Superconvergence estimates for the numerical computation of heteroclinics for maps

Jan–Martin Kleinkauf*

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Abstract

In [5] a method for the approximation of heteroclinic orbits in parameterized time–discrete systems is given which applies both to transversal and quadratic tangential heteroclinics. Solutions of a boundary value problem are used to approximate the heteroclinic on some finite interval as well as the parameter. It turns out that the error in the parameter decays much faster than that one in the orbit. In this paper we support these numerical findings by proving superconvergence estimates for the error in the parameter. Examples show that these estimates are sharp in case of projection boundary conditions but not in case of periodic boundary conditions.

Key words. dynamical systems – homoclinic bifurcations – heteroclinic orbits for maps – numerical methods – convergence estimates

AMS subject classifications. 65L12, 58F08

1 Introduction

We consider a parameterized time–discrete system

$$x_{n+1} = f(x_n, \lambda), \quad \lambda \in \mathbb{R}^r, \quad n \in \mathbb{Z}, \tag{1}$$

where $f \in \mathcal{C}^{l+1}(\mathbb{R}^k \times \mathbb{R}^r, \mathbb{R}^k)$ and $l \geq 2$. In [5] we consider the approximation of tangential heteroclinic orbits by finite boundary value problems. The purpose of this paper is to prove superconvergence of the parameter. We first review the basic setting as well as the results from [5].

Let $\bar{x}_\mathbb{Z} = (\bar{x}_n)_{n \in \mathbb{Z}}$ be a *heteroclinic* $f(\cdot, \bar{\lambda})$*–orbit connecting the hyperbolic fixed points* $\bar{\xi}_-$ *and* $\bar{\xi}_+$ *of* $f(\cdot, \bar{\lambda})$. This means that $(\bar{x}_\mathbb{Z}, \bar{\lambda})$ is a solution of (1) satisfying

$$\lim_{n \to \pm\infty} \bar{x}_n = \bar{\xi}_\pm$$

*SFB 343, Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D–33501 Bielefeld, Germany, e-mail: jan@Mathematik.Uni-Bielefeld.DE

where $\bar\xi_\pm = f(\bar\xi_\pm,\bar\lambda)$ and $f_x(\bar\xi_\pm,\bar\lambda)$ has no eigenvalue on the unit circle. In case $\bar\xi_- = \bar\xi_+$ the orbit is called *homoclinic.*

We assume that $f(\cdot,\lambda)$ is a diffeomorphism for every $\lambda \in U_\epsilon(\bar\lambda)$ and some $\epsilon > 0$. Let

$$\begin{aligned} W_u(\bar\xi_-,\bar\lambda) &= \{x \in \mathbb{R}^k : \lim_{n\to-\infty} f(\cdot,\bar\lambda)^n(x) = \bar\xi_-\},\\ W_s(\bar\xi_+,\bar\lambda) &= \{x \in \mathbb{R}^k : \lim_{n\to+\infty} f(\cdot,\bar\lambda)^n(x) = \bar\xi_+\} \end{aligned}$$

be the unstable and stable manifold of $\bar\xi_-$ and $\bar\xi_+$ with respect to $f(\cdot,\bar\lambda)$. It is well known that these sets are injectively $\mathcal{C}^l$–immersed submanifolds. Furthermore, $\dim W_u(\bar\xi_-,\bar\lambda) = k_{-u} = k - k_{-s}$ and $\dim W_s(\bar\xi_+,\bar\lambda) = k_{+s} = k - k_{+u}$ hold where $k_{\pm s}$ and $k_{\pm u}$ denote the dimension of the stable and unstable subspace of $f_x(\bar\xi_\pm,\bar\lambda)$. We set

$$p = k_{-s} - k_{+s}. \tag{2}$$

By definition every point $\bar x_n$, $n \in \mathbb{Z}$, of the heteroclinic orbit $\bar x_\mathbb{Z}$ lies in the intersection of $W_u(\bar\xi_-,\bar\lambda)$ and $W_s(\bar\xi_+,\bar\lambda)$. We call $\bar x_\mathbb{Z}$ *t–tangential* if the tangent spaces $T_{\bar x_0}W_u(\bar\xi_-,\bar\lambda)$ and $T_{\bar x_0}W_s(\bar\xi_+,\bar\lambda)$ intersect along a subspace of dimension $t \geq 0$. This property can be characterized in terms of the $\mathcal{C}^l$–operator $\Gamma : S_Z \times \mathbb{R}^r \to S_\mathbb{Z}$ defined by

$$\Gamma(x_\mathbb{Z},\lambda) = (x_{n+1} - f(x_n,\lambda))_{n\in\mathbb{Z}}, \quad (x_\mathbb{Z},\lambda) \in S_\mathbb{Z} \times \mathbb{R}^r,$$

where $S_\mathbb{Z}$ is the Banach space of all bounded sequences in $\mathbb{R}^k$ indexed by $\mathbb{Z}$ and endowed with the supremum norm $\|x_\mathbb{Z}\|_\infty = \sup_{n\in\mathbb{Z}} \|x_n\|$. It turns out that the $f(\cdot,\bar\lambda)$–heteroclinic $\bar x_\mathbb{Z}$ is t–tangential if and only if the nullspace $\mathcal{N}(\Gamma_x(\bar x_\mathbb{Z},\bar\lambda))$ is of dimension t, i.e.

$$\dim \mathcal{N}(\Gamma_x(\bar x_\mathbb{Z},\bar\lambda)) = t. \tag{3}$$

Here Γ_x denotes the partial derivative of Γ with respect to the first variable.

To stabilize a tangential orbit we assume that $r = p + t$ holds and that the extended manifolds

$$\begin{aligned} \mathcal{W}_{u,\epsilon}(\bar\xi_-,\bar\lambda) &= \{(x,\lambda) : \lambda \in U_\epsilon(\bar\lambda), x \in W_u(\xi_-(\lambda),\lambda)\},\\ \mathcal{W}_{s,\epsilon}(\bar\xi_+,\bar\lambda) &= \{(x,\lambda) : \lambda \in U_\epsilon(\bar\lambda), x \in W_s(\xi_+(\lambda),\lambda)\} \end{aligned}$$

are transversal at their common point $(\bar x_0,\bar\lambda)$, i.e.

$$T_{(\bar x_0,\bar\lambda)}\mathcal{W}_{u,\epsilon}(\bar\xi_-,\bar\lambda) + T_{(\bar x_0,\bar\lambda)}\mathcal{W}_{s,\epsilon}(\bar\xi_+,\bar\lambda) = \mathbb{R}^{k+r}.$$

By $\xi_\pm(\lambda)$, $\lambda \in U_\epsilon(\bar\lambda)$ we denote a fixed point of $f(\cdot,\lambda)$ close to $\bar\xi_\pm$. Taking $\epsilon > 0$ sufficiently small, this fixed point is locally unique, hyperbolic and $\xi_\pm$ is as smooth as f. A tangential heteroclinic satisfying both assumptions is called *nondegenerate.*

In [5] we approximate nondegenerate t–tangential heteroclinics as follows. For $-n_-, n_+ \in \mathbb{N}$ we consider solutions of the finite boundary value problem

$$\begin{aligned} x_{n+1} - f(x_n,\lambda) &= 0, \quad n = n_-,\dots,n_+ - 1, &(4)\\ b(x_{n_-},x_{n_+},\lambda) &= 0, &(5)\\ \Psi_{[n_-,n_+]}(x_{n_-},\dots,x_{n_+},\lambda) &= 0. &(6) \end{aligned}$$

Here $b \in \mathcal{C}^l(\mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^r, \mathbb{R}^{k+p})$ defines boundary conditions and $\Psi_J = \Psi_{[n_-,n_+]} \in \mathcal{C}^l(S_J \times \mathbb{R}^r, \mathbb{R}^t)$ some regularizing constraint where S_J denotes the Banach space of sequences in $\mathbb{R}^k$ indexed by J and endowed with the supremum norm. We assume that $b(\bar{\xi}_-, \bar{\xi}_+, \bar{\lambda}) = 0$ holds and the linear map

$$X_{-s} \times X_{+u} \to \mathbb{R}^{k+p}, (x_s, x_u) \mapsto b'(\bar{\xi}_-, \bar{\xi}_+, \bar{\lambda})(x_s, x_u, 0)$$

is nonsingular. For the constraint Ψ_J we assume that

$$\lim_{\min\{-n_-, n_+\} \to \infty} \Psi_J(\bar{x}_{|J}, \bar{\lambda}) = 0,$$

and that, for $-n_-, n_+$ sufficiently large, the $t \times t$–matrix

$$[D_x \Psi_J(\bar{x}_{|J}, \bar{\lambda})(\bar{u}^1_{|J}), \ldots, D_x \Psi_J(\bar{x}_{|J}, \bar{\lambda})(\bar{u}^t_{|J})]$$

is nonsingular with uniformly bounded inverse, and the derivatives $\Psi_J^{(i)}$, $i = 1, \ldots, l$, are uniformly bounded in a closed ball with fixed radius centered at $(\bar{x}_{|J}, \lambda)$. Here we used the abbreviation $x_{|J} = (x_{n_-}, \ldots, x_{n_+})$ for $x_{\mathbb{Z}} \in S_{\mathbb{Z}}$. Furthermore $\{\bar{u}^1_{\mathbb{Z}}, \ldots, \bar{u}^t_{\mathbb{Z}}\}$ denotes a basis of the nullspace $\mathcal{N}(\Gamma_x(\bar{x}_{\mathbb{Z}}, \bar{\lambda}))$. If these assumptions are satisfied, the boundary conditions and the constraints are called *regular*.

The general approximation result says that for regular boundary conditions and constraints the problem (4)—(6) has a unique solution (x_J, λ_J) in a closed ball around $(\bar{x}_{|J}, \bar{\lambda})$ of fixed radius for $-n_-, n_+$ sufficiently large.

Our method applies both to transversal and quadratic 1–tangential heteroclinics. Here a *transversal* heteroclinics is a 0–tangential which is nondegenerate. Approximating such a heteroclinic, the constraint (6) is absent.

