SECOND ORDER AND BOOTSTRAP APPROXIMATION TO STUDENT'S t STATISTIC

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Abstract. We prove the validity of the one-term Edgeworth expansion for Student's t statistic under minimal conditions: the distribution of observations is non-lattice and has finite third moment. As a corollary we obtain the second-order correctness for the bootstrap of Student's t statistic. If the distribution satisfies Cramer's condition and has finite fourth moment the rate of the second order approximation is $O(N^{-1})$.

1. Introduction and Results

Let $X_1, \ldots, X_N, \ldots$ be independent identically distributed random variables. Write $\mathbf{E}\, X_1 = \mu$. Let

$$\mathbf{t} = \mathbf{t}(X_1, \ldots, X_N) = (\overline{X} - \mu)/\hat{\sigma}$$

denote the Student statistic, where

$$\overline{X} = N^{-1}(X_1 + \cdots + X_N), \qquad \hat{\sigma}^2 = N^{-1} \sum\nolimits_{i=1}^{N} (X_i - \overline{X})^2.$$

Assume that $\sigma^2 := \mathbf{E}\, (X_1 - \mu)^2$ is finite and $\sigma^2 \neq 0$. Then the statistics $T_N = \sqrt{N}\mathbf{t}$ is asymptotically standard normal as $N \to \infty$.

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The accuracy of the normal approximation to the distribution function

$$F(x) = \mathbf{P}\{\sqrt{N}\mathbf{t} \le x\}$$

was studied by a number of authors, Helmers and van Zwet (1982), Helmers (1985), Slavova (1985), Hall (1988), Praskova (1989), Friedrich (1989), Bentkus and Götze (1996), Bentkus, Bloznelis and Götze (1996), etc. The Berry–Esseen bound $O(N^{-1/2})$ under the minimal moment condition $\mathbf{E}\,|X_1|^3 < \infty$ was obtained by Slavova (1985). Bentkus and Götze (1996) proved the bound $\sup_x |F(x) - \Phi(x)| < cN^{-1/2}\beta_3/\sigma^3$, thus extending the classical result of Esseen (1945) to the Studentized sums. Here $\Phi(x)$ denotes the standard normal distribution function and $\beta_3 := \mathbf{E}\,|X_1 - \mu|^3$.

A higher order approximation to the Student test was considered by Chung (1946), Bhattacharya and Ghosh (1978), Chibisov (1980), Babu and Singh (1985), Hall (1987), Putter and van Zwet (1997), Bentkus, Götze and van Zwet (1997), etc. Although increasingly general and precise, none of these results is optimal in the sense of expansions being established under minimal conditions. Hall (1987) proved the validity of a k-term Edgeworth expansion for Student's t statistic with remainder $o(N^{-k/2})$, for every integer k, provided that $\mathbf{E}\,|X_1|^{(k+2)/2} < \infty$ and the distribution F_0 of X_1 is non-singular. The moment conditions in Hall (1987) are the minimal ones, but the smoothness condition on the distribution F_0 of the observations is too restrictive. What is perhaps more important, Hall's result is only valid for a fixed underlying distribution function. As a result, it cannot be applied to the bootstrap.

The aim of the present paper is to prove the validity of an expansion under minimal conditions, in such a way that an extension to the bootstrap is straightforward. To this end, we approximate $F(x)$ by the one-term Edgeworth expansion

$$G(x) = \Phi(x) + \frac{\kappa_3}{6\sqrt{N}}(2\,x^2 + 1)\,\Phi'(x), \qquad \kappa_3 = \mathbf{E}\;X_1^3/\sigma^3$$

and construct bounds for the remainder

$$\Delta_N = \sup_x |F(x) - G(x)|.$$

The minimal smoothness condition which allows to prove the validity of one-term Edgeworth expansion, i.e., to prove the bound $\Delta_N = o(N^{-1/2})$, is the non-latticeness of the distribution F_0. We shall assume that F_0 is non-lattice.

Theorem 1. *Assume that $\mathbf{E}\,|X_1|^3 < \infty$ and the distribution of X_1 is nonlattice. Then*

$$\Delta_N = o(N^{-1/2}) \qquad as \quad N \to \infty. \tag{1.1}$$

Theorem 1 follows from Proposition 2.1, the formulation of which is rather technical and is deferred to Section 2.

Bootstrap. Given $\mathbb{X} := \{X_1, \dots, X_N\}$, let $X_1^*, \dots, X_N^*$ denote independent random variables uniformly distributed in $\mathbb{X}$. Write

$$T_N^* = \sqrt{N}\,\frac{\overline{X_N^*} - \overline{X}_N}{\hat\sigma_N^*}, \qquad {\sigma_N^*}^2 = N^{-1}\sum_{i=1}^N (X_i^* - \overline{X_N^*})^2, \qquad \overline{X_N^*} = (X_1^* + \dots + X_N^*)/N.$$

Theorem 2. *Suppose that* $\mathbf{E}\,|X_1|^3 < \infty$ *and that the distribution of* X_1 *is nonlattice. Then*

$$\text{(A2)}\quad \sup_{x\in R}\Big|\,\mathbf{P}\,\{T_N \le x\} - \mathbf{P}\,\{T_N^* \le x\,|\mathbb{X}\}\Big| = o(N^{-1/2}), \qquad a.\ s. \qquad as \quad N \to \infty.$$

Theorem 2 improves earlier results of Babu and Singh (1985), Helmers (1991) and Putter and van Zwet (1998) where the bound (A2) was established assuming that F_0 is non-lattice and increasingly sharp moment conditions, the sharpest to date being $\mathbf{E}\,|X_1|^{3+\varepsilon} < \infty$, for some $\varepsilon > 0$, obtained in the latter paper.

The best rate that can be achieved by the second order approximation is $O(N^{-1})$.

Assume that $\mathbf{E}\ X_1^4 < \infty$. Write $\beta_s = \mathbf{E}\,|X_1 - \mu|^s$, for $s \ge 1$, and

$$\rho_x = 1 - \sup\big\{|\,\mathbf{E}\,\exp\{i\,t\,(X_1 - \mu)\}|:\ \sigma^2/(9\beta_3) \le |t| \le x/\sigma\big\}, \qquad \rho = \rho_{\sqrt{N}}$$

Theorem 3. *There exists an absolute constant* $c > 0$ *such that, for every* $N = 2, 3, \dots,$

$$\Delta_N \le \frac{c}{\rho^2 N}\,\frac{\beta_4}{\sigma^4}, \tag{1.3}$$

whenever $\rho > 0$.

Note that the Cramer condition

$$\text{(C)}\qquad \limsup_{|t|\to\infty}|\,\mathbf{E}\,\exp\{itX_1\}| < 1$$

implies $\liminf_N \rho_{\sqrt{N}} > 0$. Therefore, under Cramer's (C) condition Theorem 3 implies $\Delta_N = O(N^{-1})$. This result was conjectured by Bentkus, Götze and van Zwet (1997). In their fundamental work Bentkus, Götze and van Zwet (1997) constructed a second order approximation to a general (nonlinear) symmetric statistic with the remainder $O(N^{-1})$. This general result was applied to a number of important statistics and established the validity of one term Edgeworth expansion with the remainder $O(N^{-1})$ under optimal conditions for each case considered with the sole exception of the Student stastistic. For this particular statistic the bound $\Delta_N = O(N^{-1})$ was obtained under Cramer's (C) condition provided that $\mathbf{E}\,|X_1|^{4+\varepsilon} < \infty$, for some $\varepsilon > 0$, see Bentkus, Götze and van Zwet (1997).

Remark. If $\mu = 0$, then the bounds (1.1) and (1.3) remain valid if we replace $F(x)$ by the distribution function $\tilde F(x) = \mathbf{P}\,\{T \le x\}$ of the selfnormalized sum

$$T = \frac{X_1 + \dots + X_N}{(X_1^2 + \dots + X_N^2)^{1/2}}.$$

2. Proofs

In proofs we apply and extend the approach developed in Bentkus and Götze (1996) and Bentkus, Götze and van Zwet (1997).

We may and shall assume that $\mathbf{E}\,X_1 = 0$ and $\sigma^2 = 1$.

In what follows c, c_1, c_2,... denote generic absolute constants. We write $c(a, b, ...)$ to denote a constant that depends only on the quantities $a, b, ...$. We shall write $A \ll B$ to denote the fact that $A \le c\,B$. Furthermore, $\exp\{itx\}$ is abbreviated by $\mathrm{e}\{x\}$.

Proof of Theorem 1. Theorem 1 is an easy consequence of Proposition 2.1, formulated below.

Proof of Theorem 2. Theorem follows from Proposition 2.1, see below, and the law of large numbers, see also Singh (1981).

Proposition 2.1. *There exists an absolute constant $c > 0$ such that for each $N = 2, 3, ...$ and each $1 < x < N^{1/6}$,*

$$\Delta_N \le c\Big(\frac{1}{x\sqrt{N}} + \frac{1}{\rho_x^2 N} + \frac{\beta_3^2}{\sigma^6 N} + \frac{\mathbf{E}\,X^4\mathbb{I}\{X^2 < \sigma^2\,N\}}{\sigma^4\rho_x^2 N} + \frac{\mathbf{E}\,|X|^3\mathbb{I}\{X^2 \ge \sigma^2\,N\}}{\sigma^3\sqrt{N}}\Big). \tag{2.1}$$

It remains to prove Proposition 2.1 and Theorem 3. Before the proof we introduce some more notation.

In what follows θ_1, θ_2, ... denote i.i.d. random variables which are uniformly distributed in $[0, 1]$. For a vector valued smooth function H, the Taylor expansion may be written as

$$H(x) = H(0) + H'(0)\,x + \cdots + H^{(n)}(0)\,\frac{x^n}{n!} + \mathbf{E}\,H^{(n+1)}(\theta_1\,x)\,(1-\theta_1)^n\,\frac{x^{n+1}}{n!}.$$

In particular, we have the mean value formula, $H(x) - H(0) = \mathbf{E}\,H'(\theta_1\,x)\,x$. Here $\mathbf{E}_{\,\theta_i}$ denotes the expectation taken with respect to the random variable θ_i. Given a function h write $\|h\| = \sup_x |h(x)|$.

Let $g : R \to R$ denote a function which is infinitely many times differentiable with bounded derivatives and such that

$$\frac{8}{9} \le g(x) \le \frac{8}{7}, \quad \text{for all} \quad x \in R, \quad \text{and} \quad g(x) = \frac{1}{\sqrt{x}}, \quad \text{for} \quad \frac{7}{8} \le x \le \frac{9}{8}.$$

Write $c_g = \|g\| + \|g'\| + \|g''\| + \|g'''\|$.

Let a denote the largest nonnegative solution of the equation

$$a^2 = \mathbf{E}\,X_1^2\,\mathbb{I}\,\{X_1^2 \le a^2 N\}.$$

For $1 \le i, j \le N$ and $1 \le k \le 4$, write $Y_i = a^{-1}N^{-1/2}X_i\mathbb{I}\{X_i^2 \le a^2N\}$ and

$$\begin{aligned}
&Y = Y_1 + \cdots + Y_N, \qquad \eta = \eta_1 + \cdots + \eta_N, \qquad D = d_1 + \cdots + d_N,\\
&\eta_i = Y_i^2 - \mathbf{E}\, Y_i^2, \qquad d_i = Y_i\,\eta_i, \qquad Q_{i,j} = Y_i^2 d_j,\\
&b_k = \mathbf{E}\,|Y_1|^k, \quad \mathcal{M} = N b_4, \quad \gamma = N|\,\mathbf{E}\, Y_1|, \quad \gamma_0 = N^{-1/2}\,\mathbf{E}\,|X_1|^3\mathbb{I}\{X_1^2 \ge N/2\}.
\end{aligned}$$

Note that $|Y_i| \le 1$. By Hölder's inequality, $\beta_3^2 \le \sigma^2\beta_4 = \beta_4$ and

$$b_3 \ge b_2^{3/2} = N^{-3/2}, \qquad \mathcal{M} \ge N\, b_2^2 = N^{-1}, \qquad (b_3N)^2 \le N^2 b_2 b_4 = \mathcal{M}. \tag{2.2}$$

If Q denotes the sum $q_1 + \cdots + q_k$ then write $Q^{(i,j)} = Q - q_i - q_j$. Similarly, $Q^{(i)} = Q - q_i$. Given a subset $A \subset \{1, \dots, k\}$ write $Q_A = \sum_{j\in A} q_j$. Given $A = \{i_1, \dots, i_m\} \subset \{1, \dots, N\}$ we write $\mathbf{E}_A$ or $\mathbf{E}_{i_1,\dots,i_m}$ to denote the conditional expectation given $\{X_j,\, j \notin A\}$.

Let $\xi, \xi_1, \xi_2, \dots$ be a sequence of independent standard normal r.v. We assume that this sequence is independent of $X_1, \dots, X_N$. By $\tilde{\xi}$ we denote the sum $\tilde{\xi}_1 + \cdots + \tilde{\xi}_N$, $\tilde{\xi}_i = N^{-1/2}\xi_i$.

Proof of Proposition 2.1. In the proof we use some ideas from Bentkus and Götze (1996). We prove the proposition in few steps. First we truncate the random variables $X_1, \dots, X_N$. Then replace $\mathbf{t}$ by a statistic which is a smooth function of the observations. Expanding this function in powers of observations we obtain the statistic S_0 given by (2.5), see below. Next, we apply Esseen's (1945) smoothing lemma to the distribution function $F_0(y) = \mathbf{P}\{S_0 \le y\}$ and then estimate the diffrence between the Fourier transforms $\hat{F}_0$ and $\hat{G}$ of F_0 and G respectively.

We may assume that for a sufficiently small $c_0 > 0$,

$$\beta_3/\sqrt{N} \le c_0, \qquad \mathcal{M} \le c_0, \qquad \rho_x^{-2}N^{-1} < c_0, \qquad \gamma_0 < c_0, \quad x^{-1}N^{-1/2<c_0}. \tag{2.3}$$

Indeed, if the first inequality fails, the bound (2.1) follows from the simple inequality $\Delta_N \ll 1 + \beta_3/\sqrt{N} \ll \beta_3^2/N$. Hence, without loss of generality we may assume that $\beta_3/\sqrt{N} \le c_0$. Then $\Delta_N \ll 1 + \beta_3/\sqrt{N} \ll 1$ and the inequality $\mathcal{M} \ge c_0$ implies $\Delta_N \ll \mathcal{M}$. We obtain (2.1) again. Finally, each of the inequalities $\rho_x^{-2}N^{-1} > c_0$ and $x^{-1}N^{-1/2} > c_0$ in combination with $\Delta \ll 1$ implies (2.1). Thus, we may and shall assume that (2.3) holds. Note that, by Lemma 4.2, $a > 3/4$. Therefore (2.3) implies

$$N b_3 \le (4/3)^3\beta_3 N^{-1/2} \le 3c_0, \qquad \gamma \le (4/3)\gamma_0. \tag{2.4}$$

Fix an integer $m = 4\rho_x^{-1}\ln N$. Write $A = \{1, \dots, m\}$ and $B = \{m+1, \dots, N\}$. Introduce the statistics

$$S = Y g(1+\eta-Y^2/N), \qquad S_0 = Y g(V_B) + \eta_A Y_B g'(V_B), \quad V_B = 1+\eta_B-Y_B^2/N. \tag{2.5}$$

By Lemma 4.3, see below, the probability $\mathbf{P}\{|S_0-\mathbf{t}|\geq N^{-2/3}\}$ is not greater than $c\mathcal{M}/\rho_x^2+c\gamma_0$. Then Slucky's argument gives

$$\Delta_N\leq\sup_x|F_0(x)-G(x)|+N^{-2/3}\max_x|G'(x)|+c\mathcal{R},\qquad \mathcal{R}=\mathcal{M}/\rho_x^2+\gamma_0.$$

But $\max_x|G'(x)|\leq c$, by (2.3). Hence, in order to prove (2.1) it remains to show

$$\sup_x|F_0(x)-G(x)|\ll\mathcal{R}+T^{-1},\qquad T=x\sqrt{N}.$$

We are going to apply Esseen's (1945) smoothing lemma. This lemma implies

$$\sup_x|F_0(x)-G(x)|\ll I(0,T)+T^{-1},\quad \text{where}\quad I(a,b):=\int_{a\leq|t|\leq b}\frac{|\hat{F}_0(t)-\hat{G}(t)|}{|t|}\,dt.$$

Write $I_1=I(0,100)$, $I_2=I(100,L)$ and $I_3=I(L,T)$, where $L^{-1}=4Nb_3$. It remains to estimate I_1, I_2 and I_3.

An application of Lemma 3.1 (with $H(u)=\mathrm{e}\{u\}$) gives $I_1\ll\mathcal{R}$.

