhood of a transversal homoclinic orbit there exists a Cantor set on which the dynamics of the system is conjugate to a subshift of finite type.

More complex phenomena occur near a homoclinic bifurcation which is a generic unfolding of a quadratic homoclinic tangency. For a class of one–parametric surface diffeomorphisms, Mora and Viana have shown in [15] that

*SFB 343, Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D–33501 Bielefeld, Germany, e-mail: jan@Mathematik.Uni-Bielefeld.DE

the existence of a homoclinic bifurcation at some parameter value, say $\bar{\lambda}$, implies the appearance of a strange attractor for every parameter value in some positive Lebesgue measure set near $\bar{\lambda}$. Other results on more general diffeomorphisms were published by Viana later on. A historical review of the discovery of homoclinic phenomena is given by Palis and Takens in [17]. This monograph collects results mainly from western literature. Earlier references to these topics in the east are the papers [8], [9] by Gavrilov and Shilnikov.

Consider a time–discrete system

$$x_{n+1} = f(x_n, \lambda), \quad n \in \mathbb{Z}, \tag{1}$$

where $f : \mathbb{R}^k \times \mathbb{R}^r \to \mathbb{R}^k$ is a $\mathcal{C}^l$–map with $l \geq 1$, and $f(\cdot, \lambda)$ is diffeomorphic for every λ in the open ball $U_\epsilon(\bar{\lambda})$ of some radius $\epsilon > 0$ centered at some $\bar{\lambda} \in \mathbb{R}^r$. In case $r = 0$ all assumptions on the parameter are neglected throughout this paper. Furthermore, let $\bar{\xi}_-$, $\bar{\xi}_+$ be *hyperbolic fixed points of* $f(\cdot, \bar{\lambda})$, i.e. $\bar{\xi}_\pm = f(\bar{\xi}_\pm, \bar{\lambda})$ and $f_x(\bar{\xi}_\pm, \bar{\lambda})$ has no eigenvalue on the unit circle. A sequence $(\bar{x}_n)_{n\in\mathbb{Z}} = \bar{x}_\mathbb{Z}$ that solves (1) for $\lambda = \bar{\lambda}$, i.e. a $f(\cdot, \bar{\lambda})$*–orbit*, is called *heteroclinic connecting* $\bar{\xi}_-$ *and* $\bar{\xi}_+$ if

$$\lim_{n\to-\infty} \bar{x}_n = \bar{\xi}_-, \quad \lim_{n\to+\infty} \bar{x}_n = \bar{\xi}_+ \tag{2}$$

hold. In case $\bar{\xi}_- = \bar{\xi}_+$ such an orbit is called *homoclinic with respect to* $\bar{\xi}_\pm$.

Obviously, every point $\bar{x}_n$, $n \in \mathbb{Z}$, of a heteroclinic $f(\cdot, \bar{\lambda})$–orbit $\bar{x}_\mathbb{Z}$ connecting $\bar{\xi}_-$ and $\bar{\xi}_+$ lies in the intersection of the *unstable manifold* $W_u(\bar{\xi}_-, \bar{\lambda}) = \{x : \lim_{n\to-\infty} f(\cdot, \bar{\lambda})^n(x) = \bar{\xi}_-\}$ and the *stable manifold* $W_s(\bar{\xi}_+, \bar{\lambda}) = \{x : \lim_{n\to+\infty} f(\cdot, \bar{\lambda})^n(x) = \bar{\xi}_+\}$. We call the heteroclinic orbit $\bar{x}_\mathbb{Z}$ t*–tangential* if the tangent spaces of these manifolds at $\bar{x}_0$ have a common subspace of dimension t. In case $t = 0$ (and $r = p + t$ where p is defined in (3)) it is called *transversal.* A transversal heteroclinic together with parts of the manifolds $W_u(\bar{\xi}_-, \bar{\lambda})$ and $W_s(\bar{\xi}_+, \bar{\lambda})$ is shown on the left side of Figure 1.

A 1–tangential heteroclinic is called *quadratic* if the manifolds $W_u(\bar{\xi}_-, \bar{\lambda})$ and $W_s(\bar{\xi}_+, \bar{\lambda})$ have a quadratic tangency at $\bar{x}_0$. Such an orbit, together with parts of these manifolds, is drawn on the right side of Figure 1.

Furthermore, consider the *extended unstable* and *stable manifolds*

$$\begin{aligned}
\mathcal{W}_{u,\epsilon}(\bar{\xi}_-, \bar{\lambda}) &= \{(x, \lambda) : \lambda \in U_\varepsilon(\bar{\lambda}), x \in W_u(\xi_-(\lambda), \lambda)\}, \\
\mathcal{W}_{s,\epsilon}(\bar{\xi}_+, \bar{\lambda}) &= \{(x, \lambda) : \lambda \in U_\varepsilon(\bar{\lambda}), x \in W_s(\xi_+(\lambda), \lambda)\}.
\end{aligned}$$

These sets are injectively $\mathcal{C}^l$–immersed submanifolds of $\mathbb{R}^{k+r}$, cf. Appendix B. Let

$$p = k_{-s} - k_{+s} \tag{3}$$

where $k_{\pm s}$ is the dimension of $W_s(\bar{\xi}_\pm, \bar{\lambda})$ and $\xi_\pm(\lambda)$ is the locally unique fixed point of $f(\cdot, \lambda)$ near $\bar{\xi}_\pm$ for $\lambda \in U_\epsilon(\bar{\lambda})$ and $\epsilon > 0$ sufficiently small. We call a t–tangential heteroclinic *nondegenerate* if $r = p + t$ holds and the extended manifolds $\mathcal{W}_{u,\epsilon}(\bar{\xi}_-, \bar{\lambda})$ and $\mathcal{W}_{s,\epsilon}(\bar{\xi}_+, \bar{\lambda})$ are transversal at $(\bar{x}_0, \bar{\lambda})$. In Figure 2 we illustrate the intersection of $\mathcal{W}_{u,\epsilon}(\bar{\xi}_-, \bar{\lambda})$ and $\mathcal{W}_{s,\epsilon}(\bar{\xi}_+, \bar{\lambda})$ near $(\bar{x}_0, \bar{\lambda})$ in case of a nondegenerate quadratic heteroclinic $\bar{x}_\mathbb{Z}$.

In case $r = 1$, a generic unfolding of a quadratic heteroclinic tangency occurs exactly at a nondegenerate quadratic 1–tangential homoclinic. To compute such tangencies we will state and analyze a method for the approximation of

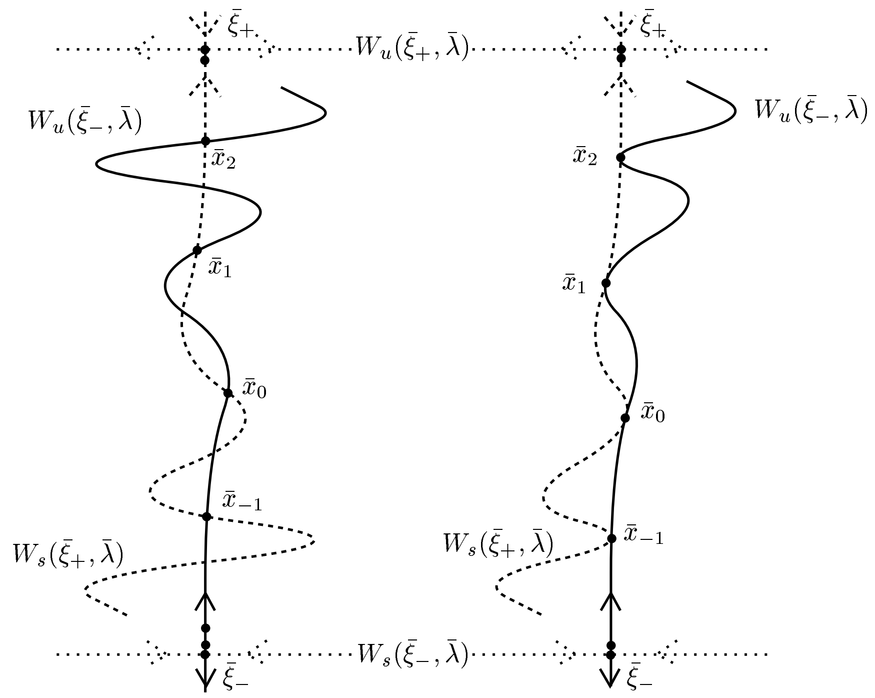

Figure 1: Parts of $W_u(\bar{\xi}_-,\bar{\lambda})$ and $W_s(\bar{\xi}_+,\bar{\lambda})$ together with a heteroclinic $\bar{x}_{\mathbb{Z}}$.

nondegenerate heteroclinics. It generalizes the method for transversal heteroclinics developed in the joint paper [5]. This method will be applied to the computation of quadratic tangencies. In [13] the results of this paper where derived for the homoclinic case.

In [5] we approximate the boundary value problem (1), (2) by

$$\begin{aligned} x_{n+1} - f(x_n,\lambda) &= 0, \quad n = n_-,\dots,n_+ - 1, && (4)\\ b(x_{n_-},x_{n_+},\lambda) &= 0 && (5)\end{aligned}$$

where $-n_-, n_+$ are positive integers and $b : \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^r \to \mathbb{R}^{k+p}$ is a $\mathcal{C}^l$–map. To formulate our method for quadratic tangencies we consider bounded solutions of (1) and of (4), (5) as zeros of the $\mathcal{C}^l$–operators

$$\Gamma = \Gamma_f : S_{\mathbb{Z}} \times \mathbb{R}^r \to S_{\mathbb{Z}},\ (x_{\mathbb{Z}},\lambda) \mapsto (x_{n+1} - f(x_n,\lambda))_{n\in\mathbb{Z}} \qquad (6)$$

and

$$\Gamma_J = \Gamma_{f,b,J} : \begin{array}{rcl} S_J \times \mathbb{R}^r & \to & S_{\hat{J}} \times \mathbb{R}^{k+p}, \\ (x_J,\lambda) & \mapsto & ((x_{n+1} - f(x_n,\lambda))_{n\in\hat{J}}, b(x_{n_-},x_{n_+},\lambda)) \end{array} \qquad (7)$$

for $J = [n_-, n_+]$ and $n_-, n_+ \in \mathbb{Z}$. Here S_I denotes the Banach space of bounded sequences in $(\mathbb{R}^k)^I$ for an interval $I \subset \mathbb{Z}$. This space is equipped with the supremum norm. For an interval I as above, the interval $\hat{I}$ is the set of all elements of I without its upper bound if this bound is finite.

In case $r = 1$, nondegenerate quadratic heteroclinics are quadratic turning points of Γ. These turning points are approximated by turning points of Γ_J. Notice that $S_J \cong \mathbb{R}^{(n_+ - n_- + 1)k}$ is of finite dimension.

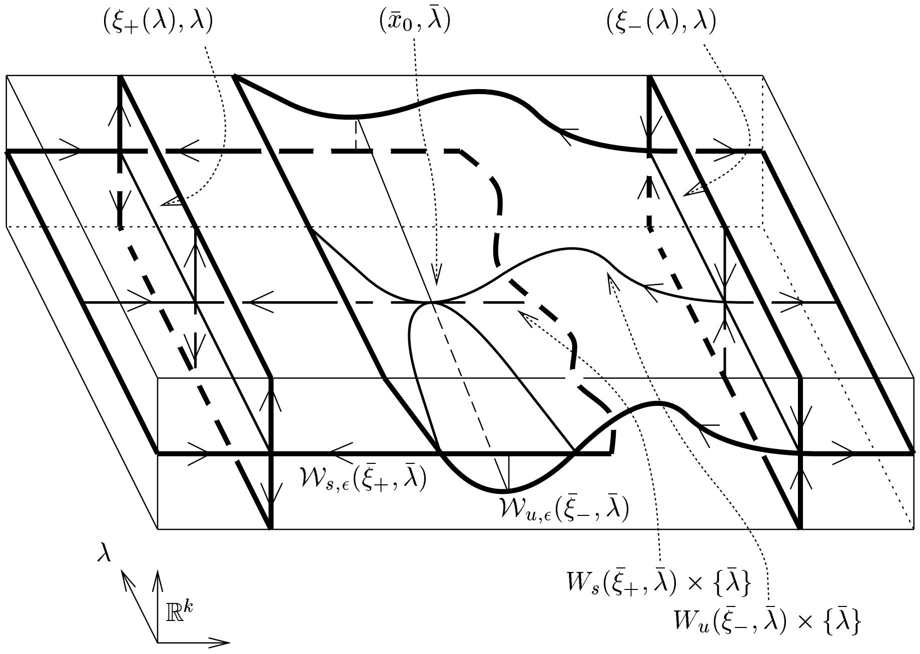

Figure 2: Part of $\mathcal{W}_{u,\epsilon}(\bar{\xi}_-,\bar{\lambda})$ and $\mathcal{W}_{s,\epsilon}(\bar{\xi}_+,\bar{\lambda})$ in case of a nondegenerate quadratic $f(\cdot,\bar{\lambda})$–tangential $\bar{x}_{\mathbb{Z}}$.

Throughout this paper we will not directly use the geometric definitions of the properties of heteroclinics given above. We will rather give analytic definitions with respect to the partial derivatives of Γ. In Appendix B we show that both kinds of definition are equivalent.

In Section 2 we review and extend results on exponential dichotomies of linear difference equations. Bounded solutions of these equations occur as nullspaces of the partial derivative $\Gamma_x(\bar{x}_{\mathbb{Z}},\bar{\lambda})$.

In Section 3 we state and analyze our general method for the approximation of nondegenerate tangential heteroclinics. We consider single orbits as well as branches of such orbits. For these branches we derive estimates on the derivatives as in the paper [7] of Descloux and Rappaz. We will show that this general method applies directly to the approximation of transversal heteroclinics. This means that our paper includes and extends the results given in [5, Section 3].

In Section 4 we state and analyze our method for the approximation of quadratic 1–tangentials. This is done by applying our general method stated in Section 3.

In Section 5 we apply our approximation method to Hénon's and to Duffing's map. In both examples we observe the interesting global phenomena of closed loops of (transversal) homoclinics. Following the branches of the corresponding tangencies we analyze where these loops rise from.

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General notations

Throughout this paper we use the notations and settings introduced in the previous subsection. In case $r=0$ the parameter λ, and in case $p=-k$ the map b, are to be neglected. Furthermore, for $\lambda \in U_\epsilon(\bar{\lambda})$ and $\kappa \in \{s,u\}$ let $X_{\pm\kappa}(\lambda)$ be the stable (in case $\kappa = s$) and the unstable (in case $\kappa = u$) subspace of $f_x(\xi_\pm(\lambda),\lambda)$. Here the stable (unstable) subspace of a square matrix is the sum of all generalized eigenspaces corresponding to eigenvalues inside (outside) the unit circle. Taking $\epsilon > 0$ sufficiently small, the upper semi continuity of spectra of matrices implies that $\dim X_\kappa(\lambda)$ is constant for all $\lambda \in U_\epsilon(\bar{\lambda})$ and fixed $\kappa \in \{-s,-u,+s,+u\}$. Notice, $k_\kappa = \dim X_\kappa(\bar{\lambda})$ holds for every κ as above.

Let $\alpha_s, \alpha_u, \alpha_{\pm s}, \alpha_{\pm u} > 0$ be constants such that

$$\sigma(f_x(\xi_\pm(\lambda),\lambda)) \cap \{c \in \mathbb{C} : e^{-\alpha_{\pm s}} \le |c| \le e^{\alpha_{\pm u}}\} = \emptyset$$

holds where $\sigma(A)$ denotes the spectrum of a square matrix A. Sometimes we will omit the indices $+,-$ which means that we consider the case $\bar{\xi}_- = \bar{\xi}_+$ where the maps $\xi_\pm$, the subspaces $X_{\pm s}$, $X_{\pm u}$ and the constants $\alpha_{\pm s}, \alpha_{\pm u}$ coincide.

Throughout we consider the Banach space $S_J(X)$ of bounded sequences on J given by

$$S_J(X) = \{(x_n)_{n\in J} = x_J \in X^J : \|x_J\|_\infty = \sup_{n\in J} \|x_n\| < \infty\}$$

where $J \subset \mathbb{Z}$ is a discrete interval and $X = (X, \|\cdot\|)$ is a Banach space. For $x_J \in S_J(X)$ and $I \subset J$ we denote $x_{|I} = (x_n)_{n\in I}$. We write S_J for $S_J(\mathbb{R}^k)$. In this paper we use the maximum norm in products of normed spaces.

For normed spaces X, Y and $s \in \mathbb{N}\setminus\{0\}$ let $\mathcal{L}^s(X,Y)$ be the space of bounded s–linear maps from X^s to Y, equipped with the standard operator norm. Denote $\mathcal{L}^1(X,Y)$ by $\mathcal{L}(X,Y)$. For $x_1,\dots,x_s \in X$ let $[x_1,\dots,x_s] \in \mathcal{L}(\mathbb{R}^s, X)$ be defined by $[x_1,\dots,x_s](v_1,\dots,v_s) = \sum_{i=1}^s v_i x_i$, $(v_1,\dots,v_s) \in \mathbb{R}^s$.

For X as above we denote the open and closed ball of radius $\delta > 0$ centered at $x \in X$ by $U_\delta(x)$ and $B_\delta(x)$, respectively.

For a map g depending on some variables, say (x,y), we denote the partial derivative of g with respect to the first variable x by $g_x = D_x g = D_1 g = \frac{\partial}{\partial x} g$. Other partial derivatives are written analogously. We will denote the range of g by $\mathcal{R}(g)$ and its nullspace (pre–image of 0) by $\mathcal{N}(g)$.

Let us close this section with an remark on the properties of f and b. For the sake of convenience we assume that f is a globally defined $\mathcal{C}^l$–map and $f(\cdot,\lambda)$ is a global diffeomorphism for every $\lambda \in U_\epsilon(\bar{\lambda})$. Both assumptions are only used when we consider global stable and unstable manifolds in Appendix B. For our approximation theory developed in the Sections 3 and 4 we only need that f has these properties in some neighborhood U of $\{\bar{x}_n : n \in \mathbb{Z}\} \times \{\bar{\lambda}\}$. Therefore, it suffices to assume that $f_x(\bar{x}_n,\bar{\lambda})$ is nonsingular for every $n \in \mathbb{Z}$ and that f is of type $\mathcal{C}^l$ in U. Moreover we only need that b is of type $\mathcal{C}^l$ in some neighborhood of $(\bar{\xi}_-, \bar{\xi}_+, \bar{\lambda})$.

2 Preliminaries on linear difference equations

Let us review some basic tools from the theory of exponential dichotomies first. Here we use two exponents (α_s, α_u) which will give us better error estimates later on. With the exception of the roughness statements in Lemma 2.3 the results in this section are proved similarly to [18], [5] where only one exponent α is used. Therefore we do not carry out these proofs in detail.

For a discrete interval $J = [n_-, n_+]$, $n_\pm \in \mathbb{Z} \cup \{\pm\infty\}$, consider the homogeneous linear difference equation

$$u_{n+1} = A_n u_n, \quad n \in \hat{J}, \tag{8}$$

with $A_n \in \mathbb{R}^{k,k}$ for $n \in J$. If all matrices A_n, $n \in J$, are nonsingular, then

$$\Phi(n,m) = \begin{cases} A_{n-1} \cdot \ldots \cdot A_m & \text{if } n > m, \\ I & \text{if } n = m, \\ A_n^{-1} \cdot \ldots \cdot A_{m-1}^{-1} & \text{if } n < m, \end{cases}$$

defines the solution operator $\Phi : J^2 \to \mathbb{R}^{k,k}$ of equation (8).

Definition 2.1 *Equation* (8) *has an* exponential dichotomy *on J, if all A_n, $n \in J$, are nonsingular and there exist constants $K, \alpha_s, \alpha_u > 0$ and projectors $P_n \in \mathbb{R}^{k,k}$, $n \in J$, such that for all $n, m \in J$*

$$\Phi(n,m) P_m = P_n \Phi(n,m) \tag{9}$$

and

$$\begin{aligned} \|\Phi(n,m)P_m\| &\leq Ke^{-\alpha_s(n-m)} && \text{in case } n \geq m, \\ \|\Phi(n,m)(I - P_m)\| &\leq Ke^{-\alpha_u(m-n)} && \text{in case } m \geq n \end{aligned}$$

hold.

We call the tuple $(K, \alpha_s, \alpha_u, P_J)$ the data of the exponential dichotomy, K its constant, (α_s, α_u) its exponents, and P_J its projectors. If $P = P_n$, $n \in J$, is constant, we just write P for the projectors.

Let equation (8) have an exponential dichotomy on J with projectors P_J and let Φ be its solution operator. Then we call the map $G : J^2 \to \mathbb{R}^{k,k}$, defined by

$$G(n,m) = \begin{cases} \Phi(n,m) P_m & \text{if } n \geq m, \\ \Phi(n,m)(P_m - I) & \text{if } n < m, \end{cases}$$

the *Green's function of* (8) *with respect to the projectors* P_J. Using these functions, we can describe the solutions of inhomogeneous difference equations as stated in the following lemma. It can be read from the proof of [18, Lemma 2.7].

Lemma 2.2 *Let* (8) *have an exponential dichotomy on $J = [n_-, n_+]$, $n_\pm \in \mathbb{Z} \cup \{\pm n_\pm\}$, with data $(K, \alpha_s, \alpha_u, P_J)$, and let the Green's function G be defined as above. Then for any $r_J \in S_J$, $\mu_- \in \mathcal{R}(P_{n_-})$ and $\mu_+ \in \mathcal{N}(P_{n_+})$ the system*

$$\begin{aligned} u_{n+1} - A_n u_n &= r_n, \quad n \in \hat{J}, \\ (P_{n_-} u_{n_-}, (I - P_{n_+}) u_{n_+}) &= (\mu_-, \mu_+) \end{aligned}$$

has a unique solution $\bar{u}_J$ in S_J given by

$$\bar{u}_n = \Phi(n,n_-)P_{n_-}\mu_- + \Phi(n,n_+)(I - P_{n_+})\mu_+ + \sum_{m\in\hat{J}} G(n,m+1)r_m, \quad n \in J.$$

Moreover,

$$\begin{aligned}\|\bar{u}_n\| &\leq K(e^{-\alpha_s(n-n_-)}\|\mu_-\| + e^{-\alpha_u(n_+-n)}\|\mu_+\| \\ &\quad + \sum_{m=n_-}^{n-1} e^{-\alpha_s(n-1-m)}\|r_m\| + \sum_{m=n}^{n_+-1} e^{-\alpha_u(m+1-n)}\|r_m\|) \\ &\leq K(e^{-\alpha_s(n-n_-)}\|\mu_-\| + e^{-\alpha_u(n_+-n)}\|\mu_+\| + \frac{1+e^{-\alpha}}{1-e^{-\alpha}}\|r_J\|_\infty)\end{aligned}$$

holds for every $n \in J$ with $\alpha = \min\{\alpha_s, \alpha_u\}$.

In the previous statement all terms which refer to μ_- and μ_+ are to be neglected in case $n_- = -\infty$ and $n_+ = +\infty$, respectively.

Consider the linear equation

$$u_{n+1} = (A_n + E_n)u_n, \quad n \in J, \tag{10}$$

with a perturbation $E_J = (E_n)_{n\in J} \in S_J(\mathbb{R}^{k,k})$. It is well known that for sufficiently small E_J the perturbed equation (10) keeps its exponential dichotomy, cf. Palmer's roughness result [18, Proposition 2.10].

In the following lemma we show that the solutions of the homogeneous equation (10) are smooth with respect to the perturbation E_J. Moreover, the derivatives of these solutions with respect to E_J satisfy the same exponential estimates as the solutions do. The following lemma is a discrete analogue of Sandstede's result [20, Lemma 1] for continuous systems. In our proof we directly obtain a Green's function of the perturbed equation (10) as a fixed point of a single operator. This simplifies Sandstede's proof which employs the composition of two operators.

Lemma 2.3 *Suppose (8) has an exponential dichotomy on $J = [n_-, n_+]$, $n_\pm \in \mathbb{Z} \cup \{\pm n_\pm\}$, with data $(K, \alpha_s, \alpha_u, P_J)$. Let $\beta_s \in (0, \alpha_s)$, $\beta_u \in (0, \alpha_u)$, and let $\varepsilon_0 > 0$ be such that $A_n + E$ is nonsingular for every $n \in J$ and every $E \in \mathbb{R}^{k,k}$ with $\|E\| \leq \varepsilon_0$. Take $\tau = K(\frac{1}{e^{\beta_u}-e^{-\alpha_s}} + \max\{\frac{1}{e^{-\beta_s}-e^{-\alpha_s}}, \frac{1}{e^{\alpha_u}-e^{\beta_u}}\} + \frac{1}{e^{\alpha_u}-e^{-\beta_s}})$, $\varepsilon = \min\{\frac{1}{2\tau}, \varepsilon_0\}$. Then for any $E_J \in S_J(\mathbb{R}^{k,k})$ with $\|E_n\| < \varepsilon$, $n \in J$, the perturbed equation (10) has an exponential dichotomy on J with data $(K_0, \beta_s, \beta_u, Q_J(E_J))$. The projectors $Q_J(E_J)$ satisfy*

$$\mathcal{R}(Q_{n_+}(E_J)) = \mathcal{R}(P_{n_+}), \quad \mathcal{N}(Q_{n_-}(E_J)) = \mathcal{N}(P_{n_-}) \tag{11}$$

in case $n_- \in \mathbb{Z}$ and $n_+ \in \mathbb{Z}$, respectively. If $\Psi(E_J, \cdot, \cdot)$ is the solution operator of (10), then $\Psi(\cdot, n, m), Q_n(\cdot) : S_J(\mathbb{R}^{k,k}) \to \mathbb{R}^{k,k}$ are $\mathcal{C}^\infty$–maps for every $n, m \in J$, *and*

$$\begin{aligned}
\|\frac{\partial^i}{\partial(E_J)^i}(\Psi(E_J,n,m)Q_m(E_J))\| &\leq K_i e^{-\beta_s(n-m)}, && (12)\\
\|\frac{\partial^i}{\partial(E_J)^i}(\Psi(E_J,m,n)(I-Q_n(E_J)))\| &\leq K_i e^{-\beta_u(n-m)} && (13)
\end{aligned}$$

hold for every $i = 0,1,2,\dots$ *and* $n,m \in J$ *with* $n \geq m$*, where* $K_0 = 2K+1$ *and* $K_i = 2^{i+1} i! \tau^i K$ *for* $i = 1,2,\dots$.