We call the $f(\cdot, \bar{\lambda})$–heteroclinic $\bar{x}_{\mathbb{Z}}$ connecting $\bar{\xi}_-$ and $\bar{\xi}_+$ *quadratic* if it is 1–tangential and nondegenerate, if $p \geq 0$ holds, and if $W_u(\bar{\xi}_-, \bar{\lambda})$, $W_s(\bar{\xi}_+, \bar{\lambda})$ have a quadratic tangency at $\bar{x}_0$ (cf. [5, Definition B.4]). Such an orbit, together with one component of the parameter vector $\bar{\lambda} \in \mathbb{R}^{p+1}$, is a turning point of the operator Γ. For its approximation, we compute a turning point of the operator defined by the right hand of (4), (5). The constraint (6) is used to obtain it as solutions of the whole problem (4)—(5).

Approximating transversal or quadratic heteroclinics the estimate

$$\|(x_J, \lambda_J) - (\bar{x}_{|J}, \bar{\lambda})\|_\infty = \mathcal{O}(e^{l_- \alpha_{-u} n_-} + e^{-l_+ \alpha_{+s} n_+}) \tag{7}$$

holds uniformly in n_-, n_+ if b defines boundary conditions of (smooth) order (l_-, l_+) (cf. [5, Definition 3.6, 3.10]). The constants $\alpha_{\pm s}, \alpha_{\pm u} > 0$ are chosen such that $f_x(\bar{\xi}_\pm, \bar{\lambda})$ has no eigenvalues in the closed ring $\{z \in \mathbb{Z} : e^{-\alpha_{\pm s}} \leq |z| \leq e^{\alpha_{\pm u}}\}$.

In numerical examples we find superconvergence of the parameter for projection boundary conditions. For these conditions $l_\pm = 2$ holds and we observe

$$\|\lambda_J - \bar{\lambda})\| = \mathcal{O}(e^{(2\alpha_{-u} + \alpha_{-s})n_-} + e^{-(2\alpha_{+s} + \alpha_{+u})n_+}),$$

whereas (7) describes the error $\|x_J - \bar{x}_{|J}\|_\infty$ correctly. Projection boundary conditions are defined such that $b(x, y, \lambda) = 0$ holds for (x, y, λ) near $(\bar{\xi}_-, \bar{\xi}_+, \bar{\lambda})$ if and only if x and y lie in the linearizations of $W_u(\xi_-(\lambda), \lambda)$ and $W_s(\xi_+(\lambda), \lambda)$

at $\xi_\pm(\lambda)$, i.e. $x \in T_{\xi_-(\lambda)} W_u(\xi_-(\lambda),\lambda)$ and $y \in T_{\xi_+(\lambda)} W_s(\xi_+(\lambda),\lambda)$. For a precise definition see [5, Equation (23)].

Our method to approximate time–discrete heteroclinics using boundary value problems mimics the computation of connecting orbits for continuous systems, cf. [1]. For this computation, in case of homoclinic orbits and projection boundary conditions, Sandstede proves superconvergence of the parameter $\lambda \in \mathbb{R}$, cf. [7]. In his proof he refines Lin's method using suitable parameterizations of the stable and unstable manifold near their intersection.

In this paper we will adopt and generalize Sandstede's approach to the case of discrete systems. In contrast to [7] our results cover heteroclinic orbit, both transversal and quadratic, and general boundary conditions. We get superconvergence for the parameter vector $\lambda \in \mathbb{R}^r$. This paper generalizes the results from the dissertation [6] where the homoclinic case of tangentials is considered only.

Notice, by the proof of our main theorem, Sandstede's results can also be generalized to the approximation of continuous heteroclinics. In this case, the superconvergence estimates can be proved for the parameter vector $\lambda \in \mathbb{R}^p$, cf. [1]. The case of general boundary conditions can be treated similarly to our approach.

Our paper is organized as follows. In Section 2 we state our main theorem on superconvergence of the parameter $\lambda \in \mathbb{R}^r$. This theorem is formulated for the general approximation of t–tangential heteroclinics which satisfy a verticality condition.

This verticality condition is trivially satisfied for transversal heteroclinics. In Section 3 we show that it is, after some transformation, also fulfilled for quadratic tangencies. Therefore we obtain superconvergence in both cases.

Section 4 contains the proof of our main theorem from Section 2. Step by step the boundary value problem (4)—(6) is solved and the behavior of the solution is analyzed.

In Section 5 we apply our method to two examples. It turns out that our superconvergence results are sharp for projection boundary conditions. In case of homoclinic orbits and periodic boundary conditions, which are defined by $b(x, y, \lambda) = x - y$, we observe faster convergence than covered by our results.

In this paper we use the notations introduced in [5]. As outlined there, the results of this paper keep valid if we consider the case that f is $\mathcal{C}^{l+1}$–diffeomorphic in some neighborhood of $\{\bar{x}_n : n \in \mathbb{Z}\} \times \{\bar{\lambda}\}$ and b is of type $\mathcal{C}^l$ in some neighborhood of $(\bar{\xi}_-, \bar{\xi}_+, \bar{\lambda})$.

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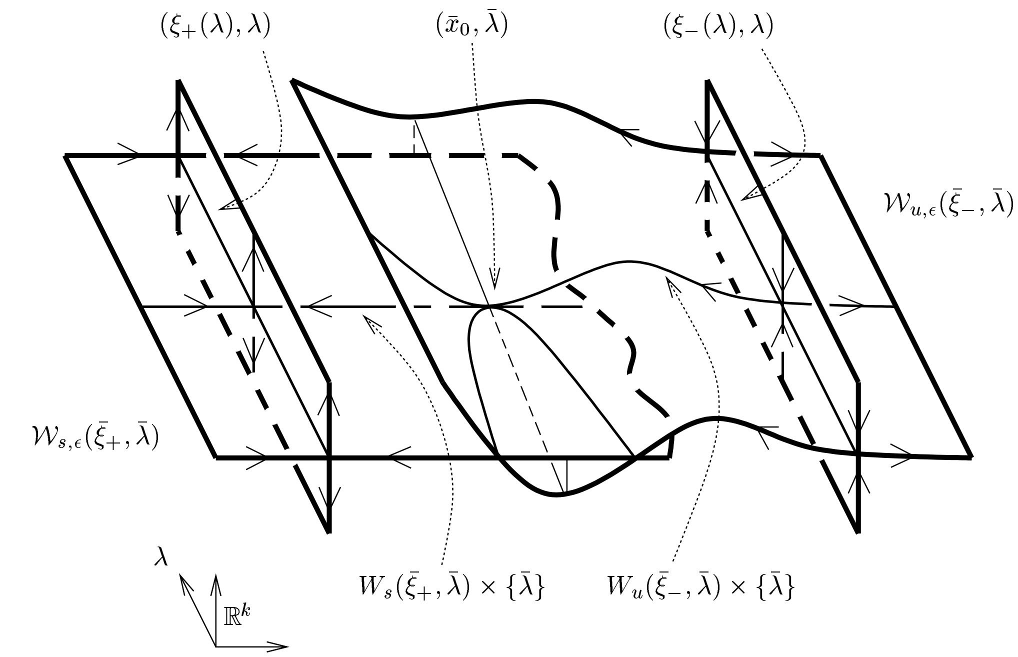

Figure 1: Part of $\mathcal{W}_{u,\epsilon}(\bar{\xi}_-,\bar{\lambda})$ and $\mathcal{W}_{s,\epsilon}(\bar{\xi}_+,\bar{\lambda})$ in case of a nondegenerate quadratic $f(\cdot,\bar{\lambda})$–tangential $\bar{x}_{\mathbb{Z}}$.

2 The main theorem

Before we state our main theorem on the superconvergence of the parameters, let us give some definitions.

Definition 2.1 *We call the t–tangential $f(\cdot,\bar{\lambda})$–heteroclinic $\bar{x}_{\mathbb{Z}}$* vertical *if there exists a submanifold Q of $\mathbb{R}^k$ of dimension t which contains $\bar{x}_0$ and lies in the intersection of the unstable and stable manifolds $W_u(\bar{\xi}_-,\bar{\lambda})$ and $W_s(\bar{\xi}_+,\bar{\lambda})$. In formulae this means*

$$\bar{x}_0 \in Q \subset W_u(\bar{\xi}_-,\bar{\lambda}) \cap W_s(\bar{\xi}_+,\bar{\lambda}), \quad \dim Q = t.$$

Obviously, transversal heteroclinics are vertical whereas this is not the case for quadratic tangencies. This is illustrated in Figure 1, where the intersection of the extended manifolds $\mathcal{W}_{u,\epsilon}(\bar{\xi}_-,\bar{\lambda})$ and $\mathcal{W}_{s,\epsilon}(\bar{\xi}_+,\bar{\lambda})$ in case of a nondegenerate quadratic tangency is drawn schematically. But in Subsection 3.2 we will show that these tangencies correspond to vertical 1–tangentials of an extended time–discrete system. Such an orbit is schematically drawn in Figure 2.