Let us show $I_2\ll\rho_x^{-2}\mathcal{M}+N^{-2/3}$. We have $I_2\leq I_{2,1}+I_{2,2}$, where

$$I_{2,1}=\int_{100\leq|t|\leq L}\frac{|\hat{F}_0(t)-\mathbf{E}\,\mathrm{e}\{S\}|}{|t|}\,dt,\quad I_{2,2}=\int_{100\leq|t|\leq L}\frac{|\,\mathbf{E}\,\mathrm{e}\{S\}-\hat{G}(t)|}{|t|}\,dt.$$

Lemmas 3.1 and 3.2 imply $I_{2,2}\ll\mathcal{R}$. It remains to estimate $I_{2,1}$. By (4.9) and (4.10), see below,

$$|\hat{F}_0(t)-\mathbf{E}\,\mathrm{e}\{S\}|\leq|t|\,|S_0-S|\ll|t|\,\mathbf{E}\,R,\quad R=R_1+\cdots+R_5,$$

where the r.variables R_i are given by (4.9) and (4.10). A simple calculation shows $\mathbf{E}\,R\ll N^{-1/2}(\rho_x^{-2}\mathcal{M}+N^{-2/3})$. Finally,

$$I_{2,1}\ll L\,\mathbf{E}\,R\ll\sqrt{N}\,\mathbf{E}\,R\ll\rho_x^{-2}\mathcal{M}+N^{-2/3}.$$

It remains to show that $I_3\ll\mathcal{R}$. We have

$$I_3\leq I_4+I_5,\qquad I_4=\int_{L\leq|t|\leq T}|\hat{F}_0(t)|/|t|dt,\quad I_5=\int_{L\leq|t|\leq T}|\hat{G}(t)|/|t|dt.$$

By (2.3), $|\hat{G}(t)|\leq\exp\{-t^2/2\}(1+|t|^3)$. Therefore, $I_5\leq\exp\{-(cL)^2\}$. The inequality $\exp\{-u^2\}\leq u^{-1}$ (which holds for sufficiently large u) applied to $u=cL$ gives $I_5\ll(Nb_3)^2\ll\mathcal{M}$, see (2.2). Let us show

$$I_4\ll N^{-1}. \tag{2.6}$$

Write $\mathbb{I} = \mathbb{I}\{100\,|Y_B| \le N^{1/20}\}$. We have

$$|\hat{F}_0(t)| \le \mathbf{E}\,|\psi(t)| + \mathbf{P}\,\{|Y_B| > N^{1/20}\}, \qquad \psi(t) = \mathbf{E}_A\,\mathrm{e}\{S_0\}\mathbb{I}.$$

By Chebyshev's inequality, the probability is bounded by $\mathbf{E}\,|Y_B|^{20}/N^2 \ll N^{-2}$, see Lemma 4.1 below. Furthermore,

$$|\psi(t)| \le |\varkappa|^m, \qquad \varkappa = \mathbf{E}_1\,\mathrm{e}\{Y_1 g(V_B) + \eta_1 Y_B g'(V_B)\}.$$

We complete the proof of (2.6) by showing

$$|\varkappa| \le 1 - \rho/2. \tag{2.7}$$

Indeed, this inequality implies $|\varkappa|^m \le \exp\{-m\rho/2\} \le N^{-2}$, by the choice of m. Let us prove (2.7). Expanding in powers of $it\eta_1 Y_B g'(V_B)$ we get

$$\varkappa = \mathbf{E}_1\,\mathrm{e}\{Y_1 g(V_B)\} + R, \quad |R| \le 2T\,\mathbf{E}_1|\eta_1 Y_B \mathbb{I}| \le T N^{1/20-1} \le N^{-1/4} < \rho/4. \tag{2.8}$$

In the last step we used (2.3). Write $Z = N^{-1/2}a^{-1}X_1 g(V_B)$. We have

$$\big|\mathbf{E}_1\,\mathrm{e}\{Y_1 g(V_B)\} - \mathbf{E}_1\,\mathrm{e}\{Z\}\big| \le 2\,T\,\mathbf{E}\,|Y_1 g(V_B)\} - Z| \le 2a^{-3}\|g\|(T/N)\gamma_0. \tag{2.9}$$

By (2.3), the right-hand side is less than $\rho/4$. But $\mathbf{E}_1\,\mathrm{e}\{Z\} \le \rho$. Collecting (2.8) and (2.9) we obtain (2.7) thus completing the proof of the proposition.

Proof of Theorem 3. In the proof we use the smoothing technique "data depending smoothing" introduced in Bentkus, Götze and van Zwet (1997). We were not able to prove the bound $\Delta_N = O(N^{-1})$ under Cramer's (C) condition (the minimal smoothness condition) using the conventional Esseen's (1945) smoothing lemma.

We may assume that for a sufficiently small $c_0 > 0$,

$$\beta_3/\sqrt{N} \le c_0, \qquad \beta_4/N \le c_0, \qquad \rho_N^{-1}\ln N < c_0 N. \tag{2.10}$$

Indeed, if the first inequality fails, the bound (1.3) follows from the simple inequality $\Delta_N \ll 1+\beta_3/\sqrt{N} \ll \beta_3^2/N \le \beta_4/N$. Hence, without loss of generality we may assume that $\beta_3/\sqrt{N} \le c_0$. Then $\Delta_N \ll 1+\beta_3/\sqrt{N} \ll 1$ and the inequality $\beta_4/N \ge c_0$ implies $\Delta_N \ll \beta_4/N$. We obtain (1.3) again. Finally, if the last inequality of (2.10) fails we have $\rho_N^{-2}N^{-1} > c_0$ and this in combination with $\Delta \ll 1$ and $\beta_4 \ge 1$ implies (1.3). Thus, we may and shall assume that (2.10) holds. Note that, by Lemma 4.2, $a > 3/4$. Therefore (2.10) implies

$$\gamma \le (4/3)^4\beta_4/N, \quad \mathcal{M} \le (4/3)^4\beta_4/N \le 4c_0, \quad Nb_3 \le (4/3)^3\beta_3 N^{-1/2} \le 3c_0. \tag{2.11}$$

Let m be the smallest integer greater than $18\rho^{-1}\ln N$. By (2.10), $m < N$. Put $A = \{1,\dots,m\}$ and $B = \{m+1,\dots,N\}$ and split $Y = Y_A + Y_B$ and $\eta = \eta_A + \eta_B$. Write

$$S_1 = Yg(V_B) + Y\eta_A g'(V_B) + 2^{-1}Y_B\eta_A^2 g''(V_B), \qquad V_B = 1 + \eta_B - Y_B^2/N, \tag{2.12}$$

and let $F_1(x) = \mathbf{P}\{S_1 \le x\}$ denote the distribution function of S_1. By Lemma 4.3, see below, the probability $\mathbf{P}\{|S_1 - \mathbf{t}| \ge \mathcal{M}\}$ is not greater than $c\rho^{-2}\beta_4/N$. Then Slucky's argument gives

$$\Delta_N \le \sup_x |F_1(x) - G(x)| + c\mathcal{M}\max_x |G'(x)| + c\rho^{-2}\beta_4/N.$$

But $\max_x |G'(x)| \le c$, by (2.10). Hence in order to prove (1.3) it remains to show that

$$\sup_x |F_1(x) - G(x)| \ll \rho^{-2}\beta_4/N. \tag{2.13}$$

We are going to apply the data depending smoothing. This smoothing procedure was introduced by Bentkus, Götze and van Zwet (1997). To make the proof shorter we shall refer to Lemma 5.1 of Bentkus, Götze and van Zwet (1997). An inspection of the proof of this lemma shows that Pravitz's (1972) smoothing lemma applied to the conditional distribution function of S_1 given $Y_{m+1},\dots,Y_N$, yields

$$\sup_x |F_1(x) - G(x)| \le c\,\big(\mathbf{E}\,\mathcal{J}_1 + |\,\mathbf{E}\,\mathcal{J}_2| + \mathcal{J}_3 + \mathbf{E}\,\mathcal{J}_4 + \mathcal{J}_5\big),$$

$$\mathcal{J}_1 = \int_{H_1\le|t|\le H} \frac{|\hat F_1^*(t)|}{|t|}\,dt, \quad \mathcal{J}_2 = \int_{|t|\le H_1} \frac{\mathrm{e}\{-x\}\hat F_1^*(t)}{H}\,dt, \quad \mathcal{J}_3 = \int_{|t|\ge H_1} \frac{|\hat G(t)|}{|t|}\,dt,$$
$$\mathcal{J}_4 = \int_{|t|\le H_1} \frac{|t|}{H^2}\,dt, \qquad \mathcal{J}_5 = \int_{|t|\le H_1} \frac{|\hat F_1(t) - \hat G(t)|}{|t|}\,dt.$$

Here

$$\hat G(t) = \exp\{-t^2/2\} - \frac{\kappa_3}{6\sqrt{N}}\big(2(it)^3 + 3it\big)\exp\{-t^2/2\} \qquad \text{and} \qquad \hat F_1(t) = \mathbf{E}\,\mathrm{e}\{S_1\}$$

denote the Fourier transforms of $G(x)$ and $F_1(x)$ respectively,

$$\hat F_1^*(t) = \mathbf{E}_A\,\mathrm{e}\{S_1\} \quad \text{and} \quad H_1 = 1/(4N\,b_3), \quad H = \rho N/\big(16\beta_4^{1/4}c_g(1+\Theta_1+\Theta_2)\big),$$

where $\Theta_1 = |Y_{m+1} + \dots + Y_k|$ and $\Theta_2 = |Y_{k+1} + \dots + Y_N|$ and $k \approx m + (N-m)/2$.

It remains to estimate $\mathbf{E}\,\mathcal{J}_k$, for $k = 1,2,\dots,5$.

Estimation of $\mathcal{J}_3$ and $\mathbf{E}\,\mathcal{J}_4$. It follows from the inequality $3/4 \le a \le 1$ that

$$H_1 \ge N^{1/2}/(\mathbf{E}\,|X_1|^3 \mathbb{I}\{X_1^2 \le a^2 N\}) \ge y, \qquad y := N^{1/2}/\beta_3.$$

Furthermore, by (2.10), $|\hat{G}(t)| \le \exp\{-t^2/2\}(1+|t|^3)$ and, therefore, $\mathcal{J}_3 \le \exp\{-cy^2\}$. Note that y is large, by (2.10). The inequality $\exp\{-u\} < u^{-1}$ (which holds for sufficiently large u) applied to $u = cy^2$ implies $\mathcal{J}_3 \ll \beta_3^2/N \le \beta_4/N$.

Consider $\mathcal{J}_4$. By (2.2),

$$H_1 \le 4^{-1}/(N b_2^{3/2}) = 4^{-1} N^{1/2}. \tag{2.14}$$

Therefore, $\mathbf{E}\,\mathcal{J}_4 = H_1^2\,\mathbf{E}\,H^{-2} \ll \rho^{-2} N^{-1}$.

Estimation of $\mathbf{E}\,\mathcal{J}_1$. We shall show that

$$|\,\mathbf{E}\,\mathcal{J}_1| \ll \rho^{-2}\mathcal{M}. \tag{2.15}$$

Write $Y_A\eta_A = D + U$, where $D = d_1 + \cdots + d_m$, and $\eta_A^2 = \tilde{D} + \tilde{U}$, where $\tilde{D} = \eta_1^2 + \cdots + \eta_m^2$. Denote $L = L_1 + \cdots + L_m$, where $L_i = l_{i,1} + l_{i,2} + l_{i,3} + l_{i,4}$,

$$l_{i,1} = Y_i g(V_B), \quad l_{i,2} = \eta_i Y_B g'(V_B), \quad l_{i,3} = d_i g'(V_B), \quad l_{i,4} = \eta_i^2 Y_B g''(V_B).$$

Furthermore, write $Z = L + Ug'(V_B) + W$, where $W = Y_B g(V_B)$. Then $S_1 = Z + X$, where $X = \tilde{U} Y_B g''(V_B)$. Expanding $\mathrm{e}\{Z + X\}$ in powers of itX we obtain

$$|\,\mathbf{E}\,\mathcal{J}_1| \le \mathbf{E}\,I_1 + \mathbf{E}\,I_2 + R, \qquad I_1 = \int_{H_1 \le |t| \le H} \frac{|\,\mathbf{E}_A\,\mathrm{e}\{Z\}|}{|t|}\,dt,$$
$$I_2 = \int_{H_1 \le |t| \le H} |\,\mathbf{E}_A\,\mathrm{e}\{Z\} X|\,dt, \qquad R = 2^{-1}\,\mathbf{E}\,X^2 \int_{H_1 \le |t| \le H} |t|\,dt.$$

A simple calculation shows $\mathbf{E}_A \tilde{U}^2 \ll m^2 b_4^2$. Hence,

$$R \ll \rho^{-2} N^2 \beta_4^{-1/2} m^2 b_4^2 \ll \mathcal{M}^{3/2} \ll \mathcal{M}.$$

Let us show $\mathbf{E}\,I_2 \ll \rho^{-2}\mathcal{M}$. Expanding $\mathrm{e}\{Z\} = \mathrm{e}\{L + W + Ug'(V_B)\}$ in powers of $itUg'(V_B)$ we get $\mathbf{E}\,I_2 \le \mathbf{E}\,I_3 + R$, where

$$I_3 = \int_{H_1 \le |t| \le H} |\,\mathbf{E}_A\,\mathrm{e}\{L + W\} X|\,dt, \qquad R = \mathbf{E}\,|XUg'(V_B)| \int_{H_1 \le |t| \le H} |t|\,dt.$$

We have $R \ll \rho^2 N^2 \beta_4^{-1/2}\,\mathbf{E}\,|XU| \ll \mathcal{M}$. Here we estimated $\mathbf{E}\,|XU| \ll \mathbf{E}\,|\tilde{U}U| \ll m^2 N^{-2} \mathcal{M}^{3/2}$. To prove the last inequality we combine the bounds

$$\mathbf{E}\,\tilde{U}^2 \ll m^2 b_4^2, \qquad \mathbf{E}\,U^2 \ll m^2 b_4/N, \tag{2.16}$$

see Lemma 4.1, and the inequality $ab \le a^2c + b^2c^{-1}$ applied to $a = |\tilde{U}|$, $b = |U|$ and $C = \mathcal{M}^{-1/2}$.

Let us show $\mathbf{E}\, I_3 \ll N^{-1}$. By the symmetry,

$$\mathbf{E}_A\, \mathrm{e}\{L\}X = m(m-1)Y_B g''(V_B)\, \mathbf{E}_A\, \mathrm{e}\{L_2 + L_3\}\eta_2\eta_3 h^{m-2}, \quad h = \mathbf{E}_1\, \mathrm{e}\{L_1\}.$$

We shall prove that

$$|h| \le 1 - \rho/2. \tag{2.17}$$

Then $|\,\mathbf{E}_A\, \mathrm{e}\{L + W\}X| \ll c_g m^2 |Y_B| N^{-2}(1-\rho/2)^{m/2}$ and estimating

$$(1-\rho/2)^{m-2} \le \exp\{-\rho(m-2)/2\} \le e^2 \exp\{-\rho m/2\} \le e^2 N^{-3}$$

we obtain $\mathbf{E}\, I_3 \ll N^{-1}$.

Let us prove (2.17). Expanding the exponent in powers of $it(l_{1,2} + l_{1,3} + l_{1,4})$ we get

$$|h - \mathbf{E}_1\, \mathrm{e}\{l_{1,1}\}| \ll c_g H\big(\mathbf{E}_1(|\eta_1| + \eta_1^2)|Y_B| + \mathbf{E}\,|d_1|\big) \le \rho/4. \tag{2.18}$$

In the last step we estimated $\mathbf{E}_1|\eta_1| \le 2/N$ and

$$\mathbf{E}_1\eta_1^2 \le b_4 = N^{-1}\mathcal{M} \le N^{-1}, \qquad \mathbf{E}\,|d_1| \le (\,\mathbf{E}\, Y_1^2)^{1/2}(\,\mathbf{E}\,\eta_1^2)^{1/2} \le N^{-1},$$

see (2.11). In (2.19) below we show that $\mathbf{E}\,|\,\mathrm{e}\{l_{1,1}\} - \mathrm{e}\{z_1\}| \le \rho/4$, where $z_1 = a^{-1}N^{-1/2}X_1 g(V_B)$. Furthermore, the inequalities $3/4 \le a \le 1$ and $8/9 \le |g| \le 8/7$ imply $|\,\mathbf{E}_1\, \mathrm{e}\{z_1\}| \le 1 - \rho$, for $H_1 \le |t| \le H$ and invoking (2.18) we obtain (2.17). For $|t| \le H$, we have

$$\begin{aligned} \big|\,\mathbf{E}_1\, \mathrm{e}\{l_{1,1}\} - \mathbf{E}_1\, \mathrm{e}\{z_1\}\big| &\le a^{-1}N^{-1/2}|tg(V_B)|\, \mathbf{E}_1|X_1|\mathbb{I}\{X_1^2 > a^2N\} \\ &\le (3/2)a^{-4}N^{-2}H\beta_4 \le (3/2)(4/3)^4 16^{-1}\rho\beta_4/N \le (c_0/3)\rho. \end{aligned} \tag{2.19}$$

Recall that c_0 is (sufficiently) small absolute constant.