Proof. We consider the following Banach space $Z_J^{\beta_s,\beta_u}$ with an exponentially weighted norm $\|\cdot\|_*$:

$$Z_J^{\beta_s,\beta_u} = \{X \in (\mathbb{R}^{k,k})^{J\times J} \quad : \quad \|X\|_* = \max\{\sup_{n\geq m} e^{\beta_s(n-m)}\|X(n,m)\|, \\ \sup_{n<m} e^{\beta_u(m-n)}\|X(n,m)\|\} < \infty\}.$$

For $X \in Z_J^{\beta_s,\beta_u}$ and $E_J \in S_J(\mathbb{R}^{k,k})$ let

$$\begin{aligned}
T_1(X,E_J) &= (\sum_{l\in\hat{J}} G(n,l+1)E_l X(l,m))_{(n,m)\in J\times J},\\
T(X,E_J) &= G + T_1(X,E_J)
\end{aligned}$$

where G is the Green's function of (8) with respect to the projectors P_J. Obviously, T_1 is bilinear. A calculation shows that the maps T_1, T are of type $\mathcal{C}^\infty(Z_J^{\beta_s,\beta_u} \times S_J(\mathbb{R}^{k,k}), Z_J^{\beta_s,\beta_u})$, and the norm of the bilinear map T_1 satisfies $\|T_1\| \leq \tau$. This means that for fixed E_J with $\|E_J\| \leq \varepsilon$ the operator $T(\cdot,E_J)$ is a global contraction with uniform rate $\frac{1}{2}$. By [6, Theorem 2.2] this operator $T(\cdot,E_J)$ has a unique fixed point $\bar{X}(E_J) \in Z_J^{\beta_s,\beta_u}$, and $\bar{X} \in \mathcal{C}^\infty(U_\varepsilon(0), Z_J^{\beta_s,\beta_u})$ holds. From

$$\bar{X}(E_J) = G + T_1(\bar{X}(E_J),E_J) \tag{14}$$

the estimates

$$\|\bar{X}^{(i)}(E_J)\|_* \leq 2^{i+1} i! \tau^i K, \quad i = 0,1,2,\dots \tag{15}$$

follow by induction.

It remains to show that $\bar{X}$ is a Green's function of the perturbed equation (10). Let $H(n,m) = \delta_{n,m-1}I$, $n,m \in J$, where $\delta_{n,m}$ is the Kronecker symbol. Notice that Lemma 2.2 applies to matrix equations also, i.e. with $u_n, r_J \in S_J(\mathbb{R}^{k,k})$, $\mu_- \in (\mathcal{R}(P_{n_-}))^k$, $\mu_+ \in (\mathcal{N}(P_{n_+}))^k$ and some adapted constant K. Applying the lemma in this way for every pair $(X,E_J) \in Z_J^{\beta_s,\beta_u} \times S_J(\mathbb{R}^{k,k})$, the function X is a fixed point of $T(\cdot,E_J)$ if and only if

$$\begin{aligned}
X(n+1,m) &= (A_n+E_n)X(n,m) + H(n,m), && (16)\\
P_{n_-}X(n_-,m) &= P_{n_-}G(n_-,m) \quad \text{in case } n_- \in \mathbb{Z}, && (17)\\
(I-P_{n_+})X(n_+,m) &= 0 \quad \text{in case } n_+ \in \mathbb{Z} && (18)
\end{aligned}$$

hold for all $(n,m) \in \hat{J} \times J$. For the proof of this equivalence rewrite (16) as $X(n+1,m) = A_nX(n,m) + r_n$ where $r_n = E_nX(n,m) + H(n,m)$, and apply Lemma 2.2 for each fixed $m \in J$ with $u_n = X(n,m)$, $n \in J$.

For $E_J \in U_\varepsilon(0)$ the function $\bar{X}(E_J)$ is the unique fixed point of $T_1(\cdot, E_J)$, and hence it is the unique solution of (16)–(18). Palmer shows in the proof of [18, Proposition 2.10] that this solution is the Green's function of (10) with respect to the projectors $Q_J(E_J)$ which are defined by

$$Q_n(E_J) = \bar{X}(E_J, n, n), \quad n \in J.$$

Therefore $\Psi(E_J, n, m)Q_m(E_J) = \bar{X}(E_J, n, m)$, $n \geq m$, and $\Psi(E_J, n, m)(I - Q_m(E_J)) = \bar{X}(E_J, n, m)$, $n < m$, hold for $n, m \in J$. By definition of the weighted norm $\|\cdot\|_*$, the estimates (12), (13) follow from (15). The rank of $Q_n(E_J)$ equals that of P_n, because $Q_n(0) = P_n$ holds by definition of $\bar{X}$ and $Q_n(E_J)$ depends continuously on E_J. Finally, the equalities in (11) for the range and nullspace of $Q_{n_+}(E_J)$ and $Q_{n_-}(E_J)$ follow from the fixed point equation (14) and the definition of T_1 and G. $\square$

The simplest case of a linear difference equation which has an exponential dichotomy occurs if $A = A_n$ is constant and A has no eigenvalue on the unit circle. A nonsingular matrix having this property is called *hyperbolic*. By Palmer's proof of [18, Remark 2.5] the following result holds.

Proposition 2.4 *The equation*

$$u_{n+1} = Au_n \tag{19}$$

with some matrix $A \in \mathbb{R}^{k,k}$ has an exponential dichotomy on $\mathbb{Z}$ if and only if A is hyperbolic. In this case let $\alpha_s, \alpha_u > 0$ satisfy $\sigma(A) \cap \{z \in \mathbb{C} : e^{-\alpha_s} \leq |z| \leq e^{\alpha_u}\} = \emptyset$, and let $P \in \mathbb{R}^{k,k}$ be the projector with the stable and unstable subspace of A as range and nullspace, respectively. Then there exists a constant $K > 0$ such that $(K, \alpha_s, \alpha_u, P)$ are the data of the exponential dichotomy of (19).

As Palmer shows in [18, Remark 2.2] exponential dichotomies can be extended over finite intervals J without changing the exponents. Furthermore, for an exponential dichotomy on $\mathbb{Z}_+$ with projectors $P_{\mathbb{Z}_+}$, the range of P_n, $n \in \mathbb{Z}_+$, is uniquely determined and its nullspace can be chosen freely as long as (9) is not violated. The same holds for dichotomies on $\mathbb{Z}_-$ where the nullspaces of the projectors are uniquely determined whereas its ranges are free, see [18, Proposition 2.3]. By the sharper version [13, Lemma 1.1.2] of [5, Lemma 2.2] such variations of the projectors do not change the exponents (α_s, α_u) in the dichotomy data — they just change the constant K. Therefore these results and Lemma 2.3 imply the following collection of statements. For the convergence of the projectors we refer to [5, Propostion 2.5].

Lemma 2.5 *Let $A^\pm \in \mathbb{R}^{k,k}$ be hyperbolic and $\sigma(A^\pm) \cap \{z \in \mathbb{C} : e^{-\alpha_{\pm s}} \leq |z| \leq e^{\alpha_{\pm u}}\} = \emptyset$ hold for some $\alpha_{\pm s}, \alpha_{\pm u} > 0$. Suppose the matrices $A_n^\pm$, $n \in \mathbb{Z}_\pm$, to be nonsingular with $\lim_{n\to\pm\infty} A_n^\pm = A^\pm$. Furthermore let the projectors $P^\pm \in \mathbb{R}^{k,k}$ have the stable and unstable subspace of $A^\pm$ as range and nullspace, respectively. Then the equation*

$$u_{n+1} = A_n^\pm u_n \tag{20}$$

has an exponential dichotomy on $\mathbb{Z}_\pm$ *with data* $(K^\pm, \alpha_{\pm s}, \alpha_{\pm u}, P^\pm_{\mathbb{Z}_\pm})$. *The projectors* $P^\pm_{\mathbb{Z}_\pm}$ *are characterized only by* $P^\pm_n = \Phi^\pm(n,0)P^\pm_0\Phi^\pm(0,n)$, $n \in \mathbb{Z}_\pm$, *and*

$$\begin{aligned}\mathcal{N}(P_0^-) &= \{x \in \mathbb{R}^k : \sup_{n\in\mathbb{Z}_-} \|\Phi^-(n,0)x\| < \infty\},\\ \mathcal{R}(P_0^+) &= \{x \in \mathbb{R}^k : \sup_{n\in\mathbb{Z}_+} \|\Phi^+(n,0)x\| < \infty\}\end{aligned}$$

where $\Phi^\pm$ *is the solution operator of* (20). *Moreover* $\lim_{n\to\pm\infty} P^\pm_n = P^\pm$ *holds, and* $P^\pm$, $P^\pm_n$ *are of equal rank for every* $n \in \mathbb{Z}_\pm$.

We close this section on linear systems with the following lemma from [5, Lemma 2.3]. Here we refer also to the internals of the proof given there. The additional result on L^* follows by an application of [5, Lemma 2.3] to the adjoint equation.

Lemma 2.6 *Suppose that* $u_{n+1} = A_n u_n$ *has exponential dichotomies on* $\mathbb{Z}_+$ *and on* $\mathbb{Z}_-$ *with data* $(K^\pm, \alpha_{\pm s}, \alpha_{\pm u}, P^\pm_{\mathbb{Z}_\pm})$. *Let* Φ *be the solution operator of this equation. Then the adjoint equation* $\varphi_{n+1} = (A_n^{-1})^T\varphi_n$ *has exponential dichotomies on* $\mathbb{Z}_\pm - 1$ *with data* $(CK^\pm, \alpha_{\pm u}, \alpha_{\pm s}, (I - (P^\pm_{n+1})^T)_{n\in\mathbb{Z}_\pm-1})$ *where the constant* $C > 0$ *depends only on the norm chosen in* $\mathbb{R}^k$. *The solution operator of this equation is* $\Phi^*(n,m) = \Phi(m+1,n+1)^T$, $n, m \in \mathbb{Z}$.

If A_n *and* A_n^{-1} *are bounded uniformly with respect to* $n \in \mathbb{Z}$, *the following statements hold.*

(i) *The linear operator*

$$L : S_\mathbb{Z} \to S_\mathbb{Z},\ u_\mathbb{Z} \mapsto (u_{n+1} - A_n u_n)_{n\in\mathbb{Z}}$$

is Fredholm of index $\mathrm{rank}(P_0^+) - \mathrm{rank}(P_0^-)$. *Moreover*

$$\mathcal{N}(L) = \{(\Phi(n,0)x)_{n\in\mathbb{Z}} : x \in \mathcal{R}(P_0^+) \cap \mathcal{N}(P_0^-)\}$$

and $\mathrm{codim}\mathcal{R}(L) = \mathrm{codim}(\mathcal{R}(P_0^+) + \mathcal{N}(P_0^-))$ *hold.*

(ii) *The operator*

$$L^* : S_\mathbb{Z} \to S_\mathbb{Z},\ \varphi_\mathbb{Z} \mapsto (\varphi_{n+1} - (A_n^{-1})^T\varphi_n)_{n\in\mathbb{Z}}$$

is Fredholm of index $\mathrm{rank}(P_0^-) - \mathrm{rank}(P_0^+)$, *and* $\dim\mathcal{N}(L^*) = \mathrm{codim}\mathcal{R}(L)$,

$$r_\mathbb{Z} \in \mathcal{R}(L) \quad \Longleftrightarrow \quad \sum_{n\in\mathbb{Z}} \varphi_n^T r_n = 0 \text{ for all } \varphi_\mathbb{Z} \in \mathcal{N}(L^*) \qquad (21)$$

hold for $r_\mathbb{Z} \in S_\mathbb{Z}$.

3 Approximation of nondegenerate tangencies

3.1 Approximation of single tangencies

We consider the $\mathcal{C}^l$–maps f and Γ, and the $f(\cdot,\bar\lambda)$–heteroclinic $\bar x_{\mathbb{Z}}$ introduced in (6) with $l \geq 1$.

Definition 3.1 *The heteroclinic $\bar x_{\mathbb{Z}}$ is called* t–tangential *for some $t \in \mathbb{N}$ if*

$$\dim \mathcal{N}(\Gamma_x(\bar x_{\mathbb{Z}},\bar\lambda)) = t$$

holds.

Furthermore, a t–tangential heteroclinic $\bar x_{\mathbb{Z}}$ is called nondegenerate *if*

$$r = p + t \tag{22}$$

(for the definition of p see (3)*) and*

$$\dim \mathcal{N}(\Gamma_{(x,\lambda)}(\bar x_{\mathbb{Z}},\bar\lambda)) = t$$

hold. If $r > p+t$ and the parameter has the form $\lambda = (\lambda_1,\lambda_2) \in \mathbb{R}^{p+t} \times \mathbb{R}^{r-p-t}$ then $\bar x_{\mathbb{Z}}$ is called nondegenerate with respect to λ_1 if $\dim \mathcal{N}(\Gamma_{(x,\lambda_1)}(\bar x_{\mathbb{Z}},\bar\lambda_1,\bar\lambda_2)) = t$ holds.

Notice, by Lemma 2.6(i) the derivative $\Gamma_x(\bar x_{\mathbb{Z}},\bar\lambda)$ is Fredholm of index $-p$. Therefore, in case of a t–tangential $\bar x_{\mathbb{Z}}$, the Fredholm property of $\Gamma_x(\bar x_{\mathbb{Z}},\bar\lambda)$ implies that the range of this operator is of codimension $p+t$. Hence $p+t \geq 0$ is valid.

By the bordering lemma ([3, Lemma 2.3]) the derivative $\Gamma'(\bar x_{\mathbb{Z}},\bar\lambda)$ is Fredholm of index t if $\bar x_{\mathbb{Z}}$ is t–tangential and nondegenerate. Adding another t conditions to $\Gamma'(\bar x_{\mathbb{Z}},\bar\lambda)$, we get a Fredholm operator of index 0, and therefore the following result holds.

Theorem 3.2 *Let $\bar x_{\mathbb{Z}}$ be t–tangential and $r = p+t$. Suppose $\Psi \in \mathcal{C}^1(S_{\mathbb{Z}} \times \mathbb{R}^r, \mathbb{R}^t)$ satisfies $\Psi(\bar x_{\mathbb{Z}},\bar\lambda) = 0$. Then the following statements are equivalent.*

(i) *The orbit $\bar x_{\mathbb{Z}}$ is nondegenerate and $\Psi'(\bar x_{\mathbb{Z}},\bar\lambda)_{|\mathcal{N}(\Gamma'(\bar x_{\mathbb{Z}},\bar\lambda))}$ is nonsingular.*

(ii) *The pair $(\bar x_{\mathbb{Z}},\bar\lambda)$ is a regular zero of (Γ,Ψ), i.e. $(\Gamma,\Psi)(\bar x_{\mathbb{Z}},\bar\lambda) = 0$ and $(\Gamma,\Psi)'(\bar x_{\mathbb{Z}},\bar\lambda)$ is one–to–one.*

For the approximation of a nondegenerate t–tangential $\bar x_{\mathbb{Z}}$ and the parameter value $\bar\lambda$, consider the $\mathcal{C}^l$–operator $\Gamma_J = \Gamma_{f,b,J}$ defined in (7) with $b \in \mathcal{C}^l(\mathbb{R}^{2k+r},\mathbb{R}^{k+p})$.

Definition 3.3 *The map b defines* regular *boundary conditions at $(\bar\xi_-,\bar\xi_+,\bar\lambda)$ if $b(\bar\xi_-,\bar\xi_+,\bar\lambda) = 0$ holds and the linear map*

$$X_{-s}(\bar\lambda) \times X_{+u}(\bar\lambda) \to \mathbb{R}^{k+p},\ (x_s,x_u) \mapsto b'(\bar\xi_-,\bar\xi_+,\bar\lambda)(x_s,x_u,0)$$

is nonsingular.

It follows immediately that *periodic boundary conditions* defined by

$$b(x,y,\lambda) = b_{per}(x,y,\lambda) = x - y, \quad x,y \in \mathbb{R}^k, \lambda \in \mathbb{R}^r,$$

are regular at $(\bar{\xi},\bar{\xi},\bar{\lambda})$ in the homoclinic case.

Next we construct *projection boundary conditions.* Due to [3, Appendix C] there exist $\mathcal{C}^{l-1}$–maps $Q_{-s} : V \to \mathbb{R}^{k,k-s}$ and $Q_{+u} : V \to \mathbb{R}^{k,k+u}$ for some neighborhood $V \subset U_\epsilon(\bar{\lambda})$ of $\bar{\lambda}$ such that $\mathcal{R}(Q_{-s}(\lambda))$ forms the stable subspace of $f_x(\xi_-(\lambda),\lambda)^T$ and $\mathcal{R}(Q_{+u}(\lambda))$ the unstable subspace of $f_x(\xi_+(\lambda),\lambda)^T$ for every $\lambda \in V$. Projection boundary conditions are defined by

$$b(x,y,\lambda) = b_{proj}(x,y,\lambda) = (Q_{-s}(\lambda)^T(x - \xi_-(\lambda)), Q_{+u}(\lambda)^T(y - \xi_+(\lambda))) \quad (23)$$

for $x,y \in \mathbb{R}^k$, $\lambda \in V$. Notice that $x \in \xi_-(\lambda) + X_{-u}(\lambda)$, $y \in \xi_+(\lambda) + X_{+s}(\lambda)$ hold if and only if $b_{proj}(x,y,\lambda) = 0$ for $\lambda \in V$. By the same proof as in [5, Section 3] projection boundary conditions are regular at $(\bar{\xi}_-,\bar{\xi}_+,\bar{\lambda})$ if $l \geq 2$.

Due to the theorem given above, it is not reasonable to approximate $(\bar{x}_{\mathbb{Z}},\bar{\lambda})$ by zeros of Γ_J without any additional constraint. Hence we consider the operator (Γ_J,Ψ_J) with $\mathcal{C}^{\max\{2,l\}}$–maps $\Psi_J : S_J \times \mathbb{R}^r \to \mathbb{R}^t$ for $J = [n_-,n_+]$, $n_\pm \in \mathbb{Z}_\pm$.

Definition 3.4 *We call a family of maps Ψ_J as above* regular at $(\bar{x}_{\mathbb{Z}},\bar{\lambda})$ *if the following three conditions are satisfied for some constants $C,N,\vartheta > 0$ and every $J = [n_-,n_+]$ with $-n_-,n_+ \geq N$.*

(i) $\lim_{\min\{-n_-,n_+\}\to\infty} \Psi_J(\bar{x}_{|J},\bar{\lambda}) = 0$

(ii) *The $t \times t$–matrix $[D_x\Psi_J(\bar{x}_{|J},\bar{\lambda})(\bar{u}^1_{|J}),\ldots,D_x\Psi_J(\bar{x}_{|J},\bar{\lambda})(\bar{u}^t_{|J})]$ is nonsingular with uniformly bounded inverse*

$$\|[D_x\Psi_J(\bar{x}_{|J},\bar{\lambda})(\bar{u}^1_{|J}),\ldots,D_x\Psi_J(\bar{x}_{|J},\bar{\lambda})(\bar{u}^t_{|J})]^{-1}\| \leq C.$$

Here $\{\bar{u}^1_{\mathbb{Z}},\ldots,\bar{u}^t_{\mathbb{Z}}\}$ denotes some fixed basis of $\mathcal{N}(\Gamma_x(\bar{x}_{\mathbb{Z}},\bar{\lambda}))$.

(iii) *The derivatives Ψ'_J and Ψ''_J are uniformly bounded by the constant C in the closed ball $B_\vartheta(\bar{x}_{|J},\bar{\lambda})$.*

By the following theorem, computing zeros of the operator (Γ_J,Ψ_J) is a regular method to approximate $(\bar{x}_{\mathbb{Z}},\bar{\lambda})$. Remember that $(\Gamma_J,\Psi_J)(x_J,\lambda) = 0$, $(x_J,\lambda) \in S_J \times \mathbb{R}^r$, reads

$$\begin{aligned} x_{n+1} - f(x_n,\lambda) &= 0, \quad n = n_-,\ldots,n_+ - 1, && (24)\\ b(x_{n_-},x_{n_+},\lambda) &= 0, && (25)\\ \Psi_J(x_J,\lambda) &= 0. && (26) \end{aligned}$$

Theorem 3.5 *Assume that $\bar{x}_{\mathbb{Z}}$ is a nondegenerate t–tangential. Furthermore, let b and the family Ψ_J be regular. Then there exist constants $\delta,C,N > 0$ such that for every $J = [n_-,n_+]$ with $-n_-,n_+ > N$ the problem (24)—(26) has a* *unique solution* (x_J, λ_J) *in the closed ball* $B_\delta(\bar{x}_{|J}, \bar{\lambda})$*. This solution satisfies the error estimate*

$$\|(x_J, \lambda_J) - (\bar{x}_{|J}, \bar{\lambda})\|_\infty \leq C \max\{\|b(\bar{x}_{n_-}, \bar{x}_{n_+}, \bar{\lambda})\|, \|\Psi_J(\bar{x}_{|J}, \bar{\lambda})\|\}. \qquad (27)$$

Moreover

$$\begin{aligned} \|(\Gamma_J, \Psi_J)'(y_J, \mu)^{-1}\|_\infty &\leq C, \\ \|(y_J, \mu) - (z_J, \nu)\|_\infty &\leq C\|(\Gamma_J, \Psi_J)(y_J, \mu) - (\Gamma_J, \Psi_J)(z_J, \nu)\|_\infty \end{aligned}$$

hold for every $(y_J, \mu), (z_J, \nu) \in B_\delta(\bar{x}_{|J}, \bar{\lambda})$.

Proof. By the regularity of b and Ψ_J

$$\lim_{\min\{-n_-, n_+\}\to\infty} \|(\Gamma_J, \Psi_J)(\bar{x}_{|J}, \bar{\lambda})\|_\infty = 0$$

holds. Moreover, the fact that Ψ_J'' is bounded uniformly in $B_\vartheta(\bar{x}_{|J}, \bar{\lambda})$ implies that for some $N_0 \in \mathbb{N}$ and every $\kappa > 0$ there exists a $\rho > 0$ such that

$$\|(\Gamma_J, \Psi_J)'(y_J, \mu) - (\Gamma_J, \Psi_J)'(\bar{x}_{|J}, \bar{\lambda})\|_\infty \leq \kappa$$

holds for every $-n_-, n_+ \geq N_0$ and $(y_J, \mu) \in B_\rho(\bar{x}_{|J}, \bar{\lambda})$. It remains to show that $(\Gamma_J, \Psi_J)'(\bar{x}_{|J}, \bar{\lambda})$ is nonsingular for sufficiently large $-n_-, n_+$ with some uniformly bounded inverse. Then the statements follow by the approximate inverse function theorem [5, Proposition 3.3] as in [5].

To show that $\|(\Gamma_J, \Psi_J)'(\bar{x}_{|J}, \bar{\lambda})^{-1}\|_\infty$ is uniformly bounded with respect to J we apply Lemma A.1 from Appendix A with $A_n = f_x(\bar{x}_n, \bar{\lambda})$, $A_\pm = f_x(\bar{\xi}_\pm, \bar{\lambda})$, $(B_J, C_J) = b'(\bar{x}_{n_-}, \bar{x}_{n_+}, \bar{\lambda})$, $(D_J, E_J) = \Psi_J'(\bar{x}_{|J}, \bar{\lambda})$ and $F_n = f_\lambda(\bar{x}_n, \bar{\lambda})$ for $n \in \hat{J}$. □

New we derive an estimate for the term $\|b(\bar{x}_{n_-}, \bar{x}_{n_+}, \bar{\lambda})\|$ in (27). For this purpose we introduce orders of boundary conditions similar to [5]. Consider the *local stable* and *unstable manifolds*

$$\begin{aligned} W_s^U(\bar{\xi}, \bar{\lambda}) &= \{x \in \mathbb{R}^k : f(\cdot, \bar{\lambda})^n(x) \in U, n \in \mathbb{N}, \text{ and } \lim_{n\to\infty} f(\cdot, \lambda)^n(x) = \bar{\xi}\}, \\ W_u^U(\bar{\xi}, \bar{\lambda}) &= \{x \in \mathbb{R}^k : f(\cdot, \bar{\lambda})^{-n}(x) \in U, n \in \mathbb{N}, \text{ and } \lim_{n\to\infty} f(\cdot, \lambda)^{-n}(x) = \bar{\xi}\} \end{aligned}$$

for some neighborhood U of $\bar{\xi}$. If we want to stress their dependency on f we write $W_\kappa^U(\bar{\xi}, f(\cdot, \bar{\lambda}))$ instead of $W_\kappa^U(\bar{\xi}, \bar{\lambda})$ for $\kappa = s, u$.

Definition 3.6 *Boundary conditions defined by* b *are called* of order (l_-, l_+) at $(\bar{\xi}_-, \bar{\xi}_+, \bar{\lambda})$ *with* $l_\pm \in \mathbb{N}$ *if*

$$\|b(x, y, \bar{\lambda})\| \leq C \max\{\|x - \bar{\xi}_-\|^{l_-}, \|y - \bar{\xi}_+\|^{l_+}\}$$

holds for some $C > 0$, *some neighborhoods* $U_\pm$ *of* $\bar{\xi}_\pm$ *and every* $x \in W_u^{U_-}(\bar{\xi}_-, \bar{\lambda})$, $y \in W_s^{U_+}(\bar{\xi}_+, \bar{\lambda})$.

By the same proofs as in [5, Section 3], periodic boundary conditions are of order $(1,1)$ at $(\bar{\xi},\bar{\xi},\bar{\lambda})$ and, in case $l \geq 2$, projection boundary conditions are of order $(2,2)$ at $(\bar{\xi}_-,\bar{\xi}_+,\bar{\lambda})$. In general, every regular boundary condition is (at least) of order $(1,1)$ due to Taylor's formula.

Using the order of a boundary condition and the fact that

$$\|\bar{x}_{n_-} - \bar{\xi}_-\| \leq Ce^{\alpha_{-u} n_-}, \quad \|\bar{x}_{n_+} - \bar{\xi}_+\| \leq Ce^{-\alpha_{+s} n_+}$$

hold for some $C > 0$ and $n_\pm \in \mathbb{Z}_\pm$ (see Lemma 3.12 in the next subsection) we get the following convergence estimates for our method.

Corollary 3.7 *Let b define boundary conditions of order (l_-,l_+). Then the statements of Theorem 3.5 are valid with the improved estimate*

$$\|(x_J,\lambda_J) - (\bar{x}_{|J},\bar{\lambda})\|_\infty \leq C\max\{e^{l_-\alpha_{-u}n_-}, e^{-l_+\alpha_{+s}n_+}, \|\Psi_J(\bar{x}_{|J},\bar{\lambda})\|\}.$$

3.2 Approximation of curves with tangencies

In this section let us consider the case $r = p+t+s$ with $s \geq 1$. We write the parameter $\lambda \in \mathbb{R}^r$ as $\lambda = (\lambda_r,\lambda_c) \in \mathbb{R}^{t+p} \times \mathbb{R}^s$. The component λ_r of the parameter will be used to get a regular problem, i.e. a nondegenerate t–tangential heteroclinic $\bar{x}_\mathbb{Z}$, while λ_c is used for the continuation of this orbit.

Throughout this section we assume that $\bar{x}_\mathbb{Z}$ is t–tangential and nondegenerate with respect to λ_r, cf. Definition 3.1. Moreover, let $\Psi : S_\mathbb{Z}\times\mathbb{R}^r \to \mathbb{R}^t$ be a $\mathcal{C}^l$–map such that $\Psi(\bar{x}_\mathbb{Z},\bar{\lambda}_r,\bar{\lambda}_c) = 0$ holds and $\Psi_x(\bar{x}_\mathbb{Z},\bar{\lambda}_r,\bar{\lambda}_c)_{|\mathcal{N}(\Gamma_x(\bar{x}_\mathbb{Z},\bar{\lambda}_r,\bar{\lambda}_c))}$ is nonsingular. Due to Theorem 3.2, the implicit function theorem and Lemma 3.12 given below, the following result is valid.