Definition 2.2 *The map $b : \mathbb{R}^{2k+r} \to \mathbb{R}^{k+p}$ defines*

(i) *boundary conditions of $\mathcal{C}^q$–*order *(l_-,l_+) at $(\bar{\xi}_-,\bar{\xi}_+,\bar{\lambda})$ for some $q,l_-,l_+ \in \mathbb{N}$ if there exist neighborhoods $V_\pm$ of $\bar{\xi}_\pm$, U of $\bar{\lambda}$ and $\mathcal{C}^q$–maps $B_\pm : V_\pm \times U \to \mathcal{L}^{l_\pm}(\mathbb{R}^k,\mathbb{R}^{k+p})$ such that*

$$b(x,y,\lambda) = B_-(x,\lambda)(x-\xi_-(\lambda))^{l_-} + B_+(y,\lambda)(y-\xi_+(\lambda))^{l_+}$$

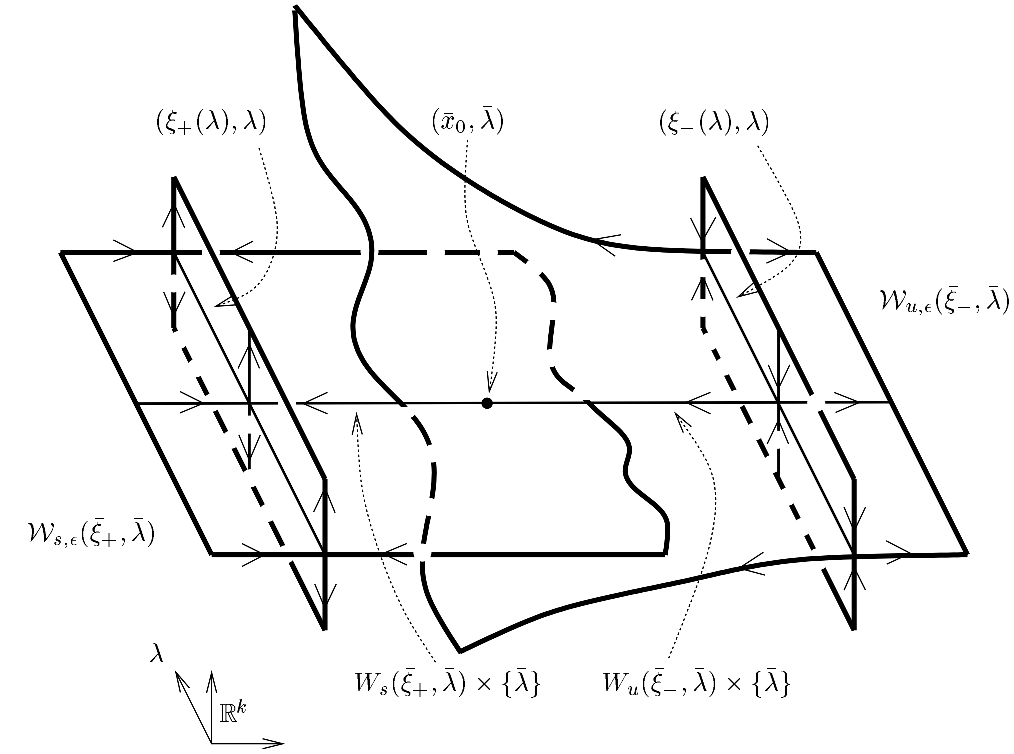

Figure 2: Part of $\mathcal{W}_{u,\epsilon}(\bar{\xi}_-,\bar{\lambda})$ and $\mathcal{W}_{s,\epsilon}(\bar{\xi}_+,\bar{\lambda})$ in case of a vertical, nondegenerate 1–tangential $f(\cdot,\bar{\lambda})$–heteroclinic $\bar{x}_{\mathbb{Z}}$.

and $\xi_\pm(\lambda) \in V_\pm$ *hold for every* $x \in W_u^{V_-}(\xi_-(\lambda),\lambda)$, $y \in W_s^{V_+}(\xi_+(\lambda),\lambda)$ *and* $\lambda \in U$.

(ii) *It defines* separated *boundary conditions if there exist maps* $b_\pm : \mathbb{R}^{k+r} \to \mathbb{R}^{k+p}$ *and subspaces* $X_\pm(\lambda) \subset \mathbb{R}^k$, $\lambda \in U_\epsilon(\bar{\lambda})$, *such that*

$$
\begin{aligned}
b_-(x,\lambda) + b_+(y,\lambda) &= b(x,y,\lambda), \\
X_-(\lambda) \oplus X_+(\lambda) &= \mathbb{R}^k, \\
\mathcal{R}(b_\pm(\cdot,\lambda)) &\subset X_\pm(\lambda)
\end{aligned}
$$

hold for every $x, y \in \mathbb{R}^k$ *and* $\lambda \in U_\epsilon(\bar{\lambda})$.

In the previous definition $\mathcal{L}^{l_\pm}(\mathbb{R}^k,\mathbb{R}^{k+p})$ denotes the Banach space of $l_\pm$–linear maps from $(\mathbb{R}^k)^{l_\pm}$ into $\mathbb{R}^{k+p}$ endowed with the usual operator norm. Furthermore $\xi_\pm$ is defined as in the introduction, and the local unstable and stable manifolds $W_u^{V_-}(\xi_-(\lambda),\lambda)$, $W_s^{V_+}(\xi_+(\lambda),\lambda)$ are given by

$$
\begin{aligned}
W_u^{V_-}(\xi_-(\lambda),\lambda) &= \{x \in W_u(\xi_-(\lambda),\lambda) : f(\cdot,\lambda)^{-n}(x) \in V_- \text{ for all } n \in \mathbb{N}\}, \\
W_s^{V_+}(\xi_+(\lambda),\lambda) &= \{x \in W_s(\xi_+(\lambda),\lambda) : f(\cdot,\lambda)^{n}(x) \in V_- \text{ for all } n \in \mathbb{N}\}.
\end{aligned}
$$

By [5, Proposition 3.14] periodic boundary conditions are of $\mathcal{C}^\infty$–order $(1,1)$ and of type $\mathcal{C}^\infty$ in the homoclinic case. Projection boundary conditions are of $\mathcal{C}^{l-1}$–order $(2,2)$ and of type $\mathcal{C}^l$, because f is of type $\mathcal{C}^{l+1}$. Furthermore, these boundary conditions are separated.

Theorem 2.3 *Let $\bar{x}_{\mathbb{Z}}$ be a heteroclinic $f(\cdot,\bar{\lambda})$–orbit connecting $\bar{\xi}_-$ and $\bar{\xi}_+$ which is t–tangential, nondegenerate and vertical with $t \geq 0$. Assume that $b \in \mathcal{C}^2(\mathbb{R}^{2k+r},\mathbb{R}^{k+p})$ defines regular boundary conditions of $\mathcal{C}^1$–order (l_-,l_+). Suppose that $\Psi_J \in \mathcal{C}^2(S_J \times \mathbb{R}^r,\mathbb{R}^t)$, $J=[n_-,n_+]$, defines a family of regular constraints.*

Then the boundary value problem (4)—(6) has a unique solution (x_J,λ_J) in a closed ball of fixed radius centered at $(\bar{x}_{|J},\bar{\lambda})$ for $-n_-,n_+$ sufficiently large. Furthermore, there exists a uniform constant $C>0$ such that

$$\|\lambda_J-\bar{\lambda}\| \leq C(e^{\alpha_{-s}n_-}+e^{-\alpha_{+u}n_+})(e^{l_-\alpha_{-u}n_-}+e^{-l_+\alpha_{+s}n_+}) \tag{8}$$

holds. If b defines separated boundary conditions, the improved estimate

$$\|\lambda_J-\bar{\lambda}\| \leq C(e^{(l_-\alpha_{-u}+\alpha_{-s})n_-}+e^{-(l_+\alpha_{+s}+\alpha_{+u})n_+}) \tag{9}$$

is valid.

The proof of this theorem is carried out in Section 4. Notice, the estimate (8) holds for periodic boundary conditions with $l_\pm=1$ in the homoclinic case. Estimate (9) is valid for projection boundary conditions with $l_\pm=2$.

Remark 2.4 *As in* [6, *Section* 4.3] *the statements of the previous theorem can be extended to the approximation of curves of vertical tangentials.*

3 Conclusions

3.1 Transversal heteroclinics

The approximation of transversal heteroclinics is covered by Theorem 2.3, since they are 0–tangential and Definition 2.1 is trivially satisfied.

3.2 Quadratic heteroclinics

Consider a quadratic heteroclinic $\bar{x}_{\mathbb{Z}}$, cf. the introduction. Furthermore, let $\bar{u}_{\mathbb{Z}}$ span the one–dimensional nullspace $\mathcal{N}(\Gamma_x(\bar{x}_{\mathbb{Z}},\bar{\lambda}))$, cf. (3). Due to $r=p+t=p+1\geq 1$ we can write the parameter $\lambda \in \mathbb{R}^r$ as $\lambda=(\lambda_p,\lambda_2)\in\mathbb{R}^p\times\mathbb{R}$.

The tuple $(\bar{x}_{\mathbb{Z}},\bar{\lambda}_p,\bar{\lambda}_2)$ is a turning point of the operator Γ. In [5, Section 4.2] we show that

$$\bar{z}_{\mathbb{Z}}=(\bar{x}_n,\bar{u}_n)_{n\in\mathbb{Z}}$$

is a 1–tangential, nondegenerate $F(\cdot,\bar{\lambda}_p,0,\bar{\lambda}_2)$–heteroclinic connecting $(\bar{\xi}_-,0)$ and $(\bar{\xi}_+,0)$, where $F\in\mathcal{C}^{l-1}(\mathbb{R}^{2k+2p+1},\mathbb{R}^{2k})$ is defined by

$$F(x,u,\lambda_p,\mu_p,\lambda_2)=(f(x,\lambda_p,\lambda_2),f'(x,\lambda_p,\lambda_2)(u,\mu_p,0))$$

for $x,u\in\mathbb{R}^k$, $\lambda_p,\mu_p\in\mathbb{R}^p$, $\lambda_2\in\mathbb{R}$. Here $(\bar{\xi}_\pm,0)$ are fixed points of $F(\cdot,\bar{\lambda}_p,0,\bar{\lambda}_2)$, and the spectra of $F_{(x,u)}(\bar{\xi}_\pm,0,\bar{\lambda}_p,0,\bar{\lambda}_2)$ and $f_x(\bar{\xi}_\pm,\bar{\lambda}_p,\bar{\lambda}_2)$ are equal.

Obviously, the orbit $\bar{z}_{\mathbb{Z}}$ is vertical because $(\bar{x}_n, \gamma\bar{u}_n)_{n\in\mathbb{Z}}$ is an $F(\cdot, \bar{\lambda}_p, 0, \bar{\lambda}_2)$–orbit for each $\gamma \in \mathbb{R}$.