Let us show that $\mathbf{E}\, I_1 \ll \beta_4/N$. Split $A = A_1 \cup A_2 \cup A_3$, so that $A_p \cap A_q = \emptyset$, for $p \ne q$, and $|A_p| \approx m/3$, for every p. Write

$$U_{p,q} = \sum_{i \in A_p,\, j \in A_q,\, i \ne j} y_i \eta_j g'(V_B), \qquad U^* = \sum_{1 \le p,q \le 3,\, p \ne q} U_{p,q}.$$

Then $Ug'(V_B) = U_{1,1} + U_{2,2} + U_{3,3} + U^*$ and $\mathrm{e}\{Z\} = \mathrm{e}\{W + L + U^*\}g_1g_2g_3$, where we denote $g_p = \mathrm{e}\{U_{p,p}\}$. By the mean value theorem, $g_p = 1 + \varkappa_p$, where $\varkappa_p = i\,t\,U_{p,p}\,\mathbf{E}_\theta\, \mathrm{e}\{\theta U_{p,p}\}$. Write

$$a_1 = 1, \quad a_2 = -g_1, \quad a_3 = -g_2, \quad a_4 = -g_3, \quad a_5 = g_1g_2, \quad a_6 = g_1g_3, \quad a_7 = g_2g_3.$$

The identity $g_1 g_2 g_3 = (a_1 + \cdots + a_7) + \varkappa_1\varkappa_2\varkappa_3$ implies

$$\mathbf{E}\, I_1 \le (\,\mathbf{E}\, I_{1,1} + \cdots + \,\mathbf{E}\, I_{1,7}) + R, \qquad I_{1,i} = \int_{H_1 \le |t| \le H} \frac{|\,\mathbf{E}_{\,A}\, \mathrm{e}\{L + W + U^*\} a_i|}{|t|}\, dt,$$

$$R \le \,\mathbf{E} \int_{H_1 \le |t| \le H} \frac{|\varkappa_1\varkappa_2\varkappa_3|}{|t|} \le \big(N\rho\beta_4^{-1/4}\big)^3\, \mathbf{E}\, |U_{1,1}U_{2,2}U_{3,3}| \ll \beta_4/N.$$

In the last step we used the inequalities

$$\mathbf{E}_{\,A} \prod_{1\le p\le 3} |U_{p,p}| = \prod_{1\le p\le 3} \mathbf{E}_{\,A}|U_{p,p}| \le \prod_{1\le p\le 3} (\,\mathbf{E}_{\,A} U_{p,p}^2)^{1/2}, \qquad \mathbf{E}_{\,A} U_{p,p}^2 \le c_g m^2 b_4/N,$$

cf. (2.16). In order to complete the proof of (I.1) it remains to show $|\,\mathbf{E}\, I_{3,i}| \ll \mathcal{M}$, for $i = 1, 2, \ldots, 7$. We shall prove that $|\,\mathbf{E}\, I_{3,7}| \ll \rho^{-2}\mathcal{M}$ since the proof for the rest $i = 1, \ldots, 6$ is almost the same or simpler. Write

$$\eta^* = \eta_A - \eta_{A_1}, \qquad Y^* = Y_A - Y_{A_1}, \qquad U_{A_1} = Y_{A_1}\eta^* + \eta_{A_1} Y^*.$$

We have

$$|\,\mathbf{E}_{\,A}\, \mathrm{e}\{L + W + U^*\} a_7| \le \,\mathbf{E}_{\,A}|\,\mathbf{E}_{\,A_1} \psi|, \qquad \psi = \mathrm{e}\{L_{A_1} + U_{A_1} g'(V_B)\}. \tag{2.20}$$

Write $\mathbb{I}_\eta = \mathbb{I}\{100 c_g |\eta^*| < 1\}$ and $\mathbb{I}_Y = \mathbb{I}\{100 c_g |Y^*| < 1\}$. Then

$$\mathbf{E}_{\,A}|\,\mathbf{E}_{\,A_1}\psi| = \,\mathbf{E}_{\,A}|\,\mathbf{E}_{\,A_1}\psi \mathbb{I}_\eta \mathbb{I}_y| + R, \qquad R \le \,\mathbf{E}_{\,A}|1 - \mathbb{I}_\eta| + \,\mathbf{E}_{\,A}|1 - \mathbb{I}_Y|. \tag{2.21}$$

By Chebyshev's inequality, $R \ll \,\mathbf{E}\,(\eta^*)^2 + \,\mathbf{E}\,(Y^*)^4 \ll m^2 b_4$. Therefore,

$$\mathbf{E} \int_{H_1 \le |t| \le H} \frac{R}{|t|}\, dt \ll m^2 b_4 \ln N \ll \rho^{-2}\mathcal{M}. \tag{2.22}$$

It remains to estimate $\mathbf{E}_{\,A_1}\psi\mathbb{I}_\eta\mathbb{I}_Y$. Write Write $m_1 = |A_1|$. By the symmetry

$$\mathbf{E}_{\,A_1}\psi\mathbb{I}_\eta\mathbb{I}_Y = h_1^{m_1}, \qquad h_1 = \,\mathbf{E}_{\,1}\, \mathrm{e}\{L_1 + (Y_1\eta^* + \eta_1 Y^*) g'(V_B)\}\mathbb{I}_\eta\mathbb{I}_Y.$$

We shall show that

$$|h_1| \le 1 - \rho/2. \tag{2.23}$$

Then

$$\big|\,\mathbf{E}_{\,A_1}\psi\mathbb{I}_\eta\mathbb{I}_Y\big| \le \exp\{-m_1\rho/2\} \ll \exp\{-\rho m/6\} \le N^{-3}$$

and this inequality in combination with (2.20), (2.21) and (2.22) gives $\mathbf{E}\, I_{3,7} \ll \rho^{-2}\mathcal{M}$. It remains to prove (2.23). Proceeding as in (2.18) we obtain

$$|h_1 - h_2| \le \rho/4, \qquad \text{where} \quad h_2 = \mathbf{E}_1\, \mathrm{e}\{l_{1,1} + (Y_1\eta^* + \eta_1 Y^*)g'(V_B)\}\mathbb{I}_\eta \mathbb{I}_Y.$$

Furthermore, expanding the exponent in h_2 in powers of $it\eta_1 Y^*$ we obtain

$$|h_2 - h_3| \le c_g H\, \mathbf{E}_1 |\eta_1 Y^*| \mathbb{I}_Y \le \rho/100, \quad h_3 = \mathbf{E}_1\, \mathrm{e}\{l_{1,1} + Y_1\eta^* g'(V_B)\}\mathbb{I}_\eta \mathbb{I}_Y.$$

We complete the proof of (2.23) by showing that $|h_3| \le 1 - (4/5)\rho$. We have

$$\big|\, \mathbf{E}_1\, \mathrm{e}\{z_2\}\mathbb{I}_\eta \big| \le 1 - \rho, \quad z_2 = a^{-1}N^{-1}X_1\big(g(V_B) + \eta^* g'(V_B)\big).$$

But $|h_3 - \mathbf{E}_1\, \mathrm{e}\{z_2\}| \le \rho/5$, cf. (2.19). Hence, $|h_3| \le 1 - (4/5)\rho$ and this completes the proof of (2.15).

Estimation of $\mathbf{E}\,\mathcal{J}_5$. Write

$$\varphi(t) = \exp\{-t^2/2\} - (N/6)\,\mathbf{E}\, Y_1^3\big(3it + 2(it)^3\big)\exp\{-t^2/2\}$$
$$f(t) = \mathbf{E}\ \mathrm{e}\{S\}, \qquad S = Yg(V), \qquad V = 1 + \eta - Y^2/N.$$

We have

$$\mathbf{E}\,\mathcal{J}_5 \le I_1 + I_2 + I_3, \qquad I_k = \int_{|t|\le H_1} \frac{|\delta_k(t)|}{|t|}\, dt,$$
$$\delta_1(t) = f(t) - \varphi(t), \qquad \delta_2(t) = \hat{F}_1(t) - f(t), \qquad \delta_3(t) = \varphi(t) - \hat{G}(t).$$

The inequality $I_1 \ll \beta_4/N$ is proved in Lemmas 3.1 and 3.2. Furthermore, the bound $I_3 \ll \beta_4/N$ is a consequence of the inequalities $a^{-3}N^{-3/2}\,\mathbf{E}\,|X_1|^3 \ll N^{-3/2}\beta_3 \ll N^{-1}$, see (2.10), and

$$\mathbf{E}\,\big(|Y_1|^3 - a^{-3}N^{-3/2}|X_1|^3\big) = a^{-3}N^{-3/2}\,\mathbf{E}\,|X_1|^3\mathbb{I}\{X_1^2 > a^2N\} \ll N^{-2}\beta_4.$$

Let us show that $I_2 \ll \rho^{-2}\mathcal{M}$. Write

$$W_1 = -Y_A^2/N, \qquad W_2 = -2\,Y_AY_B/N. \tag{2.24}$$

Then $V = V_B + \eta_A + W_1 + W_2$. Expanding g in powers of $W_1 + W_2$ and then in powers of η_A we obtain $S = S_1 + (Q_1 + \cdots + Q_4)$, where

$$Q_1 := 2^{-1}Y_A\eta_A^2\,\mathbf{E}_\theta g''(V_B + \theta\eta_A), \qquad Q_2 := 6^{-1}Y_B\eta_A^3\,\mathbf{E}_\theta g'''(V_B + \theta\eta_A),$$
$$Q_3 := Q_{3.1} + Q_{3.2}, \quad Q_{3.i} := Y_AW_i\zeta \quad Q_4 := Q_{4.1} + Q_{4.2}, \quad Q_{4.i} := Y_BW_i\zeta, \quad i = 1, 2,$$

where $\zeta = \mathbf{E}_\theta g'(V_B + \eta_A + \theta(W_1 + W_2))$. The inequality

$$|\operatorname{e}\{x\} - 1| \le 2|x|^\alpha, \qquad \text{for} \quad x \in R, \quad 0 \le \alpha \le 1, \tag{2.25}$$

implies

$$|f(t) - \mathbf{E}\operatorname{e}\{S_1 + Q_4\}| \ll R_1, \qquad R_1 = \mathbf{E}\left(|tQ_1|^{4/5} + \left|t(Q_2 + Q_{3.1})\right|^{2/3} + |tQ_{3.2}|\right).$$

A simple calculation shows that

$$R_1 \le |t|^{4/5} m^2 b_4 + |t|^{2/3}\bigl(m b_4 + m^2 N^{-5/3}\bigr) + |t|\, m\, N^{-2}.$$

Write $h = \mathbf{E}\operatorname{e}\{S_1\} itQ_{4.2}$. Expanding $\operatorname{e}\{Q_4\}$ in powers of $itQ_{4.1}$ and then in powers of $itQ_{4.2}$ we get

$$\mathbf{E}\operatorname{e}\{S_1 + Q_4\} = \hat{F}_1(t) + h + R_2, \qquad \text{where} \quad |R_2| \ll \mathbf{E}\,|tQ_{4.1}| + \mathbf{E}\,(tQ_{4.2})^2.$$

A calculation shows that $|R_2| \le m|t|N^{-2} + mt^2N^{-3}$. It remains to estimate $|h|$. Write $S_2 = Yg(V_B)$ and use (2.25) to get

$$|\operatorname{e}\{S_1\} - \operatorname{e}\{S_2\}| \ll |tY_A\eta_A| + |tY_B\eta_A^2|^{3/5}.$$

Then, for $h_1 = \mathbf{E}\operatorname{e}\{S_2\} itQ_{4.2}$, we have $|h - h_1| \ll R_3$ where

$$R_3 = \mathbf{E}\,|Q_{4.2}|\bigl(t^2|Y\eta_A| + |t|^{7/4}|Y_B\eta_A^2|^{3/4}\bigr) \le m^2\mathcal{M}(t^2N^{-3/2} + |t|^{7/4}N^{-2}).$$

To estimate h_1 write

$$h_1 = h_{1,1} + h_{1,2}, \quad h_{1,i} = (-2it/N)\,\mathbf{E}\,Y_{A_i}Y_B^2\operatorname{e}\{S_2\},$$

where $A_1 = \{1, \dots, k\}$ and $A_2 = A \setminus A_1$ and where $k \approx m/2$. We shall show that, for $i = 1, 2$,

$$|h_{1,i}| \ll R_4, \qquad R_4 = (m/N)^{1/2}|t|\exp\{-c(m/N)t^2\}, \quad \text{for} \quad |t| < H_1. \tag{2.26}$$

Then $|f(t) - \hat{F}_1(t)| \ll R_1 + R_2 + R_3 + 2\,R_4$. Collecting estimates for R_i we get

$$I_2 \ll \int_{0 \le |t| \le H_1} \frac{R_1 + R_2 + R_3 + 2\,R_4}{|t|}\, dt \ll \rho^{-2}\mathcal{M}.$$

It remains to prove (2.26). Let us show $|H_{1,1}| \ll R_4$. We have

$$|h_{1,1}| \le (2/N)|t|\ \mathbf{E}\,Y_B^2\zeta_1\zeta_2, \qquad \zeta_1 = \mathbf{E}_{A_1}|Y_{A_1}|, \quad \zeta_2 = \bigl|\mathbf{E}_{A_2}\operatorname{e}\{S_2\}\bigr|.$$

By Hölder's inequality, $\zeta_1^2 \le k/N < m/N$. Furthermore, by the symmetry, $\zeta_2 \le |u|^n$, where $u = \mathbf{E}_m\, \mathrm{e}\{Y_m g(V_B)\}$ and $n := |A_2|$. The inequality

$$|\,\mathbf{E}\, \mathrm{e}\{W\}|^2 \le 1 - t^2\, \mathbf{E}\,(W - \mathbf{E}\, W)^2 + (4/3)|t|^3\, \mathbf{E}\, |W|^3,$$

see, e.g., Petrov (1987), applied to the random variable $Y_m g(V_B)$ conditionally given $Y_{m+1}, \ldots, Y_N$, implies

$$|u|^2 \le 1 - t^2 g^2(V_B)\Big(\frac{1}{N} - (\,\mathbf{E}\, Y_m)^2\Big) + \frac{4}{3}\,|t|^3 b_3 |g(V_B)|^3 \le 1 - \frac{t^2}{4N}, \tag{2.27}$$

for $|t| \le H_1$, provided that c_0 is sufficiently small. Here we estimated $|\,\mathbf{E}\, Y_m| \le a^{-4}N^{-2}\beta_4 \le 4c_0/N$, see (2.10), and used the inequalities $8/9 \le g \le 8/7$. We obtain

$$\zeta_2 \le (1 - t^2/(4N))^{n/2} \le \exp\{-c(m/N)t^2\}$$

thus completing the proof of the bound $|h_{1,1}| \ll R_4$. Clearly, the same bound holds for $h_{1,2}$ as well. We arrive to (2.26).

Estimation of $\mathbf{E}\, \mathcal{J}_2$. We shall show that $|\,\mathbf{E}\, \mathcal{J}_2| \ll \rho^{-1}\beta_4/N$. Write $D = \{1, \ldots, r\}$ and $\Theta = |Y_{r+1} + \cdots + Y_N|$. It suffices to show that for any $r \ge N/2$,

$$\mathbf{E}\, I_1 \le c\rho^{-1}\beta_4/N, \qquad \text{where} \quad I_1 = \frac{\beta_4^{1/4}}{\rho N} \int_{20}^{H_1} |\,\mathbf{E}_D\, \mathrm{e}\{S_1\}|(1+\Theta)dt. \tag{2.28}$$

Let us prove this inequality. For this purpose we first replace S_1 by S. It follows from (4.11), see below, and (2.25) that

$$|\,\mathrm{e}\{S\} - \mathrm{e}\{S_1\}| \ll |t|(R_1 + \cdots + R_4) + |t|^{2/5}R_5^{2/5} + |t|^{1/3}R_6^{1/3} =: L_1,$$

where the random variables R_i are defined in the proof of Lemma 4.3, see below. A simple calculation gives

$$\begin{aligned}\mathbf{E}\, L_1(1+\Theta) &\ll |t|N^{-1}\,\mathbf{E}\,\big(|Y_A|^3 + Y_A^2 + |Y_A|\big) + |t|^{2/5}\,\mathbf{E}\,(Y_A^2 + |\eta_A|) + |t|^{1/3}\,\mathbf{E}\,|\eta_A| \\ &\ll |t|N^{-1}\big(\beta_4^{1/2}(m/N)^{3/2} + (m/N)^{1/2}\big) + \big(|t|^{2/5} + |t|^{1/3}\big)(m/N).\end{aligned}$$

Furthermore, using the inequality $H_1 \le cN^{1/2}$ we obtain

$$\frac{\beta_4^{1/4}}{\rho N}\int_{20}^{H_1} \mathbf{E}\, L_1(1+\Theta)dt \ll \frac{\beta_4^{3/4} + \beta_4^{1/4}}{\rho N} \le \rho^{-1}\beta_4/N.$$

It remains to show that

$$\frac{\beta_4^{1/4}}{\rho N}\int_{20}^{H_1} |\,\mathbf{E}_D\, \mathrm{e}\{S\}|(1+\Theta)dt \ll \rho^{-1}\beta_4/N.$$

Let m_0 be the smallest integer greater than $20\,N\,t^{-2}\ln|t|$. Note taht (2.10) implies that the number N is sufficiently large. Therefore we may and shall assume that $m_0 = m_0(t) \le N/2$, for $20 \le |t| \le H_1$. Write

$$A = \{1, \dots, m\}, \quad B = \{m+1, \dots, N\}, \quad \text{and} \quad V_B = 1 + \eta_B - Y_B^2/N,$$

Then $V = V_B + \eta_A - \tilde W$, where $\tilde W = Y_A^2/N + 2 Y_A Y_B/N$. Expanding g in powers of $\tilde W$ and then in powers of η_A we get

$$S = S_3 + r_1 + r_2 + r_3 + r_4, \qquad S_3 = Y_A g(V_B) + \eta_A Y_B g'(V_B)$$

where

$$|r_1| \ll |Y \tilde W|, \qquad |r_2| \ll |Y_A \eta_A|, \qquad |r_3| \ll |Y_A| \eta_A^2, \qquad |r_4| \ll |Y_B| \eta_A^2.$$