Theorem 3.8 *Under the assumptions given above and in the introduction, there exist constants $C,\rho_0,\delta_0 > 0$ and a map*

$$(x_\mathbb{Z}^\infty,\lambda_r^\infty) \in \mathcal{C}^l(U_{\rho_0}(\bar{\lambda}_c), U_{\delta_0}(\bar{x}_\mathbb{Z},\bar{\lambda}_r))$$

such that for every $\lambda_c \in U_{\rho_0}(\bar{\lambda}_c)$ the pair $(x_\mathbb{Z}^\infty,\lambda_r^\infty)(\lambda_c)$ is the unique zero of $(\Gamma,\Psi)(\cdot,\cdot,\lambda_c)$ in $U_{\delta_0}(\bar{x}_\mathbb{Z},\bar{\lambda}_r)$. Moreover $x_\mathbb{Z}^\infty(\lambda_c)$ is a $f(\cdot,\lambda_r^\infty(\lambda_c),\lambda_c)$–heteroclinic connecting $\xi_-(\lambda_r^\infty(\lambda_c),\lambda_c)$ and $\xi_+(\lambda_r^\infty(\lambda_c),\lambda_c)$. Furthermore, for $(x_\mathbb{Z},\lambda_r) \in U_{\delta_0}(\bar{x}_\mathbb{Z},\bar{\lambda}_r)$, the estimate $\|D_{(x,\lambda_r)}(\Gamma,\Psi)(x_\mathbb{Z},\lambda_r,\lambda_c)^{-1}\|_\infty \leq C$ holds.

For the local approximation of the branch $x_\mathbb{Z}^\infty(\lambda_c)$, $\lambda_c \in U_{\rho_0}(\bar{\lambda}_c)$, near $\lambda_c = \bar{\lambda}_c$, let us assume $l \geq 2$ in the remaining part of this section. Let b be regular at $(\bar{\xi}_-,\bar{\xi}_-,\bar{\lambda}_r,\bar{\lambda}_c)$ and $\Psi_J \in \mathcal{C}^l(S_J\times\mathbb{R}^r,\mathbb{R}^t)$, $J = [n_-,n_+]$ with $n_\pm \in \mathbb{Z}_\pm$, be regular at $(\bar{x}_\mathbb{Z},\bar{\lambda})$. Furthermore, assume that for some $N,\vartheta > 0$ and every $i = 0,\dots,l$, $J = [n_-,n_+]$ with $-n_-,n_+ \geq N_0$ the derivatives $\Psi_J^{(i)}$ are uniformly bounded in the ball $U_\vartheta(\bar{x}_{|J},\bar{\lambda}_r,\bar{\lambda}_c)$.

Due to the assumptions given above

$$\lim_{\min\{-n_-,n_+\}\to\infty} \|(\Gamma_J,\Psi_J)(\bar{x}_{|J},\bar{\lambda}_r,\bar{\lambda}_c)\|_\infty = 0$$

holds. By Theorem 3.5 the inverses of $D_{(x,\lambda_r)}(\Gamma_J, \Psi_J)(\bar{x}_{|J}, \bar{\lambda}_r, \bar{\lambda}_c)$ as well as $(\Gamma_J, \Psi_J)^{(i)}(x_J, \lambda_r, \lambda_c)$, $i = 0, \ldots, l$, $(x_J, \lambda_r, \lambda_c) \in U_\vartheta(\bar{x}_{|J}, \bar{\lambda}_r, \bar{\lambda}_c)$, are uniformly bounded. Therefore, the qualitative version [7, Theorem 2.1, 2.2] of the implicit function theorem (applied with $G = (\Gamma_J, \Psi_J)$, $x_0 = \bar{\lambda}_c$, $y_0 = (\bar{x}_{|J}, \bar{\lambda}_r)$, $\delta = \vartheta$ and $s = (x^\infty_{|J}, \lambda^\infty_r)$, $\alpha = id$) implies the following result.

Theorem 3.9 *Let the assumptions given above and the setting of Theorem 3.8 hold. Then there exist constants $\rho \in (0, \rho_0]$, $\delta \in (0, \delta_0]$, $C > 0$, $N \in \mathbb{N}$, and for every $J = [n_-, n_+]$ with $-n_-, n_+ \geq N$ a $\mathcal{C}^l$–map $(x_J, \lambda_{r,J}) : U_\rho(\bar{\lambda}_c) \to U_\delta(\bar{x}_{|J}, \bar{\lambda}_r)$ such that for every $\lambda_c \in U_\rho(\bar{\lambda}_c)$ the pair $(x_J, \lambda_{r,J})(\lambda_c)$ is the unique zero of $(\Gamma_J, \Psi_J)(\cdot, \cdot, \lambda_c)$ in $U_\delta(\bar{x}_{|J}, \bar{\lambda}_r)$. Moreover*

$$\|D_{(x,\lambda_r)}(\Gamma_J, \Psi_J)(y_J, \lambda_r, \lambda_c)^{-1}\|_\infty \leq C, \quad (y_J, \lambda_r) \in U_\delta(\bar{x}_{|J}, \bar{\lambda}_r),$$

and

$$\|\frac{\partial^i}{\partial \lambda_c^i}((x_J, \lambda_{r,J}) - (x^\infty_{|J}, \lambda^\infty_r))(\lambda_c)\|_\infty \leq C \sum_{j=0}^{i} \|H_J^{(i)}(\lambda_c)\|, \quad i = 0, \ldots, l-1, \tag{28}$$

hold where the $\mathcal{C}^l$–map $H_J : U_\rho(\bar{\lambda}_c) \to \mathbb{R}^{k+p+t}$ is defined by

$$H_J(\lambda_c) = (b(x^\infty_{n_-}(\lambda_c), x^\infty_{n_+}(\lambda_c), \lambda^\infty_r(\lambda_c), \lambda_c), \Psi_J(x^\infty_{|J}(\lambda_c), \lambda^\infty_r(\lambda_c), \lambda_c))$$

for $\lambda_c \in U_\rho(\bar{\lambda}_c)$.

In the remaining part of this section we will analyze the right side of (28). More precisely, we will derive estimates for the term

$$\frac{\partial^i}{\partial \lambda_c^i} b(x^\infty_{n_-}(\lambda_c), x^\infty_{n_+}(\lambda_c), \lambda^\infty_r(\lambda_c), \lambda_c)$$

where $i \in \{0, \ldots, l-1\}$ and $\lambda_c \in U_\rho(\bar{\lambda}_c)$. For this purpose we generalize the notion of the order of a boundary condition. In contrast to the assumption $r = p + t + s$ used above the following definition applies for every $r \geq 0$.

Definition 3.10 *The map $b : \mathbb{R}^{2k+r} \to \mathbb{R}^{k+p}$ defines boundary conditions of $\mathcal{C}^q$–*order *(l_-, l_+) at $(\bar{\xi}_-, \bar{\xi}_+, \bar{\lambda})$ for some $q, l_-, l_+ \in \mathbb{N}$ if there exist neighborhoods $V_\pm$ of $\bar{\xi}_\pm$, U of $\bar{\lambda}$ and $\mathcal{C}^q$–maps $B_\pm : V_\pm \times U \to \mathcal{L}^{l_\pm}(\mathbb{R}^k, \mathbb{R}^{k+p})$ such that*

$$b(x, y, \lambda) = B_-(x, \lambda)(x - \xi_-(\lambda))^{l_-} + B_+(y, \lambda)(y - \xi_+(\lambda))^{l_+} \tag{29}$$

and $\xi_\pm(\lambda) \in V_\pm$ hold for every $\lambda \in U$, $x \in W_u^{V_-}(\xi_-(\lambda), \lambda)$, $y \in W_s^{V_+}(\xi_+(\lambda), \lambda)$.

Suppose

$$\|\frac{\partial}{\partial \lambda_c^i}(x^\infty_{n_\pm}(\lambda_c) - \xi_\pm(\lambda^\infty_r(\lambda_c), \lambda_c))\| \leq C e^{-\alpha_\pm |n_\pm|}, \quad i = 0, \ldots, l-1, \tag{30}$$

hold for all $\lambda_c \in U_\rho(\bar{\lambda}_c)$, $\pm n_\pm \geq N$, some fixed $N \in \mathbb{N}$ and $\alpha_- = \alpha_{-u}$, $\alpha_+ = \alpha_{+s}$. Then the following corollary of Theorem 3.9 is valid. As we will show later (see Proposition 3.14) it applies with $l_\pm = 1$ in case of periodic boundary conditions. If projection boundary conditions are used and f is of type $\mathcal{C}^{l+1}$, it applies with $l_\pm = 2$.

Corollary 3.11 *Let b define boundary conditions of $\mathcal{C}^{l-1}$–order (l_-, l_+) for some $l_\pm \in \mathbb{N}$. Then the statements of Theorem 3.9 are valid with the improved estimates*

$$\begin{aligned} &\|\frac{\partial^i}{\partial \lambda_c^i}((x_J, \lambda_{r,J}) - (x^\infty_{|J}, \lambda_r^\infty))(\lambda_c)\|_\infty \\ \leq\ & C \max\{e^{l_- \alpha_{-u} n_-}, e^{-l_+ \alpha_{+s} n_+}, \sum_{j=0}^{i} \|h_J^{(i)}(\lambda_c)\|\}, \quad i = 0, \ldots, l-1, \end{aligned}$$

where the $\mathcal{C}^l$–map $h_J : U_\rho(\bar\lambda_c) \to \mathbb{R}^t$ is defined by

$$h_J(\lambda_c) = \Psi_J(x^\infty_{|J}(\lambda_c), \lambda_r^\infty(\lambda_c), \lambda_c), \quad \lambda_c \in U_\rho(\bar\lambda_c).$$

Proof. Due to (28) it is sufficient to prove

$$\|\frac{\partial^i}{\partial \lambda_c^i} b(x^\infty_{n_-}(\lambda_c), x^\infty_{n_+}(\lambda_c), \lambda_r^\infty(\lambda_c), \lambda_c)\| \leq C \max\{e^{l_- \alpha_{-u} n_-}, e^{-l_+ \alpha_{+s} n_+}\} \tag{31}$$

for $-n_-, n_+$ sufficiently large and λ_c in some neighborhood of $\bar\lambda_c$. By the definition of a $\mathcal{C}^{l-1}$–order

$$\begin{aligned} & b(x^\infty_{n_-}(\lambda_c), x^\infty_{n_+}(\lambda_c), \lambda_r^\infty(\lambda_c), \lambda_c) \\ =\ & B_-(x^\infty_{n_-}(\lambda_c), \lambda_r^\infty(\lambda_c), \lambda_c)(x^\infty_{n_-}(\lambda_c) - \xi_-(\lambda_r^\infty(\lambda_c), \lambda_c))^{l_-} \\ & + B_+(x^\infty_{n_+}(\lambda_c), \lambda_r^\infty(\lambda_c), \lambda_c)(x^\infty_{n_+}(\lambda_c) - \xi_+(\lambda_r^\infty(\lambda_c), \lambda_c))^{l_+} \end{aligned} \tag{32}$$

holds in this case. Here $B_\pm$ are $\mathcal{C}^{l-1}$–maps as in Definition 3.10. Differentiating both sides of (32) i–times with respect to λ_c and using (30) yields the estimate (31). □

For the proof of the estimates in (30) we use the following lemma. It generalizes well known results (without parameter) on the convergence of trajectories near hyperbolic fixed points. Here we also assume just $r \geq 1$.

Lemma 3.12 *Let $J \in \{\mathbb{Z}_-, \mathbb{Z}_+\}$, $\alpha = \alpha_s$ in case $J = \mathbb{Z}_+$ and $\alpha = \alpha_u$ in case $J = \mathbb{Z}_-$, and $C_0 > 0$. Then there exist constants $\varepsilon_0, \delta, C > 0$ such that the following statement holds:*

For $\varepsilon \in (0, \varepsilon_0]$ let $y_J \in \mathcal{C}^{l-1}(U_\varepsilon(\bar\lambda), S_J)$ satisfy $\|y_J^{(i)}(\lambda)\|_\infty \leq C_0$ and $y_J(\lambda)$ be a $f(\cdot, \lambda)$–orbit in $U_\delta((\bar\xi)_{n\in J})$ for every $\lambda \in U_\varepsilon(\bar\lambda)$ and $i = 0, \ldots, l-1$. Then

$$\|\frac{\partial}{\partial \lambda^i}(y_n - \xi)(\lambda)\| \leq C e^{-\alpha|n|}, \quad i = 0, \ldots, l-1, \tag{33}$$

hold for every $n \in J$ and $\lambda \in U_\varepsilon(\bar\lambda)$. If $y_J : M \to S_J$, $M \subset U_{\varepsilon_0}(\bar\lambda)$, is just a map satisfying the conditions on y_J as above with $i = 0$, then (33) holds for $i = 0$ and $\lambda \in M$.

Proof. For $\varepsilon \in (0, \varepsilon_0]$ consider a branch $y_J(\lambda)$, $\lambda \in U_\varepsilon(\bar\lambda)$, as in the assumption and let $u_J(\lambda) = (y_n(\lambda) - \xi(\lambda))_{n\in J} \in S_J$. Then Taylor's formula implies

$$u_{n+1}(\lambda) = f_x(\xi(\lambda), \lambda)u_n(\lambda) + E_n(\lambda)u_n(\lambda), \quad \lambda \in U_\varepsilon(\bar\lambda), n \in \hat J, \tag{34}$$

with

$$E_n(\lambda) = \int_0^1 f_x(\xi(\lambda) + tu_n(\lambda), \lambda) - f_x(\xi(\lambda), \lambda) dt.$$

For the analysis of $y_n - \xi(\lambda) = u_n(\lambda)$ we reformulate (34) as a fixed point problem in some Banach space with exponentially weighted norms. Differentiating both sides of the resulting fixed point equation will give us the desired exponential estimates.

By Proposition 2.4 the equation $u_{n+1} = f_x(\bar{\xi}, \bar{\lambda}) u_n$ has an exponential dichotomy on $\mathbb{Z}$ with data $(\tilde{K}, \tilde{\alpha}_s, \tilde{\alpha}_u, \tilde{P})$ for some constants $\tilde{K} > 0$, $\tilde{\alpha}_s > \alpha_s$, $\tilde{\alpha}_u > \alpha_u$ and a unique projector $\tilde{P} \in \mathbb{R}^{k,k}$. Taking $\varepsilon_0 > 0$ sufficiently small, Lemma 2.3 and Proposition 2.4 imply that

$$u_{n+1} = f_x(\xi(\lambda), \lambda) u_n \tag{35}$$

has an exponential dichotomy on $\mathbb{Z}$ with data $(K, \tilde{\alpha}_s, \tilde{\alpha}_u, P(\lambda))$ for some fixed $K > 0$ and every $\lambda \in U_{\varepsilon_0}(\bar{\lambda})$. Here $\tilde{\alpha}_s > \alpha_s$ and $\tilde{\alpha}_u > \alpha_u$ have to be taken slightly smaller than above. Furthermore $P : U_{\varepsilon_0}(\bar{\lambda}) \to \mathbb{R}^{k,k}$ is of type $\mathcal{C}^{l-1}$. For $\lambda \in U_{\varepsilon_0}(\bar{\lambda})$ let $\Phi(\cdot, \cdot, \lambda)$ denote the solution operator and $G(\cdot, \cdot, \lambda)$ the Green's function of (35) with respect to $P(\lambda)$.

Similar to the proof of Lemma 2.3 we use Banach spaces with exponentially weighted norms in the sequel. For $\beta \in \mathbb{R}$ consider $(Z_J^\beta, \|\cdot\|_*)$ with

$$Z_J^\beta = \{x_J \in (\mathbb{R}^k)^J : \|x_J\|_* = \sup_{n \in J} e^{\beta|n|} \|x_n\| < \infty\}.$$

Notice that $(Z_J^0, \|\cdot\|_*) = (S_J^0, \|\cdot\|_\infty)$ holds. Let

$$\begin{aligned}
T_J^\beta(x_J, E_J, \lambda) &= \Big(\sum_{m \in \hat{J}} G(n, m+1, \lambda) E_m x_m\Big)_{n \in J}, \\
R_{\mathbb{Z}_-}^\beta(x_0, \lambda) &= (\Phi(n, 0, \lambda)(I - P(\lambda)) x_0)_{n \in \mathbb{Z}_-}, \\
R_{\mathbb{Z}_+}^\beta(x_0, \lambda) &= (\Phi(n, 0, \lambda) P(\lambda) x_0)_{n \in \mathbb{Z}_+}
\end{aligned}$$

for $x_J \in Z_J^\beta$, $E_J \in S_J(\mathbb{R}^{k,k})$, $\lambda \in U_{\varepsilon_0}(\bar{\lambda})$. Taking $\beta \in (-\tilde{\alpha}_u, \tilde{\alpha}_s)$ in case $J = \mathbb{Z}_+$ and $\beta \in (-\tilde{\alpha}_s, \tilde{\alpha}_u)$ in case $J = \mathbb{Z}_-$, the properties of P and G (see Lemma 2.3) imply

$$T_J^\beta \in \mathcal{C}^{l-1}(Z_J^\beta \times S_J(\mathbb{R}^{k,k}) \times U_{\varepsilon_0}(\bar{\lambda}), Z_J^\beta), \quad R_J^\beta \in \mathcal{C}^{l-1}(\mathbb{R}^k \times U_{\varepsilon_0}(\bar{\lambda}), Z_J^\beta).$$

Moreover, $T_J^\beta(\cdot, \cdot, \lambda)$ is bilinear for every $\lambda \in U_{\varepsilon_0}(\bar{\lambda})$ with

$$\|T_J^\beta(\cdot, \cdot, \lambda)\|_* \leq \tau_J(\beta)$$

where $\tau_J(\beta) = K\big(\frac{e^{-\tilde{\alpha}_u}}{1 - e^{-(\tilde{\alpha}_u + \beta)}} + \frac{e^\beta}{1 - e^{-(\tilde{\alpha}_s - \beta)}}\big)$ in case $J = \mathbb{Z}_+$ and $\tau_J(\beta) = K\big(\frac{e^{\tilde{\alpha}_u}}{1 - e^{-(\tilde{\alpha}_u - \beta)}} + \frac{e^{-\beta}}{1 - e^{-(\tilde{\alpha}_s + \beta)}}\big)$ in case $J = \mathbb{Z}_-$.

By Lemma 2.2 the sequence $u_J(\lambda)$ defined in the beginning of this proof is a fixed point of $A_J^\beta(\cdot, \lambda)$ where

$$A_J^\beta(x_J, \lambda) = T_J^\beta(x_J, E_J(\lambda), \lambda) + R_J^\beta(u_0(\lambda), \lambda), \quad x_J \in Z_J^\beta, \lambda \in U_\varepsilon(\bar{\lambda}).$$

Notice that $E_J \in \mathcal{C}^{l-1}(U_\varepsilon(\bar{\lambda}), S_J(\mathbb{R}^{k,k}))$ and $A_J^\beta \in \mathcal{C}^{l-1}(Z_J^\beta \times U_\varepsilon(\bar{\lambda}), Z_J^\beta)$ hold.

We decrease $\varepsilon_0, \delta > 0$ such that

$$\|f_x(x,\lambda) - f_x(\xi(\lambda),\lambda)\| \le \frac{1}{2\max\{\tau_J(0),\tau_J(\alpha)\}}$$

and $\|\xi(\lambda) - \bar{\xi}\| \le \delta$ hold for every $x \in U_\delta(\bar{\xi})$ and $\lambda \in U_{\varepsilon_0}(\bar{\lambda})$. Then $A_J^0(\cdot,\lambda)$ as well as $A_J^\alpha(\cdot,\lambda)$ are uniform contractions with rate $\frac{1}{2}$. By the contraction principle (see [6, Theorem 3.3]) there exist $\mathcal{C}^{l-1}$–maps $x_J^0 : U_\varepsilon(\bar{\lambda}) \to S_J$ and $x_J^\alpha : U_\varepsilon(\bar{\lambda}) \to Z_J^\alpha$ such that $x_J^0(\lambda)$ and $x_J^\alpha(\lambda)$ are the unique fixed points of $A_J^0(\cdot,\lambda)$ and $A_J^\alpha(\cdot,\lambda)$ for $\lambda \in U_\varepsilon(\bar{\lambda})$, respectively. According to the construction of $u_J(\lambda)$ and the relation $S_J = Z_J^0 \subset Z_J^\alpha$ this implies $u_J(\lambda) = x_J^0(\lambda) = x_J^\alpha(\lambda)$. Hence $u_J \in \mathcal{C}^{l-1}(U_\varepsilon(\bar{\lambda}), Z_J^\alpha)$ and

$$u_J(\lambda) = T_J^\alpha(u_J(\lambda), E_J(\lambda), \lambda) + R_J^\alpha(u_0(\lambda),\lambda), \quad \lambda \in U_\varepsilon(\bar{\lambda}),$$

hold. Differentiating both sides of this equality repeatedly, and using the properties of the weighted norm $\|\cdot\|_*$ as well as the bilinearity of $T_J^\alpha(\cdot,\cdot,\lambda)$, we get the first part of the statement.

Notice that we did not use the smoothness of y_J to prove that $u_J(\lambda)$ is the unique fixed point of $A_J^\alpha(\cdot,\lambda)$. Therefore the second part of the statement holds also. □

Let us close this section with an analysis of the orders of the boundary conditions used in this paper. Notice that every boundary condition of $\mathcal{C}^0$–order (l_-, l_+) is of order (l_-, l_+). Furthermore, by Taylor's formula a boundary condition is (at least) of $\mathcal{C}^l$–order $(1,1)$ if b is of type $\mathcal{C}^{l+1}$ and $b(\xi_-(\lambda), \xi_+(\lambda), \lambda) = 0$ holds for every λ in some neighborhood of $\bar{\lambda}$.

Periodic boundary conditions are of $\mathcal{C}^\infty$–order $(1,1)$ at $(\bar{\xi},\bar{\xi},\bar{\lambda})$. This follows from the identity $b_{per}(x,y,\lambda) = x - y = (x - \xi(\lambda)) - (y - \xi(\lambda))$ which holds for every $x, y \in \mathbb{R}^k$, $\lambda \in U_\epsilon(\bar{\lambda})$.

To get a sharp statement on the smooth order of projection boundary conditions we use the following lemma. It generalizes the well known graph representation of local stable and unstable manifolds, see [12, Theorem 6.5]. The proof is given in Subsection B.1 of Appendix B, see page 39. Here we also assume just $r \ge 0$ for the number of parameters.

Lemma 3.13 *Let $l \ge 2$, $J(s) = \mathbb{Z}_+$, $J(u) = \mathbb{Z}_-$ and $(\kappa,\tilde{\kappa}) = (s,u)$ or $(\kappa,\tilde{\kappa}) = (u,s)$ hold. Then there exist constants $\varepsilon_1 \in (0,\epsilon]$, $\delta_1 > 0$ and for every $\varepsilon \in (0,\varepsilon_1]$, $\delta \in (0,\delta_1]$ with $\xi(U_\varepsilon(\bar{\lambda})) \subset U_\delta(\bar{\xi})$ an open set $V_\kappa \subset \mathbb{R}^{k+r}$ with $\{0\} \times U_\varepsilon(\bar{\lambda}) \subset V_\kappa$ and a $\mathcal{C}^{l-1}$–map $q_\kappa : V_\kappa \to \mathbb{R}^k$ such that*

$$W_\kappa^{U_\delta(\bar{\xi})}(\xi(\lambda),\lambda) = \{\xi(\lambda) + z + q_\kappa(z,\lambda) : (z,\lambda) \in V_\kappa \cap (X_\kappa(\lambda) \times \mathbb{R}^r)\}, \tag{36}$$

$$T_x W_\kappa^{U_\delta(\bar{\xi})}(\xi(\lambda),\lambda) = \{y \in \mathbb{R}^k : (Df(\cdot,\lambda)^n(x)y)_{n\in J(\kappa)} \in S_{J(\kappa)}\} \tag{37}$$

hold for every $\lambda \in U_\varepsilon(\bar{\lambda})$, $x \in W_\kappa^{U_\delta(\bar{\xi})}(\xi(\lambda),\lambda)$. Furthermore $\mathcal{R}(q_\kappa(\cdot,\lambda)) \subset X_{\tilde{\kappa}}(\lambda)$, $q_\kappa(0,\lambda) = 0$, $D_1 q_\kappa(0,\lambda) = 0$ are valid and $D_1 q_\kappa$ is of type $\mathcal{C}^{l-1}$.

We use this lemma to prove that projection boundary conditions are of $\mathcal{C}^{l-2}$–order $(2,2)$. Consider the projectors $P_\kappa(\lambda)$, $\lambda \in U_\epsilon(\bar{\lambda})$, defined by $\mathcal{R}(P_\kappa(\lambda)) =$

$X_\kappa(\lambda)$, $\mathcal{N}(P_\kappa(\lambda)) = X_{\tilde\kappa}(\lambda)$ for $\{\kappa,\tilde\kappa\} = \{s,u\}$. Due to Lemma 2.3 and Proposition 2.4 the map $P_\kappa : U_\epsilon(\bar\lambda) \to \mathbb{R}^{k,k}$ is of type $\mathcal{C}^{l-1}$. Moreover, Lemma 3.13 implies

$$\begin{aligned}
&(I - P_\kappa(\lambda))(x-\xi(\lambda))\\
&\quad= q_\kappa(P_\kappa(\lambda)(x-\xi(\lambda)),\lambda)\\
&\quad= \int_0^1 (1-t) D_1^2 q_\kappa(tP_\kappa(\lambda)(x-\xi(\lambda)),\lambda)dt\,(P_\kappa(\lambda)(x-\xi(\lambda)))^2
\end{aligned}$$

for every $x \in U_\delta(\bar\xi)$ and $\lambda \in U_\varepsilon(\bar\lambda)$; here we used the setting of the previous lemma. Let the $\mathcal{C}^{l-2}$–map $A_\kappa : U_\delta(\bar\xi) \times U_\varepsilon(\bar\lambda) \to \mathcal{L}^2(\mathbb{R}^k,\mathbb{R}^k)$ be defined by

$$A_\kappa(x,\lambda)[y_1,y_2] = \int_0^1 (1-t) D_1^2 q_\kappa(tP_\kappa(\lambda)(x-\xi(\lambda)),\lambda)dt\,[P_\kappa(\lambda)y_1, P_\kappa(\lambda)y_2]$$

for $x \in U_\delta(\bar\xi), \lambda \in U_\varepsilon(\bar\lambda), y_1,y_2 \in \mathbb{R}^k$.

Consider the map b_{proj} defining projection boundary conditions as in (23). Let $A_{\pm\kappa}$, $\kappa = s,u$, be defined as above with $\bar\xi = \bar\xi_\pm$ and some constants $\delta = \delta_\pm$, $\varepsilon = \varepsilon_\pm$. Then

$$\begin{aligned}
&B_-(x,\lambda)[y_1,y_2] + B_+(\tilde x,\lambda)[\tilde y_1,\tilde y_2]\\
&\quad= (Q_{-s}(\lambda)^T A_{-u}(x,\lambda)[y_1,y_2], Q_{+u}(\lambda)^T A_{+s}(\tilde x,\lambda)[\tilde y_1,\tilde y_2],)
\end{aligned}$$

with $x \in U_{\delta_-}(\bar\xi_-), \tilde x \in U_{\delta_-}(\bar\xi_+), \lambda \in U_{\varepsilon_0}(\bar\lambda), y_1,y_2,\tilde y_1,\tilde y_2 \in \mathbb{R}^k$, defines $\mathcal{C}^{l-2}$–maps $B_\pm : U_{\delta_\pm}(\bar\xi_\pm) \times U_{\varepsilon_0}(\bar\lambda) \to \mathcal{L}^2(\mathbb{R}^k,\mathbb{R}^k)$; here we have set $\varepsilon_0 = \min\{\varepsilon_-,\varepsilon_+\}$. These maps $B_\pm$ satisfy the condition (29) of Definition 3.10 for $b = b_{proj}$ with $l_\pm = 2$ and $q = l-2$. Hence b_{proj} is of $\mathcal{C}^{l-2}$–order $(2,2)$.