To approximate $(\bar{x}_{\mathbb{Z}}, \bar{\lambda}_p, \bar{\lambda}_2)$, turning points of the operator Γ_J are used. This operator is defined by

$$\Gamma_J(x_J, \lambda_p, \lambda_2) = \left((x_{n+1} - f(x_n, \lambda_p, \lambda_2))_{n\in\hat{J}}, b(x_{n_-}, x_{n_+}, \lambda_p, \lambda_2)\right)$$

for $(x_J, \lambda_p, \lambda_2) \in S_J \times \mathbb{R}^p \times \mathbb{R}$. By [5, Section 4.2] the computation turning points of Γ_J is equivalent to solving the extended boundary value problem

$$\begin{aligned} (x_{n+1}, u_{n+1}) - F(x_n, u_n, \lambda_p, \mu_p, \lambda_2) &= 0, \quad n = n_-, \ldots, n_+ - 1, \\ B(x_{n_-}, u_{n_-}, x_{n_+}, u_{n_+}, \lambda_p, \mu_p, \lambda_2) &= 0, \\ \bar{u}_0^T(x_0 - \bar{x}_0) &= 0 \end{aligned}$$

where $B \in \mathcal{C}^{l-1}(\mathbb{R}^{4k+2p+1}, \mathbb{R}^{2(k+p)})$ is defined by

$$B(x, u, y, v, \lambda_p, \mu_p, \lambda_2) = (b(x, y, \lambda_p, \lambda_2), b'(x, y, \lambda_p, \lambda_2)(u, v, \mu_p, 0))$$

for $x, y, u, v \in \mathbb{R}^k$, $\lambda_p, \mu_p \in \mathbb{R}^p$ and $\lambda_2 \in \mathbb{R}$. The extended boundary condition B defines regular boundary conditions of $\mathcal{C}^{q-1}$–order (l_-, l_+) at the tuple $((\bar{\xi}_-, 0), (\bar{\xi}_+, 0), \bar{\lambda}_p, 0, \bar{\lambda}_2)$ if the boundary conditions defined by b are of $\mathcal{C}^q$–order $(l_-, l_+$ at $(\bar{\xi}_-, \bar{\xi}_+, \bar{\lambda}_p, \bar{\lambda}_2)$ and $q \geq 1$ holds. Furthermore, B defines separated boundary conditions if b does.

This means that we approximate turning points of Γ_J in some neighborhood of $(\|\bar{x}\|_J, \bar{\lambda}_p, \bar{\lambda}_2)$ by approximating the vertical $F(\cdot, \bar{\lambda}_p, 0, \bar{\lambda}_2)$–heteroclinic $(\bar{x}_n, \bar{u}_n)_{n\in\mathbb{Z}}$ using the method stated in the introduction. Hence Theorem 2.3 applies, and we end up with the following result.

Corollary 3.1 *Assume that $\bar{x}_{\mathbb{Z}}$ is a quadratic $f(\cdot, \bar{\lambda}_p, \bar{\lambda}_2)$–heteroclinic where f is of type $\mathcal{C}^4$. Suppose $b \in \mathcal{C}^3$ defines boundary conditions of $\mathcal{C}^2$–order (l_-, l_+). Then there exists a unique turning point $(x_J, \lambda_{p,J}, \lambda_{2,J})$ of Γ_J in a closed ball of fixed radius centered at $(\bar{x}_{|J}, \lambda_{p,J}, \lambda_{2,J})$ for $-n_-, n_+$ sufficiently large. Furthermore, there exists a uniform constant $C > 0$ such that*

$$\|(\lambda_{p,J}, \lambda_{2,J}) - (\bar{\lambda}_p, \bar{\lambda}_2)\| \leq C(e^{\alpha_{-s}n_-} + e^{-\alpha_{+u}n_+})(e^{l_-\alpha_{-u}n_-} + e^{-l_+\alpha_{+s}n_+}) \quad (10)$$

is satisfied. If b defines separated boundary conditions, the improved estimate

$$\|(\lambda_{p,J}, \lambda_{2,J}) - (\bar{\lambda}_p, \bar{\lambda}_2)\| \leq C(e^{(l_-\alpha_{-u}+\alpha_{-s})n_-} + e^{-(l_+\alpha_{+s}+\alpha_{+u})n_+}) \quad (11)$$

is valid.

The previous corollary applies to homoclinic orbits and periodic boundary conditions with $l_\pm = 1$ in (10). In case of projection boundary conditions the estimate (11) is valid with $l_\pm = 2$. This follows by [5, Proposition 3.14].

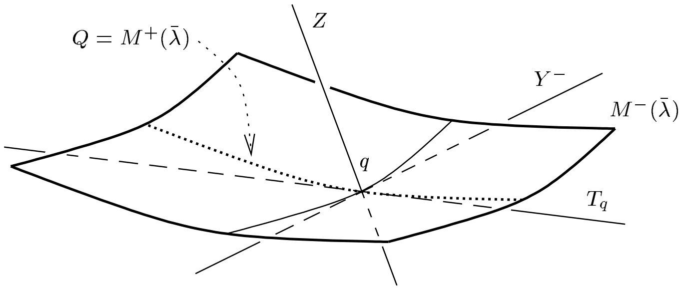

Figure 3: Schematic drawing of the manifolds $M^+(\bar\lambda)$ and $M^-(\bar\lambda)$.

4 Proof of the main theorem

4.1 A special parameterization of two submanifolds

Consider two $\mathcal{C}^{l+1}$–submanifolds $M^+(\lambda)$ and $M^-(\lambda)$ of $\mathbb{R}^k$ depending $\mathcal{C}^{l+1}$–smoothly on $\lambda \in U_\epsilon(\bar\lambda)$, and set $m_\pm = \dim M^\pm(\bar\lambda)$. Here $\bar\lambda \in \mathbb{R}^r$ and $\epsilon > 0$. In case $r = 0$ all statements on the parameter λ are to be neglected.

Assume that there exists a $\mathcal{C}^{l+1}$–submanifold Q of dimension t in the intersection of $M^+(\bar\lambda)$ and $M^-(\bar\lambda)$. For some $q \in Q$ let

$$(T_q M^+(\bar\lambda) + T_q M^-(\bar\lambda)) \oplus Z = \mathbb{R}^k$$

hold for a subspace $Z \subset \mathbb{R}^k$ of dimension $\dim Z = r$. In particular this means

$$m_+ + m_- - t = k - r. \tag{12}$$

In case $t = 0$ let $Q = \{q\} \subset M^+(\bar\lambda) \cap M^-(\bar\lambda)$ and $T_q = \{0\}$. Let subspaces $Y^+, Y^- \subset \mathbb{R}^k$ be given with

$$T_q M^+(\bar\lambda) = T_q \oplus Y^+, \quad T_q M^-(\bar\lambda) = T_q \oplus Y^-.$$

This situation is schematically drawn in Figure 3 for the case $k = 3$, $m_+ = t = r = 1$ and $m_- = 2$ (which means $M^+(\bar\lambda) = Q$ locally near q and $Y^+ = \{0\}$).

Furthermore, assume that the extended manifolds $\{(x, \lambda) \in \mathbb{R}^k \times U_\epsilon(\bar\lambda) : x \in M^+(\lambda)\}$ and $\{(x, \lambda) \in \mathbb{R}^k \times U_\epsilon(\bar\lambda) : x \in M^-(\lambda)\}$ are transversal at their common point $(q, \bar\lambda)$, cf. (12).

In his paper [7] Sandstede constructs parameterizing maps $q_\pm \in \mathcal{C}^l(U^\pm_{\varepsilon_0}, \mathbb{R}^k)$ of $M^\pm(\lambda)$ where $U^\pm_{\varepsilon_0} = U_{\varepsilon_0}(0, 0, \bar\lambda) \subset T_q Q \times Y^\pm \times \mathbb{R}^r$. These maps satisfy

$$q_\pm(a, \beta^\pm, \lambda) \in M^\pm(\lambda), \quad (a, \beta^\pm, \lambda) \in U^\pm_{\varepsilon_0},$$

and

$$q_\pm(0, 0, \bar\lambda) = q, \quad D_1 q_+(0, 0, \bar\lambda) = D_1 q_-(0, 0, \bar\lambda), \quad \mathcal{R}(D_1 q_\pm(0, 0, \bar\lambda)) = T_q Q. \tag{13}$$

Furthermore, consider an index family of maps

$$\omega^\pm_\nu \in \mathcal{C}^l(U_{\delta_0}, Y^\mp + Z), \quad \nu \in S,$$

where $U_{\delta_0} = U_{\delta_0}(0,0,0,\bar{\lambda}) \subset T_qQ \times Y^+ \times Y^- \times \mathbb{R}^r$ and $\delta_0 > 0$. Suppose that the derivatives of these maps up to order l are uniformly bounded, and define

$$S_\gamma = \{\nu \in S : \max_{\kappa=+,-,i=0,1} \|D^i\omega_\nu^\kappa(0,0,0,\bar{\lambda})\| \le \gamma\}, \quad \gamma > 0.$$

Then the equation

$$q_+(a,\beta^+,\lambda) + \omega_\nu^+(a,\beta^+,\beta^-,\lambda) = q_-(a,\beta^-,\lambda) + \omega_\nu^-(a,\beta^+,\beta^-,\lambda)$$

can be solved for β^+,β^-,λ as follows.

Lemma 4.1 *For $q_\pm$ and $\omega_\nu^\pm$, $\nu \in S$, as above there exist constants $C,\gamma > 0$, $\rho \in (0,\delta_0]$ such that the following statements are valid for every $\nu \in S_\gamma$.*

(i) *In case $t > 0$ there exists a map $(\lambda_\nu,\beta_\nu^+,\beta_\nu^+) \in \mathcal{C}^l(U_\rho,U_{\delta_0})$ satisfying the following for every $a \in U_\rho$, where $U_\rho = U_\rho(0) \subset T_qQ$.*

- *The identity*

$$\begin{aligned} & q_+(a,\beta_\nu^+(a),\lambda_\nu(a)) + \omega_\nu^+(a,\beta_\nu^+(a),\beta_\nu^-(a),\lambda_\nu(a)) \\ &= \quad q_-(a,\beta_\nu^-(a),\lambda_\nu(a)) + \omega_\nu^-(a,\beta_\nu^+(a),\beta_\nu^-(a),\lambda_\nu(a)) \end{aligned}$$

  holds.