An application of (2.25) gives

$$|\,\mathrm{e}\{S\} - \mathrm{e}\{S_3\}| \ll |tY\tilde W| + |tY_A\eta_A|^{2/3} + |tY_B\eta_A^2|^{1/2} + |tY_A\eta_A^2|^{2/5}.$$

Finally, combining the inequalities

$$\begin{aligned}
&\mathbf{E}\,|Y\tilde W|(1+\Theta) \ll N^{-1}, \quad \mathbf{E}\,|Y_B\eta_A^2|^{1/2}(1+\Theta) \ll m/N \ll t^{-2}\ln|t|,\\
&\mathbf{E}\,|Y_A\eta_A|^{2/3}(1+\Theta) \ll \mathbf{E}\,(Y_A^2 + |\eta_A|) \ll m/N \ll t^{-2}\ln|t|,\\
&\mathbf{E}\,|Y_A\eta_A|^{2/5}(1+\Theta) \ll \mathbf{E}\,(Y_A^2 + |\eta_A|) \ll m/N \ll t^{-2}\ln|t|,
\end{aligned}$$

we obtain

$$\frac{\beta_4^{1/4}}{\rho N}\int_{20}^{H_1} \mathbf{E}\,|\,\mathrm{e}\{S\} - \mathrm{e}\{S_3\}|(1+\Theta)\,dt \ll \rho^{-1}\beta_4/N.$$

We complete the proof of (2.28) by showing

$$|\,\mathbf{E}_A\,\mathrm{e}\{S_3\}| \ll |t|^{-2} + c\,c_g N^{-3/4}\beta_4^{3/4}|Y_B|^{3/2}. \tag{2.29}$$

Let us prove (2.29). Write $\mathbb{I}_B = \mathbb{I}\{100 c_g N Y_B^2 \beta_4 < 1\}$. We have

$$|\,\mathbf{E}_A\,\mathrm{e}\{S_3\}| \le \zeta + (1 - \mathbb{I}_B), \qquad \zeta = |\,\mathbf{E}_A\,\mathrm{e}\{S_3\}\mathbb{I}_B|.$$

But

$$1 - \mathbb{I}_B \ll c_g N^{-3/4}\beta_4^{3/4}|Y_B|^{3/2}.$$

Therefore, it suffices to estimate ζ. By the symmetry

$$\zeta \le |v\mathbb{I}_B|^{m_0}, \qquad v = \mathbf{E}_1\,\mathrm{e}\{Y_1 g(V_B) + \eta_1 Y_B g'(V_B)\}.$$

Expanding the exponent in powers of $it\eta_1 Y_B g'(V_B)$ we obtain

$$v = u + R, \qquad u = \mathbf{E}_1 \mathrm{e}\{Y_1 g(V_B)\} \quad \text{with} \quad |R| \le t^2\, \mathbf{E}_1 \eta_1^2 Y_B^2 g'(V_B) \le t^2 b_4 Y_B^2 c_g.$$

Note that $|R|\mathbb{I}_B \le t^2 N^{-1}/100$. Then

$$|v\mathbb{I}_B| \le |u| + |R|\mathbb{I}_B \le 1 - \frac{t^2}{8N} + \frac{t^2}{100N} \le 1 - \frac{t^2}{9N}. \tag{2.30}$$

Here we used the inequality $|u| \le 1 - t^2/(8N)$ which follows from the inequality (2.27), for $|t| \le H_1 \le 4^{-1} N^{1/2}$, see (2.14). It follows from (2.30) that

$$\zeta \le \bigl(1 - t^2/(9N)\bigr)^{m_0} \le \exp\{-t^2 m_0/(9N)\} \le |t|^{-20/9} < t^{-2}.$$

We arrive to (2.29) thus completing the proof of (2.28).

Collecting the estimates of $\mathbf{E}\,\mathcal{J}_k$, for $k = 1, \dots, 5$ we arrive to (2.13) thus completing the proof of the theorem.

3. Expansions

Throughout this section we assume that at least one set of inequalities (2.3) or (2.10) hold.

Lemma 3.1. *Let $H : \mathbf{R} \to \mathbb{C}$ be a bounded infinitely many times differentiable function with bounded derivatives. Then*

$$\bigl|\mathbf{E}\ H(S) - \mathbf{E}\ H(\xi) - \Gamma\bigr| \ll c_H \mathcal{R}, \qquad \Gamma = -\frac{N}{6} b_3\bigl(3\,\mathbf{E}\,H'(\xi) + 2\,\mathbf{E}\,H'''(\xi)\bigr),$$

where $\mathcal{R} = \mathcal{M} + \gamma$ and $c_H = \|H'\| + \cdots + \|H^{vi}\|$.

Proof of Lemma 3.1. We shall write $g \sim h$ if $|g - h| \ll c_H \mathcal{R}$. A simple calculation shows that $\Gamma \sim \Gamma_1 + \Gamma_2 + \Gamma_3$,

$$\Gamma_1 = \frac{N}{6}\ \mathbf{E}\,H'''(\xi) Y_1^3, \quad \Gamma_2 = -\frac{N}{2}\ \mathbf{E}\,H'(\xi) d_1, \quad \Gamma_3 = -\frac{N(N-1)}{2}\ \mathbf{E}\,H'''(\xi) Y_1^2 d_2.$$

Therefore, the lemma is a consequence of the following two facts,

$$\mathbf{E}\,H(Y) \sim \mathbf{E}\,H(\xi) + \Gamma_1, \qquad\qquad \mathbf{E}\,H(S) \sim \mathbf{E}\,H(Y) + \Gamma_2 + \Gamma_3. \tag{3.1}$$

For the proof of the first part of (3.1) we refer to Bentkus, Götze, Paulauskas and Račkauskas (1990), where the inequality $|\,\mathbf{E}\,H(Y_1 + \cdots + Y_N) - \mathbf{E}\,H(\xi)| \ll c_H(\mathcal{M} + (b_3 N)^2)$ was proved in the case of centered summands Y_i. It remains to prove the second part of (3.1).

Expanding in powers of Y^2/N and using the bound $\mathbf{E}\,|Y|^3 \ll 1$, see Lemma 4.1, we get $\mathbf{E}\ H(S) \sim h_1$, where $h_1 = \mathbf{E}\ H(Y g(1+\eta))$. Write

$$g_1(\eta) = \mathbf{E}_{\,\theta} g'(1+\theta\eta), \qquad Y\eta = U + D, \qquad U = \sum_{i\neq j} Y_i \eta_j, \qquad D = \sum_i d_i.$$

By the mean value theorem, $g(1+\eta) = 1 + g_1(\eta)\,\eta$. Then $Y\,g(1+\eta) = Y + (U+D)g_1(\eta)$. Expanding H in powers of $U\,g_1(\eta)$ we get $h_1 = h_2 + h_3 + R$,

$$h_2 = \mathbf{E}\ H(W), \qquad h_3 = \mathbf{E}\ H'(W) U g_1(\eta) \quad \text{where} \quad W = Y + D g_1(\eta)$$

and where $|R| \ll c_H\ \mathbf{E}\,U^2$. By Lemma 4.1, $\mathbf{E}\,U^2 \ll \mathcal{R}$. Therefore, $h_1 \sim h_2 + h_3$.

Let us show that $h_3 \sim \Gamma_3$. The inequality $|g_1(\eta) - g'(1)| \ll |\eta|$ implies

$$|h_3 - h_4| \ll c_H\,\mathbf{E}\,|U\eta|, \quad h_4 = \mathbf{E}\,H'(W)\,U\,g'(1) = -\frac{N(N-1)}{2}\,\mathbf{E}\,H'(W)\,Y_1\eta_2.$$

Furthermore, the inequalities $|U\eta| \le U^2 + \eta^2$, $\mathbf{E}\,U^2 \ll \mathcal{R}$ and $\mathbf{E}\,\eta^2 \ll \mathcal{M}$, see Lemma 4.1, imply $h_3 - h_4 \sim 0$. Hence, $h_3 \sim h_4$. Next we show that

$$h_4 \sim -\frac{N(N-1)}{2}\,h_5, \qquad h_5 = \mathbf{E}\,H'''(Y^{(1,2)}) Y_1^2 d_2. \tag{3.2}$$

By the mean value theorem, $g_1(\eta) - g_1(\eta^{(1)}) = w\,\eta_1$ with some $|w| \ll 1$. Then

$$D\,g_1(\eta) = D^{(1)} g_1(\eta) + d_1 g_1(\eta) = W_1 + W_2,$$

where we denote $W_1 = D^{(1)} g_1(\eta^{(1)})$ and $W_2 = d_1 g_1(\eta) + D^{(1)} w \eta_1$. We have $W = Y + W_1 + W_2$. Expanding H' in powers of W_2 we get

$$|\,\mathbf{E}\,H'(W) Y_1 \eta_2 - h_6| \ll c_H R_1, \quad h_6 = \mathbf{E}\,H'(Y + W_1) Y_1 \eta_2, \quad R_1 = \mathbf{E}\,|Y_1 \eta_2 W_2|.$$

Furthermore, expanding H' in powers of Y_1 we obtain $|h_6 - h_7 - h_8| \ll c_H R_2$, where

$$h_7 = \mathbf{E}\,H''(Y^{(1)} + W_1) Y_1^2 \eta_2, \qquad h_8 = \mathbf{E}\,H'''(Y^{(1)} + W_1) Y_1^3 \eta_2,$$

and $R_2 = \mathbf{E}\,|\eta_2|(\,\mathbf{E}\,Y_1^4 + |\,\mathbf{E}\,Y_1|)$. A simple calculation shows that $N^2|R_i| \ll \mathcal{R}$, $i = 1, 2$. We obtain

$$h_4 \sim -\frac{N(N-1)}{2}\,(h_7 + h_8).$$

We complete the proof of (3.2) by showing that

$$N^2 h_8 \sim 0 \qquad \text{and} \qquad N^2(h_7 - h_5) \sim 0. \tag{3.3}$$

By the mean value theorem, $g_1(\eta^{(1)}) - g_1(\eta^{(1,2)}) = w\,\eta_2$, with some $w \ll 1$. Then we can write $W_1 = W_3 + W_4$, where

$$W_3 = D^{(1,2)} g_1(\eta^{(1,2)}) \qquad \text{and} \qquad W_4 = d_2 g_1(\eta^{(1)}) + D^{(1,2)}\, w\, \eta_2.$$

Expanding H'' and H''' in powers of W_4 we get

$$|h_7 - h_9| \ll c_H R_1, \quad h_9 = \mathbf{E}\, H''(Y^{(1)} + W_3) Y_1^2 \eta_2, \quad R_1 = \mathbf{E}\, Y_1^2 |\eta_2 W_4|,$$
$$|h_8 - h_{10}| \ll c_H R_2, \quad h_{10} = \mathbf{E}\, H'''(Y^{(1)} + W_3) Y_1^3 \eta_2, \quad R_2 = \mathbf{E}\, Y_1^3 |\eta_2 W_4|.$$

Furthermore, expanding in powers of Y_2 we get $|h_{10}| \ll c_H R_3$, with $R_3 = \mathbf{E}\, |Y_1^3 d_2|$. Finally, expanding H'' in powers of Y_2 and then in powers of W_3 we obtain

$$|h_9 - h_5| \ll c_H R_4, \qquad \text{with} \qquad R_4 = \big(\mathbf{E}\, Y_1^2 Y_2^2 |\eta_2| + \mathbf{E}\, Y_1^2 |d_2 W_3|\big).$$

A straightforward calculation shows $N^2 R_i \ll \mathcal{R}$, $i = 1,2,3,4$. Hence, $N^2 h_8 \sim N^2 h_{10} \sim 0$ and $N^2 h_7 \sim N^2 h_9 \sim N^2 h_5$. We arrive to (3.3) thus completing the proof of (3.2).

Next we replace $Y^{(1,2)}$ by $\tilde{\xi}^{(1,2)}$ in h_5. The error of this replacement,

$$\big|\mathbf{E}\, H'''(Y_3 + \cdots + Y_N) - \mathbf{E}\, H'''(\tilde{\xi}^{(1,2)})\big| \ll c_H(b_3 N + N\gamma). \tag{3.4}$$

For the proof of this inequality we refer to Bentkus, Götze, Paulauskas and Račkauskas (1990), where a similar bound was proved in the case of centered summands Y_i. Combining (3.4) and the inequalities

$$|\mathbf{E}\, H'''(\tilde{\xi}^{(1,2)}) - \mathbf{E}\, H'''(\xi)| \ll c_H\, \mathbf{E}\, |\tilde{\xi}_1 + \tilde{\xi}_2| \ll N^{-1/2}, \quad \mathbf{E}\, Y_1^2 |d_2| \ll b_3/N \tag{3.5}$$

we obtain

$$N^2 \big| h_5 - \mathbf{E}\, H'''(\xi) Y_1^2 d_2 \big| \ll c_H(b_3 N + N\gamma + N^{-1/2}) b_3 N \ll c_H \mathcal{R}.$$

Hence, $-N(N-1)h_5/2 \sim \Gamma_3$ and this completes the proof of $h_3 \sim \Gamma_3$.

It remains to show that $h_2 \sim \mathbf{E}\, H(Y) + \Gamma_2$. We start by showing

$$h_2 \sim \mathbf{E}\ H(Y) + h_{11}, \qquad \text{where} \quad h_{11} = -\frac{N}{2}\ \mathbf{E}\ H'(Y^{(1)}) d_1. \tag{3.6}$$

Expanding g_1 and then H in powers of η we get

$$|h_2 - h_{12}| \ll c_H R, \qquad h_{12} = \mathbf{E}\ H(Y + D\, g'(1)), \qquad |R| = \mathbf{E}\ |\eta D|.$$

The inequalities $|\eta D| \le \eta^2 + D^2$, $\mathbf{E}\,\eta^2 \ll \mathcal{M}$ and $\mathbf{E}\,D^2 \ll \mathcal{R}$, see Lemma 4.1, imply $R \sim 0$. Therefore, $h_2 \sim h_{12}$. Furthermore, expanding H in powers of $Dg'(1)$ we get

$$h_{12} = \mathbf{E}\,H(Y) + h_{13} + R, \qquad h_{13} = \mathbf{E}\,H'(Y)\,D\,g'(1), \tag{3.7}$$

where $|R| \ll c_H \mathbf{E}\,D^2 \sim 0$. Hence, $h_{12} \sim \mathbf{E}\,H(Y) + h_{13}$. By the symmetry, $h_{13} = -2^{-1}N\,\mathbf{E}\,H'(Y)d_1$. Now, expanding H' in powers of Y_1 we obtain $h_{13} \sim h_{11}$. This together with (3.7) yields (3.6). Finally, we replace $Y^{(1)}$ in h_{11} by $\tilde{\xi}^{(1)}$. The error this replacement, $|h_{11} - \Gamma_2| \ll c_H \mathcal{R}$, cf (3.4) and (3.5). We obtain $h_{11} \sim G_2$ thus completing the proof of the lemma.

Lemma 3.2. *The following inequality is true,*

$$\int_{100\le|t|\le L} \frac{|\,\mathbf{E}\ \mathrm{e}\{S\} - \hat{G}(t)|}{|t|}\,dt \ll \mathcal{M} + \gamma_0, \qquad L = (4b_3N)^{-1}. \tag{3.8}$$

Proof of Lemma 3.2. Note, that in view of (2.3) we may asume that $L > 100$ and therefore the integral in (3.8) is nonempty.

The proof is technical and laborous. Behind the proof is a simple idea. We split the sample in two parts $\{X_1, \dots, X_m\}$ and $\{X_{m+1}, \dots, X_N\}$, where the first part is relatively small. Expanding the statistics S and then the exponent, $\exp\{itS\}$, in powers of observations $X_1, \dots, X_m$ we replace $f(t)$ in (3.8) by its partial expansion which contains as a factor the (conditional) ch. f. of a linear statistics of $X_1, \dots, X_m$. This factor will ensure the integrability (with respect to the measure $dt/|t|$ in the region $|t| \ge 100$) of the remainders of further expansions. The idea to separate a piece (corresponding to a relatively small part of the sample) of the linear part of the statistic and then use the ch. f. of it as a factor ensuring the integrability is not new, see e.g. Helmers and van Zwet (1982), van Zwet (1984), Bentkus and Götze (1996), Bentkus, Götze and van Zwet (1997).