We collect all the statements on periodic and projection boundary conditions given above in the following proposition. It shows that Corollary 3.11 applies to these boundary conditions as mentioned there.

Proposition 3.14 (i) *Periodic boundary conditions are regular and of $\mathcal{C}^\infty$–order $(1,1)$ at $(\bar\xi,\bar\xi,\bar\lambda)$. The defining map b_{per} is of type $\mathcal{C}^\infty$.*

(ii) *In case $l \geq 2$ projection boundary conditions are regular and of $\mathcal{C}^{l-2}$–order $(2,2)$ at $(\bar\xi_-,\bar\xi_+,\bar\lambda)$. The defining map b_{proj} is of type $\mathcal{C}^{l-1}$.*

3.3 On the approximation of transversal heteroclinics

The $f(\cdot,\bar\lambda)$–heteroclinic $\bar x_{\mathbb{Z}}$ is called *transversal* if it is 0–tangential and nondegenerate. Notice that this definition is equivalent to that used in [5].

For a transversal heteroclinic we can apply all results from Subsections 3.1 and 3.2 with $t = 0$. In this case these results hold without the regularization maps Ψ and Ψ_J. Hence, for transversal heteroclinics, the results of Subsection 3.1 are equal to those in [5, Section 3]. The results from Subsection 3.2 extend those of [5] to the parameter dependent case.

4 Approximation of quadratic tangencies

4.1 Approximation of quadratic heteroclinics as turning points

In this section we suppose that the heteroclinic $\bar{x}_{\mathbb{Z}}$ is 1–tangential and $\bar{u}_{\mathbb{Z}} \in \mathcal{N}(\Gamma_x(\bar{x}_{\mathbb{Z}}, \bar{\lambda}))$ is nontrivial. Furthermore, let $l \geq 2$ and $r \geq p \geq 0$, and write the parameters $\lambda \in \mathbb{R}^r$ as $\lambda = (\lambda_p, \lambda_2) \in \mathbb{R}^p \times \mathbb{R}^{r-p}$.

Definition 4.1 *We call the* 1*–tangential* $\bar{x}_{\mathbb{Z}}$ quadratic *if*

$$\dim \mathcal{N}(\Gamma_{(x,\lambda_p)}(\bar{x}_{\mathbb{Z}}, \bar{\lambda}_p, \bar{\lambda}_2)) = 1$$

and

$$\Gamma_{xx}(\bar{x}_{\mathbb{Z}}, \bar{\lambda}_p, \bar{\lambda}_2))\bar{u}_{\mathbb{Z}}^2 \notin \mathcal{R}(\Gamma_{(x,\lambda_p)}(\bar{x}_{\mathbb{Z}}, \bar{\lambda}_p, \bar{\lambda}_2))$$

hold where $\bar{u}_{\mathbb{Z}}$ *spans* $\mathcal{N}(\Gamma_x(\bar{x}_{\mathbb{Z}}, \bar{\lambda}_p, \bar{\lambda}_2))$.

As we will show in Appendix B, the heteroclinic $\bar{x}_{\mathbb{Z}}$ is quadratic 1–tangential and nondegenerate if and only if the the quadratic tangency of $W_u(\bar{\xi}_-, \bar{\lambda})$ and $W_s(\bar{\xi}_+, \bar{\lambda})$ at $\bar{x}_0$ unfolds generically, cf. the introduction.

Our definition of quadratic 1–tangentials relates directly to the notation of quadratic turning points of Γ, which are defined as follows. Consider real Banach spaces X, Y, a map $g \in \mathcal{C}^1(X \times \mathbb{R}, Y)$ and some $(x, \mu) \in X \times \mathbb{R}$. Furthermore let $g_x(x, \mu)$ be a Fredholm operator of index 0. Then (x, μ) is called a *turning point of* g *with respect to* μ if the conditions

$$g(x, \mu) = 0, \quad \dim \mathcal{N}(g_x(x, \mu)) = 1, \quad g_\mu(x, \mu) \notin \mathcal{R}(g_x(x, \mu))$$

are satisfied. In case $g \in \mathcal{C}^2$, a turning point (x, λ) is called *quadratic* if

$$g_{xx}(x, \mu)u^2 \notin \mathcal{R}(g_x(x, \mu))$$

holds for some (and hence for every) nontrivial $u \in \mathcal{N}(g_x(x, \mu))$.

In case $r = p + 1$ and $\mathcal{R}(\Gamma_x(\bar{x}_{\mathbb{Z}}, \bar{\lambda}_p, \bar{\lambda}_2)) \cap \mathcal{R}(\Gamma_{\lambda_p}(\bar{x}_{\mathbb{Z}}, \bar{\lambda}_p, \bar{\lambda}_2)) = \{0\}$, the definitions above imply that the heteroclinic $\bar{x}_{\mathbb{Z}}$ is 1–tangential and nondegenerate if and only if $(\bar{x}_{\mathbb{Z}}, \bar{\lambda}_p, \bar{\lambda}_2)$ is a turning point of Γ with respect to λ_2. Furthermore, this turning point $(\bar{x}_{\mathbb{Z}}, \bar{\lambda}_p, \bar{\lambda}_2)$ is quadratic if and only if the heteroclinic $\bar{x}_{\mathbb{Z}}$ is quadratic. Notice that the operator $\Gamma_{(x,\lambda_p)}(\bar{x}_{\mathbb{Z}}, \bar{\lambda}_p, \bar{\lambda}_2)$ is Fredholm of index 0 by the choice of p, see Lemma 2.6.

By these arguments the computation of quadratic heteroclinics is locally equivalent to the computation of turning points. Therefore we will use turning points of the approximate operator Γ_J as approximation of these tangentials. This approach to approximate singularities of operators by singularities of approximate operators is well known, see [7] for example.

By standard methods we are able to compute turning points of Γ_J numerically. The following theorem gives our approximation method. The proofs of that theorem given in the next subsection are an application of the results from Section 3.

Theorem 4.2 *Suppose, in case $r = p+1$, that the 1–tangential $\bar{x}_{\mathbb{Z}}$ is quadratic and nondegenerate. Assume that b is of type $\mathcal{C}^2$ and defines regular boundary conditions at $(\bar{\xi}_-, \bar{\xi}_+, \bar{\lambda}_p, \bar{\lambda}_2)$. Then there exist constants $C, N, \delta > 0$ such that for every $J = [n_-, n_+]$ with $-n_-, n_+ \geq N$ the operator Γ_J has a unique quadratic turning point $(x_J, \lambda_{p,J}, \lambda_{2,J})$ with respect to λ_2 in $B_\delta(\bar{x}_{|J}, \bar{\lambda}_p, \bar{\lambda}_2)$. It is the only zero in that ball at which $D_{(x,\lambda_p)}\Gamma_J$ is not one–to–one, and it satisfies*

$$\begin{aligned} &\|(x_J, \lambda_{p,J}, \lambda_{2,J}) - (\bar{x}_{|J}, \bar{\lambda}_p, \bar{\lambda}_2)\|_\infty \\ &\quad \leq \; C \max\{\|b(\bar{x}_{n_-}, \bar{x}_{n_+}, \bar{\lambda}_p, \bar{\lambda}_2)\|, \|b'(\bar{x}_{n_-}, \bar{x}_{n_+}, \bar{\lambda}_p, \bar{\lambda}_2)(\bar{u}_{n_-}, \bar{u}_{n_+}, 0, 0)\|\} \end{aligned} \tag{38}$$

for $\bar{u}_{\mathbb{Z}}$ as in Definition 4.1.

Corollary 4.3 *Assume that b defines boundary conditions of $\mathcal{C}^1$–order (l_-, l_+) for some $l_-, l_+ \in \mathbb{N}$. Then the statements of Theorem 4.2 are valid with the improved estimate*

$$\|(x_J, \lambda_{p,J}, \lambda_{2,J}) - (\bar{x}_{|J}, \bar{\lambda}_p, \bar{\lambda}_2)\|_\infty \leq C \max\{e^{l_- \alpha_{-u} n_-}, e^{-l_+ \alpha_{+s} n_+}\}.$$

Let us consider the case $r = p + 1 + s$ with $p \geq 0$ and $s \geq 1$ in the following and formulate our results on curves of quadratic tangencies. We decompose the parameters $\lambda \in \mathbb{R}^r$ as $\lambda = (\lambda_p, \lambda_2, \lambda_c) \in \mathbb{R}^p \times \mathbb{R} \times \mathbb{R}^s$. Let $\bar{x}_{\mathbb{Z}}$ be a quadratic 1–tangential which is nondegenerate with respect to (λ_p, λ_2). Furthermore, we assume that

$$\dim \mathcal{N}(\Gamma_{(x,\lambda_p)}(x_{\mathbb{Z}}, \lambda_p, \lambda_2, \lambda_c)) = 1 \tag{39}$$

holds for each turning point $(x_{\mathbb{Z}}, \lambda_p, \lambda_2, \lambda_c)$ of Γ in a small neighborhood of $(\bar{x}_{\mathbb{Z}}, \bar{\lambda}_p, \bar{\lambda}_2, \bar{\lambda}_c)$. Notice, this assumption does not follow from our previous definitions, but due to the Fredholm property of $\Gamma_x(\bar{x}_{\mathbb{Z}}, \bar{\lambda}_p, \bar{\lambda}_2, \bar{\lambda}_c)$ it seems generic.

Theorem 4.4 *Under the assumptions given above there exist constants $\rho_0, \delta_0 > 0$ and a $\mathcal{C}^l$–map $(x^\infty_{\mathbb{Z}}, \lambda^\infty_p, \lambda^\infty_2, u^\infty_{\mathbb{Z}}) : U_{\rho_0}(\bar{\lambda}_c) \to S_{\mathbb{Z}} \times \mathbb{R}^p \times \mathbb{R} \times S_{\mathbb{Z}}$ such that the following statements hold for every $\lambda_c \in U_{\rho_0}(\bar{\lambda}_c)$:*

(i) *The tuple $(x^\infty_{\mathbb{Z}}, \lambda^\infty_p, \lambda^\infty_2)(\lambda_c)$ is the unique quadratic turning point of the operator $\Gamma(\cdot,\cdot,\cdot,\lambda_c)$ with respect to λ_2 in the ball $U_{\delta_0}(\bar{x}_{\mathbb{Z}}, \bar{\lambda}_p, \bar{\lambda}_2)$. It is the only zero in that ball at which $\Gamma_{(x,\lambda_p)}$ is not one–to–one. The orbit $x^\infty_{\mathbb{Z}}(\lambda_c)$ is a quadratic 1–tangential heteroclinic connecting $\xi_-(\lambda^\infty_p(\lambda_c), \lambda^\infty_2(\lambda_c), \lambda_c)$ and $\xi_+(\lambda^\infty_p(\lambda_c), \lambda^\infty_2(\lambda_c), \lambda_c)$. Furthermore it is nondegenerate with respect to (λ_p, λ_2).*

(ii) *The sequence $u^\infty_{\mathbb{Z}}(\lambda_c)$ is uniquely determined by*

$$\Gamma_x(x^\infty_{\mathbb{Z}}(\lambda_c), \lambda^\infty_p(\lambda_c), \lambda^\infty_2(\lambda_c), \lambda_c) u^\infty_{\mathbb{Z}}(\lambda_c) = 0, \quad \bar{u}_0^T(u^\infty_0(\lambda_c) - \bar{u}_0) = 0$$

and satisfies

$$\lim_{n\to\pm\infty} u^\infty_n(\lambda_c) = 0$$

uniformly with respect to λ_c.

Theorem 4.5 *Consider the setting of Theorem 4.4 and the case $l \geq 3$. Let b define regular boundary conditions at $(\bar\xi_-, \bar\xi_+, \bar\lambda_p, \bar\lambda_2, \bar\lambda_c)$. Then there exist constants $C, N > 0$, $\rho \in (0, \rho_0]$, $\delta \in (0, \delta_0]$, and for every $J = [n_-, n_+]$ with $-n_-, n_+ \geq N$ a $\mathcal{C}^{l-1}$–map $(x_J, \lambda_{p,J}, \lambda_{2,J}) : U_\rho(\bar\lambda_c) \to S_J \times \mathbb{R}^p \times \mathbb{R}$ satisfying the following statements for every $\lambda_c \in U_\rho(\bar\lambda_c)$:*

The tuple $(x_J, \lambda_{p,J}, \lambda_{2,J})(\lambda_c)$ is the unique quadratic turning point of the operator $\Gamma_J(\cdot,\cdot,\cdot,\lambda_c)$ with respect to λ_2 in the ball $U_\delta(\bar x_{|J}, \bar\lambda_p, \bar\lambda_2)$. It is the only zero in that ball at which $D_{(x,\lambda_p)}\Gamma_J$ is not one–to–one, and it satisfies

$$\|\frac{\partial^i}{\partial\lambda_c^i}((x_J, \lambda_{p,J}, \lambda_{2,J}) - (x^\infty_{|J}, \lambda^\infty_p, \lambda^\infty_2))(\lambda_c)\| \leq C \sum_{j=0}^{i} \|H_J^{(i)}(\lambda_c)\|$$

for $i = 0, \ldots, l-2$ with the $\mathcal{C}^{l-1}$–map $H_J : U_\rho(\bar\lambda_c) \to \mathbb{R}^{2(k+p)}$ defined by

$$\begin{aligned} H_J(\lambda_c) \;=\; & (b(x^\infty_{n_-}(\lambda_c), x^\infty_{n_+}(\lambda_c), \lambda^\infty_p(\lambda_c), \lambda^\infty_2(\lambda_c), \lambda_c), \\ & b'(x^\infty_{n_-}(\lambda_c), x^\infty_{n_+}(\lambda_c), \lambda^\infty_p(\lambda_c), \lambda^\infty_2(\lambda_c), \lambda_c)(u^\infty_{n_-}(\lambda_c), u^\infty_{n_+}(\lambda_c), 0, 0, 0)) \end{aligned}$$

for $\lambda_c \in U_\rho(\bar\lambda_c)$.

Corollary 4.6 *Assume that b defines boundary conditions of $\mathcal{C}^l$–order (l_-, l_+) at $(\bar\xi_-, \bar\xi_+, \bar\lambda_p, \bar\lambda_2, \bar\lambda_c)$ for some $l_\pm \in \mathbb{N}$. Then the statements of Theorem 4.5 hold with the improved estimate*

$$\|\frac{\partial^i}{\partial\lambda_c^i}((x_J, \lambda_{p,J}, \lambda_{2,J}) - (x^\infty_{|J}, \lambda^\infty_p, \lambda^\infty_2))(\lambda_c)\| \leq C \max\{e^{l_-\alpha_{-u}n_-}, e^{-l_+\alpha_{+s}n_+}\}.$$

4.2 Proofs of the approximation results

For the proof of the results we use extended systems, cf. [25].

Proposition 4.7 *Consider $g \in \mathcal{C}^2(X \times \mathbb{R}, Y)$, $l \in \mathcal{L}(X, R)$, $c \in \mathbb{R} \setminus \{0\}$ and*

$$G : X \times X \times \mathbb{R} \to Y \times Y \times \mathbb{R}, (y, v, \eta) \mapsto (g(y, \eta), g_y(y, \eta)v, lv - c)$$

with real Banach spaces X, Y. Suppose that $g_y(\bar y, \bar\eta)$ is Fredholm of index 0 for some $(\bar y, \bar\eta) \in Y \times \mathbb{R}$ and let $\bar v \in Y$.

(i) *Then $(\bar y, \bar\eta)$ is a quadratic turning point of g if $(\bar y, \bar v, \bar\eta)$ is a regular zero of G, i.e. $G(\bar y, \bar v, \bar\eta) = 0$ and $G'(\bar y, \bar v, \bar\eta) = 0$ is one-to-one.*

(ii) *Furthermore, $G'(\bar y, \bar v, \bar\eta)$ is nonsingular if $(\bar y, \bar\eta)$ is a quadratic turning point of g and $\bar v \in \mathcal{N}(g_y(\bar y, \bar\eta))$ satisfies $l\bar v = c$.*

Due to the theorem of Hahn–Banach (see [19, Theorem 3.5]) the definition of turning points given in Subsection 4.1 is equivalent to that used in [25]. In that paper (cf. [25, Theorem 2.1(a), 2.2]) the previous proposition is shown in case $X = Y$. Analogously it holds in our general case.

In the following let the assumptions of Theorem 4.2 hold. Our aim is to apply the proposition given above with $g = \Gamma$, $X = S_{\mathbb{Z}} \times \mathbb{R}^p$, $Y = S_{\mathbb{Z}}$, l defined by $l(x_{\mathbb{Z}}, \lambda_p) = \bar{u}_0^T x_0$, $(x_{\mathbb{Z}}, \lambda_p) \in S_{\mathbb{Z}} \times \mathbb{R}^p$, $c = \bar{u}_0^T \bar{u}_0$, and $\bar{y} = (\bar{x}_{\mathbb{Z}}, \bar{\lambda}_p)$, $\bar{\eta} = \bar{\lambda}_2$, $\bar{v} = (\bar{u}_{\mathbb{Z}}, 0)$. Then the operator $G = G_f : (S_{\mathbb{Z}} \times \mathbb{R}^p)^2 \times \mathbb{R} \to S_{\mathbb{Z}}^2 \times \mathbb{R}$ in Proposition 4.7 is defined by

$$G_f(x_{\mathbb{Z}}, \lambda_p, u_{\mathbb{Z}}, \mu_p, \lambda_2) = (\Gamma(x_{\mathbb{Z}}, \lambda_p, \lambda_2), \Gamma'(x_{\mathbb{Z}}, \lambda_p, \lambda_2)(u_{\mathbb{Z}}, \mu_p, 0), \bar{u}_0^T(u_0 - \bar{u}_0))$$

for $(x_{\mathbb{Z}}, \lambda_p, u_{\mathbb{Z}}, \mu_p, \lambda_2) \in (S_{\mathbb{Z}} \times \mathbb{R}^p)^2 \times \mathbb{R}$.

For the application of the theory of Section 3 we consider the map

$$F : \begin{array}{rcl} \mathbb{R}^{2k} \times \mathbb{R}^{2p} \times \mathbb{R} & \to & \mathbb{R}^{2k}, \\ (x, u, \lambda_p, \mu_p, \lambda_2) & \mapsto & (f(x, \lambda_p, \lambda_2), f'(x, \lambda_p, \lambda_2)(u, \mu_p, 0)) \end{array}$$

and the extended discrete system

$$z_{n+1} = F(z_n, \eta_1, \eta_2), \quad n \in \mathbb{Z}, \quad (\eta_1, \eta_2) \in \mathbb{R}^{2p} \times \mathbb{R},$$

where $z_n \in \mathbb{R}^{2k}$.

Let us sum up some properties of F first.

Lemma 4.8 *The map F is of type C^{l-1} and the following statements hold for every $(x, u, \lambda_p, \mu_p, \lambda_2) \in \mathbb{R}^{2k+2p+1}$:*

(i) *The map $F(\cdot, \cdot, \lambda_p, \mu_p, \lambda_2)$ is diffeomorphic. More explicitly, the spectra of $F_{(x,u)}(x, u, \lambda_p, \mu_p, \lambda_2)$ and $f_x(x, \lambda_p, \lambda_2)$ are equal.*

(ii) *The point x is a fixed point of $f(\cdot, \lambda_p, \lambda_2)$ if and only if $(x, 0)$ is a fixed point of $F(\cdot, \cdot, \lambda_p, 0, \lambda_2)$. Moreover, if x is a fixed point as above and Y_s and Y_u denote the stable and unstable subspace of $f_x(x, \lambda_p, \lambda_2)$ then $Y_s \times Y_s$ and $Y_u \times Y_u$ form the stable and unstable subspace of $F_{(x,u)}(x, 0, \lambda_p, 0, \lambda_2)$.*

(iii) *There exist $\varepsilon \in (0, \epsilon]$ and a neighborhood V of $\bar{\xi} = \xi(\bar{\lambda}_p, \bar{\lambda}_2)$ such that*

$$\begin{aligned} & W_\kappa^{V \times \mathbb{R}^k}((\xi(\lambda_p, \lambda_2), 0), F(\cdot, \cdot, \lambda_p, 0, \lambda_2)) \\ & = \{(y, v) \in W_\kappa^V(\xi(\lambda_p, \lambda_2), f(\cdot, \lambda_p, \lambda_2)) \times \mathbb{R}^k : \\ & \qquad v \in T_y W_\kappa^V(\xi(\lambda_p, \lambda_2), f(\cdot, \lambda_p, \lambda_2))\} \end{aligned}$$

holds for every $(\lambda_p, \lambda_2) \in U_\varepsilon(\bar{\lambda}_p, \bar{\lambda}_2)$ and $\kappa = s, u$.

The proof of the previous lemma follows from the definition of F and, in case (iii), from the identity (37) in Lemma 3.13.

For the 1–tangential heteroclinic $\bar{x}_{\mathbb{Z}}$ and $\bar{u}_{\mathbb{Z}} \in \mathcal{N}(\Gamma_x(\bar{x}_{\mathbb{Z}}, \bar{\lambda}_p, \bar{\lambda}_2)) \setminus \{0\}$ the extended sequence

$$\bar{z}_{\mathbb{Z}} = (\bar{x}_n, \bar{u}_n)_{n \in \mathbb{Z}}$$

has the following properties.

Lemma 4.9 *The sequence $\bar{z}_{\mathbb{Z}}$ is a 1–tangential heteroclinic $F(\cdot,\cdot,\bar{\lambda}_p,0,\bar{\lambda}_2)$–orbit connecting $(\bar{\xi}_-,0)$ and $(\bar{\xi}_+,0)$. Furthermore, it is nondegenerate and*

$$\mathcal{N}(\Gamma'_F(\bar{z}_{\mathbb{Z}},\bar{\lambda}_p,0,\bar{\lambda}_2)) = \mathbb{R}(0,\bar{u}_n)_{n\in\mathbb{Z}} \times \{(0,0,0)\} \tag{40}$$

holds.

Proof. By definition $\bar{z}_{\mathbb{Z}}$ is an $F(\cdot,\cdot,\bar{\lambda}_p,0,\bar{\lambda}_2)$–orbit. The exponential dichotomy of $u_{n+1} = f_x(\bar{x}_n,\bar{\lambda}_p,\bar{\lambda}_2)u_n$ implies $\lim_{n\to\pm\infty}\bar{u}_n = 0$. Hence $\bar{z}_{\mathbb{Z}}$ connects $(\bar{\xi}_-,0)$ and $(\bar{\xi}_+,0)$.

For the proof that $\bar{z}_{\mathbb{Z}}$ is nondegenerate and 1–tangential, we consider the operators Γ_f and Γ_F, cf. (6). By the definition of F

$$\Gamma'_F((\bar{x}_n,\bar{u}_n)_{n\in\mathbb{Z}},\bar{\lambda}_p,0,\bar{\lambda}_2)((y_n,v_n)_{n\in\mathbb{Z}},\eta_p,\omega_p,\eta_2)$$

and

$$(\Gamma'_f(\bar{x}_{\mathbb{Z}},\bar{\lambda}_p,\bar{\lambda}_2),\Gamma'_f(\bar{x}_{\mathbb{Z}},\bar{\lambda}_p,\bar{\lambda}_2)(v_{\mathbb{Z}},\omega_p,0)+\Gamma''_f(\bar{x}_{\mathbb{Z}},\bar{\lambda}_p,\bar{\lambda}_2)[(\bar{u}_{\mathbb{Z}},0,0),(y_{\mathbb{Z}},\eta_p,\eta_2)])$$

are conjugate through some fixed permutation for every $(y_{\mathbb{Z}},v_{\mathbb{Z}},\eta_p,\omega_p,\eta_2) \in S^2_{\mathbb{Z}} \times \mathbb{R}^{2p+1}$. Hence, by the quadratic property of $\bar{x}_{\mathbb{Z}}$, equation (40) holds. Furthermore, $2p+1 = 2k_{-s}-2k_{+s}+1$ is valid due to (3). By Lemma 4.8(ii) this means that the condition (22) on the number of parameters of nondegenerate $F(\cdot,\bar{\lambda}_p,0,\bar{\lambda}_2)$–orbits is satisfied. □

Putting all these results together, the task to approximate the turning point $(\bar{x}_{\mathbb{Z}},\bar{\lambda}_p,\bar{\lambda}_2)$ of Γ can be done by approximating the $F(\cdot,\cdot,\bar{\lambda}_p,0,\bar{\lambda}_2)$–heteroclinic $\bar{z}_{\mathbb{Z}}$. Notice, the operators G_f and (Γ_F,Ψ_F) are conjugate through some permutation, where

$$\Psi_F((x_n,u_n)_{n\in\mathbb{Z}},\lambda_p,\mu_p,\lambda_2) = \bar{u}_0^T(u_0-\bar{u}_0) \tag{41}$$

for $((x_n,u_n)_{n\in\mathbb{Z}},\lambda_p,\mu_p,\lambda_2) \in S_{\mathbb{Z}}(\mathbb{R}^{2k}) \times \mathbb{R}^{2p+1}$. Therefore, Theorem 3.2 applies to the orbit $\bar{z}_{\mathbb{Z}}$ and (Γ_F,Ψ_F).

For the computation of turning points of $\Gamma_{f,b,J}$ let us consider the map

$$B: \begin{array}{rcl} \mathbb{R}^{4k}\times\mathbb{R}^{2p}\times\mathbb{R} & \to & \mathbb{R}^{2(k+p)}, \\ (x,u,y,v,\lambda_p,\mu_p,\lambda_2) & \mapsto & (b(x,y,\lambda_p,\lambda_2),b'(x,y,\lambda_p,\lambda_2)(u,v,\mu_p,0)). \end{array}$$

By the following lemma B defines the correct boundary conditions to approximate the heteroclinic $\bar{z}_{\mathbb{Z}}$.

Lemma 4.10 *The map B is of type $\mathcal{C}^{l-1}$, and for $q \geq 1$ and $l_\pm \in \mathbb{N}$ the following statements hold:*

(i) *The boundary conditions defined by B are regular (with respect to F) at $((\bar{\xi}_-,0),(\bar{\xi}_+,0),\bar{\lambda}_p,0,\bar{\lambda}_2)$.*

(ii) *If b defines boundary conditions of $\mathcal{C}^q$–order (l_-,l_+) at $(\bar{\xi}_-,\bar{\xi}_+,\bar{\lambda}_p,\bar{\lambda}_2)$ (with respect to f), then the boundary conditions defined by B are of $\mathcal{C}^{q-1}$–order (l_-,l_+) at $((\bar{\xi}_-,0),(\bar{\xi}_+,0),\bar{\lambda}_p,0,\bar{\lambda}_2)$ (with respect to F).*

Proof. Assertion (i) follows from the definition of B and Lemma 4.8(ii).