- *The estimate*

$$\|\frac{\partial^i}{\partial a^i}((\lambda_\nu,\beta_\nu^+,\beta_\nu^-)(a) - (\bar{\lambda},0,0))\| \le C\sum_{j=0}^{i} \|H_\nu^{(j)}(a)\|$$

  is satisfied for $i = 0,\dots,l-1$ with a $\mathcal{C}^l$–map $H_\nu : U_\rho \to \mathbb{R}^k$ defined by

$$H_\nu(\tilde{a}) = (\omega_\nu^+ - \omega_\nu^-)(\tilde{a},0,0,\bar{\lambda}), \quad \tilde{a} \in U_\rho.$$

(ii) *In case $t = 0$ there exists $(\lambda_\nu,\beta_\nu^+,\beta_\nu^+) \in U_{\delta_0}$ such that*

$$q_+(\beta_\nu^+,\lambda_\nu) + \omega_\nu^+(\beta_\nu^+,\beta_\nu^-,\lambda_\nu) = q_-(\beta_\nu^-,\lambda_\nu) + \omega_\nu^-(\beta_\nu^+,\beta_\nu^-,\lambda_\nu)$$

and

$$\max\{\|\lambda_\nu - \bar{\lambda}\|, \|\beta_\nu^+\|, \|\beta_\nu^-\|\} \le C\|(\omega_\nu^+ - \omega_\nu^-)(0,0,\bar{\lambda})\|$$

hold.

Notice that Sandstede proves part (i) of the previous lemma in the case $m_+ + m_- = k$ and $t = r = 1$ where $\omega_\nu^\pm$ depends only on $(a,\beta^\pm,\lambda)$. But it is straightforward to adapt his ideas to the general case considered in (i). This is carried out in case $m_+ + m_- = k$ in the proof of [6, Lemma 4.3.5]. That proof shows that the assumption $m_+ + m_- = k$ can also be dropped.

Part (ii) of the previous lemma follows analogously to part (i) using the approximate inverse function theorem [2, Proposition 3.3] instead of the implicit function theorem.

4.2 Stepwise solving the boundary value problem

Let the assumptions of Theorem 2.3 hold. To apply the parameterization considered in the previous lemma let

$$\begin{aligned} M^-(\lambda) &= \{x \in W_u(\xi_-(\lambda), \lambda) : f(\cdot,\lambda)^n(x) \in U_\delta(\bar{x}_n) \text{ for every } n \in \mathbb{Z}_-\} \\ M^+(\lambda) &= \{x \in W_s(\xi_+(\lambda), \lambda) : f(\cdot,\lambda)^n(x) \in U_\delta(\bar{x}_n) \text{ for every } n \in \mathbb{Z}_+\} \end{aligned}$$

for $\lambda \in U_\epsilon(\bar{\lambda})$ and some $\delta > 0$. Taking $\epsilon, \delta > 0$ sufficiently small with $\xi_\pm(U_\epsilon(\bar{\lambda})) \subset U_\delta(\bar{\xi}_\pm)$, the results of [5, Appendix B.1] imply that these sets $M_\pm(\lambda)$, $\lambda \in U_\epsilon(\bar{\lambda})$, satisfy the assumptions of Lemma 4.1 with $q = \bar{x}_0$ and the set Q from Definition 2.1. Notice, due to [5]

$$T_{\bar{x}_0} M_-(\bar{\lambda}) = T_{\bar{x}_0} W_u(\bar{\xi}_-, \bar{\lambda}), \quad T_{\bar{x}_0} M_+(\bar{\lambda}) = T_{\bar{x}_0} W_s(\bar{\xi}_+, \bar{\lambda})$$

hold.

In the following construction let $c_\pm = (a, \beta_\pm, \lambda) \in V^\pm$ be elements in generic neighborhoods $V^\pm = V_a \times V_{\beta_\pm} \times V_\lambda$ of $(0, 0, \bar{\lambda}) \in T_{\bar{x}_0} Q \times Y^\pm \times \mathbb{R}^r$. By $y_n^\pm(c_\pm)$, $n \in \mathbb{Z}_\pm$, we denote the unique $f(\cdot,\lambda)$–orbit satisfying

$$y_0^\pm(c_\pm) = q_\pm(c_\pm).$$

The sequences $y_{\mathbb{Z}_\pm}^\pm(c_\pm)$ are elements of $S_{\mathbb{Z}_\pm} = \{x_{\mathbb{Z}_\pm} \in (\mathbb{R}^k)^{\mathbb{Z}_\pm} : \|x_{\mathbb{Z}_\pm}\|_\infty = \sup_{n\in\mathbb{Z}_\pm} \|x_n\| < \infty\}$, and by the construction of $M_\pm(\lambda)$ in [5] the maps $y_{\mathbb{Z}_\pm}^\pm : V^\pm \to S_{\mathbb{Z}_\pm}$ are of type $\mathcal{C}^l$.

We consider the variational equation

$$u_{n+1} = f_x(y_n^\pm(c_\pm), \lambda) u_n, \quad n \in \hat{\mathbb{Z}}_\pm, \tag{14}$$

along the $f(\cdot,\lambda)$–orbits $y_{\mathbb{Z}_\pm}^\pm(c_\pm)$ where $\hat{\mathbb{Z}}_+ = \mathbb{Z}_+$ and $\hat{\mathbb{Z}}_- = \mathbb{Z}_- - 1$. Let $\Phi_\pm(\cdot,\cdot,c_\pm) : \mathbb{Z}_\pm^2 \to \mathbb{R}^{k,k}$ denote its solution operator. By the theory of exponential dichotomies we get a splitting of solutions of the linear difference equation (14) into exponentially growing and decaying parts. Using [5, Lemma 2.5] and the roughness theorem [5, Lemma 2.3] the following statements hold.

Lemma 4.2 *There exist a constant $K > 0$ and projector valued $\mathcal{C}^l$–maps $P_n^\pm : V^\pm \to \mathbb{R}^{k,k}$, $n \in \mathbb{Z}_\pm$, such that*

$$\begin{aligned} \|\frac{\partial^i}{\partial c_\pm^i} \Phi_\pm(n,m,c_\pm) P_m^\pm(c_\pm)\| &\le K e^{-\alpha_{\pm s}(n-m)}, \\ \|\frac{\partial^i}{\partial c_\pm^i} \Phi_\pm(m,n,c_\pm)(I - P_n^\pm(c_\pm))\| &\le K e^{-\alpha_{\pm u}(n-m)} \end{aligned}$$

hold for $i = 0, \ldots, l$ and every $n, m \in \mathbb{Z}^\pm$ with $n \ge m$. Furthermore, the equality

$$\Phi_\pm(n,m,c_\pm) P_m^\pm(c_\pm) = P_n^\pm(c_\pm) \Phi_\pm(n,m,c_\pm)$$

is valid for every $n, m \in \mathbb{Z}_\pm$ and

$$\mathcal{R}(P_0^-(c_-)) = Z \oplus Y^+, \quad \mathcal{N}(P_0^+(c_+)) = Z \oplus Y^-$$

hold.

To find a solution of the boundary value problem (4)—(6) let us parameterize $f(\cdot,\lambda)$–orbits staying near $\bar{x}_{|J_\pm}$ by the maps $y^\pm_{\mathbb{Z}_\pm}$. Let $J_- = [n_-, 0]$ and $J_+ = [0, n_+]$ for some $-n_-, n_+ \in \mathbb{N}$.

Obviously, the sequence

$$z_n^\pm = y_n^\pm(c_\pm) + v_n, \quad n \in J_\pm, \tag{15}$$

is an $f(\cdot,\lambda)$–orbit for $v_{J_\pm} \in S_{J_\pm}$ if and only if

$$v_{n+1} = f_x(y_n^\pm(c_\pm),\lambda)v_n + g_n^\pm(v_n, c_\pm), \quad n \in \hat{J}_\pm, \tag{16}$$

holds where

$$g_n^\pm(v_n, c_\pm) = f(y_n^\pm(c_\pm) + v_n, \lambda) - f(y_n^\pm(c_\pm),\lambda) - f_x(y_n^\pm(c_\pm),\lambda)v_n.$$

Notice that $g^\pm_{\mathbb{Z}_\pm} \in \mathcal{C}^l(\mathbb{R}^k \times V^\pm, S_{\mathbb{Z}_\pm})$ is valid because f is of type $\mathcal{C}^{l+1}$ and $y^\pm_{\mathbb{Z}_\pm}$ is of type $\mathcal{C}^l$. Furthermore, by Taylor's formula

$$g_n^\pm(v, c_\pm) = \int_0^1 (1-t) f_{xx}(y_n^\pm(c_\pm) + tv, \lambda)dt\, v^2, \quad v \in \mathbb{R}^k, n \in \mathbb{Z}_\pm,$$

is satisfied.

By [5, Lemma 2.2], the solutions of

$$\begin{aligned}
v_n^- &= \Phi_-(n, n_-, c_-)P^-_{n_-}(c_-)x_- \\
&\quad - \sum_{m=n_-}^{n-1} \Phi_-(n, m+1, c_-)(I - P^-_{m+1}(c_-))g_m^-(v_m^-, c_-) \\
&\quad + \sum_{m=n}^{-1} \Phi_-(n, m+1, c_-)P^-_{m+1}(c_-)g_m^-(v_m^-, c_-), \quad n \in J_-, \qquad (17) \\
v_n^+ &= \Phi_+(n, n_+, c_+)(I - P^+_{n_+}(c_+))x_+ \\
&\quad - \sum_{m=0}^{n-1} \Phi_+(n, m+1, c_+)(I - P^+_{m+1}(c_+))g_m^+(v_m^+, c_+) \\
&\quad + \sum_{m=n}^{n_+-1} \Phi_+(n, m+1, c_+)P^+_{m+1}(c_+)g_m^+(v_m^+, c_+), \quad n \in J_+ \qquad (18)
\end{aligned}$$

solve the system (16). The vectors x_- and x_+ are arbitrary elements of the stable subspace X_- of $f_x(\bar{\xi}_-, \bar{\lambda})$ and the unstable subspace X_+ of $f_x(\bar{\xi}_+, \bar{\lambda})$.

To obtain estimates for the solutions of (17), (18) we consider Banach spaces with exponentially weighted norms. For a finite interval $J = [n_-, n_+]$ and $v_J \in (\mathbb{R}^k)^J$ let

$$\|v_J\|_- = \max_{n \in J} e^{\alpha_{-s}(n-n_-)}\|v_n\|, \quad \|v_J\|_+ = \max_{n\in J} e^{\alpha_{+u}(n_+-n)}\|v_n\|.$$

We denote the space $(\mathbb{R}^k)^J$ endowed with the norm $\|\cdot\|_\pm$ by $S_J^\pm$. Notice, the norms $\|\cdot\|_\pm$ and $\|\cdot\|_\infty$ are equivalent because J is finite.