Given two complex functions $F(t)$ and $H(t)$ write $F \sim H$ if

$$\int_{c_1\le|t|\le L} \frac{|F(t) - H(t)|}{|t|}\,dt \ll \mathcal{R}, \qquad \mathcal{R} := \mathcal{M} + \gamma_0.$$

Write $f = \mathbf{E}\ \mathrm{e}\{S\}$. Then (3.8) is equivalent to $f \sim \hat{G}$. A simple calculation shows (use Lemma 4.2) that

$$\hat{G} \sim \mathbf{E}\ \mathrm{e}\{\xi\} + \frac{N}{6}G_1 - \frac{N}{2}G_2 - \frac{N(N-1)}{2}G_3,$$
$$G_1 = (it)^3\,\mathbf{E}\ \mathrm{e}\{\xi\}Y_1^3, \qquad G_2 = it\,\mathbf{E}\ \mathrm{e}\{\xi\}d_1, \qquad G_3 = (it)^3\,\mathbf{E}\ \mathrm{e}\{\xi\}Y_1^2 d_2.$$

In order to prove (3.8) we shall show that

$$f(t) \sim \mathbf{E}\, \mathrm{e}\{\xi\} + \frac{N}{6} G_1 - \frac{N}{2} G_2 - \frac{N(N-1)}{2} G_3.$$

Fix $m \approx 400\, N\, t^{-2} \ln|t|$. We may and shall assume that

$$100 < m < N/2. \tag{3.9}$$

Indeed, for $t > 100$, the function $t \to t^{-2} \ln t$ is decreasing and therefore, (3.9) is a consequence of the inequalities

$$100^{-2} 400 \ln 100 < 2 \qquad \text{and} \qquad 400\, N\, L^{-2} \ln L > 100.$$

The first inequality is obvious. The second one follows from Hölder's inequality $b_3 \geq b_2^{3/2} = N^{-3/2}$.

Write $A = \{1, \ldots, m\}$ and $B = \{m+1, \ldots, N\}$. Then

$$S = Y g\big(V - W + \eta_A - Y_A^2/N\big), \quad V = 1 + \eta_B - Y_B^2/N, \quad W = 2\, N^{-1} Y_A Y_B.$$

Let us show that

$$f \sim f_1, \quad f_1 = \mathbf{E}\, \mathrm{e}\{S_1\}, \quad \text{where} \quad S_1 = Y g(V) + Y \eta_A g'(V) - Y_B W g'(V). \tag{3.10}$$

Expanding g in powers of Y_A^2/N and then in powers of η_A we get

$$S = Y g(V - W) + Y \eta_A g'(V - W) + R_1, \qquad |R_1| \ll |Y| \big(\eta_A^2 + Y_A^2/N\big).$$

Next we expand g and g' in powers of W to get

$$\begin{aligned} Y g(V - W) &= Y g(V) - Y W g'(V) + R_2, \qquad && |R_2| \ll |Y|\, W^2, \\ Y\, \eta_A g'(V - W) &= Y\, \eta_A g'(V) + R_3, && |R_3| \ll |Y\, \eta_A W|. \end{aligned}$$

Then $|S - S_1| \ll |R_1| + |R_2| + |R_3| + R_4$, where $R_4 = \mathbf{E}\, |Y_A W|$. Now, the inequality $|f - f_1| \ll |t|\, \mathbf{E}\, |S - S_1|$ combined with the bounds $\mathbf{E}\, |R_i| \ll m\, b_4$, for $i = 1, 2, 3, 4$, see Lemma 4.1, gives

$$\big|f(t) - f_1(t)\big| \ll \frac{\ln|t|}{|t|} \,(N^{-1} + \mathcal{M}) \sim 0$$

and we obtain (3.10).

Write $Y_A \eta_A = D_A + U_A$, where $D_A = \sum_{j=1}^m d_j$ denotes the diagonal of the product $Y_A \eta_A$. Furthermore, write $S_1 = S_2 + \Gamma$, where

$$S_2 = Y g(V) + \big(D_A + \eta_A Y_B\big) g'(V), \qquad \Gamma = U_A g'(V) - Y_B W g'(V).$$

Then $f_1 = \mathbf{E}\, \mathrm{e}\{S_2+\Gamma\}$ and expanding in powers of $it\Gamma$ we get $f_1 = f_2+f_3+f_4+R$, where

$$f_2 = \mathbf{E}\, \mathrm{e}\{S_2\}, \quad f_3 = \mathbf{E}\, \mathrm{e}\{S_2\} itU_A g'(V), \quad f_4 = -\,\mathbf{E}\, \mathrm{e}\{S_2\} itY_B W g'(V),$$

and where

$$|R| \ll t^2\, \mathbf{E}\, \Gamma^2 \ll t^2\, \mathbf{E}\, U_A^2 + t^2\, \mathbf{E}\, Y_B^2 W^2 \sim 0,$$

by Lemma 4.1. Therefore, $R \sim 0$. We obtain $f_1 \sim f_2 + f_3 + f_4$.

The rest of the proof is divided in two steps. It the first step we show that

$$f_2 \sim \mathbf{E}\, \mathrm{e}\{Y\} - \frac{N}{2}\tilde{G}_2 - \big(C_N^2 - C_m^2\big)\tilde{G}_3, \tag{3.11}$$

$$f_3 \sim -C_m^2 \tilde{G}_3, \tag{3.12}$$

and $f_4 \sim 0$. Here C_k^2 denotes $k(k-1)/2$ and

$$\tilde{G}_1 = (it)^3\, \mathbf{E}\, \mathrm{e}\{Y^{(1)}\}Y_1^3, \quad \tilde{G}_2 = it\, \mathbf{E}\, \mathrm{e}\{Y^{(1)}\}d_1, \quad \tilde{G}_3 = (it)^3\, \mathbf{E}\, \mathrm{e}\{Y^{(1,2)}\}Q_{1,2}.$$

In the second step we replace $\tilde{G}_i$ by G_i, $i = 2, 3$ in (3.11) and (3.12) and show that $\mathbf{E}\, \mathrm{e}\{Y\} \sim \mathbf{E}\, \mathrm{e}\{\xi\} + \frac{N}{6} G_1$.

Let us introduce some more notation. For $j = 1, 2, \ldots, m$ write

$$\begin{aligned} &y_j = Y_j\, g(V) + \eta_j\, Y_B g'(V), \qquad z_j = Y_j\, g(V) + \big(d_j + \eta_j\, Y_B\big)g'(V), \\ &u(t) = \mathbf{E}_1\, \mathrm{e}\{Y_1\, g(V)\}, \quad v(t) = \mathbf{E}_1\, \mathrm{e}\{y_1\}, \quad w(t) = \mathbf{E}_2\, \mathrm{e}\{z_2\}. \end{aligned} \tag{3.13}$$

Step 1. We start by showing that $f_4 \sim 0$. By the symmetry,

$$f_4 = -2it\,\frac{m}{N}\, \mathbf{E}\, \mathrm{e}\{S_2\} Y_B^2 Y_1 g'(V) = -2it\,\frac{m}{N}\, \mathbf{E}\, h_1 Y_B^2 w^{m-1}(t) W_1,$$

where we denote $h_1 = \mathrm{e}\{Y_B g(V)\}g'(V)$ and $W_1 = \mathbf{E}_1\, \mathrm{e}\{z_1\}Y_1$. Expanding $\mathrm{e}\{z_1\}$ in powers of itz_1 we get

$$|W_1| \le |t|\, \mathbf{E}_1 |Y_1 z_1| + \gamma \ll N^{-1}|t|(1+|Y_B|) + \gamma.$$

This inequality in combination with (4.13) and (4.1) implies

$$|f_4| \ll |t|\, \mathbf{E}\, |w(t)|^{m-1} Y_B^2 |W_1| \ll \big(t^{-8}N^{-1} + |t|^{-9}\gamma\big) \sim 0.$$

Hence, $f_4 \sim 0$.

Let us prove (3.12). The proof is long and involved. The "backbone" of the proof is the sequence of equivalence relations,

$$f_3 \sim f_5, \quad f_5 \sim f_6, \quad f_6 \sim f_7, \quad f_7 \sim f_9, \quad f_9 \sim f_{10}, \quad f_{10} \sim f_{12}, \quad f_{12} \sim -C_m^2 \tilde{G}_3,$$

where the functions $f_5,\dots,f_{12}$ are introduced below.

By the symmetry,

$$f_3 = m(m-1)it\,\mathbf{E}\,\mathrm{e}\{S_2\}g'(V)Y_1\eta_2 = m(m-1)it\,\mathbf{E}\,h_1w^{m-2}(t)W_1W_2,$$

where we denote $W_2 = \mathbf{E}_2\,\mathrm{e}\{z_2\}\eta_2$. Let us show that $f_3 \sim f_5$, where

$$f_5 = m(m-1)(it)^3\,\mathbf{E}\,h_1w^{m-2}(t)\zeta, \qquad \zeta = Y_1z_1\eta_2z_2.$$

Expanding exponents in powers of itz_1 and itz_2 we get $W_1 = W_{1,1}+W_{1,2}+W_{1,3}$ and $W_2 = W_{2,1} = W_{2,2}+W_{2,3}$, where

$$\begin{aligned} &W_{1,1} = \mathbf{E}_1Y_1, \qquad W_{1,2} = \mathbf{E}_1Y_1itz_1, \qquad W_{1,3} = \mathbf{E}_{1,\theta_1}Y_1(itz_1)^2\,\mathrm{e}\{\theta_1z_1\}/2,\\ &W_{2,1} = \mathbf{E}_{2,\theta_2}\eta_2itz_2\,\mathrm{e}\{\theta_2z_2\}, \qquad W_{2,2} = \mathbf{E}_2\eta_2itz_2, \qquad W_{2,3} = W_{2,1}-W_{2,2}.\end{aligned}$$

Combining these expansions we obtain $W_1W_2 = (it)^2\,\mathbf{E}_{1,2}\zeta + R_1+R_2+R_3$, where

$$R_1 = W_{1,1}W_{2,1}, \quad R_2 = W_{1,3}W_{2,1} \quad \text{and} \quad R_3 = W_{1,2}W_{2,3}.$$

Thus, we have $f_3 = f_5 + H_1 + H_2 + H_3$, where

$$H_i = m(m-1)it\;\mathbf{E}\,h_1w^{m-2}(t)R_i, \qquad i=1,2,3.$$

and we shall prove that $H_i \sim 0$, $i=1,2,3$. A straightfoward calculation gives

$$|R_1| \ll |t|N^{-1}\gamma b_3(1+|Y_B|) \qquad \text{and} \qquad |R_2| \ll |t|^3b_3^2(1+|Y_B|^3).$$

Furthermore, the inequality $|e^{ix}-1| \le |x|$ implies $|W_{2,3}| \ll t^2\,\mathbf{E}_2z_2^2|\eta_2|$ and thus

$$|R_3| \ll |t|^3b_4N^{-1}(1+|Y_B|^3).$$

Combining these bounds with (4.13) and using (4.1) we obtain $H_i \sim 0$, $i=1,2,3$. Hence, $f_3 \sim f_5$.

Next we replace ζ in f_5 by $Q_{1,2}g^2(V)$. Write

$$f_6 = m(m-1)(it)^3\,\mathbf{E}\,h_1g^2(V)w^{m-2}(t)Q_{1,2} \qquad \text{and} \qquad r = \zeta - Q_{1,2}g^2(V).$$

Then $f_5 = f_6 + R$, where $|R| \ll m^2|t|^3\,\mathbf{E}\,|r|$. A simple calculation shows that $\mathbf{E}_{1,2}|r| \ll N^{-2}\mathcal{M}(1+Y_B^2)$. Hence, $|R| \ll |t|^3(m/N)^2\mathcal{M} \sim 0$ and therefore, $f_5 \sim f_6$.

Let us show that $f_6 \sim f_7$, where

$$f_7 = m(m-1)(it)^3\,\mathbf{E}\,h_2Q_{1,2}, \qquad h_2 = \mathrm{e}\{Y^{(1,2)}g(V)\}g_2(V), \quad g_2 = g^2g'.$$

By the symmetry, we can write

$$w^{m-2}(t) = \mathbf{E}_A \, \mathrm{e}\big\{Y_A^{(1,2)} g(V) + D^{(1,2)} g'(V) + \eta_A^{(1,2)} Y_B g'(V)\big\}. \tag{3.14}$$

Expanding the exponent in (3.14) in powers of $itD^{(1,2)}g'(V)$ and then in powers of $it\eta_A^{(1,2)}Y_B g'(V)$ we show that the right-hand side of (3.14) equals to

$$\mathbf{E}_A \, \mathrm{e}\{Y_A^{(1,2)} g(V)\}\big(1 + it\eta_A^{(1,2)} Y_B g'(V)\big) + \mathbf{E}_A (r_1 + r_2)\big),$$
$$|r_1| \ll |t\, D^{(1,2)}|, \qquad |r_2| \ll \big(t\, \eta_A^{(1,2)} Y_B\big)^2.$$

Substituting this expansion in f_6 we get $f_6 = f_7 + f_8 + R_1 + R_2$, where

$$f_8 = m(m-1)(it)^4 \, \mathbf{E}\, h_2 Q_{1,2} \eta_A^{(1,2)} Y_B g'(V), \qquad |R_i| \ll m^2 |t|^3 \, \mathbf{E} \left|Q_{1,2} r_i\right|, \quad i = 1, 2.$$

We show that $R_i \sim 0$, $i = 1, 2$ and $f_8 \sim 0$. By the symmetry,

$$\mathbf{E} \left|Q_{1,2} D^{(1,2)}\right| \le m \, \mathbf{E} \, |Q_{1,2} d_3| \ll m \, b_3^2 / N.$$

Therefore, $|R_1| \ll (m/N)^3 t^4 (N b_3)^2 \sim 0$. To show $R_2 \sim 0$ write

$$\mathbf{E}\, |Q_{1,2}| \big(\eta_A^{(1,2)}\big)^2 = (m-2)\, \mathbf{E}\, |Q_{1,2}| \eta_3^2 \ll m \, N^{-2} b_3 \mathcal{M},$$

Then, $|R_2| \ll (m/N)^3 |t|^5 \mathcal{M} b_3 N \sim 0$ and we obtain $f_6 \sim f_7 + f_8$. Furthermore, write

$$f_8 = m(m-1)(m-2)(it)^4 \, \mathbf{E}\, h_2 Q_{1,2} \eta_3 Y_B g'(V).$$

Expanding the exponent in h_2 in powers of $itY_3 g(V)$ we get

$$|f_8| \ll m^3 |t|^5 \, \mathbf{E}\, |Q_{1,2} d_3| \ll (m/N)^3 |t|^5 (N b_3)^2 \sim 0.$$

We obtain $f_6 \sim f_7$.

Next we shall replace h_2 by $\mathrm{e}\{Y^{(1,2)}\} g_2(1)$ in f_7. Firstly we shall prove that $f_7 \sim f_9$, where

$$f_9 = m(m-1)(it)^3 \, \mathbf{E}\, h_3 Q_{1,2}, \quad h_3 = \mathrm{e}\{Y^{(1,2)} g(V_1)\} g_2(V_1), \quad V_1 = 1 + \eta_B.$$

We have $V = V_1 - Y_B^2/N$. Expanding in powers of Y_B^2/N we get

$$|h_2 - h_3| \ll (1 + |tY^{(1,2)}|) Y_B^2 / N.$$

Then

$$|f_7 - f_9| \ll R, \quad \text{where} \quad R = m^2 |t|^3 \, \mathbf{E}\, \big(1 + |t\, Y^{(1,2)}|\big) |Q_{1,2}| Y_B^2 / N.$$

A simple calculation shows that $R \ll (m/N)^2 t^4 b_3$. Since $Nb_3 \ll |t|^{-1}$, we obtain $R \sim 0$. Hence, $f_7 \sim f_9$.

Now we replace h_3 by $h_4 g_2(1)$, $h_4 = \mathrm{e}\{Y^{(1,2)} g(V_1)\}$ in f_9. Expanding g_2 in powers of η_B we get $f_9 = f_{10} + f_{11} + R$, where

$$f_{10} = -C_m^2 (it)^3 \,\mathbf{E}\, h_4 Q_{1,2}, \qquad f_{11} = m(m-1)(it)^3 \,\mathbf{E}\, h_4 Q_{1,2} g_2'(1) \eta_B,$$

and where

$$|R| \ll m^2 |t|^3 \,\mathbf{E}\, |Q_{1,2}| \eta_B^2 \ll m^2 |t|^3 N^{-1} b_3 \mathcal{M} \sim 0.$$

Hence, $f_9 \sim f_{10} + f_{11}$. We shall show that $f_9 \sim f_{10}$. To show $f_{11} \sim 0$ write

$$f_{11} = (N-m) m(m-1)(it)^3 g_2'(1) \,\mathbf{E}\, h_4 \eta_N Q_{1,2}. \tag{3.15}$$

Furthermore, write $Y^{(1,2)} g(V_1) = S_3 + r_1 + r_2$, where

$$S_3 = Y^{(1,2,N)} g(1 + \eta_B^{(N)}), \qquad r_1 = Y_N g(V_1), \qquad r_2 = Y^{(1,2,N)} \big(g(V_1) - g(1 + \eta_B^{(N)}) \big).$$

Expanding the exponent in h_4 in powers of itr_2 and then in powers of itr_1 we obtain

$$\mathbf{E}_N h_4 \eta_N = it \,\mathrm{e}\{S_3\} \,\mathbf{E}_N \eta_N r_1 \varkappa_1 + it \,\mathbf{E}_N \,\mathrm{e}\{S_3 + r_1\} \eta_N r_2 \varkappa_2,$$

where $\varkappa_i$ depends only on r_i, and $|\varkappa_i| \ll 1$. Substituting this expression in (3.15) we get

$$|f_{11}| \ll N m^2 t^4 \,\mathbf{E}\, |\,\mathbf{E}_{A'} \,\mathrm{e}\{S_3\}| |Q_{1,2} d_N| + N m^2 t^4 \,\mathbf{E}\, |Q_{1,2} \eta_N r_2|, \quad A' = A \setminus \{1,2\}.$$

Now combining the inequalities $|r_2| \ll |Y^{(1,2,N)} \eta_N|$ and (4.1) and

$$E_{1,2,N} |Q_{1,2}| \eta_N^2 \ll N^{-1} b_3 b_4, \qquad \mathbf{E}_{1,2,N} |Q_{1,2} d_N| \ll N^{-1} b_3^2$$

and $|\,\mathbf{E}_A \,\mathrm{e}\{S_3\}| \ll t^{-10}$, see (EXP.O), we obtain

$$|f_{11}| \ll (m/N)^2 t^{-6} (b_3 N)^2 + (m/N)^2 t^4 b_3 N \mathcal{M} \sim 0,$$

thus completing the proof of $f_9 \sim f_{10}$.