We proof (ii). The equality (29) holds on the local unstable and stable manifold of $\bar{\xi}_-$ and $\bar{\xi}_+$. These sets are differential manifolds. The assertion follows by differentiating both sides of the equality (29) and using Lemma 4.8(iii). $\square$

The last step in the proof of Theorem 4.2 is the following technical lemma on the local uniqueness of nonregular zeros near turning points.

Lemma 4.11 *Let the assumptions of Proposition 4.7 hold. Take $\varepsilon > 0$ such that $g_y(y,\eta)$ is Fredholm of index 0 for every $(y,\eta) \in B_\varepsilon(\bar{y},\bar{\eta})$. For $(y,\eta) \in X \times \mathbb{R}$ define*

$$A(y,\eta)(v,\omega) = (g'(y,\eta)(v,\omega), lv), \quad (v,\omega) \in X \times \mathbb{R},$$

and assume that $A(y,\eta)$ is nonsingular for every $(y,\eta) \in B_\varepsilon(\bar{y},\bar{\eta})$. Let $(\bar{y},\bar{v},\bar{\eta}) \in X^2 \times \mathbb{R}$, $C, \rho > 0$ and $\delta \in (0, \min\{\varepsilon, \frac{\rho}{2}, \frac{\rho}{4C^2(\|\bar{v}\|+1)}]$ be chosen such that

$$\|A(y,\eta)^{-1}\|, \|g''(y,\eta)\| \leq C, \quad (y,\eta) \in B_\delta(\bar{y},\bar{\eta}),$$

hold. Furthermore, suppose that G has a unique zero $(\tilde{y},\tilde{v},\tilde{\eta})$ in $B_\delta(\bar{y},\bar{v},\bar{\eta})$. Then $(\tilde{y},\tilde{\eta})$ is the only zero of g in $B_\delta(\bar{y},\bar{\eta})$ at which g_y is not one–to–one.

Proof. Assume that $(y,\eta) \in B_\delta(\bar{y},\bar{\eta})$ is a zero of g such that $g_y(y,\eta)$ is not one–to–one. It suffices to show $(y,\eta) = (\tilde{y},\tilde{\eta})$.

By the Fredholm property of $g_y(y,\eta)$ there exists a nontrivial element v of the nullspace $\mathcal{N}(g_y(y,\eta))$. Hence $g'(y,\eta)(\tilde{v}-v,0) = (g'(y,\eta) - g'(\tilde{y},\tilde{\eta}))(\tilde{v},0)$ holds. Moreover, lv is nonzero because $A(y,\eta)$ is nonsingular. Hence we can scale v such that $lv = l\tilde{v}$ holds. This implies $A(y,\eta)(\tilde{v}-v,0) = ((g'(y,\eta) - g'(\tilde{y},\tilde{\eta}))(\tilde{v},0),0)$, and we get

$$\|\tilde{v}-v\| \leq \|A(y,\eta)^{-1}\| \, \|g'(y,\eta) - g'(\tilde{y},\tilde{\eta})\| \, \|\tilde{v}\| \leq 2C^2\delta(\|\bar{v}\|+\delta) \leq \frac{\rho}{2}.$$

Due to $\|v-\bar{v}\| \leq \delta \leq \frac{\rho}{2}$ the triple (y,v,η) is a zero of G in $B_\delta(\bar{y},\bar{v},\bar{\eta})$. Therefore, by the local uniqueness of this zero, $(y,\eta) = (\tilde{y},\tilde{\eta})$ follows. $\square$

In the remaining part of this subsection we carry out the proofs of the Theorems and Corollaries 4.2—4.6. The idea is that we apply the approximation results of Section 3 to the 1–tangential nondegenerate $F(\cdot,\bar{\lambda}_p,0,\bar{\lambda}_2)$–heteroclinic $\bar{z}_\mathbb{Z}$ which connects $(\bar{\xi}_-,0)$ and $(\bar{\xi}_+,0)$. This is possible by the previous lemmata.

Proof of Theorem 4.2 and Corollary 4.3. Let

$$\Psi_{F,J}((x_n,u_n)_{n\in J},\lambda_p,\eta_p,\lambda_2) = \bar{u}_0^T(u_0-\bar{u}_0)$$

for $x_J, u_J \in S_J, (\lambda_p,\eta_p,\lambda_2) \in \mathbb{R}^{2p+1}$ and $J = [n_-,n_+]$ with $-n_-, n_+ \geq 0$. Notice that $\Psi_{F,J}(\bar{z}_{|J},\bar{\lambda}_p,0,\bar{\lambda}_2) = 0$ holds. Then Theorem 3.5 applies to the heteroclinic $\bar{z}_\mathbb{Z}$ with boundary conditions defined by B and the constraints defined by $\Psi_{F,J}$. Hence the map $(\Gamma_{F,B,J},\Psi_{F,J})$ has a locally unique zero $((x_n,u_n)_{n\in J},\lambda_{p,J},\eta_{p,J},\lambda_{2,J})$ in some ball $B_{\delta,J}$ around $(\bar{z}_{|J},\bar{\lambda}_p,0,\bar{\lambda}_2)$ of fixed radius δ for sufficiently large $-n_-,n_+$. This zero is regular, and by (27) the estimate (38) holds. Notice that the right side of (38) gives an upper bound for $\|\eta_{p,J}\|$ also. Furthermore, the tuple $(x_J, \lambda_{p,J}, \lambda_{2,p})$ is a quadratic turning point of Γ_J by Proposition 4.7(i). This means that $D_{(x,\lambda_p)}\Gamma_{f,b,J}(x_J, \lambda_{p,J}, \lambda_{2,p})$ is not one–to–one.

It remains to show that the turning point $(x_J, \lambda_{p,J}, \lambda_{2,p})$ is the only zero of Γ_J in $B_{\delta,J}$ at which $D_{(x,\lambda_p)}\Gamma_{f,b,J}$ is not one–to–one. Define

$$\Psi_{f,J}(y_J, \lambda_p, \lambda_2) = \bar{u}_0^T(y_0 - \bar{x}_0), \quad y_J \in S_J, (\lambda_p, \lambda_2) \in \mathbb{R}^{p+1}.$$

Taking $-n_-, n_+$ sufficiently large and $\delta > 0$ sufficiently small, the estimates

$$\|(\Gamma_{f,b,J}, \Psi_{f,J})'(y_J, \lambda_p, \lambda_2)^{-1}\|_\infty \le C, \quad (y_J, \lambda_p, \lambda_2) \in B_\delta(\bar{x}_{|J}, \bar{\lambda}_p, \bar{\lambda}_2),$$

are valid by Theorem 3.5 for some constant $C > 0$. Furthermore, the tuple $((x_n, u_n)_{n\in J}, \lambda_{p,J}, \eta_{p,J}, \lambda_{2,J})$ is the unique zero of $(\Gamma_{F,B,J}, \Psi_{F,J})$ in the ball $B_{\delta,J}$. Hence Lemma 4.11 applies with $g = \Gamma_{f,b,J}$, $\bar{y} = (\bar{x}_{|J}, \bar{\lambda}_p)$, $\bar{\eta} = \bar{\lambda}_2$ $\bar{v} = (\bar{u}_{|J}, 0)$ and $l = D_{(x,\lambda_p)}\Psi_{f,J}$. This proofs the theorem.

Obviously, Corollary 3.7 and Lemma 4.10(ii) yield the assertion of the corollary. □

For the proof of the Theorems 4.4, 4.5 and Corollary 4.6 notice that the statements of the Lemmata 4.8—4.10 stay valid if we replace the parameter–component $\lambda_2 \in \mathbb{R}$ by $(\lambda_2, \lambda_c) \in \mathbb{R}^{1+s}$.

Proof of Theorem 4.4. Define Ψ_F as in (41) replacing λ_2 by (λ_2, λ_c). Theorem 3.8 applies to the $F(\cdot, \cdot, \bar{\lambda}_p, 0, \bar{\lambda}_2, \bar{\lambda}_c)$–heteroclinic $\bar{z}_\mathbb{Z}$, and gives us a locally unique branch

$$z_\mathbb{Z}^\infty(\lambda_c) = (x_n^\infty(\lambda_c), u_n^\infty(\lambda_c))_{n\in\mathbb{Z}}$$

of $F(\cdot, \cdot, \lambda_p^\infty, \mu_p^\infty, \lambda_2^\infty, \lambda_c)$–heteroclinics satisfying

$$\bar{u}_0^T(u_0^\infty(\lambda_c) - \bar{u}_0) = 0$$

for every λ_c in some neighborhood of $\bar{\lambda}_c$. Here we use the abbreviations $\lambda_p^\infty = \lambda_p^\infty(\lambda_c)$, $\mu_p^\infty = \mu_p^\infty(\lambda_c)$ and $\lambda_2^\infty = \lambda_2^\infty(\lambda_c)$. By condition (39) the component $\mu_p^\infty(\lambda_c)$ of the parameter vanishes for sufficiently small $\|\lambda_c - \bar{\lambda}_c\|$. Defining $\Psi = \Psi_f$ by

$$\Psi_f(x_\mathbb{Z}, \lambda_p, \lambda_2, \lambda_c) = \bar{u}_0^T(x_0 - \bar{x}_0), \quad x_\mathbb{Z} \in S_\mathbb{Z}, (\lambda_p, \lambda_2, \lambda_c) \in \mathbb{R}^{p+1+s},$$

Theorem 3.8 yields that $D_{(x,\lambda_p,\lambda_2)}(\Gamma_f, \Psi_f)(x_\mathbb{Z}, \lambda_p, \lambda_2, \lambda_c)$ is nonsingular for every $(x_\mathbb{Z}, \lambda_p, \lambda_2, \lambda_c)$ in some neighborhood of $(\bar{x}_\mathbb{Z}, \bar{\lambda}_p, \bar{\lambda}_2, \bar{\lambda}_c)$. Hence, Proposition 4.7 and Lemma 4.11 yield the assertion. □

The proof of Theorem 4.5 and Corollary 4.6 is obvious and will be omitted. It works analogously to that one of Theorem 4.2 and Corollary 4.3, and uses Theorem 3.9 and Corollary 3.11 instead of Theorem 3.5 and Corollary 3.7.

5 Applications

We compute turning points of the operator

$$\Gamma_J(x_J, \lambda_p, \lambda_2) = ((x_{n+1} - f(x_n, \lambda_p, \lambda_2))_{n\in\hat{J}}, b(x_{n_-}, x_{n_+}, \lambda_p, \lambda_2))$$

as approximations of quadratic tangentials. This will be done by using test function as proposed by Griewank and Reddien, cf. [10], and is justified by Theorem 4.2. In this section we will only consider the case of homoclinic tangencies. Hence $p = 0$ holds and the parameter λ_p disappears.

To obtain an initial guess for a turning point of Γ_J we follow a branch of homoclinics (in general: heteroclinics) as in [5] which starts at a transversal one. This is justified by Theorem 3.8. We use a standard path–following algorithm (cf. [1]), and compute a point of the branch where the λ_2–component of its tangential vector changes the sign. Taking this point as starting vector, we calculate the turning point as zero of (Γ_J, Ψ_J) where $\Psi_J : S_J \times \mathbb{R}^r \to \mathbb{R}$ is a test function defined as in [4, Section 5.3.3].

We use projection boundary conditions (see (23)) where the spectral decomposition of $f_x(\bar{\xi}_\pm, \bar{\lambda})^T$ is done numerically by the code "HQR3" described in [26]. The zeros of Γ_J and (Γ_J, Ψ_J) are approximated by (Gauss–)Newton's method with tolerance 10^{-15} on a computer with machine precision of about 10^{-16}. In (Gauss)–Newton's method we use LU decompositions with absolute column pivoting and take advantage of the sparsity pattern of $\Gamma'_J(x_J, \lambda)$ and $(\Gamma_J, \Psi_J)'(x_J, \lambda)$. The Euclidean norm is used in $\mathbb{R}^k$.

For the presentation to follow we approximate parts of the unstable and stable manifolds of $W_u(\bar{\xi}_\pm, \bar{\lambda})$ and $W_s(\bar{\xi}_\pm, \bar{\lambda})$ by an algorithm based on path–following methods described in [13, Chapter 7].

Example 1: Hénon's map

We consider Hénon's map

$$f(x, y, a, b) = (1 + y - ax^2, bx).$$

It is globally of type $\mathcal{C}^\infty$, and for $b \neq 0$ the map $f(\cdot, \cdot, a, b)$ it $\mathcal{C}^\infty$–diffeomorphic. For $(b-1)^2 > 4a \neq 0$ this map has two fixed points $\xi_\pm(a, b) = (x_\pm, bx_\pm)$ where

$$x_\pm = \frac{b - 1 \pm \sqrt{(b-1)^2 + 4a}}{2a}.$$

We consider homoclinic orbits with respect to the fixed point $\xi(a, b) = \xi_+(a, b)$. In all computations to follow this fixed point is hyperbolic. For more details on the map see the references given in [5, Section 4].

In [5] the approximation of a homoclinic orbit for the parameter values

$$\bar{a} = 1.4, \quad b = 0.3$$

is shown, see [5, Figure 1]. Varying the parameter b, we get a branch of approximate homoclinics. For the interval $J = [-25, 25]$ this branch is shown in Figure 3, where the *amplitude* of an $f(\cdot, a, b)$–orbit x_J is defined as

$$\mathrm{ampl}(x_J) = \sqrt{\sum_{n \in J} \|x_n - \xi(a, b)\|^2}.$$

This branch has four turning points which correspond to quadratic tangencies. At each of these points and at $b \approx 0.3$ the shape of the unstable and stable manifolds $W_u(\xi(\bar{a}, b), \bar{a}, b)$ and $W_s(\xi(\bar{a}, b), \bar{a}, b)$ is sketched.

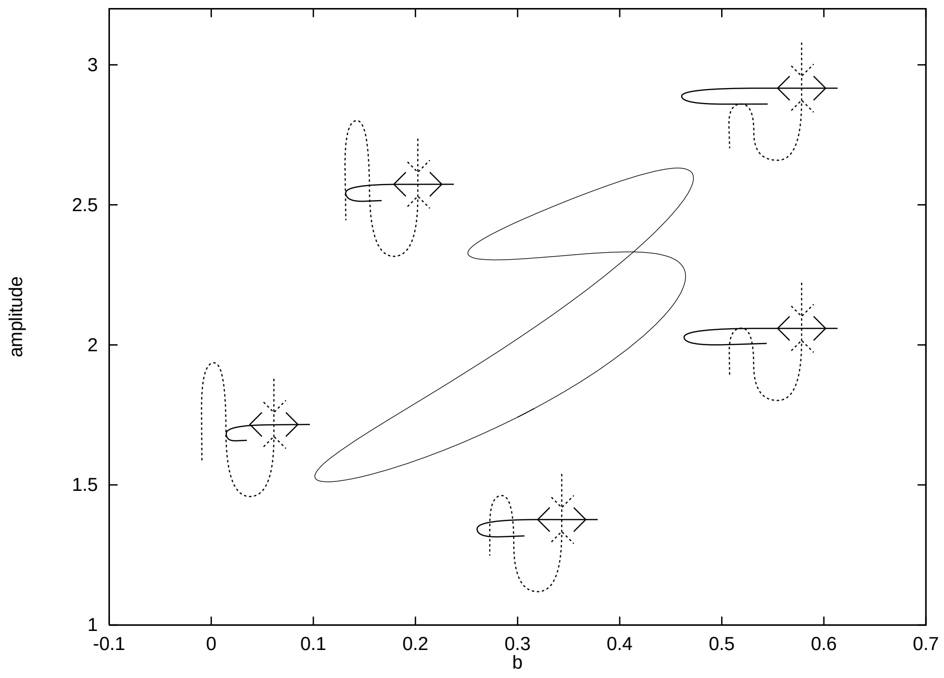

Figure 3: Branch of homoclinics with a qualitative sketch of the unstable and the stable manifolds for Hénon's map.

We consider the turning point at $b \approx 0.464527$ with an amplitude of approximately 2.2446. Taking this point as starting vector (filled with $\xi(\bar{a},\bar{b})$'s on both sides) we compute a turning point of Γ_J for the large interval $J = [-100, 100]$. The calculated orbit $\bar{x}_{[-100,100]}$ is considered as "exact" tangential homoclinic, the corresponding value $\bar{b}$ is taken as "exact" value of b. This orbit and parts of $W_u(\xi(\bar{a},\bar{b}),\bar{a},\bar{b})$ and $W_s(\xi(\bar{a},\bar{b}),\bar{a},\bar{b})$ are shown in Figure 4.

The spectrum of $f_x(\xi(\bar{a},\bar{b}),\bar{a},\bar{b})$ is

$$\sigma(f_x(\xi(\bar{a},\bar{b}),\bar{a},\bar{b})) = \{0.220066, -2.11085\}.$$

Let

$$\alpha_s = \log_{10}(0.220066) = 0.657447, \quad \alpha_u = \log_{10}(2.11085) = 0.324458.$$

We compute turning points (x_J, b_J) of $\Gamma_J(\cdot,\bar{a},\cdot)$ for intervals $J = [n_-, n_+]$ with moderate values of n_-, n_+, and consider the logarithmic global error

$$\log_{10}(\max\{\max_{n\in J} \|x_n - \bar{x}_n\|, |b_J - \bar{b}|\}).$$

This global error together with the logarithmic error

$$\log_{10}(|b_J - \bar{b}|)$$

in the critical parameter b is shown in Figure 5. For the logarithmic global error we find approximately the slopes $2\alpha_u$ and $-2\alpha_s$ in n_- and n_+ on the two parts of the corresponding surface. Therefore, the global error behaves like

$$\max\{C_1 10^{2\alpha_u n_-}, C_2 10^{-2\alpha_s n_+}\}$$

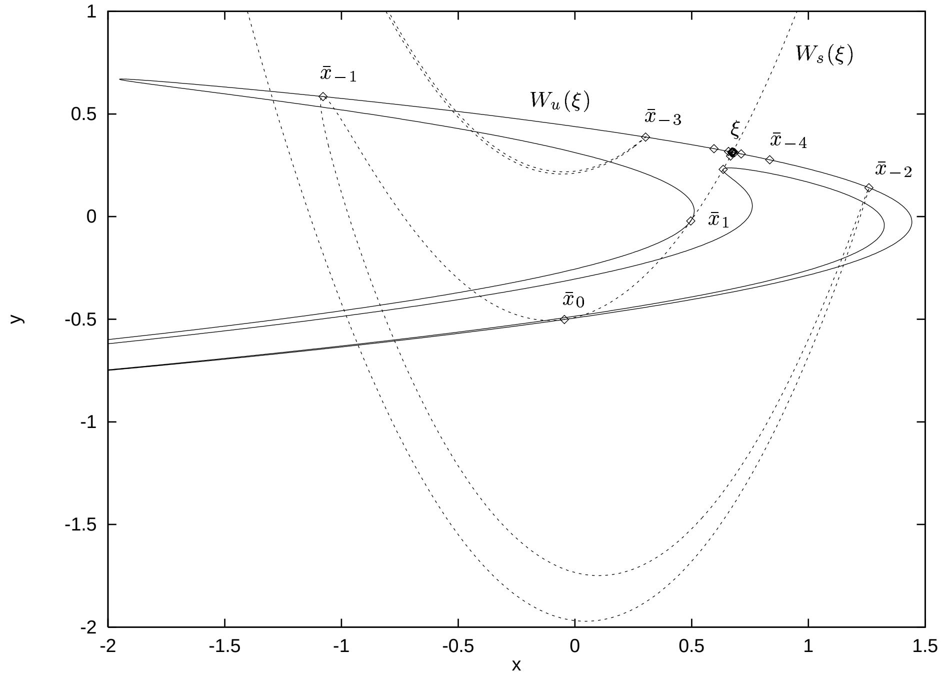

Figure 4: Parts of $W_u(\xi(\bar{a},\bar{b}),\bar{a},\bar{b})$ and $W_s(\xi(\bar{a},\bar{b}),\bar{a},b)$ together with the "exact" tangential homoclinic $\bar{x}_{[-100,100]}$ for Hénon's map.

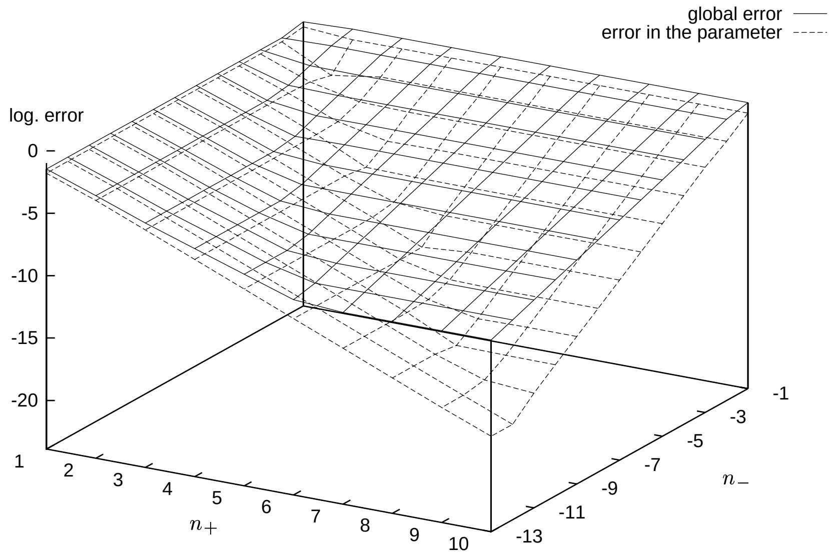

Figure 5: Global error and error in the critical parameter b for Hénon's map.

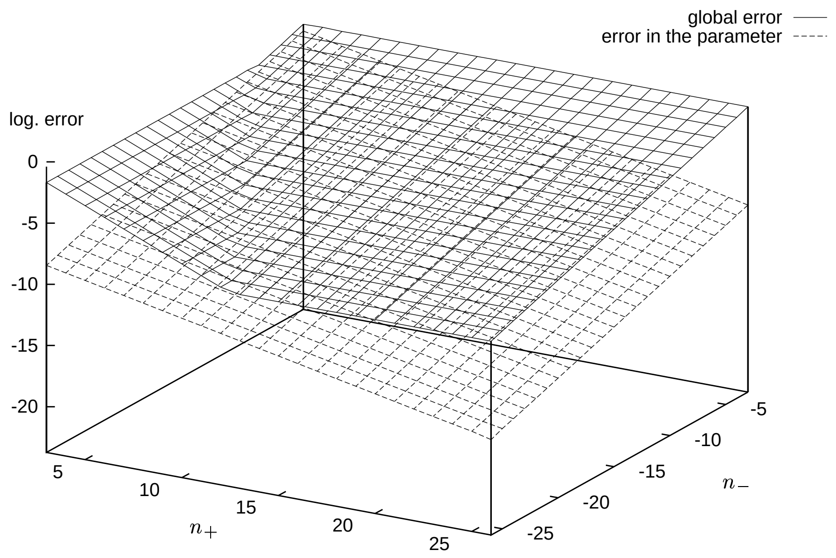

Figure 6: Global error and error in the critical parameter b for Hénon's map and periodic boundary conditions.

for some $C_1, C_2 > 0.$, This is exactly the rate from Corollary 4.3.

For the error in the parameter we observe superconvergence. Analyzing the slopes we find that in our example an error of the form

$$\max\{C_3 10^{(2\alpha_u+\alpha_s)n_-}, C_4 10^{-(2\alpha_s+\alpha_u)n_+}\}$$

occurs for some $C_3, C_4 > 0$. This superconvergence phenomenon is well known and analyzed for continuous systems, cf. Sandstede's paper [21]. It is shown in [13, Section 4.3] that the ideas of [21] can be modified and applied to quadratic homoclinics of discrete systems. This gives an upper bound for the error in the critical parameter as observed above, see [13, Section 5.1.1].

For periodic boundary conditions, the global error and the error in the parameter b are shown in Figure 6. We observe that the global error has a shape like

$$\max\{C_1 10^{\alpha_u n_-}, C_2 10^{-\alpha_s n_+}\}.$$

This is again the rate from Corollary 4.3. Similar to projection boundary conditions we find a superconvergence for the error in the critical parameter given by

$$|b - \bar{b}| \approx C\, 10^{-\alpha_u(n_+ - n_-)}.$$

This is faster than the upper bound $C(10^{\alpha_s n_-} + 10^{-\alpha_u n_+})(10^{\alpha_u n_-} + 10^{-\alpha_s n_+})$ obtained in [13, Section 5.1.1].

Finally, we continue the branches of the four tangentials shown in Figure 3 by varying a. This is justified by Theorem 4.5. The result of this calculations is shown in Figure 7. The following scenario takes place when we start at $a = 1.4$

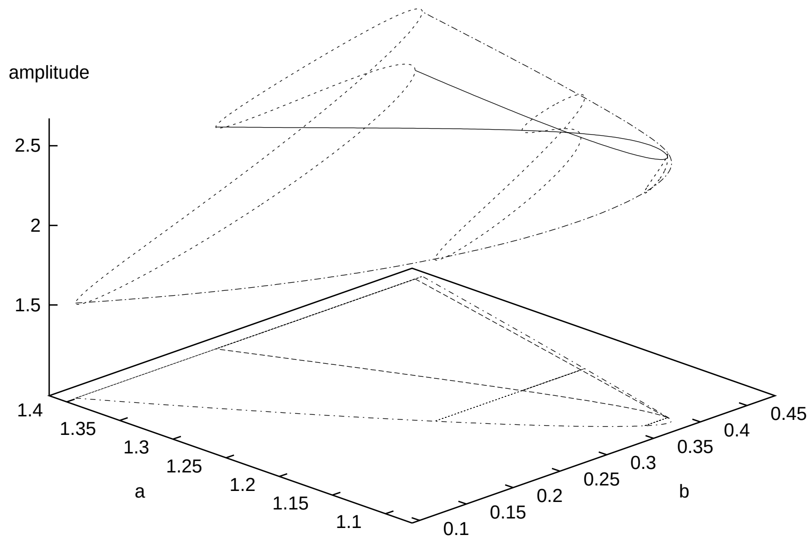

Figure 7: Two branches of tangencies (——, –·–) and three closed loops of homoclinics (– – –) with their projections on the parameter plane for Hénon's map.

and decrease the value of a. First two of the four tangentials shown in Figure 3 collide and vanish at a cusp. Then the remaining two tangentials collide and vanish at an isola formation point. (For cusps and isola formation points see [24].) The shape of the unstable and stable manifolds, and the tangential orbits at these singularities are shown in [13, Figures 7.39, 7.40].

Example 2: Duffing's map

As in [5] we consider the forced Duffing equation

$$\dot{u} = v, \quad \dot{v} = u - u^3 - \delta v + \gamma \cos(t). \tag{42}$$

Duffing's map is defined as period map of this equation, i.e. $f(x, y, \delta, \gamma)$ is the solution of (42) at time $t = 2\pi$ with the initial value $(u, v)(0) = (x, y)$. Thus f is of type $\mathcal{C}^\infty$ and $f(\cdot, \cdot, \delta, \gamma)$ is $\mathcal{C}^\infty$–diffeomorphic for every (δ, γ). The evaluation of f, f', as well as the computation of the fixed point $\xi(\delta, \gamma)$ of $f(\cdot, \cdot, \delta, \gamma)$, is done numerically as described in [5].