Lemma 4.3 *There exist zero neighborhoods $W^\pm \subset X_\pm$, constants $N, C, \varepsilon > 0$ and for every $J_\pm$ as above with $-n_-, n_+ \geq N$ certain $\mathcal{C}^l$–maps $\bar{v}^\pm_{J_\pm} : W^\pm \times V^\pm \to U_\varepsilon^\pm$ such that for every $x_\pm \in W^\pm$ the following statements are valid. Here $U_\varepsilon^\pm$ is the open ball $U_\varepsilon(0) \subset S^\pm_{J_\pm}$ with respect to the norm $\|\cdot\|_\pm$.*

(i) *The sequence* $\bar{v}^{\pm}_{J_\pm}(x_\pm, c_\pm)$ *is a solution of* (16) *which satisfies*

$$\bar{v}^{\pm}_0(x_\pm, c_\pm) \in Y^{\pm} \oplus Z.$$

(ii) *The estimates*

$$\|D^i_{c_\pm} \bar{v}^{\pm}_{J_\pm}(x_\pm, c_\pm)\|_\pm \le C\|x_\pm\|, \quad i = 0, \dots, l-1,$$

hold and the total derivatives of $\bar{v}^{\pm}_{J_\pm}$ *are uniformly bounded up to order* l.

Proof. Obviously, the solutions of (17), (18) are zeros of the operators

$$\Pi^{\pm}_{J_\pm} : S^{\pm}_{J_\pm} \times X_\pm \times V^{\pm} \to S^{\pm}_{J_\pm}$$

which are defined by taking the right sides of (17), (18) to the left. Due to $g^{\pm}_{\mathbb{Z}_\pm} \in \mathcal{C}^l$ and Lemma 4.2, these operators are of type $\mathcal{C}^l$.

By the definition of $g^{\pm}_{\mathbb{Z}_\pm}$ and the previous lemma

$$\Pi^{\pm}_{J_\pm}(0, 0, 0, 0, \bar{\lambda}) = 0$$

holds and the derivatives of $\Pi^{\pm}_{J_\pm}$ up to order l are uniformly bounded in some neighborhood of that zero. Furthermore, $D_1 \Pi^{\pm}_{J_\pm}(0, 0, 0, 0, \bar{\lambda})$ is the identity map and hence an isomorphism. Therefore part (i) follows by the implicit function theorem.

For the proof of part (ii) notice that

$$\begin{aligned}
\Pi^{-}_{J_-}(0, x_-, c_-) &= -(\Phi_-(n, n_-, c_-) P^{-}_{n_-}(c_-) x_-)_{n \in J_-}, \\
\Pi^{+}_{J_+}(0, x_+, c_+) &= -(\Phi_+(n, n_+, c_+)(I - P^{+}_{n_+}(c_+)) x_+)_{n \in J_+}
\end{aligned}$$

hold. By Lemma 4.2 and the qualitative version of the implicit function theorem (cf. [3, Theorem 2.1, 2.2]) the statement follows. □

In the following we shrink the zero neighborhoods $W^{\pm}$ if necessary. Let $x_\pm \in W^{\pm}$ be arbitrary and

$$z^{\pm}_n(x_\pm, c_\pm) = y^{\pm}_n(c_\pm) + \bar{v}^{\pm}_n(x_\pm, c_\pm), \quad n \in J_\pm,$$

with $\bar{v}^{\pm}_{J_\pm}$ and $J_\pm$ as in the previous lemma. By the equivalence of (15) and (16), the sequence $z^{\pm}_{J_\pm}(x_\pm, c_\pm)$ is an $f(\cdot, \lambda)$–orbit.

As the next step we parameterize $x_\pm$ with respect to $c_\pm$ such that the tuple $(z^{-}_{n_-}(x_-, c_-), z^{+}_{n_+}(x_+, c_+), \lambda)$ solves the boundary conditions. This means that we solve

$$\theta_J(x_-, x_+, a, \beta_-, \beta_+, \lambda) := b(z^{-}_{n_-}(x_-, c_-), z^{+}_{n_+}(x_+, c_+), \lambda) = 0. \tag{19}$$

In the following let $c = (a, \beta_-, \beta_+, \lambda)$ and $V = V_a \times V_{\beta_-} \times V_{\beta_+} \times V_\lambda$.

Suppose that b is of type $\mathcal{C}^l$ and that it defines boundary conditions of $\mathcal{C}^q$–order (l_-, l_+) for some $q \in \{1, \dots, l-1\}$. Notice, for the proof of Theorem 2.3 we need $l = 2$ and $q = 1$.

Lemma 4.4 *There exist constants $C, N > 0$ and for every $J = [n_-, n_+]$ with $-n_-, n_+ \geq N$ certain $\mathcal{C}^l$–maps $\tilde{x}_J^{\pm} : V \to W^{\pm}$ such that $(\tilde{x}_J^-(c), \tilde{x}_J^+(c), c)$ solve* (19) *and*

$$\max_{\kappa=-,+} \|D^i \tilde{x}_J^\kappa(c)\| \leq C(e^{l_- \alpha_- {}_u n_-} + e^{-l_+ \alpha_+ {}_s n_+}), \quad i = 0, \ldots, q, \tag{20}$$

hold. If b defines separated boundary conditions, the improved estimates

$$\|D^i \tilde{x}_J^-(c)\| \leq C e^{l_- \alpha_- {}_u n_-}, \quad \|D^i \tilde{x}_J^+(c)\| \leq C e^{-l_+ \alpha_+ {}_s n_+}$$

are valid. Furthermore, the total derivatives of $\tilde{x}_J^{\pm}$ up to order l are uniformly bounded.

Proof. The first part of the statement follows by the implicit function theorem applied to the equation (19) as follows. By Lemma 4.3, the definitions of $y_{\mathbb{Z}_\pm}^{\pm}$, $z_{J_\pm}^{\pm}$, and (17), (18) we obtain $z_{n_\pm}^{\pm}(0, 0, 0, \bar{\lambda}) = \bar{x}_{n_\pm}$, $D_1 z_{n_-}^-(0, 0, 0, \bar{\lambda}) = P_{n_-}^-(0, 0, \bar{\lambda})$ and $D_1 z_{n_+}^+(0, 0, 0, \bar{\lambda}) = (I - P_{n_+}^+(0, 0, \bar{\lambda}))$. Hence [5, Lemma 2.5] and the regularity of the boundary conditions (cf. the introduction) imply

$$\lim_{\min\{-n_-, n_+\} \to \infty} \theta_J(0, 0, 0, 0, 0, \bar{\lambda}) = 0.$$

Moreover, $D_{(x_-, x_+)} \theta_J(0, 0, 0, 0, 0, \bar{\lambda})$ is isomorphic with uniformly bounded inverse for sufficiently large $-n_-, n_+$, and the derivatives of θ_J are uniformly bounded. Therefore the $\mathcal{C}^l$–maps $\tilde{x}_J^{\pm} : V \to W^{\pm}$ exist, and

$$\max_{\kappa=-,+} \|D^i \tilde{x}_J^\kappa(c)\| \leq C \|\theta_J(0, 0, c)\|, \quad i = 0, \ldots, q,$$

holds uniformly for sufficiently large $-n_-, n_+$ and a fixed constant $C > 0$.

The estimate (20) follows by the identity

$$\theta_J(0, 0, c) = b(y_{n_-}^-(c_-), y_{n_+}^+(c_+), \lambda),$$

the smooth order of the boundary condition (cf. Definition 2.2), and the uniform convergence rates of $y_{n_\pm}^{\pm}(c_\pm)$ and its derivatives (cf. [5, Lemma 3.12]).

For separated boundary conditions the definition of θ_J reads

$$\theta_J(x_-, x_+, c) = b_-(z_{n_-}^-(x_-, c_-), \lambda) + b_+(z_{n_+}^+(x_+, c_+), \lambda)$$

with $b_\pm$ as in Definition 2.2. Since $(\tilde{x}_J^-(c), \tilde{x}_J^+(c))$ is constructed as locally unique zero of $\theta_J(\cdot, \cdot, c)$, this means that the term $\tilde{x}_J^-(c)$ does not depend on n_+. The same holds for $\tilde{x}_J^+(c)$ and n_-. Therefore, the second part of the statement follows from the first one enlarging n_+ and n_-, respectively. □

By the previous lemma we constructed $f(\cdot, \lambda)$–orbits $z_{J \cap \mathbb{Z}_\pm}^{\pm}(\tilde{x}_J^{\pm}(c), c_\pm)$ on both half intervals $J \cap \mathbb{Z}_\pm$ which solve the boundary conditions. To paste these two solutions together at $n = 0$, we have to solve

$$z_0^-(\tilde{x}_J^-(c), c_-) = z_0^+(\tilde{x}_J^+(c), c_+) \tag{21}$$

which is, by our construction, equivalent to

$$q_-(c_-) + \bar{v}_0^-(\tilde{x}_J^-(c), c_-) = q_+(c_+) + \bar{v}_0^+(\tilde{x}_J^+(c), c_+).$$

We obtain solutions of this problem applying Lemma 4.1 with

$$\omega_J^{\pm}(c) = \bar{v}_0^{\pm}(\tilde{x}_J^{\pm}(c), c_{\pm}).$$

This is possible due to Lemma 4.3.

To make the following estimates simple, let

$$d(n_-, n_+) = (e^{\alpha_{-s} n_-} + e^{-\alpha_{+u} n_+})(e^{l_- \alpha_{-u} n_-} + e^{-l_+ \alpha_{+s} n_+})$$

and, in case of separated boundary conditions,

$$d(n_-, n_+) = e^{(l_- \alpha_{-u} + \alpha_{-s}) n_-} + e^{-(l_+ \alpha_{+s} + \alpha_{+u}) n_+}.$$

Then the Lemmata 4.3 and 4.4 as well as the definitions of the exponentially weighted norms $\| \cdots \|_{\pm}$ imply

$$\| D^i \omega_J^{\pm}(c) \| \le C d(n_-, n_+), \quad i = 0, \ldots, q,$$

for sufficiently large $-n_-, n_+$, with a constant $C > 0$ independent of J, i and c.