Write $u_1 = \mathbf{E}_1 \,\mathrm{e}\{Y_1 g(V_1)\}$. Note that by the symmetry,

$$f_{10} = -C_m^2 (it)^3 \,\mathbf{E}\, u_1^{m-2}(t) h_5 Q_{1,2}, \qquad h_5 = \mathrm{e}\{Y_B g(1 + \eta_B)\}.$$

Now we shall replace h_5 by $\mathrm{e}\{Y_B\}$. Expanding g and then the exponent in powers of η_B we obtain $h_5 = \mathrm{e}\{Y_B\} + r$, with $|r| \ll |t Y_B \eta_B|$. Then $f_{10} = f_{12} + R$, where

$$f_{12} = -C_m^2 (it)^3 \,\mathbf{E}\, u_1^{m-2}(t) \,\mathrm{e}\{Y_B\} Q_{1,2} \quad \text{and} \quad |R| \ll m^2 t^4 \,\mathbf{E}\, |u_1(t)|^{m-2} |Q_{1,2} Y_B \eta_B|.$$

Combining (4.12) with the inequalities $\mathbf{E}\,|Q_{1,2}| \ll b_3/N$ and

$$\mathbf{E}\,|Y_B\eta_B| \le (\,\mathbf{E}\,Y_B^2\,\mathbf{E}\,\eta_B^2)^{1/2} \ll (\,\mathbf{E}\,\eta_B^2)^{1/2} \ll \mathcal{M}^{1/2},$$

see (4.1), we obtain $|R| \ll t^{-6} b_3 N \mathcal{M}^{1/2}$. But $b_3 N \mathcal{M}^{1/2} \le (b_3 N)^2 + \mathcal{M}$. Hence, $|R| \sim 0$ and therefore, $f_{10} \sim f_{12}$.

It remains to replace $h_6 := u_1^{m-2}(t)$ by $\mathbf{E}\,\mathrm{e}\{Y_A^{(1,2)}\}$ in f_{12}. Let us show that

$$\big|\,\mathbf{E}_A(h_6 - \mathrm{e}\{Y_A^{(1,2)}\})\big| \ll |t|^{-9}|\eta_B|. \tag{3.16}$$

Fix $k \approx m/2$ and write $A_1 = \{3, \dots, k\}$ and $A_2 = \{k+1, \dots, m\}$. Then $Y_A^{(1,2)} = Y_{A_1} + Y_{A_2}$ and $h_6 = h_{6,1}h_{6,2}$, where $h_{6,i} = \mathrm{e}\{Y_{A_i} g(1+\eta_B)\}$. Firstly, we replace $\mathbf{E}_A h_{6,1}h_{6,2}$ by $\mathbf{E}_A\,\mathrm{e}\{Y_{A_1}\}h_{6,2}$. The error of this replacement,

$$\big|\,\mathbf{E}_A\big(h_6 - h_{6,2}\,\mathrm{e}\{Y_{A_1}\}\big)\big| \ll |\,\mathbf{E}_{A_2}h_{6,2}|\,\mathbf{E}_{A_1}|h_{6,1} - \mathrm{e}\{Y_{A_1}\}| \ll |t|^{-9}|\eta_B|$$

Here we estimated $|\,\mathbf{E}_{A_2}h_{6,2}| \ll t^{-10}$, see (4.12), and used the inequalities

$$|h_{6,1} - \mathbf{E}\,\mathrm{e}\{Y_{A_1}\}| \ll |t\,Y_{A_1}\eta_B|$$

and (4.1). Similarly,

$$\big|\,\mathbf{E}_A\big(h_{6,2}\,\mathrm{e}\{Y_{A_1}\} - \mathrm{e}\{Y_{A_2}\}\,\mathrm{e}\{Y_{A_1}\}\big)\big| \ll |t|^{-9}|\eta_B|$$

and (3.16) follows.

An application of (3.16) gives $f_{12} = -C_m^2\tilde{G}_3 + R$, where $|R| \ll m^2 t^{-6}\,\mathbf{E}\,|Q_{1,2}\eta_B|$. Now the inequalities $\mathbf{E}\,|Q_{1,2}| \ll b_3/N$ and $\mathbf{E}\,\eta_B^2 \ll \mathcal{M}$ implies

$$|R| \ll (m/N)^2 t^{-6} b_3 N \mathcal{M}^{1/2} \sim 0.$$

We arrive to (3.12).

Let us prove (3.11). Write $S_4 = Yg(V) + \eta_A Y_B g'(V)$.

Firstly we show that

$$f_2 \sim f_{13} - f_{14}, \quad f_{13} = \mathbf{E}\,\mathrm{e}\{S_4\}, \quad f_{14} = \frac{it}{2}\,\mathbf{E}\,\mathrm{e}\{S_4\}D_A = m\,\frac{it}{2}\,\mathbf{E}\,\mathrm{e}\{S_4\}d_1, \tag{3.17}$$

To show (3.17) write $f_2 = \mathbf{E}\,\mathrm{e}\{S_4 + D_A g'(V)\}$ and $D_A g'(V) = r_1 + r_2$, where $r_j = D_{A_j} g'(V)$ and where $A_1 = \{1, \dots, k\}$ and $A_2 = \{k+1, \dots, m\}$ and the number $k \approx m/2$. Expanding in powers of itr_1 and then in powers of itr_2 we get

$$\mathrm{e}\{D_A g'(V)\} = \mathrm{e}\{r_1 + r_2\} = 1 + itr_1 + itr_2 + (it)^2\,\mathbf{E}_{\theta_1}\big(R_1 + R_2 + R_3\big), \tag{3.18}$$

$$R_1 = \frac{r_1^2}{2}\,\mathrm{e}\{\theta_1 r_1\}\,\mathrm{e}\{r_2\}, \qquad R_2 = r_1 r_2\,\mathrm{e}\{\theta_1 r_2\}, \qquad R_3 = \frac{r_2^2}{2}\,\mathrm{e}\{\theta_1 r_2\}. \tag{3.19}$$

Therefore, we can write $f_2 = f_{13} + f_{15} + H_1 + H_2 + H_3$,

$$f_{15} = it\,\mathbf{E}\,\mathrm{e}\{S_4\}D_A g'(V), \qquad H_j = (it)^2\,\mathbf{E}\,\mathrm{e}\{S_4\}R_j, \quad j = 1, 2, 3.$$

By (4.13), $|\,\mathbf{E}_{A_2}\,\mathrm{e}\{S_4 + r_2\}|$ and $|\,\mathbf{E}_{A_1}\,\mathrm{e}\{S_4\}|$ are bounded by $c(1 + Y_B^{20})t^{-10}$. These bounds and the inequalities $\mathbf{E}\,D_{A_j}^2 \ll \mathcal{R}$, see Lemma 4.1, implies $H_1 + H_3 \sim 0$. To show $H_2 \sim 0$ we estimate

$$|H_2| \ll t^2\,\mathbf{E}\,|D_{A_1} D_{A_2}| \ll t^2 m^2 b_3^2 = t^2 (m/N)^2 (N b_3)^2 \sim 0.$$

Hence, $f_2 \sim f_{13} + f_{15}$. Finally, expanding $g'(V) = g'(1 + \eta_B - Y_B^2/N)$ in powers of $\eta_B - Y_B^2/N$ we obtain $f_{15} \sim -f_{14}$. We arrive to (3.17).

Let us show that

$$f_{14} \sim \frac{m}{2}\,\tilde{G}_2 \tag{3.20}$$

Write $\mathrm{e}\{S_4\} = \mathrm{e}\{S_5\}\,\mathrm{e}\{y_1\}$, where we denote $S_5 = Y^{(1)} g(V) + \eta_A^{(1)} Y_B g'(V)$ and y_1 is defined in (3.13). Expanding the exponent in powers of ity_1 we get $f_{14} = f_{16} + R$, where

$$f_{16} = m\,\frac{it}{2}\,\mathbf{E}\,\mathrm{e}\{S_5\}d_1 \qquad \text{and} \qquad |R| \ll t^2 m\,\mathbf{E}\,r|d_1 y_1|.$$

Here $r = |\,\mathbf{E}_{A_2}\,\mathrm{e}\{S_5\}|$. By (4.13), $r \ll t^{-10}(1 + Y_B^{20})$. Hence, $|R| \ll m t^{-8} b_4 \sim 0$ and we obtain $f_{14} \sim f_{16}$.

Now we show $f_{16} \sim f_{17}$, where

$$f_{17} = m\,\frac{it}{2}\,\mathbf{E}\,\mathrm{e}\{Y^{(1)} g(V)\}d_1, \qquad f_{18} = m\,\frac{(it)^2}{2}\,\mathbf{E}\,\mathrm{e}\{Y^{(1)} g(V)\}d_1 \eta_A^{(1)} Y_B g'(V).$$

Expanding the exponent in f_{16} in powers of $it\eta_A^{(1)} Y_B g'(V)$ we get $f_{16} = f_{17} + f_{18} + R$, where

$$|R| \ll |t|^3 m\,\mathbf{E}\,|d_1|(\eta_A^{(1)})^2 \ll |t|^3 (m/N)^2 b_3 N \mathcal{M} \sim 0.$$

To show $f_{18} \sim 0$ write

$$f_{18} = 2^{-1}(it)^2 m(m-1)\,\mathbf{E}\,\mathrm{e}\{Y^{(1)} g(V)\}d_1 \eta_2 Y_B g'(V)$$

and expand the exponent in powers of $itY_2 g(V)$ to get

$$|f_{18}| \ll |t|^3 m^2\,\mathbf{E}\,|d_1 d_2| \ll |t|^3 (m/N)^2 (N b_3)^2 \sim 0.$$

Hence, $f_{16} \sim f_{17}$. We complete the proof of (3.20) by showing $f_{17} \sim (m/2)\tilde{G}_2$.

By the symmetry,

$$f_{17} = (it/2) m\,\mathbf{E}\,u^{m-1}(t)\,\mathrm{e}\{Y_B g(V)\}d_1.$$

Firstly, we replace $\mathrm{e}\{Y_B g(V)\}$ by $\mathrm{e}\{Y_B g(V_1)\}$ and then by $\mathrm{e}\{Y_B\}$. For this purpose we expand g and then the exponent, cf. the proof $f_7 \sim f_9$ and $f_{10} \sim f_{12}$ above. Secondly, proceeding as in proof of $f_{12} \sim -C_m^2 \tilde{G}_3$ above we replace $u^{m-1}(t)$ by $\mathbf{E}_A \,\mathrm{e}\{Y_A^{(1)}\}$. Thus we obtain

$$f_{17} \sim (m/2) it \,\mathbf{E}\,\{Y_A^{(1)}\}\,\mathrm{e}\{Y_B\} d_1 = (m/2)\tilde{G}_2$$

and this completes the proof of (3.20). It follows from (3.17) that $f_2 \sim f_{13} - (m/2)\tilde{G}_2$.

It remains to expand the function f_{13}. Firstly, we show that $f_{13} \sim f_{19} + f_{20}$, where

$$f_{19} = \mathbf{E}\;\mathrm{e}\{S_6\}, \qquad f_{20} = -(N-m)m\,\frac{it}{2}\,\mathbf{E}\;\mathrm{e}\{S_6\}\eta_1 Y_N, \quad S_6 = Y g(V). \tag{3.21}$$

Then we show that

$$f_{19} \sim \mathbf{E}\;\mathrm{e}\{Y\} - \frac{1}{2}(N-m)\tilde{G}_2 - \frac{1}{2}(N-m)(N-1)\,\tilde{G}_3, \tag{3.22}$$

$$f_{20} \sim -\frac{1}{2}(N-m)m\tilde{G}_3. \tag{3.23}$$

To prove (3.21) we shall use the same argument as in proof of $f_2 \sim f_{13} + f_{15}$ above. Write $A_1 = \{1, \ldots, k\}$ and $A_2 = \{k+1, \ldots, m\}$, where $k \approx m/2$. Then $\eta_A = \eta_{A_1} + \eta_{A_2}$ and $f_{13} = \mathbf{E}\;\mathrm{e}\{S_6\}\,\mathrm{e}\{r_1 + r_2\}$, where $r_i = \eta_{A_i} Y_B g'(V)$. It follows from (3.18) that

$$f_{13} = f_{19} + f_{21} + H_1 + H_2 + H_3, \quad f_{21} = it\,\mathbf{E}\;\mathrm{e}\{S_6\}\eta_A Y_B g'(V), \quad H_j = (it)^2\,\mathbf{E}\;\mathrm{e}\{S_6\}R_j,$$

$j = 1, 2, 3$, where R_j are defined as in (3.19), but with new values of r_j. Let us show

$$f_{13} \sim f_{19} + f_{21}. \tag{3.24}$$

Combining the inequalities $\mathbf{E}_{A_j}\eta_{A_j}^2 \ll \mathcal{M}$ and

$$|\,\mathbf{E}_{A_1}\,\mathrm{e}\{S_6\}| \ll t^{-10}, \qquad |\,\mathbf{E}_{A_2}\,\mathrm{e}\{S_6 + r_2\}| \ll t^{-10}(1 + Y_B^{20}),$$

(which follows from (4.12) and (4.13)) we obtain $|H_1| + |H_3| \sim 0$. It remains to show $|H_2| \sim 0$. Write $A_{1.1} = \{1, \ldots, k_1\}$ and $A_{1.2} = \{k_1 + 1, \ldots, k\}$, where the number $k_1 \approx k/2$. Then $r_1 = r_{1.1} + r_{1.2}$, where $r_{1.i} = \eta_{A_{1.i}}$ and

$$H_2 = (it)^2\,\mathbf{E}\;\mathrm{e}\{S_6 + \theta r_2\}r_1 r_2 = H_{2.1} + H_{2.2}, \qquad H_{2.i} = (it)^2\,\mathbf{E}\;\mathrm{e}\{S_6 + \theta r_2\}r_{1.i}r_2.$$

Now we show that $H_{2.2} \sim 0$. We have

$$|H_{2.2}| \ll t^2\,\mathbf{E}\,|\,\mathbf{E}_{A_{1.1}}\,\mathrm{e}\{S_6\}||r_{1.2}r_2| \ll t^{-8}\,\mathbf{E}\,|r_{1.2}r_2| \ll t^{-8}\mathcal{M} \sim 0.$$

Here we applied (4.12) and used the inequalities $\mathbf{E}\, r_{1.2}^2 \ll \mathcal{M}$ and $\mathbf{E}\, r_2^2 \ll \mathcal{M}$. The same argument gives $H_{2.1} \sim 0$. Therefore, $|H_2| \sim 0$ and this completes the proof of (3.24).

In order to prove (3.21) it remains to show $f_{21} \sim f_{20}$. Write $r = \eta_B - Y_B^2/N$. Expanding $g'(V) = g'(1+r)$ in powers of r we get $f_{21} = f_{20} + R$,

$$|R| \le |t| \big| \mathbf{E}\ \mathrm{e}\{S_6\} \eta_A r Y_B \big| \le R_1 + R_2, \quad R_j = |t|\, \mathbf{E}\, \big| \mathbf{E}_{A_j}\, \mathrm{e}\{S_6\} \big| \big| (\eta_A - \eta_{A_j}) r Y_B \big|.$$

Combining (4.12) with the inequalities $\mathbf{E}\, \eta_{A_j}^2 \ll \mathcal{M}$ and $\mathbf{E}\, (rY_B)^2 \ll \mathcal{M}$, see Lemma 4.1, we obtain $|R| \sim 0$. Hence, $f_{21} \sim f_{20}$. We arrive to (3.21).