Notice, there is a $\mathbb{Z}_2$–symmetry of the type $(x, y, \delta, \gamma) \mapsto (-x, -y, \delta, -\gamma)$ in the system. Furthermore,

$$(x_1, y_1) = f(x_0, y_0, \delta, \gamma) \quad \Longleftrightarrow \quad (x_0, -y_0) = f(x_1, -y_1, -\delta, \gamma) \tag{43}$$

follows by the definition of f for every $x_0, y_0, x_1, y_1, \delta, \gamma \in \mathbb{R}$.

By the same procedure as in [5, Section 4] we compute an "exact" homoclinic orbit $\bar{x}_{[-15,15]}$ at

$$\delta = 0.25, \quad \bar{\gamma} = 0.3.$$

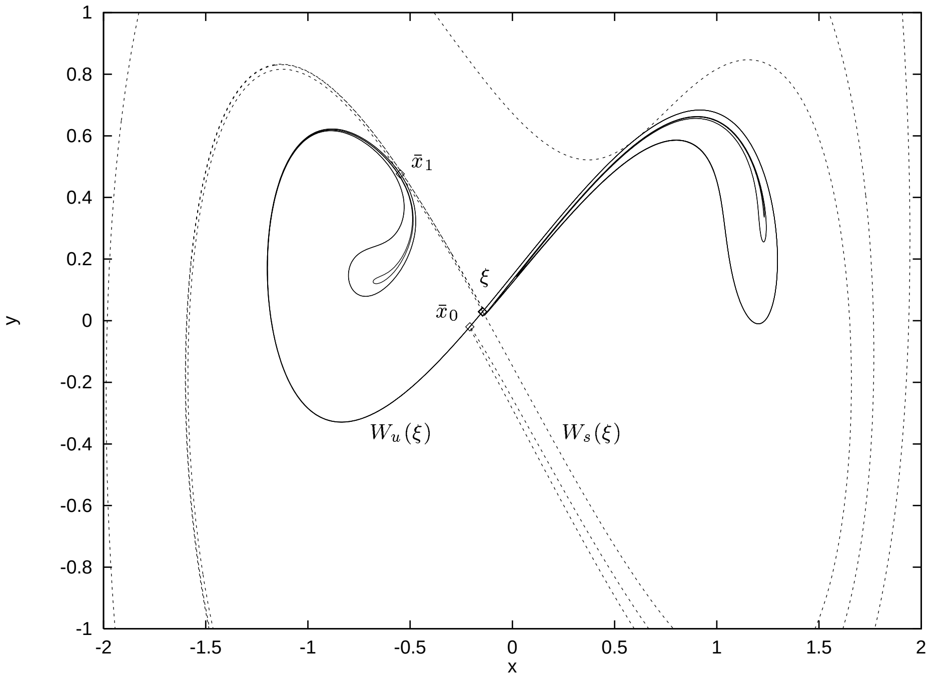

Figure 8: Parts of $W_u(\xi(\bar{\delta},\bar{\gamma}),\bar{\delta},\bar{\gamma})$ and $W_s(\xi(\bar{\delta},\bar{\gamma}),\bar{\delta},\bar{\gamma})$ together with points of the "exact" tangential $\bar{x}_{[-15,15]}$ for Duffing's map.

Varying the parameter δ we find a closed loop of homoclinics with two tangentials at

$$\bar{\delta} = 0.395809624199, \quad \tilde{\delta} = -\bar{\delta},$$

respectively. The tangential at $(\bar{\delta},\bar{\gamma})$ is drawn in Figure 8 together with parts of the unstable and stable manifold of the fixed point

$$\xi(\bar{\delta},\bar{\gamma}) = (-0.145518, 0.0291852).$$

Due to (43) the corresponding figure for $(\tilde{\delta},\bar{\gamma})$ looks like Figure 8 after a reflection and change of the names of the manifolds, see [13, Figure 7.32].

The spectrum of the Jacobian $f_x(\xi(\bar{\delta},\bar{\gamma}),\bar{\delta},\bar{\gamma})$ is given by $\{5.28\cdot 10^{-4}, 1.57\cdot 10^{2}\}$. This means that the approximation error decays extremely fast with respect to n_- and n_+, and hence we do not present any figures on the error behavior.

In Figure 9 the closed loop of homoclinics mentioned above is drawn at the parameter value $\gamma = 0.3$. Varying the parameter γ we follow the branches of the two tangentials.

In our figure these two branches have an intersection at $\delta = \gamma = 0$. But this is not the case for the curves of homoclinic orbits. The reason is that at $\delta = \gamma = 0$ the equation (42) becomes the cubic oscillator $\dot{u} = v$, $\dot{v} = u - u^3$ which is autonomous. Therefore, $W_u(\xi(0,0),0,0)$ and $W_s(\xi(0,0),0,0)$ coincide and form a curve of non–quadratic 1–tangentials. The two tangentials on the branches considered above conjugate by the reflection (43), and hence their amplitudes are equal. For a figure confer [13, Figures 7.33]. Notice, the case of non–quadratic 1–tangentials is not covered by our theory.

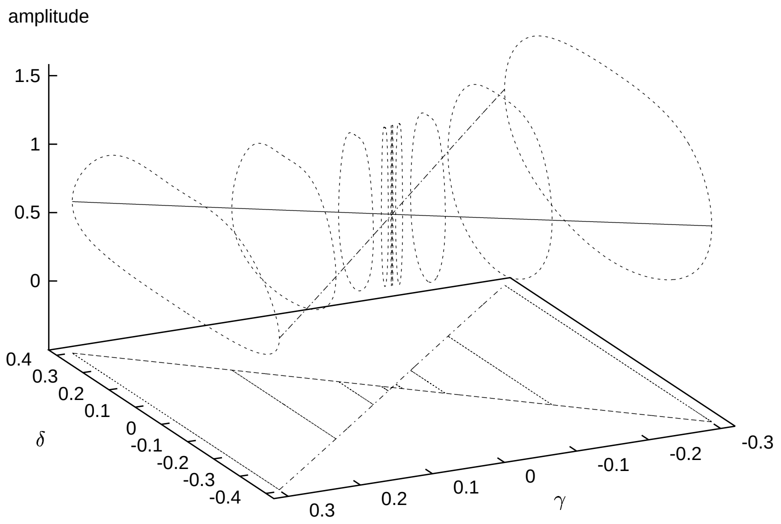

Figure 9: Two branches of tangentials (—–, $-\cdot-$) and 10 closed loops of homoclinics ($---$) with their projections on the parameter plane for Duffing's map.

In Figure 9 we also show some curves of homoclinics for fixed values of γ and varied δ.

A A linearization lemma for finite systems

The following lemma is used to prove Theorem 3.5. It is the main linearization result of this paper and generalizes the ones proved in [5, Theorem 3.4] (where $p = t = 0$) and in the diploma thesis [11] (where $t = 0$).

We consider matrices $A_n \in \mathbb{R}^{k,k}$, $n \in \mathbb{Z}$, which are nonsingular and converge to some hyperbolic matrices $A_\pm \in \mathbb{R}^{k,k}$ as $n \to \pm\infty$. Let $X_{\pm s}$, $X_{\pm u}$ be the stable and unstable subspace of $A_\pm$, respectively, and let $k_{\pm s} = \dim X_{\pm s}$. By Lemma 2.5 the equation

$$u_{n+1} = A_n u_n \tag{44}$$

has an exponential dichotomy on $\mathbb{Z}_-$ and $\mathbb{Z}_+$ with projectors $P^\pm_{\mathbb{Z}_\pm}$. Moreover, the rank of $P_0^\pm$ equals $k_{\pm s}$ by Proposition 2.4 and Lemma 2.5. Due to Lemma 2.6 the operator $L : S_\mathbb{Z} \to S_\mathbb{Z}$, $u_\mathbb{Z} \mapsto (u_{n+1} - A_n u_n)_{n\in\mathbb{Z}}$, is Fredholm of index $-p$ with $p = k_{-s} - k_{+s}$. Furthermore the nullspace $\mathcal{N}(L)$ is of dimension $t = \dim(\mathcal{R}(P_0^+) \cap \mathcal{N}(P_0^-))$. In the following let $\{\bar{u}^1_\mathbb{Z}, \ldots, \bar{u}^t_\mathbb{Z}\}$ be a basis of $\mathcal{N}(L)$. Notice, $t + p = \text{codim}\mathcal{R}(L) \geq 0$ holds by the definition of the Fredholm index of L, and $k + p \geq 0$ is satisfied since $p \in \{-k, \ldots, k\}$.

For some $M \in \mathbb{N}$ we consider operators $L_J \in \mathcal{L}(S_J \times \mathbb{R}^{t+p}, S_{\hat{J}} \times \mathbb{R}^{k+p} \times \mathbb{R}^t)$

defined by

$$L_J(y_J,\mu) = ((y_{n+1} - A_n y_n - F_n\mu)_{n\in\hat{J}}, B_J(y_{n_-}, y_{n_+}) + C_J\mu, D_J y_J + E_J\mu)$$

for $(y_J,\mu) \in S_J \times \mathbb{R}^{t+p}$ and $J = [n_-, n_+]$ with $\pm n_\pm \geq M$. The terms on the right side are defined by $F_\mathbb{Z} \in \mathcal{L}(\mathbb{R}^{t+p}, S_\mathbb{Z})$, $B_J \in \mathbb{R}^{k+p,2k}$, $C_J \in \mathbb{R}^{k+p,t+p}$, $D_J \in \mathcal{L}(S_J, \mathbb{R}^t)$ and $E_J \in \mathbb{R}^{t,t+p}$.

Lemma A.1 *Consider the operator L_J defined as above. Assume that*

$$\lim_{\min\{-n_-,n_+\}\to\infty} B_J = B$$

holds where $B \in \mathbb{R}^{k+p,2k}$ is a matrix such that the restriction $B_{|X_{-s}\times X_{+u}}$ is nonsingular. Furthermore, let

$$\dim \mathcal{N}(L, F_\mathbb{Z}) = t$$

hold for the bordered operator $(L, F_\mathbb{Z}) \in \mathcal{L}(S_\mathbb{Z} \times \mathbb{R}^{t+p}, S_\mathbb{Z})$. Suppose

$$\|C_J\|, \|D_J\|, \|E_J\|, \|[D_J(\bar{u}^1_{|J}), \ldots, D_J(\bar{u}^t_{|J})]^{-1}\| \leq C_0$$

holds for sufficiently large $-n_-, n_+$ with some constant C_0 which does not depend on J. Then there exist constants $C, N > 0$ such that for every $J = [n_-, n_+]$ with $-n_-, n_+ \geq N$ the operator L_J is homeomorphic with

$$\|L_J^{-1}\|_\infty \leq C.$$

Proof. We will prove that there exist constants $C, N > 0$ such that the following statements hold for every $J = [n_-, n_+]$ with $-n_-, n_+ \geq N$.

Statement 1. For every $r_{\hat{J}} \in S_{\hat{J}}$ the system

$$u_{n+1} - A_n u_n - F_n\omega = r_n, \quad n \in \hat{J}, \tag{45}$$

has a solution $(u_J, \omega) \in S_J \times \mathbb{R}^{t+p}$ satisfying

$$\|(u_J, \omega)\|_\infty \leq C\|r_{\hat{J}}\|_\infty.$$

Statement 2. For every $(s, \tau) \in \mathbb{R}^{k+p} \times \mathbb{R}^t$ the equation

$$L_J(v_J, \eta) = (0, s, \tau) \tag{46}$$

has a solution $(v_J, \eta) \in S_J \times \mathbb{R}^{t+p}$ satisfying

$$\|(v_J, \eta)\|_\infty \leq C\|(s, \tau)\|.$$

Then for each $(q_{\hat{J}}, s, \tau) \in S_{\hat{J}} \times \mathbb{R}^{k+p} \times \mathbb{R}^t$ we can solve

$$L_J(y_J, \mu) = (r_{\hat{J}}, s, \tau), \quad (y_J, \mu) \in S_J \times \mathbb{R}^{t+p}, \tag{47}$$

as follows. Suppose (u_J,ω) is a solution as in Statement 1, and let (v_J,η) solve $L_J(v_J,\eta) = (0, s - B_J(u_{n_-},u_{n_+}) - C_J\omega, \tau - D_J u_J - E_J\omega)$ according to Statement 2. Then $(y_J,\mu) = (u_J + v_J, \omega+\eta)$ solves (47). Moreover, by the estimates of Statement 1 and 2 we get $\|(y_J,\mu)\|_\infty \le \tilde{C}\|(r_{\hat{J}},s,\tau)\|$ for some $\tilde{C} > 0$ which is independent of J and $(r_{\hat{J}},s,\tau)$. This gives the assertion.

In the following proofs of the Statements 1 and 2 let $C, N > 0$ denote generic constants, and let $J = [n_-,n_+]$ be an interval with $-n_-, n_+ \ge N$.

Proof of Statement 1. Due to Lemma 2.6 the operator L defined there is Fredholm of index $-p$ with a nullspace of dimension t. The bordering lemma for Fredholm operators (see [3, Lemma 2.3]) and the assumption $\dim \mathcal{N}(L,F_{\mathbb{Z}}) = t$ implies that $\tilde{L} = \begin{pmatrix} L & F_{\mathbb{Z}} \\ U & 0 \end{pmatrix}$, where $Uy_{\mathbb{Z}} = [\bar{u}_0^1,\dots,\bar{u}_0^t]^T y_0$, $y_{\mathbb{Z}} \in S_{\mathbb{Z}}$, is a homeomorphism with $\|\tilde{L}^{-1}\|_\infty \le C$. For $r_{\hat{J}} \in S_{\hat{J}}$ define $r_{\mathbb{Z}} \in S_{\mathbb{Z}}$ by setting $r_n = 0$ for $n \in \mathbb{Z}\setminus J$. Let $(z_{\mathbb{Z}},\omega) = \tilde{L}^{-1}(r_{\mathbb{Z}},0)$ and define $u_J = z_{|J}$. Then (45) and

$$\|(u_J,\omega)\|_\infty \le \|(z_{\mathbb{Z}},\omega)\|_\infty \le C\|(r_{\mathbb{Z}},0)\| \le C\|r_{\hat{J}}\|_\infty$$

follow.

Proof of Statement 2. By Lemma 2.5 we can assume that

$$\begin{aligned} \mathcal{R}(P_0^+) &= Z_1 + Z_R, & \mathcal{N}(P_0^+) &= Z_N + Z_2, \\ \mathcal{N}(P_0^-) &= Z_1 + Z_N, & \mathcal{R}(P_0^-) &= Z_R + Z_2 \end{aligned}$$

hold for a decomposition $Z_1 \oplus Z_R \oplus Z_N \oplus Z_2$ of $\mathbb{R}^k$. Notice that $\dim Z_1 = t$ and $\dim Z_2 = t + p$ is satisfied by Lemma 2.6.

In the following let $\Phi : S^2_{\mathbb{Z}} \to \mathbb{R}^{k,k}$ be the solution operator of (44) and $G^\pm : S^2_{\mathbb{Z}_\pm} \to \mathbb{R}^{k,k}$ its Green's function with respect to the projectors $P^\pm_{\mathbb{Z}_\pm}$. To get an ansatz for a solution (v_J,η) of (46) we define

$$z^-_{m_-} = \sum_{l=-\infty}^{-1} G^-(m_-,l+1)F_l, \quad z^+_{m_+} = \sum_{l=0}^{\infty} G^+(m_+,l+1)F_l$$

for $m_\pm \in \mathbb{Z}_\pm$. Notice that Lemma 2.2 implies $z^\pm_{\mathbb{Z}_\pm} \in S_{\mathbb{Z}_\pm}(\mathbb{R}^{k,t+p})$. Furthermore, $v^\pm_{\mathbb{Z}_\pm} = z^\pm_{\mathbb{Z}_\pm}\eta$ is the unique solution of $v^\pm_{n+1} - A_n v^\pm_n = F_n\eta$, $n \in \hat{\mathbb{Z}}_\pm$, with $P_0^+ v_0^+ = 0$ and $(I - P_0^-)v_0^- = 0$, respectively. By definition of Z_1, Z_R, Z_N, Z_2 this implies

$$\mathcal{R}(z_0^-) \subset Z_R + Z_2, \quad \mathcal{R}(z_0^+) \subset Z_N + Z_2.$$

To obtain the general solution of the inhomogeneous equation $v_{n+1} - A_n v_n = F_n\eta$ for $n \in \{n_-,\dots,-1\}$ and $n \in \{0,\dots,n_+\}$, respectively, we add solutions of the homogeneous equation as follows. For $\theta \in Z_1$, $\zeta_- \in Z_N$, $\zeta_+ \in Z_R$, $\rho_- \in \mathcal{R}(P^-_{n_-})$, $\rho_+ \in \mathcal{N}(P^+_{n_+})$, $\mu \in \mathbb{R}^{t+p}$ and $m_\pm \in J \cap \mathbb{Z}_\pm$ let

$$\begin{aligned} v^-_{m_-} &= z^-_{m_-}\eta + \Phi(m_-,0)(I - P_0^-)(\theta + \zeta_-) + \Phi(m_-,n_-)P^-_{n_-}\rho_-, \\ v^+_{m_+} &= z^+_{m_+}\eta + \Phi(m_+,0)P_0^+(\mu + \zeta_+) + \Phi(m_+,n_+)(I - P^+_{n_+})\rho_+. \end{aligned}$$

Then we set $v_n = y_n^-$ if $n \in \{n_-,\dots,-1\}$ and $v_n = y_n^+$ if $n \in \{0,\dots,n_+\}$. Therefore, the pair (v_J,η) solves (46) if

$$y_0^+ - y_0^- = 0, \quad B_J(v_{n_-},v_{n_+}) + C_J\eta = s, \quad D_J v_J + E_J\eta = \tau \tag{48}$$

is satisfied.

As in the proof of [5, Theorem 3.4] it is convenient to change the coordinates $(\rho_-,\rho_+,\theta)\in\mathcal{R}(P^-_{n_-})\times\mathcal{N}(P^+_{n_+})\times Z_1$ to $(x_s,x_u,\kappa)\in X_{-s}\times X_{+u}\times\mathbb{R}^t$ by

$$x_s = T_{n_-}\rho_-,\quad x_u = T_{n_+}\rho_+,\quad \theta=[\bar u_0^1,\ldots,\bar u_0^t]\kappa.$$

Here $T_{n_\pm}=I\pm(P^\pm_{n_\pm}-P^\pm)$, and $P^\pm\in\mathbb{R}^{k,k}$ is the projector onto $X_{\pm s}$ along $X_{\pm u}$. By Lemma 2.5 the projectors $P^\pm$ satisfy $\lim_{n\to\pm\infty}P^\pm_n=P^\pm$. Hence the matrices $T_{n_\pm}$ are nonsingular if N is sufficiently large, and its inverses are uniformly bounded.

Using these coordinates the system (48) reads

$$\begin{pmatrix} X & \sigma_J & 0\\ \Delta_J & B+\rho_J & \nu_J\\ \Theta_J & \Lambda_J & D_J[\bar u^1_{|J},\ldots,\bar u^t_{|J}]\end{pmatrix}\begin{pmatrix}(\zeta_+,\zeta_-,\eta)\\ (x_s,x_u)\\ \kappa\end{pmatrix}=\begin{pmatrix}0\\ s\\ \tau\end{pmatrix}\tag{49}$$

with

$$\begin{aligned}
X(\zeta_+,\zeta_-,\eta) &= \zeta_+-\zeta_-+(z_0^+-z_0^-)\eta,\\
\Delta_J(\zeta_+,\zeta_-,\eta) &= B_J(z^-_{n_-}\eta+\Phi(n_-,0)(I-P_0^-)\zeta_-,z^+_{n_+}\eta+\Phi(n_+,0)P_0^+\zeta_+)\\
&\quad +C_J\eta,\\
\Theta_J(\zeta_+,\zeta_-,\eta) &= D_Jc_J(\zeta_+,\zeta_-,\eta)+E_J\eta,\\
\sigma_J(x_s,x_u) &= \Phi(0,n_+)(I-P^+_{n_+})T^{-1}_{n_+}x_u-\Phi(0,n_-)P^-_{n_-}T^{-1}_{n_-}x_s,\\
\rho_J(x_s,x_u) &= (B_J-B)(x_s,x_u)+B_J((I-P^-)T^{-1}_{n_-}x_s,P^+T^{-1}_{n_+}x_u),\\
\Lambda_J(x_s,x_u) &= D_Jd_J(x_s,x_u),\\
\nu_J(\kappa) &= B_J(\Phi(n_-,0)(I-P_0^-)[\bar u_0^1,\ldots,\bar u_0^t]\kappa,\\
&\quad \Phi(n_+,0)P_0^+[\bar u_0^1,\ldots,\bar u_0^t]\kappa)
\end{aligned}$$

and

$$\begin{aligned}
c_n(\zeta_+,\zeta_-,\eta) &= \begin{cases} z_n^-\eta+\Phi(n,0)(I-P_0^-)\zeta_- & \text{for } n\in\{n_-,\ldots,-1\},\\ z_n^+\eta+\Phi(n,0)P_0^+\zeta_+ & \text{for } n\in\{0,\ldots,n_+\},\end{cases}\\
d_n(x_s,x_u) &= \begin{cases} \Phi(n,n_-)P^-_{n_-}T^{-1}_{n_-}x_s & \text{for } n\in\{n_-,\ldots,-1\},\\ \Phi(n,n_+)(I-P^+_{n_+})T^{-1}_{n_+}x_u & \text{for } n\in\{0,\ldots,n_+\}.\end{cases}
\end{aligned}$$

The block–matrix in (49) defines a linear map from $(Z_R\times Z_N\times\mathbb{R}^{t+p})\times(X_{-s}\times X_{+u})\times\mathbb{R}^t$ into $(Z_R\times Z_N\times Z_2)\times\mathbb{R}^{k+p}\times\mathbb{R}^t$. These spaces are of the same dimension. Furthermore, $\sigma_J,\rho_J,\nu_J\to 0$ as $n_\pm\to\pm\infty$ follows by arguments similar to those in the proof of [5, Theorem 3.4], i.e. using the exponential dichotomies, the property $\lim_{n\to\pm\infty}P^\pm_n=P^\pm$ and the estimates

$$\|(I-P^-)T^{-1}_{n_-}x_s\|\le\frac{\|P^-_{n_-}-P^-\|\,\|x_s\|}{1-\|P^-_{n_-}-P^-\|},\quad \|P^+T^{-1}_{n_+}x_u\|\le\frac{\|P^+_{n_+}-P^+\|\,\|x_u\|}{1-\|P^+_{n_+}-P^+\|}.$$

Since $B_{|X_{-s}\times X_{+u}}$ is nonsingular and Δ_J, Θ_J and Λ_J are uniformly bounded, it suffices to show that the map $X\in\mathcal{L}(Z_R\times Z_N\times\mathbb{R}^{t+p},Z_R\times Z_N\times Z_2)$ is nonsingular. Then the block–matrix in (49) is also nonsingular if N is sufficiently large and its inverse is uniformly bounded by some properly chosen constant. Retransforming the coordinates (x_s,x_u,κ) and using the ansatz for (v_J,η) yields the assertion of Statement 2.

Let us first collect some properties of the adjoint operator

$$L^* : S_{\mathbb{Z}} \to S_{\mathbb{Z}}, \varphi_{\mathbb{Z}} \mapsto (\varphi_{n+1} - (A_{n+1}^{-1})^T \varphi_n)_{n\in\mathbb{Z}}.$$

By Lemma 2.6 the kernel $\mathcal{N}(L^*)$ is of dimension $t+p$, and so we find a basis $\{\bar{\varphi}_{\mathbb{Z}}^1, \ldots, \bar{\varphi}_{\mathbb{Z}}^{t+p}\}$ of $\mathcal{N}(L^*)$. The fact that $\Phi^*(n,m) = \Phi(m+1,n+1)^T$, $n,m\in\mathbb{Z}$, is the solution operator of the adjoint equation $\varphi_{n+1} = (A_{n+1}^{-1})^T\varphi_n$, $n\in\mathbb{Z}$, implies

$$\bar{\varphi}_n^i = \Phi(0,n+1)^T \bar{\varphi}_{-1}^i, \quad n\in\mathbb{Z}, \quad i=1,\ldots,t+p. \tag{50}$$

Furthermore, the adjoint equation has an exponential dichotomy on $\mathbb{Z}_\pm - 1$ with projectors $(I-(P_{n+1}^\pm)^T)_{n\in\mathbb{Z}_\pm-1}$. Due to Lemma 2.5 this implies

$$\bar{\varphi}_{-1}^i \in \mathcal{R}((I-P_0^+)^T) \cap \mathcal{N}((I-P_0^-)^T), \quad i=1,\ldots,t+p. \tag{51}$$

By definition, X acts between spaces of the same finite dimension. Therefore it remains to show that X is injective. Let $(\zeta_+,\zeta_-,\eta)\in\mathcal{N}(X)$ be arbitrary, i.e.

$$\zeta_+ - \zeta_- + (z_0^+ - z_0^-)\eta = 0. \tag{52}$$

By the choice of Z_R and Z_N we obtain $\zeta_+\in\mathcal{R}(P_0^+)$, $\zeta_-\in\mathcal{N}(P_0^-)$, and by (51)

$$[\bar{\varphi}_{-1}^1, \ldots, \bar{\varphi}_{-1}^{t+p}]^T (z_0^+ - z_0^-)\eta = 0 \tag{53}$$

follows. Furthermore, the definition of $z_0^\pm$ implies

$$z_0^+ - z_0^- = (P_0^+ - I)\sum_{n=0}^{\infty}\Phi(0,n+1)F_n - P_0^- \sum_{n=-\infty}^{-1}\Phi(0,n+1)F_n.$$

Therefore, by (50)–(53)

$$\sum_{n\in\mathbb{Z}} [\bar{\varphi}_n^1, \ldots, \bar{\varphi}_n^{t+p}]^T F_n \eta = 0 \tag{54}$$

holds. By the assumption $\dim N(L,F_{\mathbb{Z}}) = t$ we get $\dim\mathcal{R}(F_{\mathbb{Z}}) = t+p$ and $\mathcal{R}(F_{\mathbb{Z}})\cap\mathcal{R}(L) = \{0\}$. Together with property (21) of Lemma 2.6, this implies that the matrix $\sum_{n\in\mathbb{Z}}[\bar{\varphi}_n^1,\ldots,\bar{\varphi}_n^{t+p}]^T F_n \in \mathbb{R}^{t+p,t+p}$ is nonsingular. Therefore $\eta = 0$ follows by (54). From (52) and $Z_R\cap Z_N = \{0\}$ we obtain $(\zeta_+,\zeta_-) = (0,0)$. □

B Geometric characteristics of heteroclinics

In this paper we defined the properties of heteroclinic orbits in terms of the partial derivatives of Γ. Now we derive equivalent definitions for t–tangential, nondegenerate, and quadratic heteroclinic orbits in terms of the intersection of the unstable and the stable manifold of the fixed points connected by the heteroclinics. Obviously, all points of a heteroclinic orbit lie in the intersection of these manifolds.

We consider the proof of the stable manifold theorem for parameterized systems given in Lemma 3.13. This will be done in such a way that we derive charts of the stable and unstable manifolds in terms of zeros of

$$\Gamma_\pm : S_{\mathbb{Z}_\pm}\times\mathbb{R}^r \to S_{\hat{\mathbb{Z}}_\pm}, (x_{\mathbb{Z}},\lambda)\mapsto (x_{n+1} - f(x_n,\lambda))_{n\in S_{\hat{\mathbb{Z}}_\pm}}.$$

Differentiating the corresponding parameterizing maps we will derive the open equivalence results.