Lemma 4.5 *There exist constants $C, N > 0$ and for any $J = [n_-, n_+]$ with $-n_-, n_+ \ge N$ a tuple $(a_J, \beta_J^-, \beta_J^+, \lambda_J) \in V$ such that (x_J, λ_J), where x_J is defined by*

$$x_n = z_n^{\pm}(\tilde{x}_J^{\pm}(a_J, \beta_J^-, \beta_J^+, \lambda_J), a_J, \beta_J^{\pm}, \lambda_J), \quad n \in J \cap \mathbb{Z}_{\pm},$$

is a solution of the boundary value problem (4)—(6) *satisfying*

$$\| \lambda_J - \bar{\lambda} \| \le C d(n_-, n_+)$$

and

$$\lim_{\min\{-n_-, n_+\} \to \infty} \| x_J - \bar{x}_{|J} \|_\infty = 0.$$

Proof. In case $t = 0$ we obtain solutions of (21) directly by the Lemma 4.1(i). The asserted estimates follow by the Lemmata 4.3 and 4.4.

Let $t > 0$ hold. By Lemma 4.1(ii) there exist $\mathcal{C}^l$–maps $\beta_J^{\pm} : V_a \to V_{\beta_\pm}$, $\lambda_J : V_a \to V_\lambda$ such that $(x_J(a), \lambda_J(a))$ is a solution of (4), (5). Here $x_J(a)$ is defined as above with respect to $(a, \beta_J^{\pm}(a), \lambda_J(a))$ instead of $(a_J, \beta_J, \lambda_J)$. Furthermore, for some $C > 0$,

$$\| \frac{\partial^i}{\partial a^i} ((\lambda_J, \beta_J^{\pm})(a) - (\bar{\lambda}, 0)) \| \le C d(n_-, n_+), \quad i = 0, \ldots, q,$$

hold for $-n_-, n_+$ sufficiently large. It remains to find $a_J \in V_a$ such that

$$g_J(a_J) = \Psi_J(x_J(a_J), \lambda_J(a_J)) = 0$$

is satisfied. This will be done by applying an approximate inverse function theorem.

By the previous lemmata and the regularity of Ψ_J (cf. the introduction), the derivatives of g_J up to order l are uniformly bounded. Due to

$$\lim_{\min\{-n_-, n_+\} \to \infty} \| x_J(a) - \bar{x}_{|J} \|_\infty = 0$$

we obtain

$$\lim_{\min\{-n_-,n_+\}\to\infty} g_J(0) = 0.$$

Furthermore, the second identity of (13) implies

$$D_1 y^{\pm}_{\mathbb{Z}_\pm}(0,0,\bar\lambda) = [\bar u^1_{|\mathbb{Z}_\pm}, \dots \bar u^t_{|\mathbb{Z}_\pm}]$$

where $\{\bar u^1_{\mathbb{Z}}, \dots, \bar u^t_{\mathbb{Z}}\}$ denotes a basis of the kernel $\mathcal{N}(\Gamma_x(\bar x_{\mathbb{Z}}, \bar\lambda))$ (cf. (3)). By the regularity of Ψ_J this means that $g'_J(0)$ is isomorphic with uniformly bounded inverses for $-n_-, n_+$ sufficiently large.

By the approximate inverse function theorem [2, Proposition 3.3] the map g_J has a unique zero a_J in some fixed neighborhood of 0 such that

$$\lim_{\min\{-n_-,n_+\}\to\infty} \|a_J\| = 0$$

is satisfied for $-n_-, n_+$ sufficiently large. This gives the assertion. □

By the previous lemma we have obtained a solution (x_J, λ_J) of the boundary value problem (4)—(6) which satisfies

$$\lim_{\min\{-n_-,n_+\}\to\infty} \|(x_J, \lambda_J) - (\bar x_{|J}, \bar\lambda)\|_\infty = 0.$$

Moreover, this solution of the boundary value problem is locally unique by [5, Theorem 3.5] for $-n_-, n_+$ sufficiently large. Therefore, we have proved the assertion of Theorem 2.3.

5 Applications

In this section we want to illustrate the superconvergence estimates obtained in Section 3. We show that these estimates are sharp for projection boundary conditions but not for periodic boundary conditions.

As in [2] we compute zeros of Γ_J using Newton's method. Turning points of Γ_J are obtained as zeros of

$$\begin{aligned}\Gamma_J(x_J, \lambda_p, \lambda_2) &= 0,\\ \Psi_J(x_J, \lambda_p, \lambda_2) &= 0\end{aligned}$$

for $x_J \in S_J, \lambda_p \in \mathbb{R}^p, \lambda_2 \in \mathbb{R}$ by the same method. Here Ψ_J is a test function (cf. [4]) which is zero at $(x_J, \lambda_p, \lambda_2)$ if the Jacobian $D_{(x,\lambda_p)}\Gamma_J(x_J, \lambda_p, \lambda_2)$ is singular. For more details on the implementation see [5, Section 5].

Example 1: Hénon's map

We consider Hénon's map

$$f(x, y, a, b) = (1 + y - ax^2, bx).$$

It is globally of type $\mathcal{C}^\infty$, and for $b \neq 0$ the map $f(\cdot,\cdot,a,b)$ is $\mathcal{C}^\infty$–diffeomorphic. For $(b-1)^2 > 4a \neq 0$ it has two fixed points $\xi_\pm(a,b) = (x_\pm, bx_\pm)$ where

$$x_\pm = \frac{b - 1 \pm \sqrt{(b-1)^2 + 4a}}{2a}. \tag{22}$$

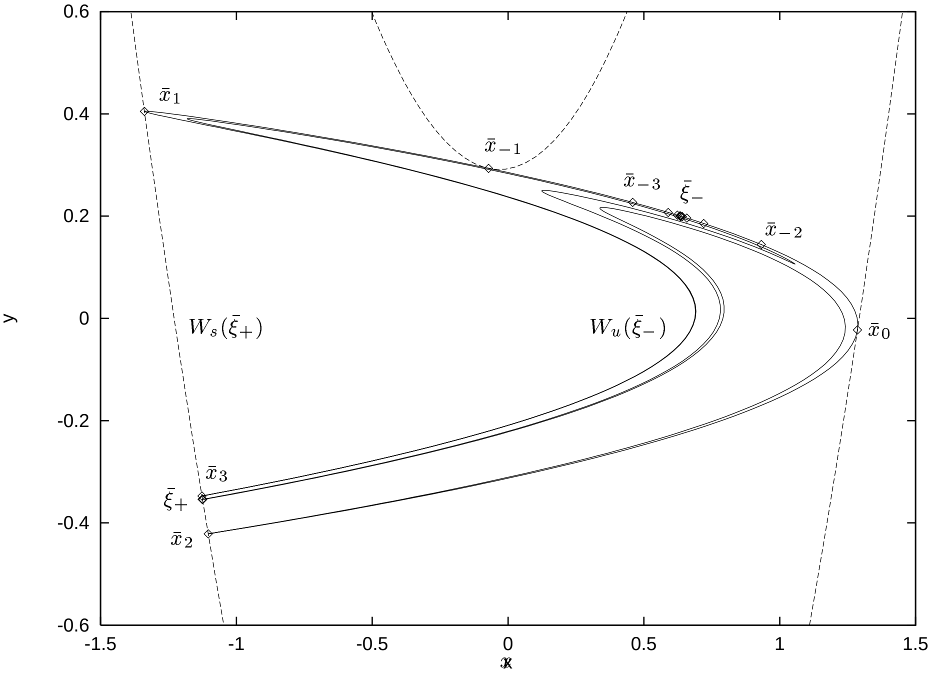

Figure 4: Parts of the unstable and stable manifolds of $\bar{\xi}_-$ and $\bar{\xi}_+$ together with the "exact" tangential heteroclinic $\bar{x}_{[-100,100]}$ for Hénon's map.

In all computations to follow, the Jacobian $D_1 f(\xi_\pm(a,b), a, b)$ has one eigenvalue inside and one outside the unit circle, i.e. one stable and one unstable eigenvalue. Hence $p = 0$ holds in this case, cf. (2).

Using the techniques stated in [5, Section 5], we approximate a transversal heteroclinic at

$$a = 1.4, \quad b = 0.4$$

which connects $\xi_+(a,b)$ and $\xi_-(a,b)$. This is done with projection boundary conditions for $J = [-100, 100]$. Varying the parameter b, we find a closed loop of heteroclinics which has two turning points. (Notice, in the homoclinic case considered in [2] we obtained four turning points.) One of these turning points occurs at

$$\bar{b} = 0.31479.$$

We denote the corresponding orbit as $\bar{x}_{[-100,100]}$ and consider it as "exact" heteroclinic connecting $\bar{\xi}_- = \xi_+(a,\bar{b})$ and $\bar{\xi}_+ = \xi_-(a,\bar{b})$. This orbit, together with parts of the unstable and stable manifolds $W_u(\bar{\xi}_-, a, \bar{b})$ and $W_s(\bar{\xi}_+, a, \bar{b})$ is shown in Figure 4.