Let us prove (3.23). Firstly we show that

$$f_{20} \sim f_{22}, \qquad f_{22} = -(N-m)m \frac{(it)^3}{2}\, \mathbf{E}\ \mathrm{e}\{Y^{(1,N)} g(V_2)\} Q_{N,1}, \tag{3.25}$$

where $V_2 = 1 + \eta_B^{(N)} - (Y_B^{(N)})^2/N$, and then replace $g(V_2)$ by $g(1) = 1$ in the exponent. Write $r_1 = Y(g(V) - g(V_2))$ and $r_2 = Y_N g(V_2)$. Then $Yg(V) = Y^{(N)} g(V_2) + r_1 + r_2$. Expanding the exponent in powers of itr_1 and then in powers of itr_2 we get $\mathrm{e}\{r_1 + r_2\} = 1 + itr_2 + R_1 + R_2$, where

$$R_1 = itr_1\, \mathrm{e}\{r_2\}\, \mathbf{E}_{\theta_1}\, \mathrm{e}\{\theta_1 r_1\}, \qquad R_2 = \frac{(itr_2)^2}{2}\, \mathbf{E}_{\theta_1}\, \mathrm{e}\{\theta_1 r_2\}. \tag{3.26}$$

Thus we can write $f_{20} = f_{23} + H_0 + H_1 + H_2$, where

$$f_{23} = -(N-m)m \frac{(it)^2}{2}\, \mathbf{E}\, h_7 r_2, \qquad \text{with} \qquad h_7 = \mathrm{e}\{Y^{(N)} g(V_2)\} \eta_1 Y_N,$$

and $H_j = -(N-m)m(it/2)\, \mathbf{E}\, h_7 R_j$, where $R_0 = 1$ and R_1, R_2 are given by (3.26). Let us show that

$$f_{20} \sim f_{23}. \tag{3.27}$$

Expanding the exponent in h_7 in powers of $itY_1 g(V_2)$ we get

$$\begin{aligned} &|H_0| \ll Nmt^2 |\, \mathbf{E}\, Y_N|\, \mathbf{E}\, |d_1 g(V_2)| \zeta_1, && |H_2| \ll Nmt^4\, \mathbf{E}\, |Y_N^3 d_1 g^3(V_2)| \zeta_1, \\ &\zeta_1 = \big| \mathbf{E}_{A'}\, \mathrm{e}\{Y^{(1,N)} g(V_2)\} \big|, && A' = \{2, \dots, m\}. \end{aligned}$$

By (4.12), $\zeta_1 \ll t^{-10}$. We obtain

$$|H_0| \ll t^{-8} \gamma m b_3 \sim 0, \qquad |H_2| \ll Nm|t|^{-6} b_3^2 \sim 0.$$

To show $H_1 \sim 0$ write $\varkappa_1 = g(V) - g(V_2)$ and $\varkappa_2 = g(V_2) + \theta \varkappa_1$. Then

$$H_1 = -2^{-1}(N-m)m(it)^2\, \mathbf{E}\ \mathrm{e}\{Y \varkappa_2\} \eta_1 Y_N r_1.$$

Furthermore, write $r_{1.1} = Y_1 \varkappa_1$ and $r_{1.2} = Y^{(1)} \varkappa_1$. Then $r_1 = r_{1.1} + r_{1.2}$ and

$$H_1 = H_{1.1} + H_{1.2}, \quad \text{where} \quad H_{1.j} = -(N-m)m \frac{(it)^2}{2} \mathbf{E}\, \mathrm{e}\{Y \varkappa_2\} \eta_1 Y_N r_{1,j}. \tag{3.28}$$

We have $|H_{1.1}| \ll Nmt^2 \mathbf{E}\, \zeta_2 |d_1 Y_N \varkappa_1|$, where

$$\zeta_2 = \big| \mathbf{E}_{A_2} \mathrm{e}\{Y \varkappa_2\} \big| \ll t^{-10}, \qquad A_2 = \{k+1, \dots, m\}, \quad k \approx m/2, \tag{3.29}$$

see (4.12). Combining (2.2) and the inequalities

$$|\varkappa_1| \ll |V - V_2| \le |\eta_N| + \frac{Y_N^2}{N} + \frac{|Y_N Y_B^{(N)}|}{N}$$

we obtain

$$\mathbf{E}\, |Y_N \varkappa_1| \ll b_3 + N^{-2} \le 2b_3. \tag{3.30}$$

It follows from (3.29) and (3.30) that

$$|H_{1.1}| \ll N\, m\, t^{-8}\, \mathbf{E}\, |d_1|\, \mathbf{E}\, |Y_N \varkappa_1| \ll t^{-8} N\, m\, b_3^2 \sim 0.$$

We complete the proof of (3.27) by showing $H_{1.2} \sim 0$. Write $A_1 = \{2, \dots, k\}$ and $A_2 = \{k+1, \dots, m\}$, where the integer $k \approx m/2$. Furthermore, write $r_{1.3} = Y_{A_1} \varkappa_1$ and $r_{1.4} = (Y_{A_2} + Y_B)\varkappa_1$. Then $r_{1.2} = r_{1.3} + r_{1.4}$ and $H_{1.2} = H_{1.3} + H_{1.4}$, where $H_{1.j}$ are given by (3.28), but with $j = 3, 4$. The proofs of $H_{1.3} \sim 0$ and $H_{1.4} \sim 0$ are almost the same and we shall prove $H_{1.3} \sim 0$ only. Expanding the exponent $\mathrm{e}\{Y\varkappa_2\} = \mathrm{e}\{Y^{(1)}\varkappa_2 + Y_1 \varkappa_2\}$ in powers of $itY_1\varkappa_2$, we get

$$|H_{1.3}| \ll N\, m\, |t|^3\, \mathbf{E}\, \zeta_2 \big| \varkappa_2 d_1 Y_N r_{1.3} \big|.$$

Using the inequalities (3.29), (3.30) and $|\varkappa_2| \ll 1$ and $\mathbf{E}\, |Y_{A_1}| \ll 1$ we obtain

$$|H_{1.3}| \ll N\, m\, |t|^{-7} b_3^2 \sim 0.$$

Thus, we have $H_{1.2} \sim 0$.

In order to complete the proof of (3.25) we expand the exponent in h_7 in powers of $itY_1 g(V_2)$ to get

$$f_{23} = -(N-m)m \frac{(it)^3}{2} \mathbf{E}\, \mathrm{e}\{Y^{(1,N)} g(V_2)\} Q_{1,N} g^2(V_2) + R,$$

where

$$|R| \ll Nmt^4\, \mathbf{E}\, |Q_{N,1} Y_1| \zeta_1 \ll m\, b_4 t^{-6} \sim 0$$

and where ζ_1 is given after (3.27) and satisfies $|\zeta_1| \ll t^{-10}$. Finally, expanding $g^2(V_2)$ in powers of $(V_2 - 1)$ we obtain $f_{23} \sim f_{22}$ thus completing the proof of (3.25).

It remains to show $f_{22} \sim -(1/2)(N-m)m\tilde{G}_3$. What we have to do is to replace $Y^{(1,N)}g(V_2)$ by $Y^{(1,N)}g(1)$ in the exponent in f_{22} (note that $g(1) = 1$). We skip the proof of this step, since it is similar to the proof of $f_7 \sim -C_m^2\tilde{G}_3$, but simpler. We arrive to (3.23).

Let us prove (3.22). Firstly, we shall show that

$$f_{19} \sim f_{24}, \qquad f_{24} = \mathbf{E}\ \mathrm{e}\{Yg(1+\eta_B)\}.$$

Write $\varkappa_1 = g(V) - g(1+\eta_B)$ and $r = Y\varkappa_1$. Then $S_6 = Yg(1+\eta_B) + r$ and expanding the exponent in prowers of itr we get

$$f_{19} = f_{24} + R, \qquad R = it\,\mathbf{E}\ \mathrm{e}\{Y\varkappa_2\}r, \qquad \varkappa_2 = g(1+\eta_B) + \theta_1\varkappa_1.$$

To show $R \sim 0$ split $R = R_1 + R_2$, where

$$R_j = it\,\mathbf{E}\ \mathrm{e}\{Y\varkappa_2\}r_j, \qquad\qquad r_1 = Y_{A_1}\varkappa_1, \quad r_2 = (Y_{A_2} + Y_B)\varkappa_1$$

and where $A_1 = \{1, \dots, k\}$, $A_2 = \{k+1, \dots, m\}$ and $k \approx m/2$. Let us show $R_2 \sim 0$. We have $|R_2| \ll |t|\,\mathbf{E}\,\zeta|Y_{A_2} + Y_B|\,|\varkappa_1|$, where we denote $\zeta = |\,\mathbf{E}_{A_1}\varepsilon\{Y\varkappa_2\}|$. By (4.12), $\zeta \ll t^{-10}$. Furthermore, by the mean value theorem, $|\varkappa_1| \ll |V - (1+\eta_B)| = Y_B^2/N$. Therefore, $|R_2| \ll |t|^{-9}N^{-1} \sim 0$. The same argument yields $R_1 \sim 0$. We obtain $R \sim 0$ and, hence, $f_{19} \sim f_{24}$.

Write $S_7 = Y_A g(1+\eta_B) + Y_B$. Now we show that

$$\begin{aligned} &f_{24} \sim f_{25} + f_{26}, \qquad f_{25} = \mathbf{E}\ \mathrm{e}\{S_7 - D_B/2\}, \\ &f_{26} = -(it/2)\,\mathbf{E}\ \mathrm{e}\{S_7 - D_B/2\}U_B, \qquad U_B = \sum_{i,j \in B,\, i \neq j} Y_j\eta_j. \end{aligned} \tag{3.31}$$

To prove (3.31) expand $g(1+\eta_B)$ in powers of η_B,

$$g(1+\eta_B) = 1 - \eta_B/2 + \eta_B^2\varkappa, \qquad \text{with} \quad |\varkappa| \ll 1, \tag{3.32}$$

and write $Y_B(1+\eta_B) = Y_B - Y_B\eta_B/2 + r\varkappa$, where $r = Y_B\eta_B^2$. Then $S_6 = S_7 - Y_B\eta_B/2 + r\varkappa$. Expanding the exponent in powers of $itr\varkappa$ we get

$$f_{24} = f_{27} + R, \qquad f_{27} = \mathbf{E}\ \mathrm{e}\{S_7 - Y_B\eta_B/2\}, \quad |R| \ll |t|\,\mathbf{E}\,\zeta|r\varkappa|,$$

where

$$\zeta = \big|\,\mathbf{E}_A\,\mathrm{e}\{Y_A g(1+\eta_B)\}\big| \ll t^{-10}, \tag{3.33}$$

by (4.12). We have $|R| \ll |t|^{-9}\,\mathbf{E}\,|r\varkappa|$. It follows from the inequalities $|\varkappa| \ll 1$ and $\mathbf{E}\,|r| \ll \mathcal{R}$, see Lemma 4.1, that $R \sim 0$. Hence, $f_{24} \sim f_{27}$.

Now we show $f_{27} \sim f_{25} + f_{26}$. Write $Y_B\eta_B = D_B + U_B$ and expand the exponent in f_{27} in powers of $-itU_B/2$ to get

$$f_{27} = f_{25} + f_{26} + R, \qquad |R| \ll t^2\,\mathbf{E}\,\zeta U_B^2,$$

with ζ given by (3.33). The inequality $\mathbf{E}\,U_B^2 \ll \mathcal{R}$, see Lemma 4.1, combined with (3.33) gives $|R| \ll t^{-8}\mathcal{R} \sim 0$. We arrive to (3.31).

Let us show that

$$f_{26} \sim -\frac{1}{2}(N-m-1)(N-m)\tilde{G}_3. \tag{3.34}$$

We prove (3.34) in three steps. Firstly, we replace the exponent $\mathrm{e}\{S_7 - D_B/2\}$ by $\mathrm{e}\{S_7\}$. For this purpose we expand the exponent in powers of $-itD_B/2$ to get $f_{26} = f_{28} + R$,

$$f_{28} = -(it/2)\,\mathbf{E}\;\mathrm{e}\{S_7\}U_B + R, \qquad |R| \ll t^2\big|\,\mathbf{E}\;\mathrm{e}\{S_7 - \theta D_B/2\}D_BU_B\big|.$$

Note that $|\,\mathbf{E}_A\,\mathrm{e}\{S_7\}| \ll \zeta \ll t^{-10}$, see (3.33). Then $|R| \ll t^{-8}\,\mathbf{E}\,|D_BU_B|$. But $\mathbf{E}\,|D_BU_B| \le \mathbf{E}\,(D_B^2 + U_B^2) \ll \mathcal{R}$, see Lemma 4.1. Hence, $R \sim 0$ and thus $f_{26} \sim f_{28}$. Secondly, we replace $\mathrm{e}\{S_7\} = \mathrm{e}\{Y_Ag(1+\eta_B) + Y_B\}$ by $\mathrm{e}\{Y_A + Y_B\}$ in f_{28}. For this purpose we apply the inequality

$$\big|\,\mathbf{E}_A\,\mathrm{e}\{Y_Ag(1+\eta_B)\} - \mathbf{E}_A\,\mathrm{e}\{Y_A\}\big| \ll |t|^{-9}|\eta_B|, \tag{3.35}$$

which is proved in the same way as (3.16). It follows from (3.35) that

$$f_{28} = f_{29} + R, \quad f_{29} = -(it/2)\,\mathbf{E}\;\mathrm{e}\{Y_A + Y_B\}U_B, \qquad |R| \ll t^{-8}\,\mathbf{E}\,|U_B\eta_B|.$$

The inequalities $\mathbf{E}\,|U_B\eta_B| \le \mathbf{E}\,(U_B^2 + \eta_B^2) \ll \mathcal{R}$ implies $R \sim 0$ and we obtain

$$f_{28} \sim f_{29} = -(N-m-1)(N-m)(it/2)\;\mathbf{E}\;\mathrm{e}\{Y\}Y_N\eta_{N-1}.$$

In the third step we expand the exponent in powers of itY_N and then in powers of itY_{N-1} to get

$$f_{29} \sim -\frac{(N-m-1)(N-m)}{2}(it)^3\,\mathbf{E}\;\mathrm{e}\{Y^{(N,N-1)}\}Q_{N,N-1} = -\frac{(N-m-1)(N-m)}{2}\tilde{G}_3.$$

We arrive to (3.34).

Let us prove that

$$f_{25} \sim \mathbf{E}\;\mathrm{e}\{Y\} - \frac{N-m}{2}\tilde{G}_2 - \frac{1}{2}(N-m)m\,\tilde{G}_3. \tag{3.36}$$

Expanding the exponent in f_{25} in powers of $itD_B/2$ we get

$$f_{25} = f_{30} + f_{31} + R, \qquad f_{30} = \mathbf{E}\,\mathrm{e}\{S_7\}, \quad f_{31} = -(N-m)(it/2)\,\mathbf{E}\,\mathrm{e}\{S_7\}d_N,$$
$$|R| \ll \mathbf{E}\,\big|\mathbf{E}_A\,\mathrm{e}\{S_7\}\big|t^2 D_B^2 \ll t^{-8}\,\mathbf{E}\,D_B^2 \sim 0.$$

Here we estimated $|\,\mathbf{E}_A\,\mathrm{e}\{S_7\}| \ll t^{-10}$, see (4.12) and $\mathbf{E}\,D_B^2 \ll \mathcal{R}$, see Lemma 4.1. We obtain $f_{25} \sim f_{30} + f_{31}$.

Let us show that

$$f_{31} \sim \frac{N-m}{2}\,\tilde{G}_2. \tag{3.37}$$

Firstly we replace S_7 by Y in the exponent in f_{31}. Using (9.ETA) we obtain

$$f_{31} = -(N-m)\,\frac{it}{2}\,\mathbf{E}\,\mathrm{e}\{Y\}d_N + R, \qquad |R| \ll t^{-8}\,\mathbf{E}\,|D_B\eta_B| \sim 0,$$

se Lemma 4.1. Finally, expanding the exponent $\mathrm{e}\{Y\} = \mathrm{e}\{Y^{(N)} + Y_N\}$ in powers of itY_N we obtain

$$f_{31} \sim -(N-m)\,\frac{it}{2}\,\mathbf{E}\,\mathrm{e}\{Y^{(N)}\}d_N = -\frac{N-m}{2}\,\tilde{G}_2,$$

thus proving (3.37).

Let us prove that

$$f_{30} \sim \mathbf{E}\,\mathrm{e}\{Y\} - \frac{1}{2}(N-m)m\tilde{G}_3. \tag{3.38}$$

Write $A_1 = \{1, \dots, k\}$ and $A_2 = \{k+1, \dots, m\}$, where $k \approx m/2$. Furthermore, write $S_8 = Y_{A_2}g(1+\eta_B) + Y_B + Y_{A_1}$. Firstly, we show

$$f_{30} \sim f_{32} + f_{33}, \quad f_{32} = \mathbf{E}\,\mathrm{e}\{S_8\}, \quad f_{33} = -\frac{it}{2}\,\mathbf{E}\,\mathrm{e}\{S_8\}Y_{A_1}\eta_B. \tag{3.39}$$

An application of (3.32) gives

$$S_7 = Y_{A_2}g(1+\eta_B) + Y_B + Y_{A_1}g(1+\eta_B) = S_8 - Y_{A_1}\eta_B/2 + Y_{A_1}\eta_B^2\varkappa, \quad |\varkappa| \ll 1.$$

Now expand the exponent in powers of $itY_{A_1}\eta_B^2\varkappa$ and then in powers of $-itY_{A_1}\eta_B/2$ to get $f_{30} = f_{32} + f_{33} + R$, with $|R| \ll \mathbf{E}\,\zeta r$, where

$$\zeta = \big|\,\mathbf{E}_{A_2}\,\mathrm{e}\{Y_{A_2}g(1+\eta_B)\}\big| \qquad \text{and} \qquad r = t^2Y_{A_1}^2\eta_B^2 + |tY_{A_1}|\eta_B^2.$$

By (4.12), $\zeta \ll t^{-10}$. Furthermore, a simple calculation gives $\mathbf{E}\,|r| \ll (t^2 + |t|)\mathcal{M}$. Hence, $R \sim 0$ and we obtain (3.39).