B.1 The stable manifold theorem for parameterized systems

In this section let $J \in \{\mathbb{Z}_-, \mathbb{Z}_+\}$, and $(\kappa, \bar{\kappa}) = (s, +)$ in case $J = \mathbb{Z}_+$ and $(\kappa, \bar{\kappa}) = (u, -)$ in case $J = \mathbb{Z}_-$. We consider a sequence $\bar{y}_J \in S_J$ such that

$$u_{n+1} = f_x(\bar{y}_n, \bar{\lambda}) u_n$$

has an exponential dichotomy on J with projectors P_J. Furthermore, let

$$\mathcal{M}(\bar{y}_J, \delta, \varepsilon) = \{(y, \lambda) \in U_\delta(\bar{y}_0) \times U_\varepsilon(\bar{\lambda}) : f(\cdot, \lambda)^n(y) \in U_\delta(\bar{y}_n) \text{ for every } n \in J\} \tag{55}$$

for $\delta, \varepsilon > 0$.

Our first aim is to show that $\mathcal{M}(\bar{y}_J, \delta, \varepsilon)$ is a submanifold of $\mathbb{R}^{k+r}$ if $\delta, \varepsilon > 0$ are sufficiently small. For this purpose let us consider the operator

$$G : S_J \times U_{\varepsilon_0}(\bar{\lambda}) \times Y \to S_{\hat{J}} \times X, (y_J, \lambda, \gamma) \mapsto (\Gamma_{\bar{\kappa}}(y_J, \lambda), g(y_0, \lambda, \gamma)). \tag{56}$$

Here Y is some Banach space, $\varepsilon_0 > 0$ some constant, $X = \mathcal{R}(P_0)$ in case $J = \mathbb{Z}_+$ and $X = \mathcal{N}(P_0)$ in case $J = \mathbb{Z}_-$. It is assumed that $g : \mathbb{R}^k \times U_{\varepsilon_0}(\bar{\lambda}) \times Y$ is of type $\mathcal{C}^l$ with $g(\bar{y}_0, \bar{\lambda}, 0) = 0$ and the linear maps $D_1 g(\bar{y}_0, \bar{\lambda}, 0)_{|X}$, $D_3 g(\bar{y}_0, \bar{\lambda}, 0)$ are one–to–one. As we will show later, the set $\mathcal{M}(\bar{y}_J, \delta, \varepsilon)$ can be considered as the extended local stable (unstable) manifold of $\bar{\xi}_+(\bar{\xi}_-)$ in case $\bar{y}_{\mathbb{Z}_+} = \bar{x}_{|\mathbb{Z}_+}$ $(\bar{y}_{\mathbb{Z}_-} = \bar{x}_{|\mathbb{Z}_-})$.

Lemma B.1 *Under the assumptions given above*

(i) *the derivative $G_y(\bar{y}, \bar{\lambda}, 0)$ is a homeomorphism.*

(ii) *There exist constants $\delta_1 > 0$ and $\varepsilon_1 \in (0, \varepsilon_0]$ such that the set $\mathcal{M}(\bar{y}_J, \delta, \varepsilon)$ is a $\mathcal{C}^l$–submanifold of $\mathbb{R}^{k+r}$ of dimension $r + \dim X$ for every $\delta \in (0, \delta_1]$ and $\varepsilon \in (0, \varepsilon_1]$.*

(iii) *Moreover, there exist neighborhoods $V \subset Y$ of 0 and $W(\delta, \varepsilon) \subset \mathbb{R}^r \times Y$ of $(\bar{\lambda}, 0)$, a constant $\tilde{\delta}_1 > \delta_1$, and a $\mathcal{C}^l$–map $\tilde{x}_J : U_\varepsilon(\bar{\lambda}) \times V \to U_{\tilde{\delta}_1}(\bar{y}_J)$ such that $\tilde{x}_J(\lambda, \gamma)$ is the unique zero of $G(\cdot, \lambda, \gamma)$ in $U_{\tilde{\delta}_1}(\bar{y}_J)$ for every $(\lambda, \gamma) \in U_\varepsilon(\bar{\lambda}) \times V$ and we have the graph representation*

$$\mathcal{M}(\bar{y}_J, \delta, \varepsilon) = \{(\tilde{x}_0(\lambda, \gamma), \lambda) : (\lambda, \gamma) \in W(\delta, \varepsilon)\}. \tag{57}$$

The value of $\sup_{(\lambda,\gamma) \in W(\delta,\varepsilon)} \|(\lambda, \gamma) - (\bar{\lambda}, 0)\|$ tends to zero as $\max\{\delta, \varepsilon\} \to 0$.

(iv) *Furthermore, there exists a $\mathcal{C}^l$–diffeomorphism $\psi : U_\psi \to V_\psi$ such that $\psi^{-1}(\lambda, \gamma, 0) = (\tilde{x}_0(\lambda, \gamma), \lambda)$ holds for every $(\lambda, \gamma) \in W(\delta, \varepsilon)$; here $U_\psi \subset \mathbb{R}^{k+r}$ is a neighborhood of $(\bar{y}_0, \bar{\lambda})$ and V_ψ is open such that $W(\delta, \varepsilon) \times \{0\} \subset V_\psi \subset \mathbb{R}^r \times Y \times \mathbb{R}^{k - \dim X}$ is satisfied.*

Proof. Due to Lemma 2.2 the map $G_y(\bar{y},\bar{\lambda},0)$ is homeomorphic. By the implicit function theorem there exist $\varepsilon_2 \in (0,\varepsilon_0]$, $\tilde{\delta}_1 > 0$ and a $\mathcal{C}^l$–map $\tilde{x}$: $U_{\varepsilon_2}(\bar{\lambda},0) \to U_{\tilde{\delta}_1}(\bar{y}_J)$ such that $\tilde{x}_J(\lambda,\gamma)$ is the unique zero of $G(\cdot,\lambda,\gamma)$ in $U_{\tilde{\delta}_1}(\bar{y}_J)$ for every $(\lambda,\gamma) \in U_{\varepsilon_2}(\bar{\lambda},0)$. Differentiating both sides of the equation

$$G(\tilde{x}_J(\lambda,\gamma),\lambda,\gamma) = 0, \quad (\lambda,\gamma) \in U_{\varepsilon_2}(\bar{\lambda},0),$$

with respect to γ, we get $D_2\tilde{x}_0(\bar{\lambda},0) = -(D_1 g(\bar{y}_0,\bar{\lambda},0)_{|J})^{-1} D_3 g(\bar{y}_0,\bar{\lambda},0)$ which is nonsingular. Let $\tilde{k} = k - \dim X$ and $B \in \mathbb{R}^{k,\tilde{k}}$ such that $\mathcal{R}(B) \oplus X = \mathbb{R}^k$. Set $\phi(\lambda,\gamma,\chi) = (\tilde{x}_0(\lambda,\gamma) + B\chi, \lambda)$ for $(\lambda,\gamma,\chi) \in U_{\varepsilon_2}(\bar{\lambda},0) \times \mathbb{R}^{\tilde{k}}$. By definition $\phi \in \mathcal{C}^l(U_{\varepsilon_2}(\bar{\lambda},0) \times \mathbb{R}^{\tilde{k}}, \mathbb{R}^{k+r})$ holds, and $\phi'(\bar{\lambda},0,0)$ is nonsingular. Therefore, ϕ is locally $\mathcal{C}^l$–diffeomorphic at $(\bar{\lambda},0,0)$, which means that there exist open neighborhoods $V_\psi \subset U_{\varepsilon_2}(\bar{\lambda},0)\times\mathbb{R}^{\tilde{k}}$ and $U_\psi \subset \mathbb{R}^{k+r}$ of $(\bar{\lambda},0)$ and $(\bar{y}_0,\bar{\lambda})$ such that $\tilde{\phi} : V_\psi \to U_\psi, (\lambda,\gamma,\chi) \mapsto \phi(\lambda,\gamma,\chi)$ is $\mathcal{C}^l$–diffeomorphic. We define $\psi = \tilde{\phi}^{-1}$. By construction $\psi^{-1}(\lambda,\gamma,0) = (\tilde{x}_0(\lambda,\gamma),\lambda)$ holds for every $(\lambda,\gamma,0) \in V_\psi$.

By assumption $D_3 g(\bar{y}_0,\bar{\lambda},0)$ is nonsingular. Due to the implicit function theorem there exist $\delta_2,\varepsilon_1 \in (0,\varepsilon_2]$ and a $\mathcal{C}^l$–map $\tilde{\gamma} : U_{\varepsilon_1}(\bar{y}_0,\bar{\lambda}) \to U_{\delta_2}(0)$ such that $\tilde{\gamma}(x,\lambda)$ is the unique zero of $g(\cdot,x,\lambda)$ in $U_{\delta_2}(0)$ for every $(x,\lambda) \in U_{\varepsilon_1}(\bar{y}_0,\bar{\lambda})$. Furthermore we can achieve $U_{\varepsilon_1}(\bar{\lambda}) \times U_{\delta_2}(0) \times \{0\} \subset V_\psi$.

Define $\delta_1 = \min\{\varepsilon_1,\tilde{\delta}_1\}$ and $V = U_{\varepsilon_2}(0) \subset Y$. Let $\delta \in (0,\delta_1]$, $\varepsilon \in (0,\varepsilon_1]$ and define $W(\delta,\varepsilon) = \tilde{x}_J^{-1}(U_\delta(\bar{y}_J)) \cap (U_\varepsilon(\bar{\lambda}) \times U_{\delta_2}(0)) \subset \mathbb{R}^r \times Y$. It is easy to verify that the identity (57) as well as the other open assertions follow. □

The first application of the previous lemma is the proof of Lemma 3.13. Under the assumptions given there we show that $\mathcal{M}((\bar{\xi})_{n\in J},\delta,\varepsilon)$ is equal to the set

$$\mathcal{W}_{\kappa,\varepsilon}^{U_\delta(\bar{\xi})}(\bar{\xi},\bar{\lambda}) = \{(x,\lambda) \in \mathbb{R}^k \times U_\varepsilon(\bar{\lambda}) : x \in W_\kappa^{U_\delta(\bar{\xi})}(\xi(\lambda),\lambda)\}$$

if $\delta,\varepsilon > 0$ are sufficiently small and $\xi(U_\varepsilon(\bar{\lambda})) \subset U_\delta(\bar{\xi})$ is satisfied. Therefore the set $\mathcal{M}((\bar{\xi})_{n\in J},\delta,\varepsilon)$ is the local extension of $W_\kappa^{U_\delta(\bar{\xi})}(\bar{\xi},\bar{\lambda})$ with respect to the parameter λ. The sets $\mathcal{W}_{s,\varepsilon}^{U_\delta(\bar{\xi})}(\bar{\xi},\bar{\lambda})$ and $\mathcal{W}_{u,\varepsilon}^{U_\delta(\bar{\xi})}(\bar{\xi},\bar{\lambda})$ are called the *extended local stable* and *unstable manifold of* $\bar{\xi}$.

Proof of Lemma 3.13. By Lemma 2.3 and Proposition 2.4 there exists a projector valued $\mathcal{C}^{l-1}$–map $P_\kappa : U_{\varepsilon_0}(\bar{\lambda}) \to \mathbb{R}^{k,k}$ such that $\mathcal{R}(P_\kappa(\lambda)) = X_\kappa(\lambda)$ and $\mathcal{N}(P_\kappa(\lambda)) = X_{\tilde{\kappa}}(\lambda)$ hold for every $\lambda \in U_{\varepsilon_0}(\bar{\lambda})$. Let

$$D(\lambda) = I - P_\kappa(\lambda) - P_\kappa(\bar{\lambda}) + 2P_\kappa(\bar{\lambda})P_\kappa(\lambda), \quad \lambda \in U_{\varepsilon_0}(\bar{\lambda}).$$

This implies that $D : U_{\varepsilon_0}(\bar{\lambda}) \to \mathbb{R}^{k,k}$ is of type $\mathcal{C}^{l-1}$ and $D(\bar{\lambda}) = I$. Taking $\varepsilon_0 > 0$ sufficiently small, $D(\lambda)$ is nonsingular and $D(\lambda)X_\sigma(\lambda) = X_\sigma(\bar{\lambda})$, $\sigma = s,u$, holds for every $\lambda \in U_{\varepsilon_0}(\bar{\lambda})$.

We consider the $\mathcal{C}^{l-1}$–map

$$g : \mathbb{R}^k \times U_{\varepsilon_0}(\bar{\lambda}) \times X_\kappa(\bar{\lambda}) \to X_\kappa(\bar{\lambda}), (x,\lambda,\gamma) \mapsto D(\lambda)P_\kappa(\lambda)(x - \xi(\lambda)) - \gamma$$

and the sequence $\bar{y}_J = (\bar{\xi})_{n\in J}$, cf. Proposition 2.4. Then there exist $\delta_1 > 0$, $\varepsilon_1 \in (0,\varepsilon_0]$ such that $\mathcal{M}((\bar{\xi})_{n\in J},\delta,\varepsilon)$ satisfies (57) for every $\delta \in (0,\delta_1]$, $\varepsilon \in (0,\varepsilon_1]$. Notice that $\tilde{x}_J$ depends on the map g. In the following let $\delta,\varepsilon > 0$ be as above.

Taking some basis of $\mathbb{R}^r \times X_\kappa(\bar{\lambda})$, Lemma 3.12 applies to the sequences $\tilde{x}_J(\lambda,\gamma)$ for sufficiently small $\|(\lambda - \bar{\lambda},\gamma)\|$. Therefore, by the definition (55) the set $\mathcal{M}((\bar{\xi})_{n\in J},\delta,\varepsilon)$ equals $\mathcal{W}_{\kappa,\varepsilon}^{U_\delta(\bar{\xi})}(\bar{\xi},\bar{\lambda})$ if $\xi(U_\varepsilon(\bar{\lambda}))\subset U_\delta(\bar{\xi})$ holds and $\delta_1,\varepsilon_1>0$ are sufficiently small. This implies

$$W_\kappa^{U_\delta(\bar{\xi})}(\xi(\lambda),\lambda)=\{\tilde{x}_0(\lambda,\gamma):(\lambda,\gamma)\in W(\delta,\varepsilon)\},\quad \lambda\in U_\varepsilon(\bar{\lambda}), \tag{58}$$

where the neighborhood $W(\delta,\varepsilon)$ is determined by Lemma 3.12. Differentiating both sides of the equality

$$G(\tilde{x}_J(\lambda,\gamma),\lambda,\gamma)=0,\quad (\lambda,\gamma)\in W(\delta,\varepsilon), \tag{59}$$

and using Lemma 2.2 we get the identity (37) if $\delta_1,\varepsilon_1>0$ are sufficiently small.

We consider the $\mathcal{C}^{l-1}$–map $\phi(x,\lambda)=(\lambda,D(\lambda)P_\kappa(\lambda)x)$, $(x,\lambda)\in\mathbb{R}^k\times U_{\varepsilon_0}(\bar{\lambda})$. By its continuity the set $V_\kappa=\phi^{-1}(W(\delta,\varepsilon))$ is a neighborhood of $(\bar{\lambda},0)\in\mathbb{R}^{r+k}$. We define the $\mathcal{C}^{l-1}$–map $q_\kappa:V_\kappa\to\mathbb{R}^k$ by

$$q_\kappa(x,\lambda)=(I-P_\kappa(\lambda))((\tilde{x}_0\circ\phi)(x,\lambda)-\xi(\lambda)),\quad (x,\lambda)\in V_\kappa.$$

Notice that $\tilde{x}_0(\lambda,\gamma)=\xi(\lambda)$ holds for $\lambda\in U_\varepsilon(\bar{\lambda})$ and $\gamma\in V$ if and only if $\gamma=0$. Together with (58) and $\xi(U_\varepsilon(\bar{\lambda}))\subset U_\delta(\bar{\xi})$, this implies $\{0\}\times U_\varepsilon(\bar{\lambda})\subset V_\kappa$.

In the following let $\lambda\in U_\varepsilon(\bar{\lambda})$ be arbitrary. By $\tilde{x}_0(\lambda,0)=\xi(\lambda)$ we get $q_\kappa(0,\lambda)=0$. Moreover $\mathcal{R}(q_\kappa(\cdot,\lambda))\subset\mathcal{N}(P_\kappa(\lambda))=X_{\tilde{\kappa}}(\lambda)$ holds. Furthermore, the relation $\mathcal{R}(D_1\tilde{x}_0(\lambda,0))\subset\mathcal{R}(P_\kappa(\lambda))$ is satisfied due to the equalities (37), (58) and Proposition 2.4, Lemma 2.5. This implies $D_1q_\kappa(0,\lambda)=0$.

Next we prove that D_1q_κ is of type $\mathcal{C}^{l-1}$. Differentiating both sides of the identity (59) we get

$$G_x(\tilde{x}_J(\lambda,\gamma),\lambda,\gamma)D_2\tilde{x}_J(\lambda,\gamma)=-G_\gamma(\tilde{x}_J(\lambda,\gamma),\lambda,\gamma),\quad (\lambda,\gamma)\in W(\delta,\varepsilon).$$

Taking $\delta_1,\varepsilon_1>0$ sufficiently small, $G_x(\tilde{x}_J(\lambda,\gamma),\lambda,\gamma)$ is a homeomorphism for every $(\lambda,\gamma)\in W(\delta,\varepsilon)$. Moreover, G_x and G_γ are of type $\mathcal{C}^{l-1}$ by the structure of g. This implies that $D_2\tilde{x}_J$ and D_1q_κ are of type $\mathcal{C}^{l-1}$.

It remains to show the identity (36). In case $y\in W_\kappa^{U_\delta(\bar{\xi})}(\xi(\lambda),\lambda)$ the identity (58) implies $y=\tilde{x}_0(\lambda,\gamma)$ for some $\gamma\in X_\kappa(\bar{\lambda})$ with $(\lambda,\gamma)\in W(\delta,\epsilon)$. By definition of $\tilde{x}_0(\lambda,\gamma)$ we get $\gamma=D(\lambda)P_\kappa(\lambda)(y-\xi(\lambda))$. Putting $x_\kappa=D(\lambda)^{-1}\gamma=P_\kappa(\lambda)(y-\xi(\lambda))$ this implies

$$\begin{aligned} y &= \xi(\lambda)+P_\kappa(\lambda)(y-\xi(\lambda))+(I-P_\kappa(\lambda))(\tilde{x}_0(\lambda,\gamma)-\xi(\lambda)) \\ &= \xi(\lambda)+x_\kappa+q_\kappa(x_\kappa,\lambda). \end{aligned}$$

Notice, $(x_\kappa,\lambda)\in V_\kappa$ holds due to $\phi(\lambda,\gamma)=(x_\kappa,\lambda)$.

Conversely, let $y=\xi(\lambda)+x_\kappa+q_\kappa(x_\kappa,\lambda)$ for some $(x_\kappa,\lambda)\in V_\kappa$ (and some $\lambda\in U_\varepsilon(\bar{\lambda})$, $x_\kappa\in X_\kappa(\lambda)$). Then $(\lambda,D(\lambda)P_\kappa(\lambda)x_\kappa)\in W(\delta,\varepsilon)$ holds. By the definition of g and $\tilde{x}_J$ we get $x_\kappa=P_\kappa(\lambda)(\tilde{x}_0(\lambda,D(\lambda)P_\kappa(\lambda)x_\kappa)-\xi(\lambda))$, which implies

$$\tilde{x}_0(\lambda,D(\lambda)P_\kappa(\lambda)x_\kappa)=\xi(\lambda)+x_\kappa+q_\kappa(x_\kappa,\lambda)=y.$$

Hence $y\in W_\kappa^{U_\delta(\bar{\xi})}(\xi(\lambda),\lambda)$ holds due to (58). □

Notice, in the proof given above, we did not use the special structure of g when we derived the identity (58). Therefore

$$\mathcal{M}((\bar{\xi})_{n\in J},\delta,\varepsilon)=\mathcal{W}_{\kappa,\varepsilon}^{U_\delta(\bar{\xi})}(\bar{\xi},\bar{\lambda})$$

holds also if $\mathcal{M}((\bar{\xi})_{n\in J},\delta,\varepsilon)$ is constructed by Lemma B.1 with $g(x,\lambda,\gamma) = P_\kappa(\bar{\lambda})(x-\xi(\lambda))-\gamma$, $(x,\lambda,\gamma)\in\mathbb{R}^k\times U_{\varepsilon_0}(\bar{\lambda})\times X_\kappa(\bar{\lambda})$; here P_κ is defined as in the proof above. Therefore the set $\mathcal{W}_{\kappa,\varepsilon}^{U_\delta(\bar{\xi})}(\bar{\xi},\bar{\lambda})$ is a $\mathcal{C}^l$–submanifold of $\mathbb{R}^{k+r}$ of dimension $r+\dim X_\kappa(\bar{\lambda})$ if $\delta,\varepsilon>0$ are sufficiently small and satisfy $\xi(U_\varepsilon(\bar{\lambda}))\subset U_\delta(\bar{\xi})$.

For the applications in the following subsections let us consider an $f(\cdot,\bar{\lambda})$–orbit $\bar{y}_J$ such that $\lim_{|n|\to\infty}\bar{y}_n=\bar{\xi}$ holds. Furthermore, define

$$\begin{aligned}\mathcal{W}_{\kappa,\varepsilon}^{U_\delta(\bar{y}_J)}(\bar{\xi},\bar{y}_0,\bar{\lambda}) \quad &= \quad \{(x,\lambda)\in\mathbb{R}^k\times U_\varepsilon(\bar{\lambda}):\\ &\qquad (f(\cdot,\lambda)^n(x))_{n\in J}\in U_\delta(\bar{y}_J),\ \lim_{|n|\to\infty}f(\cdot,\lambda)^n(x)=\xi(\lambda)\}\end{aligned} \tag{60}$$

for $\delta>0$ and $\varepsilon\in(0,\epsilon]$ satisfying $\xi(U_\varepsilon(\bar{\lambda}))\subset U_\delta(\bar{\xi})$. We call this set the *extended local stable* (in case $\kappa=s, J=\mathbb{Z}_+$) and *unstable* (in case $\kappa=u, J=\mathbb{Z}_-$) *manifold of $\bar{\xi}$ at $\bar{y}_0$*. Furthermore, define the section at $\lambda=\bar{\lambda}$ by

$$\begin{aligned}&W_\kappa^{U_\delta(\bar{y}_J)}(\bar{\xi},\bar{y}_0,\bar{\lambda})\\ &= \quad \{x\in\mathbb{R}^k:(f(\cdot,\bar{\lambda})^n(x))_{n\in J}\in U_\delta(\bar{y}_J),\ \lim_{|n|\to\infty}f(\cdot,\bar{\lambda})^n(x)=\xi(\bar{\lambda})\},\quad \delta>0.\end{aligned}$$

Due to Lemma 2.5 the equation $u_{n+1}=f_x(\bar{y}_n,\bar{\lambda})u_n$ has an exponential dichotomy on J. By a proof similar to that given above

$$\mathcal{M}(\bar{y}_J,\delta,\varepsilon)=\mathcal{W}_{\kappa,\varepsilon}^{U_\delta(\bar{y}_J)}(\bar{\xi},\bar{y}_0,\bar{\lambda}) \tag{61}$$

holds if $\delta,\varepsilon>0$ are sufficiently small and satisfy $\xi(U_\varepsilon(\bar{\lambda}))\subset U_\delta(\bar{\xi})$. Moreover, these sets are $\mathcal{C}^l$–submanifolds of $\mathbb{R}^{k+r}$ of dimension $r+\dim X_\kappa(\bar{\lambda})$.

The *extended global stable* (in case $\kappa=s, J=\mathbb{Z}_+$) and *unstable* (in case $\kappa=u, J=\mathbb{Z}_-$) *manifold of $\bar{\xi}$* is defined as

$$\mathcal{W}_{\kappa,\varepsilon}(\bar{\xi},\bar{\lambda})=\{(x,\lambda)\in\mathbb{R}^k\times U_\varepsilon(\bar{\lambda}):x\in W_\kappa(\xi(\lambda),\lambda)\},\quad \varepsilon\in(0,\epsilon].$$

It is trivial that

$$\mathcal{W}_{\kappa,\varepsilon}(\bar{\xi},\bar{\lambda})=\bigcup_{(x,\lambda)\in\mathcal{W}_{\kappa,\varepsilon}(\bar{\xi},\bar{\lambda})}\mathcal{N}(x,\lambda),\quad \varepsilon\in(0,\epsilon],$$

holds where $\mathcal{N}(x,\lambda)=\mathcal{W}_{\kappa,\tilde{\varepsilon}}^{U(x,\lambda)}(\xi(\lambda),x,\lambda)$ (here the right side is defined similarly to (60)), $U(x,\lambda)=U_\delta((f(\cdot,\lambda)^n(x))_{n\in J},\lambda)$ and $\delta=\delta(x)>0$, $\tilde{\varepsilon}=\tilde{\varepsilon}(x,\lambda)\in(0,\varepsilon-\|\lambda-\bar{\lambda}\|]$ are sufficiently small such that $\mathcal{N}(x,\lambda)$ is a submanifold of $\mathbb{R}^{k+r}$. By Lemma B.1 we can assume without loss of generality that the charts of these submanifolds are $\mathcal{C}^l$–diffeomorphisms. Therefore $\mathcal{W}_{\kappa,\varepsilon}(\bar{\xi},\bar{\lambda})$ is a $\mathcal{C}^l$–manifold with an atlas consisting of all these charts. Due to the fact that the sets $\mathcal{N}(x,\lambda)$ are submanifolds, the inclusion map of $\mathcal{W}_{\kappa,\varepsilon}(\bar{\xi},\bar{\lambda})$ in $\mathbb{R}^{k+r}$ is an injective $\mathcal{C}^l$–immersion; see [14, Chapter 2, §2] for the definition of an immersion. This means that the sets $\mathcal{W}_{\kappa,\varepsilon}(\bar{\xi},\bar{\lambda})$, $\varepsilon\in(0,\epsilon]$, are injectively $\mathcal{C}^l$–immersed submanifolds of $\mathbb{R}^{k+r}$. They are of the same dimension as $\mathcal{W}_{\kappa,\tilde{\varepsilon}(x,\lambda)}^{U_{\delta(x)}(y_J(x,\lambda))}(\xi(\lambda),x,\lambda)$, which is $r+\dim X_\kappa(\bar{\lambda})$. Moreover, to analyze geometrical properties of $\mathcal{W}_{\kappa,\varepsilon}(\bar{\xi},\bar{\lambda})$ near $(\bar{y}_0,\bar{\lambda})$ it suffices to study $\mathcal{W}_{\kappa,\varepsilon}^{U_\delta(\bar{y}_J)}(\bar{\xi},\bar{y}_0,\bar{\lambda})$ for properly chosen $\delta,\varepsilon>0$.

In the following subsections geometric properties of the global (extended) unstable and stable manifolds of $\bar{\xi}_-$ and $\bar{\xi}_+$ at $\bar{x}_0$ will be derived in this way. Therefore, if f is only defined in some neighborhood of $\{\bar{x}_n:n\in\mathbb{Z}\}$ and these global manifolds are not defined, the following results hold for the (extended) local unstable and stable manifolds.