To illustrate the superconvergence estimates from Corollary 3.1, we compute turning points (y_J, b_J) of $\Gamma_J(\cdot, a, \cdot)$ near $(\bar{x}_{|J}, \bar{b})$ for various intervals $J = [n_-, n_+]$. In Figure 5 we plot the logarithmic *global error*

$$\log_{10}(\max\{\|y_J - \bar{x}_{|J}\|_\infty, |b_J - \bar{b}|\})$$

and the logarithmic *error in the parameter b*

$$\log_{10}(|b_J - \bar{b}|)$$

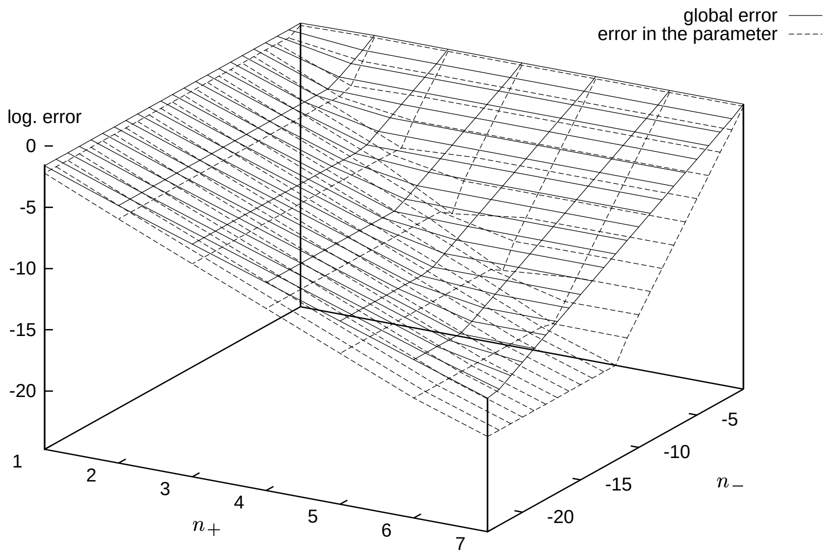

Figure 5: Global error and error in the parameter b for Hénon's map and projection boundary conditions.

versus n_- and n_+. We observe errors of the form

$$\max\{\|y_J - \bar{x}_{|J}\|_\infty, |b_J - \bar{b}|\} \approx \max\{C_1 10^{2\alpha_{-u} n_-}, C_2 10^{-2\alpha_{+s} n_+}\}$$

and

$$|b_J - \bar{b}| \approx \max\{C_3 10^{(2\alpha_{-u}+\alpha_{-s})n_-}, C_4 10^{-(2\alpha_{+s}+\alpha_{+u})n_+}\} \tag{23}$$

for some constants $C_1, \dots, C_4 > 0$. Here $\alpha_{\pm u}, \alpha_{\pm s} > 0$ are given by

$$\begin{aligned} \alpha_{\pm u} &= \log_{10}(|\min\{\lambda \in \sigma(D_1(f(\bar{\xi}_\pm, a, \bar{b}))) : |\lambda| > 1\}|), \\ \alpha_{\pm s} &= -\log_{10}(|\max\{\lambda \in \sigma(D_1(f(\bar{\xi}_\pm, a, \bar{b}))) : |\lambda| < 1\}|). \end{aligned}$$

This means that for projection boundary conditions the estimates of Corollary 3.1 are sharp in our example. Similar observations were made in [5] in the homoclinic case using these boundary conditions.

For periodic boundary conditions in the homoclinic case, the estimates of Corollary 3.1 seem not to be sharp. Computing a quadratic homoclinic at

$$\bar{b} = 0.46453$$

with respect to $\xi_+(a, \bar{b})$, we obtain an error in the parameter of the form

$$|b_J - \bar{b}| \approx C\, 10^{-\alpha(n_+ - n_-)}$$

for some $C > 0$ where $\alpha = \min\{\alpha_{\pm s}, \alpha_{\pm u}\}$, cf. Figure 6 and [5, Section 5, Example 1]. Similar observations were made in [6] for other examples.

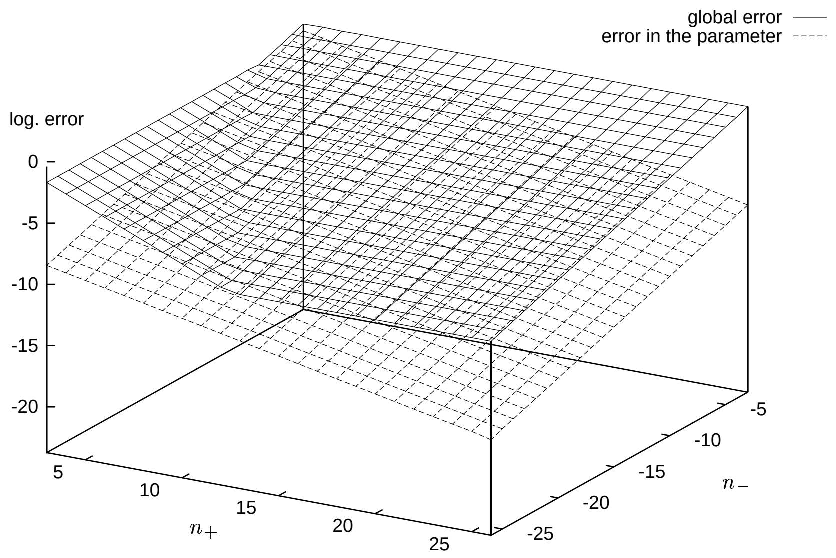

Figure 6: Global error and error in the parameter b for Hénon's map and periodic boundary conditions.

Example 2: The extended Hénon map

In a second example we want to approximate a transversal heteroclinic orbit which needs $p = 1$ parameters, cf. (2). For this purpose we add a component z and two more parameters c, d to Hénon's map as follows.

Let

$$f(x, y, z, a, b, c, d) = (1 + y - ax^2, bx, cxy + dy + ze^x).$$

be the extended Hénon map. As the original Hénon map it is of type $\mathcal{C}^\infty$, and $f(\cdot, \cdot, \cdot, a, b, c, d)$ is $\mathcal{C}^\infty$–diffeomorphic for $b \neq 0$. For $(b-1)^2 > 4a \neq 0$ it has two fixed points $(x_\pm, bx_\pm, z_pm)$ where $x_\pm$ is defined as in (22) and

$$z_\pm = \frac{bcx_\pm + bdx_\pm}{1 - e^{x_\pm}}.$$

The map f is constructed such that each heteroclinic $(\bar{u}_n, \bar{v}_n)_{n\in\mathbb{Z}}$ of Hénon's map gives a heteroclinic $(\bar{u}_n, \bar{v}_n, 0)_{n\in\mathbb{Z}}$ of the extended Hénon map in case $c = d = 0$. Suppose $(\bar{u}_n, \bar{v}_n)_{n\in\mathbb{Z}}$ connects two fixed points (x_-, y_-) and (x_+, y_+) of Hénon's map with $x_- < 0 < x_+$ where the each Jacobian at $(x_\pm, y_\pm)$ has one stable eigenvalue. Then $(\bar{u}_n, \bar{v}_n, 0)_{n\in\mathbb{Z}}$ connects the fixed points $(x_-, y_-, 0)$ and $(x_+, y_+, 0)$ of the extended map, and the Jacobian at $(x_-, y_-, 0)$ has two stable eigenvalues while the Jacobian at $(x_+, y_+, 0)$ still has one stable eigenvalue. This means that we need $p = 1$ parameters to approximate such an heteroclinic orbit.

Now we change d to $d = 1$, and hence $c = 0$ is no longer a solution. We take $a = 1.4$ and $b = 0.4$. To obtain a starting vector in this situation for a heteroclinic of the extended Hénon map, we compute a heteroclinic of Hénon's

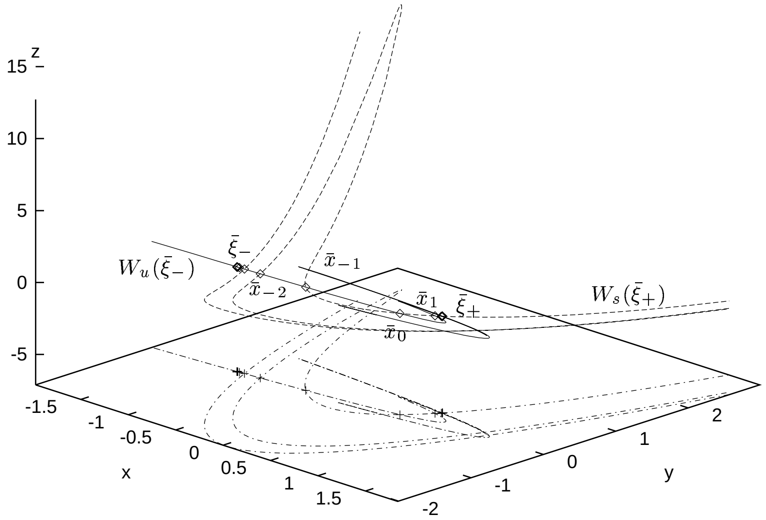

Figure 7: Parts of the one–dimensional manifolds $W_u(\bar{\xi}_-)$ and $W_s(\bar{\xi}_+)$ with points of $\bar{x}_{[-100,100]}$ for the extended Hénon map. The orthogonal projections on the (x, y)–plane are also shown.

map connecting $\xi_-(a, b)$ and $\xi_+(a, b)$ with $J = [-100, 100]$. Then we add 0 as z–component to each point of the orbit, set $c = 0$ and start Newton's method with this vector to obtain a zero of $\Gamma_J(\cdot, a, b, \cdot, d)$ using projection boundary conditions. This gives us a solution $(\bar{x}_J, \bar{c})$ for the extended Hénon map where

$$\bar{c} = 1.1818.$$

The orbit $\bar{x}_J$ and parts of the unstable and stable manifolds $W_u(\bar{\xi}_-, a, b, \bar{c}, d)$ and $W_s(\bar{\xi}_+, a, b, \bar{c}, d)$ are drawn in Figure 7. Here $\bar{\xi}_\pm$ denote the fixed points of $f(\cdot, a, b, \bar{c}, d)$ which are connected by the heteroclinic.

Analyzing the global errors and the errors in the parameter as in the previous example, we obtain rates of convergence as in the last example, see (23). Hence the estimates of Theorem 2.3 are sharp in this example for projection boundary conditions.

References

[1] W.-J. Beyn. The numerical computation of connecting orbits in dynamical systems. *IMA J. Numer. Anal.*, 9:379–405, 1990.

[2] W.-J. Beyn and J.-M. Kleinkauf. The numerical computation of homoclinic orbits for maps. *SIAM J. Numer. Anal.*, 34:1207–1236, 1997.

[3] J. Descloux and J. Rappaz. Approximation of solution branches of nonlinear equations. *RAIRO Anal. Numér.*, 16:319–349, 1982.

[4] A. Griewank and G. W. Reddien. Characterization and computation of generalized turning points. *SIAM J. Numer. Anal.*, 21:176–185, 1984.

[5] J.-M. Kleinkauf. The numerical computation and geometrical analysis of heteroclinic tangencies. Preprint 98-048, SFB 343, Universität Bielefeld, 1998.

[6] J.-M. Kleinkauf. *Numerische Analyse tangentialer homokliner Orbits.* Shaker, Aachen, 1998.

[7] B. Sandstede. Convergence estimates for the numerical approximation of homoclinic solutions. *IMA J. Numer. Anal.*, 17:463–478, 1997.