B.2 Properties of tangential heteroclinics

Theorem B.2 *Let $t \in \mathbb{N}$. Then the heteroclinic $\bar{x}_{\mathbb{Z}}$ is t–tangential if and only if*

$$\dim(T_{\bar{x}_0} W_u(\bar{\xi}_-, \bar{\lambda}) \cap T_{\bar{x}_0} W_s(\bar{\xi}_+, \bar{\lambda})) = t$$

is satisfied.

Proof. By the arguments in the previous subsection the tangent spaces of $W_u^{U_\delta(\bar{x}_{|\mathbb{Z}_-})}(\bar{\xi}_-, \bar{x}_0, \bar{\lambda})$ and $W_s^{U_\delta(\bar{x}_{|\mathbb{Z}_+})}(\bar{\xi}_+, \bar{x}_0, \bar{\lambda})$ at $\bar{x}_0$ equal those of $W_u(\bar{\xi}_-, \bar{\lambda})$ and $W_s(\bar{\xi}_+, \bar{\lambda})$ at $\bar{x}_0$ for sufficiently small $\delta > 0$.

By Lemma 2.5 the equation $u_{n+1} = f_x(\bar{x}_n, \bar{\lambda}) u_n$ has an exponential dichotomy on $\mathbb{Z}_-$ and on $\mathbb{Z}_+$ with projectors $P^{\pm}_{\mathbb{Z}_\pm}$. Assume that the submanifolds $W_u^{U_\delta(\bar{x}_{|\mathbb{Z}_-})}(\bar{\xi}_-, \bar{x}_0, \bar{\lambda})$ and $W_s^{U_\delta(\bar{x}_{|\mathbb{Z}_+})}(\bar{\xi}_+, \bar{x}_0, \bar{\lambda})$ are constructed as in Lemma B.1. Then

$$T_{\bar{x}_0} W_u^{U_\delta(\bar{x}_{|\mathbb{Z}_-})}(\bar{\xi}_-, \bar{x}_0, \bar{\lambda}) = \mathcal{N}(P_0^-), \quad T_{\bar{x}_0} W_s^{U_\delta(\bar{x}_{|\mathbb{Z}_+})}(\bar{\xi}_+, \bar{x}_0, \bar{\lambda}) = \mathcal{R}(P_0^+)$$

is satisfied. By the properties of $\mathcal{N}(P_0^-)$ and $\mathcal{R}(P_0^+)$ given in Lemma 2.5 the statement follows. □

B.3 Properties of nondegenerate tangencies

Theorem B.3 *Let the heteroclinic $\bar{x}_{\mathbb{Z}}$ be t–tangential for some $t \in \mathbb{N}$, and let $r = p + t$ hold. Then $\bar{x}_{\mathbb{Z}}$ is nondegenerate if and only if $\mathcal{W}_{u,\epsilon}(\bar{\xi}_-, \bar{\lambda})$ and $\mathcal{W}_{s,\epsilon}(\bar{\xi}_+, \bar{\lambda})$ are transversal at $(\bar{x}_0, \bar{\lambda})$, i.e.*

$$T_{(\bar{x}_0,\bar{\lambda})} \mathcal{W}_{u,\epsilon}(\bar{\xi}_-, \bar{\lambda}) + T_{(\bar{x}_0,\bar{\lambda})} \mathcal{W}_{s,\epsilon}(\bar{\xi}_+, \bar{\lambda}) = \mathbb{R}^{k+r}. \tag{62}$$

Proof. We assume $r \geq 1$; in case $r = 0$ the proof is obvious as we will see below. Let us transform the condition (62) first. At the point $(\bar{x}_0, \bar{\lambda})$ the tangent spaces of the immersed submanifolds $\mathcal{W}_{u,\epsilon}(\bar{\xi}_-, \bar{\lambda})$ and $\mathcal{W}_{s,\epsilon}(\bar{\xi}_+, \bar{\lambda})$ equal those of $\mathcal{W}_{u,\varepsilon}^{U_\delta(\bar{x}_{|\mathbb{Z}_-})}(\bar{\xi}_-, \bar{x}_0, \bar{\lambda})$ and $\mathcal{W}_{s,\varepsilon}^{U_\delta(\bar{x}_{|\mathbb{Z}_+})}(\bar{\xi}_+, \bar{x}_0, \bar{\lambda})$ for some properly chosen $\delta, \varepsilon > 0$. By (61) we can assume that

$$\begin{aligned}
\mathcal{W}_{u,\varepsilon}^{U_\delta(\bar{x}_{|\mathbb{Z}_-})}(\bar{\xi}_-, \bar{x}_0, \bar{\lambda}) &= \{(\tilde{x}_0^-(\lambda, \gamma), \lambda) : (\lambda, \gamma) \in W^-(\delta, \varepsilon)\}, \\
\mathcal{W}_{s,\varepsilon}^{U_\delta(\bar{x}_{|\mathbb{Z}_+})}(\bar{\xi}_+, \bar{x}_0, \bar{\lambda}) &= \{(\tilde{x}_0^+(\lambda, \gamma), \lambda) : (\lambda, \gamma) \in W^+(\delta, \varepsilon)\}
\end{aligned}$$

holds for some open set $W^{\pm}(\delta, \varepsilon)$ and $\mathcal{C}^l$–maps $\tilde{x}^{\pm}_{\mathbb{Z}_\pm}$ as in Lemma B.1. This implies

$$T_{(\bar{x}_0,\bar{\lambda})} \mathcal{W}_{\kappa_\pm,\varepsilon}^{U_\delta(\bar{x}_{|\mathbb{Z}_\pm})}(\bar{\xi}_\pm, \bar{x}_0, \bar{\lambda}) = \mathcal{R}\left(\begin{pmatrix} D_1 \tilde{x}_0^{\pm}(\bar{\lambda}, 0) & D_2 \tilde{x}_0^{\pm}(\bar{\lambda}, 0) \\ I_r & 0 \end{pmatrix}\right).$$

By the same arguments as in the proof of Theorem B.2 we get $\mathcal{R}(D_2 \tilde{x}_0^-(\bar{\lambda}, 0)) = \mathcal{N}(P_0^-)$ and $\mathcal{R}(D_2 \tilde{x}_0^+(\bar{\lambda}, 0)) = \mathcal{R}(P_0^+)$, where $P^{\pm}_{\mathbb{Z}_\pm}$ are projectors of the exponential dichotomy of $u_{n+1} = f_x(\bar{x}_n, \bar{\lambda}) u_n$ on $\mathbb{Z}_\pm$. Therefore, due to $\dim(\mathcal{N}(P_0^-) + \mathcal{R}(P_0^+)) = k - p - t = k - r$, the condition (62) is equivalent to

$$(D_1 \tilde{x}_0^-(\bar{\lambda}, 0) - D_1 \tilde{x}_0^+(\bar{\lambda}, 0))\mu \notin \mathcal{N}(P_0^-) + \mathcal{R}(P_0^+) \quad \text{for all } \mu \in \mathbb{R}^r \setminus \{0\}. \tag{63}$$

Notice, by similar arguments the condition (62) is always satisfied in case $r = 0$.

By Lemma 2.5 we can assume that

$$\begin{array}{rclrcl} \mathcal{R}(P_0^+) &=& Z_1 + Z_R, & \mathcal{N}(P_0^+) &=& Z_N + Z_2, \\ \mathcal{N}(P_0^-) &=& Z_1 + Z_N, & \mathcal{R}(P_0^-) &=& Z_R + Z_2 \end{array}$$

hold for a decomposition $Z_1 \oplus Z_R \oplus Z_N \oplus Z_2$ of $\mathbb{R}^k$, where $\dim Z_1 = t$. Let projectors $P, Q_R, Q_N, S \in \mathbb{R}^{k,k}$ be defined by

$$\begin{array}{rclrcl} \mathcal{R}(P) &=& Z_1, & \mathcal{N}(P) &=& Z_R + Z_N + Z_2, \\ \mathcal{R}(Q_R) &=& Z_R, & \mathcal{N}(Q_R) &=& Z_1 + Z_N + Z_2, \\ \mathcal{R}(Q_N) &=& Z_N, & \mathcal{N}(Q_N) &=& Z_1 + Z_R + Z_2, \\ \mathcal{R}(S) &=& Z_2, & \mathcal{N}(P) &=& Z_1 + Z_R + Z_N, \end{array}$$

which implies $S = I - P - Q_R - Q_N$ and $SP_0^+ = S(I - P_0^-) = 0$. Therefore the condition (63) is equivalent to

$$S(D_1\tilde{x}_0^-(\bar{\lambda}, 0) - D_1\tilde{x}_0^+(\bar{\lambda}, 0))\mu \neq 0 \quad \text{for all } \mu \in \mathbb{R}^r \setminus \{0\}. \tag{64}$$

In the following we show that (64) is false if and only if $\bar{x}_{\mathbb{Z}}$ is degenerate. For this purpose assume that the maps $\tilde{x}_{\mathbb{Z}}^{\pm}$ are constructed as in Lemma B.1 with respect to $g^-(x, \lambda, \gamma) = (I - P_0^-)(x - \bar{x}_0) - \gamma$, $\gamma \in \mathcal{N}(P_0^-)$, and $g^+(x, \lambda, \gamma) = P_0^+(x - \bar{x}_0) - \gamma$, $\gamma \in \mathcal{R}(P_0^+)$, where $(x, \lambda) \in \mathbb{R}^{k+r}$. Let $G^{\pm}$ be defined as the map G in (56) with respect to $g = g^{\pm}$. Hence $G^{\pm}(\tilde{x}_{\mathbb{Z}_\pm}^{\pm}(\lambda, 0), \lambda, 0) = 0$ holds for every sufficiently small $\|\lambda - \bar{\lambda}\|$. Differentiating both sides of this identity we get

$$D_x\Gamma_{\pm}(\bar{x}_{|\mathbb{Z}_\pm}, \bar{\lambda}) D_1\tilde{x}_{\mathbb{Z}_\pm}^{\pm}(\bar{\lambda}, 0) = -D_\lambda\Gamma_{\pm}(\bar{x}_{|\mathbb{Z}_\pm}, \bar{\lambda}) \tag{65}$$

and

$$(I - P_0^-)D_1\tilde{x}_0^-(\bar{\lambda}, 0) = 0, \quad P_0^+ D_1\tilde{x}_0^+(\bar{\lambda}, 0) = 0. \tag{66}$$

Assume that we have equality in (64) for some $\mu \in \mathbb{R}^r \setminus \{0\}$. Define $v_{\mathbb{Z}_\pm}^{\pm} = D_1\tilde{x}_{\mathbb{Z}_\pm}^{\pm}(\bar{\lambda}, 0)\mu$. By the properties of P, Q_R, Q_N, S we get $(I - P - Q_R - Q_N)(v_0^+ - v_0^-) = S(v_0^+ - v_0^-) = 0$. Therefore the sequence

$$v_n = \begin{cases} v_n^- - \Phi(n, 0)(Pv_0^- - Q_N(v_0^+ - v_0^-)), & \text{in case } n < 0, \\ v_n^+ - \Phi(n, 0)(Pv_0^+ + Q_R(v_0^+ - v_0^-)), & \text{in case } n \geq 0, \end{cases} \tag{67}$$

is well defined at $n = 0$ and bounded. Here Φ is the solution operator of $u_{n+1} = f_x(\bar{x}_n, \bar{\lambda})u_n$. From (65)

$$\Gamma_\lambda(\bar{x}_{\mathbb{Z}}, \bar{\lambda})\mu = -\Gamma_\lambda(\bar{x}_{\mathbb{Z}}, \bar{\lambda})v_{\mathbb{Z}} \in \mathcal{R}(\Gamma_\lambda(\bar{x}_{\mathbb{Z}}, \bar{\lambda}))$$

follows, and hence $\bar{x}_{\mathbb{Z}}$ is degenerate.

Conversely, assume that $\bar{x}_{\mathbb{Z}}$ is degenerate. By the assumption $r = p + t$ this means that

$$\Gamma_\lambda(\bar{x}_{\mathbb{Z}}, \bar{\lambda})\mu = -\Gamma_\lambda(\bar{x}_{\mathbb{Z}}, \bar{\lambda})v_{\mathbb{Z}}$$

holds for some $\mu \in \mathbb{R}^r \setminus \{0\}$ and some $v_{\mathbb{Z}} \in S_{\mathbb{Z}}$. Let

$$\begin{array}{rcll} w_n^- &=& v_n - \Phi(n, 0)(I - P_0^-)v_0, & n \in \mathbb{Z}_-, \\ w_n^+ &=& v_n - \Phi(n, 0)P_0^+ v_0, & n \in \mathbb{Z}_+. \end{array}$$

Then $w_{\mathbb{Z}_\pm}^{\pm} \in S_{\mathbb{Z}_\pm}$ holds due to Lemma 2.5, and we get $w_0^{\pm} = D_1\tilde{x}_0^{\pm}(\bar{\lambda}, 0)\mu$ by Lemma 2.2 and (65), (66). Furthermore,

$$S(D_1\tilde{x}_0^+(\bar{\lambda}, 0) - D_1\tilde{x}_0^-(\bar{\lambda}, 0))\mu = S(I - P_0^+ - P_0^-)v_0 = 0$$

follows, and hence (64) is false. □

B.4 Properties of quadratic heteroclinics

In this section we want to show that the unstable and the stable manifold of the fixed points $\bar{\xi}_-$ and $\bar{\xi}_+$ have a quadratic tangency if and only if the 1–tangential heteroclinic $\bar{x}_{\mathbb{Z}}$ is quadratic. As in Section 4.1 we assume $r \geq p \geq 0$ and decompose the parameters $\lambda \in \mathbb{R}^r$ as $\lambda = (\lambda_p, \lambda_2) \in \mathbb{R}^p \times \mathbb{R}^{r-p}$. Furthermore let $l \geq 2$ hold.

To be precise we define in which case we call two immersed submanifolds to have a quadratic tangency.

Definition B.4 *Consider two $\mathcal{C}^2$–submanifolds A, B of $\mathbb{R}^k$, and a point $p \in A \cap B$. Suppose that*

$$M \cap U = \{p + \gamma_M + x_M(\gamma_M) : \gamma_M \in V_M\}, \quad M = A, B,$$

hold for a neighborhood $U \subset \mathbb{R}^k$ of p where $x_A : V_A \to T_B + T_s$, $x_B : V_B \to T_A + T_s$ are $\mathcal{C}^2$–maps with neighborhoods $V_A \subset T_pA$ and $V_B \subset T_pB$ of 0. Here $T_0 = T_pA \cap T_pB$, and $T_A, T_B \subset \mathbb{R}^k$ are subspaces such that $T_pA = T_0 + T_A$, $T_pB = T_0 + T_B$ hold. Assume that $T_s = (T_pA + T_pB)^\perp$ satisfies $\dim T_s \geq 1$. Then A and B have a quadratic tangency *at p if there exists $s \in T_s$ such that the quadratic form*

$$s^T(x''_A(0)_{|T_0 \times T_0} - x''_B(0)_{|T_0 \times T_0})$$

is nondegenerate.

Two injectively $\mathcal{C}^2$–immersed submanifolds A, B have a quadratic tangency *at p if the locally induced submanifolds of A and B near p have this property.*

Notice, in the setting above $\dim T_s \geq 1$ holds if and only if $\dim(T_pA + T_pB) \leq k - 1$ is satisfied.

Remark B.5 *In case $\dim T_s = 1$ the definition given above for submanifolds A, B is equivalent to the one used in* [2, Chapter 3, §1.4] *for a simple tangency.*[1] *As outlined there, that definition is equivalent to the coordinate representation derived by Palis, Takens and Newhouse in* [16, §6]. *This result says that two smooth submanifolds A and B of a n–dimensional manifold M have a simple tangency at a point P if and only if there are coordinates $\{(x_1, \dots, x_n)\}$ in a neighborhood U of P such that $x(P) = 0$ and*

$$\begin{aligned} A \cap U &= \{x_i = 0, \dim A \leq i \leq n\}, \\ B \cap U &= \{x_i = 0, i \leq 1 \leq n - (\dim B + 1), x_n = \sum_{j=n-\dim B}^{\dim A} \varepsilon_i x_j^2, \varepsilon_j = \pm 1\} \end{aligned}$$

[1] To show the equivalence consider the map $\tilde{f}(x) = s^T(x - x_B([\phi_B(x)]_1))$, for x in some neighborhood of p, where $\phi_B : \mathbb{R}^k \to T_pB \times (T_A + T_s)$ is a $\mathcal{C}^2$–map which is locally diffeomorphic at p and satisfies $\phi_B(\gamma, 0)^{-1} = x_B(\gamma)$ for small $\gamma \in V_B$; the vector s is an element of T_s, and $[\phi_B(x)]_1$ means the component of $\phi_B(x)$ in T_pB. For more details see the proof of [13, Satz 2.3.3] and the map ϕ used there.

hold.

Theorem B.6 *Let* $\dim \mathcal{N}(\Gamma_{x,\lambda_p}(\bar{x}_{\mathbb{Z}}, \bar{\lambda}_p, \bar{\lambda}_2)) = 1$ *hold. Then the* 1*–tangential heteroclinic* $\bar{x}_{\mathbb{Z}}$ *is quadratic if and only if the immersed submanifolds* $W_u(\bar{\xi}_-, \bar{\lambda})$ *and* $W_s(\bar{\xi}_+, \bar{\lambda})$ *have a quadratic tangency at the point* $\bar{x}_0$*.*

Proof. Let the projectors $P^{\pm}_{\mathbb{Z}_\pm}, P, Q_R, Q_N, S$, as well as Φ, Z_1, Z_R, Z_N, Z_2, be as in the proof of Theorem B.3. In that proof Z_2 can be chosen such that $Z_2 \perp (Z_1 + Z_N + Z_R)$ holds. Without loss of generality we assume that $\mathcal{W}^{U_\delta(\bar{x}_{|\mathbb{Z}_-})}_{u,\varepsilon}(\bar{\xi}_-, \bar{x}_0, \bar{\lambda}) = \mathcal{M}(\bar{x}_{|\mathbb{Z}_-}, \delta, \varepsilon)$ and $\mathcal{W}^{U_\delta(\bar{x}_{|\mathbb{Z}_+})}_{s,\varepsilon}(\bar{\xi}_+, \bar{x}_0, \bar{\lambda}) = \mathcal{M}(\bar{x}_{|\mathbb{Z}_+}, \delta, \varepsilon)$ hold for some $\delta, \varepsilon > 0$. Furthermore, assume that the sets $\mathcal{M}(\bar{x}_{|\mathbb{Z}_\pm}, \delta, \varepsilon)$ are constructed as in Lemma B.1 with the maps $g^-(x, \lambda, \gamma_-) = (I - P_0^-)(x - \bar{x}_0) - \gamma_-$, $g^+(x, \lambda, \gamma_+) = P_0^+(x - \bar{x}_0) - \gamma_+$ for $(x, \lambda) \in \mathbb{R}^{k+r}$, $\gamma_- \in \mathcal{N}(P_0^-)$ and $\gamma_+ \in \mathcal{R}(P_0^+)$. Let the operators $G^\pm$ be defined as in (56) with respect to the maps $g^\pm$. Then there exist $\mathcal{C}^2$–maps $\tilde{x}^{\pm}_{\mathbb{Z}_\pm}$ as in Lemma B.1 such that (57) holds with $J = \mathbb{Z}_\pm$, $\tilde{x}_0 = \tilde{x}_0^\pm$ and $W(\delta, \varepsilon) = W^\pm(\delta, \varepsilon)$. Moreover,

$$W^{U_\delta(\bar{x}_{|\mathbb{Z}_\pm})}_{\kappa_\pm}(\bar{\xi}_\pm, \bar{x}_0, \bar{\lambda}) = \{\tilde{x}_0(\bar{\lambda}, \gamma_\pm) : (\bar{\lambda}, \gamma_\pm) \in W^\pm(\delta, \varepsilon)\} \qquad (68)$$

holds with $\kappa_- = u$, $\kappa_+ = s$ if $\delta, \varepsilon > 0$ are chosen properly.

The sets $W_u(\bar{\xi}_-, \bar{\lambda})$ and $W_s(\bar{\xi}_+, \bar{\lambda})$ are injectively immersed $\mathcal{C}^l$–submanifolds of $\mathbb{R}^k$. This follows by analogous arguments as in Subsection B.1, see [12, Theorem 6.13] also. Moreover, at the point $\bar{x}_0$, these sets are locally determined by the submanifolds $W^{U_\delta(\bar{x}_{|\mathbb{Z}_-})}_{u}(\bar{\xi}_-, \bar{x}_0, \bar{\lambda})$ and $W^{U_\delta(\bar{x}_{|\mathbb{Z}_+})}_{s}(\bar{\xi}_+, \bar{x}_0, \bar{\lambda})$. Hence it suffices to prove that these submanifolds have a quadratic tangency at $\bar{x}_0$ if and only if $\bar{x}_{\mathbb{Z}}$ is quadratic. For this purpose we consider the condition of Definition B.4 for $p = \bar{x}_0$, $A = W^{U_\delta(\bar{x}_{|\mathbb{Z}_-})}_{u}(\bar{\xi}_-, \bar{x}_0, \bar{\lambda})$ and $B = W^{U_\delta(\bar{x}_{|\mathbb{Z}_+})}_{s}(\bar{\xi}_+, \bar{x}_0, \bar{\lambda})$. Due to (68) we get

$$T_pA = \mathcal{N}(P_0^-), \quad T_pB = \mathcal{R}(P_0^+),$$

and therefore we can choose

$$T_0 = Z_1, \quad T_A = Z_N, \quad T_B = Z_R, \quad T_s = Z_2$$

where $\dim(T_0) = 1$ and $\dim(T_s) = p + 1$ hold. The maps x_A and x_B can be chosen as

$$x_A(\gamma_A) = \tilde{x}_0^-(\gamma_A) - \bar{x}_0 - \gamma_A, \quad x_B(\gamma_B) = \tilde{x}_0^-(\gamma_B) - \bar{x}_0 - \gamma_B$$

for $(\bar{\lambda}, \gamma_A) \in W^-(\delta, \varepsilon)$, $(\bar{\lambda}, \gamma_B) \in W^+(\delta, \varepsilon)$. Notice that $W^-(\delta, \varepsilon) \subset \mathbb{R}^r \times \mathcal{N}(P_0^-)$ and $W^+(\delta, \varepsilon) \subset \mathbb{R}^r \times \mathcal{R}(P_0^+)$ hold. Furthermore, $x_A(0) = 0$, $x_B(0) = 0$ follow from the definition of $\tilde{x}^{\pm}_{\mathbb{Z}_\pm}(\bar{\lambda}, 0)$. Differentiating both sides of the equality

$$G^\pm(\tilde{x}^\pm_{\mathbb{Z}_\pm}(\bar{\lambda}, \gamma_\pm), \bar{\lambda}, \gamma_\pm) = 0, \quad \gamma_\pm \text{ sufficiently small}, \qquad (69)$$

yields $\tilde{x}_0^\pm(\bar{\lambda}, 0)\gamma_\pm = \gamma_\pm$, and hence $x'_A(0) = 0$, $x'_B(0) = 0$ is valid.

For the proof of the theorem it suffices to show the following statement.

Auxiliary statement. Let $\gamma_0 \in Z_1 \setminus \{0\}$. Then

$$\Gamma_{xx}(\bar{x}_{\mathbb{Z}}, \bar{\lambda})\bar{u}_{\mathbb{Z}}^2 \in \mathcal{R}(\Gamma_x(\bar{x}_{\mathbb{Z}}, \bar{\lambda})) \tag{70}$$

is valid if and only if

$$S(D^2\tilde{x}_0^-(0)\gamma_0^2 - D^2\tilde{x}_0^+(0)\gamma_0^2) = 0 \tag{71}$$

is satisfied.

If this is proved, the choice of the maps x_A and x_B implies that $\Gamma_{xx}(\bar{x}_{\mathbb{Z}}, \bar{\lambda})\bar{u}_{\mathbb{Z}}^2 \notin \mathcal{R}(\Gamma_x(\bar{x}_{\mathbb{Z}}, \bar{\lambda}))$ holds if and only if there exists $s \in T_s$ such that $s^T(x_A''(0)\gamma_0^2 - x_B''(0)\gamma_0^2) \neq 0$ is satisfied. This is the statement of the theorem.

Proof of the auxiliary statement. Differentiating both sides of the identity (69) yields

$$\begin{aligned} D_x\Gamma_\pm(\bar{x}_{|\mathbb{Z}_\pm}, \bar{\lambda})D^2\tilde{x}_{\mathbb{Z}_\pm}^\pm(0)\gamma_0^2 &= -D_{xx}\Gamma_\pm(\bar{x}_{|\mathbb{Z}_\pm}, \bar{\lambda})\bar{u}_{|\mathbb{Z}_\pm}^2, &(72)\\ (I - P_0^-)D^2\tilde{x}_0^-(0)\gamma_0^2 &= 0, &(73)\\ P_0^+D^2\tilde{x}_0^+(0)\gamma_0^2 &= 0 &(74)\end{aligned}$$

after properly scaling $\bar{u}_{\mathbb{Z}}$.

Suppose that (70) holds, i.e. there exists $\bar{v}_{\mathbb{Z}} \in S_{\mathbb{Z}}$ such that

$$\Gamma_x(\bar{x}_{\mathbb{Z}}, \bar{\lambda})\bar{v}_{\mathbb{Z}} = -\Gamma_{xx}(\bar{x}_{\mathbb{Z}}, \bar{\lambda})\bar{u}_{\mathbb{Z}}^2$$

is satisfied. Hence, by Lemma 2.2 and (72)—(74), the sequences $w_{\mathbb{Z}_\pm}^\pm = \bar{v}_{|\mathbb{Z}_\pm} - D^2\tilde{x}_{\mathbb{Z}_\pm}^\pm(0)\gamma_0$ are the unique solutions of

$$u_{n+1}^\pm - f_x(\bar{x}_n, \bar{\lambda})u_n^\pm = 0, \quad u_{\mathbb{Z}_\pm}^\pm \in S_{\mathbb{Z}_\pm}$$

satisfying $(I - P_0^-)u_0^- = (I - P_0^-)\bar{v}_0$ and $P_0^+u_0^+ = P_0^+\bar{v}_0$, respectively. This implies $w_n^- = \Phi(n,0)(I - P_0^-)\bar{v}_0$ in case $n \leq 0$ and $w_n^+ = \Phi(n,0)P_0^+\bar{v}_0$ in case $n \geq 0$. By the choice of S we obtain

$$\begin{aligned} S(D^2\tilde{x}_0^-(0)\gamma_0 - D^2\tilde{x}_0^+(0)\gamma_0) &= S(w_0^+ - w_0^-)\\ &= S(P_0^+ + P_0^- - I)\bar{v}_0\\ &= 0.\end{aligned}$$

Conversely, let (71) hold. Set $v_{\mathbb{Z}_\pm}^\pm = D^2\tilde{x}_{\mathbb{Z}_\pm}^\pm(0)\gamma_0$. By the choice of S

$$(I - P - Q_R - Q_N)(v_0^+ - v_0^-) = 0$$

is satisfied. Let $v_{\mathbb{Z}} \in S_{\mathbb{Z}}$ be defined as in (67). Due to (72)—(74)

$$\Gamma_{xx}(\bar{x}_{\mathbb{Z}}, \bar{\lambda})\bar{u}_{\mathbb{Z}}^2 = -\Gamma_x(\bar{x}_{\mathbb{Z}}, \bar{\lambda})v_{\mathbb{Z}}$$

follows. □

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