Now we show that

$$f_{33} \sim -2^{-1}(N-m)k\tilde{G}_3. \tag{3.40}$$

For this purpose we use the inequality

$$\big| \mathbf{E}_{A_2} \, \mathrm{e}\{Y_{A_2} g(1+\eta_B)\} - \mathbf{E}_{A_2} \, \mathrm{e}\{Y_{A_2}\}\big| \ll |t|^{-9} |\eta_B|, \tag{3.41}$$

which can be proved in the same way as (3.16). Using (3.41) we replace $Y_{A_2} g(1+\eta_B)$ by Y_{A_2} in the exponent in f_{33}. Thus we have

$$f_{33} \sim f_{34}, \quad f_{34} = -\frac{it}{2} \, \mathbf{E} \, \mathrm{e}\{Y\} Y_{A_1} \eta_B = -(N-m)k \, \frac{it}{2} \, \mathbf{E} \, \mathrm{e}\{Y\} Y_1 \eta_N.$$

Finally, expanding the exponent $\mathrm{e}\{Y\} = \mathrm{e}\{Y^{(1,N)} + Y_N + Y_1\}$ in powers of itY_1 and then in powers of itY_N we obtain

$$f_{34} \sim -(N-m)k \, \frac{(it)^3}{2} \, \mathbf{E} \, \mathrm{e}\{Y^{(1,N)}\} Q_{1,N} = -\frac{1}{2}(N-m)k\tilde{G}_3 \tag{3.42}$$

thus completing the proof of (3.40).

It remains to show that

$$f_{32} \sim \mathbf{E} \, \mathrm{e}\{Y\} - \frac{1}{2}(N-m)(m-k)\tilde{G}_3. \tag{3.43}$$

By (3.32), we can write $S_8 = Y - Y_{A_2}\eta_B/2 + Y_{A_2}\eta_B^2\varkappa$, where $|\varkappa| \ll 1$ depends only on η_B. Expanding $\mathrm{e}\{S_8\}$ in powers of $itY_{A_2}\eta_B^2\varkappa$ and then in powers of $-itY_{A_2}\eta_B/2$ we get

$$f_{32} \sim \mathbf{E} \, \mathrm{e}\{Y\} - \frac{it}{2} \, \mathbf{E} \, \mathrm{e}\{Y\} Y_{A_2} \eta_B,$$

cf. the proof of (3.39). Finally, proceeding as in proof of (3.42) we obtain (3.43) thus completing the proof of (3.38). We arrive to (3.36). The proof of (3.11) is complete.

Step 2. Here we show that

$$\mathbf{E} \, \mathrm{e}\{Y\} \sim \mathbf{E} \, \mathrm{e}\{\xi\} + \frac{N}{6} G_1 \tag{3.44}$$

$$\frac{N}{2}\tilde{G}_2 \sim \frac{N}{2} G_2, \qquad \frac{N(N-1)}{2}\tilde{G}_3 \sim \frac{N(N-1)}{2} G_3. \tag{3.45}$$

There are known several proofs of (3.44), see e.g. Petrov (1975), Bentkus, Götze, Paulauskas and Račkauskas (1990), where asymptotic expansions of a linear statistic are considered. In the case where the summands $Y_1, \dots, Y_N$ are centered, Bentkus, Götze, Paulauskas and Račkauskas (1990) give the following bound

$$\Delta := \int_{c \le |t| \le L} \Big| \mathbf{E} \, \mathrm{e}\{Y_1 + \cdots + Y_N\} - \mathbf{E} \, \mathrm{e}\{\xi\} - \frac{N}{6}\tilde{G}_1 \Big| dt/|t| \ll N b_4.$$

This bound easily extends to more general situation of sums of random variables which are not necessarily centered. In this case we have

$$\Delta \ll N\, b_4 + N\, |\, \mathbf{E}\, Y_1|,$$

and thus (3.44).

In order to prove (3.45) it suffices to show that

$$\int_{100 \le |t| \le L} |\delta_i| dt/|t| \ll N\, b_3 + N|\, \mathbf{E}\, Y_1|,$$

where

$$\delta_1(t) = \mathbf{E}\ \mathrm{e}\{Y^{(1)}\} - \mathbf{E}\ \mathrm{e}\{\tilde{\xi}^{(1)}\}, \qquad \delta_2(t) = \mathbf{E}\ \mathrm{e}\{Y^{(1,2)}\} - \mathbf{E}\ \mathrm{e}\{\tilde{\xi}^{(1,2)}\}.$$

The proof of these inequalities is well known in the case where the summands $Y_1, \dots, Y_N$ are centered (then $N|\,\mathbf{E}\, Y_1| = 0$), see, e.g., Bentkus, Götze, Paulauskas and Račkauskas (1990). Their proof easily extends to more general situation where the summands are not necessarily centered. Lemma is proved

4. Auxiliary Results

We shall say that condition (A) is satisfied if at least one set of inequalities (2.3) or (2.10) hold.

Lemma 4.1. *Assume that the condition (A) is satisfied. Then*

$$\mathbf{E}\, |Y_B|^s \le c(s), \tag{4.1}$$

for each $s > 0$ and each subset $B \subset \{1, \dots, N\}$. Furthermore, for every $m \le n$,

$$\mathbf{E}\, D^2 \ll m b_4 + m^2 b_3^2 \ll \mathcal{M}, \qquad D = \sum_{1 \le j \le m} d_j,$$

$$\mathbf{E}\, U^2 \ll (m/N)^2 \mathcal{M}, \qquad U = \sum_{1 \le i,j \le m,\, i \ne j} Y_i \eta_j.$$

Write $A = \{1, \dots, m\}$, where $m < n$. We have $\mathbf{E}\, \eta_A^2 \ll m b_4$,

$$\mathbf{E}\, |Y_A|^s \eta_A^2 \ll m b_4, \qquad \text{for} \quad s = 1, 2; \tag{4.2}$$

$$\mathbf{E}\, |Y_A|^s |\eta_A| N^{-1} \le c(s) m b_4, \qquad \text{for} \quad s > 0; \tag{4.3}$$

$$\mathbf{E}\, |Y_A|^s N^{-1} \ll m b_4 \qquad \text{for} \quad s = 2, 3. \tag{4.4}$$

Proof of Lemma 4.1. The last inequality is trivial. The inequality $\mathbf{E}\,|Y_B|^s \le c(s)$ is proved in Bentkus and Götze (1996).

To estimate $\mathbf{E}\,D^2$ write

$$\mathbf{E}\,D^2 \le \mathbf{E}\,T_1 + \mathbf{E}\,|T_2|, \qquad T_1 = \sum_{1\le i\le m} d_i^2, \quad T_2 = \sum_{1\le i,j\le m,\, i\ne j} d_i d_j.$$

Using the inequality $|Y_i| \le 1$ we get $\mathbf{E}\,T_1 \ll m b_4$. Furthermore, $\mathbf{E}\,|T_2| \ll m^2 b_3^4$. Finally, by (2.2), $(N b_3)^2 \le \mathcal{M}$.

Let us estimate $\mathbf{E}\,U^2$. Write

$$U = U_1 + U_2, \qquad U_1 = \sum_{1\le i,j\le m,\, i\ne j} \mathbf{E}\,Y_i \eta_j, \quad U_2 = \sum_{1\le i,j\le m,\, i\ne j} (Y_i - \mathbf{E}\,Y_i)\eta_j.$$

A simple calculation shows

$$\mathbf{E}\,U_1^2 = \big((m-1)\,\mathbf{E}\,Y_1\big)^2\,\mathbf{E}\,(\eta_1 + \cdots + \eta_m)^2 \ll (m/N)^3 \gamma^2 \mathcal{M},$$
$$\mathbf{E}\,U_2^2 = m(m-1)\,\mathbf{E}\,(Y_1 - \mathbf{E}\,Y_1)^2\,\mathbf{E}\,\eta_2^2 \ll (m/N)^2 \mathcal{M}.$$

Clearly, the bound $\mathbf{E}\,U^2 \ll (m/N)^2\mathcal{M}$ is an easy consequnce of these inequalities.

The inequality $\mathbf{E}\,\eta_A^2 \ll m\,b_4$ is trivial. Let us prove (4.2). Write $\eta_A^2 = T_1 + T_2$, where

$$T_1 = \sum_{i\in A} \eta_i^2 \qquad \text{and} \quad T_2 = \sum_{i,j\in A,\, i\ne j} \eta_i \eta_j.$$

Note that $\mathbf{E}\,T_2^2 \ll m^2 b_4^2$. Combining the inequalities

$$\mathbf{E}\,|Y_A|^s \eta_A^2 \le \mathbf{E}\,|Y_A|^s |T_1| + \mathbf{E}\,|Y_A|^s |T_2|,$$
$$\mathbf{E}\,|Y_A|^s |T_2| \le \big(\mathbf{E}\,|Y_A|^{2s}\big)^{1/2} \big(\mathbf{E}\,T_2^2\big)^{1/2} \ll \big(\mathbf{E}\,T_2^2\big)^{1/2} \ll m b_4,$$
$$\mathbf{E}\,|Y_A|^s |T_1| \le m\,\mathbf{E}\,|Y_A^{(1)} + Y_1|^s \eta_1^2 \ll m\,\mathbf{E}\,|Y_A^{(1)}|^s\,\mathbf{E}\,\eta_1^2 + m\,\mathbf{E}\,|Y_1|^s \eta_1^2 \ll m b_4,$$

we obtain (4.2).

Let us prove (4.3). By Hölder's inequality and (4.1),

$$\mathbf{E}\,|Y_A|^s |\eta_A| \le \big(\mathbf{E}\,|Y_A|^{2s}\big)^{1/2}\big(\mathbf{E}\,\eta_A^2\big)^{1/2} \ll \big(\mathbf{E}\,\eta_A^2\big)^{1/2} \le (m b_4)^{1/2}.$$

Now (4.3) follows from the simple bound $N^{-1} \le b_4^{1/2}$, see (2.2).

Let us prove (4.4). We have

$$\mathbf{E}\,|Y_A|^s \le c(s)\,\mathbf{E}\,|Y_A - \mathbf{E}\,Y_A|^s + c(s)|\,\mathbf{E}\,Y_A|^s,$$

By (A), $|\,\mathbf{E}\,Y_A|^s = (m/N)^s \gamma^s \le (m/N)^s$. Furthermore, and application of Rozenthal's inequality gives

$$\mathbf{E}\,|Y_A - \mathbf{E}\,Y_A|^s \le c(s) m\,\mathbf{E}\,|Y_1 - \mathbf{E}\,Y_1|^s + c(s)(m\,\mathbf{E}\,(Y_1 - \mathbf{E}\,Y_1)^2)^{s/2} \ll m b_s + (m/N)^{s/2},$$

for $s = 2, 3$. Thus, we obtain

$$\mathbf{E}\,|Y_A|^s \ll (m/N)\,N b_s + (m/N)^{s/2} + (m/N)^s \ll (m/N),$$

for $s = 2, 3$. Now (4.4) follows from the inequality $(m/N)N^{-1} \le m b_4$, see (2.2).

Lemma 4.2. *Assume that the condition (A) is satisfied. Then*

$$1-\sqrt{2}\,\gamma_0 \le a \le 1, \qquad 1-2\beta_4/N \le a \le 1. \tag{4.5}$$

In particular we have $3/4 \le a \le 1$.

Proof of Lemma 4.2. Clearly, $a^2 \le \sigma^2 \le 1$. Furthermore, write $\varkappa(u) = \mathbf{E}\, X_1^2 \mathbb{I}\{X_1^2 \le u\,N\}$, for $u \ge 0$. We have

$$\tau := 1-\varkappa(1/2) = \mathbf{E}\, X_1^2 \mathbb{I}\{X_1^2 > N/2\} \le \sqrt{2}\,\gamma_0 \le 2\beta_4/N. \tag{4.6}$$

Note that $\sqrt{2}\gamma_0 < 1/4$, by (2.3), and $2\beta_4/N < 1/4$, by (2.10). In each case we have $\tau < 1/2$. The function $\varkappa(u)$ is nondecreasing. Therefore, (4.6) implies $\varkappa(1-\tau) \ge \varkappa(1/2) = 1-\tau$. But, $\varkappa(1) \le \sigma^2 = 1$. Then there exists a solution of the equation $u = \varkappa(u)$ in the interval $1-\tau \le u \le 1$. This implies $a^2 \ge 1-\tau$ and we obtain $a \ge a^2 \ge 1-\tau$. This inequality in combination with (4.6) yields (4.5) thus completing the proof of the lemma.

Lemma 4.3. *Asume that (2.3) holds. Let S_0 be given by (2.5). Then*

$$\mathbf{P}\,\{|S_0-\mathbf{t}| \ge N^{-2/3}\} \ll \mathcal{R}, \qquad \mathcal{R} = \rho_x^{-2}\mathcal{M}+\gamma_0. \tag{4.7}$$

Assume that (2.10) holds. Let S_1 be given by (2.12). Then

$$\mathbf{P}\,\{|S_1-\mathbf{t}| > \mathcal{M}\} \ll \rho^{-2}N^{-1}\beta_4. \tag{4.8}$$

Proof of Lemma 4.3. The inequalities

$$\mathbf{P}\,\Big\{\sqrt{N}\,\mathbf{t} \ne \frac{Y}{\sqrt{1+\eta-Y^2/N}}\Big\} \le N\;\mathbf{P}\,\{X_1^2 > a^2N\} \le (4/3)^3\gamma_0 \ll \beta_4/N,$$
$$\mathbf{P}\,\Big\{\frac{Y}{\sqrt{1+\eta-Y^2/N}} \ne S\Big\} \le \mathbf{P}\,\{|\eta| > 1/4\} + \mathbf{P}\,\{Y^2/N > 1/4\}$$

and

$$\mathbf{P}\,\{|\eta| > 1/4\} \le 16\,\eta^2 \le 32N\;\mathbf{E}\;Y_1^4, \qquad \mathbf{P}\,\{Y^2/N > 1/4\} \le 4\,N^{-1}\,\mathbf{E}\;Y^2 = 4/N$$

imply $\mathbf{P}\,\{\mathbf{t} \ne S\} \ll \mathcal{R}$.

In order to prove (4.7) it remains to show $\mathbf{P}\,\{|S-S_0| > N^{-2/3}\} \ll \mathcal{R}$. Write $S = Yg(V_B+\eta_A+W)$, where $W = -Y_A^2/N - 2Y_AY_B/N$. Expanding g in powers of W we get $|S - Yg(V_B+\eta_A)| \le c_g|YW|$, where $|YW| \le R_1+R_2+R_3$ and

$$R_1 = |Y_A|^3/N, \qquad R_2 = 3Y_A^2|Y_B|/N, \qquad R_3 = 2|Y_A|Y_B^2/N. \tag{4.9}$$

Expanding g in powers of η_A we obtain

$$(4.10)\qquad |Y g(V_B+\eta_A) - S_0| \le c_g(R_4+R_5), \qquad R_4 = |Y_A\eta_A|, \quad R_5 = |Y_B|\eta_A^2.$$

Then $|S-S_0| \le c_g(R_1+\dots+R_5)$ and, by Chebyshev's inequality,

$$\begin{aligned} &\mathbf{P}\left\{|S-S_0| \ge N^{-2/3}\right\} \\ &\ll N^{2/3}\,\mathbf{E}\,R_1 + N^{2/3}\,\mathbf{E}\,R_2 + \mathbf{E}\,N^{4/3}R_3^2 + \mathbf{E}\,(N^{2/3}R_4)^{4/3} + N^{2/3}\,\mathbf{E}\,R_5. \end{aligned}$$

A simple calculation shows that the right hand side is bounded by $c\mathcal{R}$ thus completing the proof of (4.7)

In order to prove (4.8) it suffices to show $\mathbf{P}\left\{|S-S_1| > \mathcal{M}\right\} \le c\rho^{-2}N^{-1}\beta_4$. We have $V = V_B+\eta_A - W_1 - W_2$, where W_1 and W_2 are given by (2.24). Expanding g in powers of W_1+W_2 and then in powers of η_A we get

$$\begin{aligned} (4.11)\quad |S-S_1| \le{}& c_g\bigl(R_1+\dots+R_6\bigr), \qquad \text{where} \quad R_1 = |Y_AW_1|, \quad R_2 = |Y_AW_2|, \\ & R_3 = |Y_BW_1|, \quad R_4 = |Y_BW_2|, \quad R_5 = |Y_A|\eta_A^2, \quad R_6 = |Y_B\eta_A^3|. \end{aligned}$$

We complete the proof by showing $P_i := \mathbf{P}\left\{|R_i| > \mathcal{M}/6\right\} \ll \rho^{-2}\mathcal{M}$, where we abbreviate ρ_N by ρ. In what follows we use the inequalities $N\mathcal{M} \ge 1$ and $\mathbf{E}\,|Y_D|^s \le c(s)$, see Lemma 4.1. By Chebyshev's inequality,

$$\begin{aligned} &P_1 \ll (\mathcal{M}\,N)^{-4/3}\,\mathbf{E}\,Y_A^4 \ll \mathbf{E}\,Y_A^4 \ll m^3b_4 \ll \rho^{-2}\mathcal{M}, \\ &P_i \ll (\mathcal{M}\,N)^{-2}\,\mathbf{E}\,Y_B^2\,\mathbf{E}\,Y_A^4 \ll \mathbf{E}\,Y_A^4 \ll \rho^{-2}\mathcal{M}, \quad i = 1,\,2, \\ &P_4 \ll (\mathcal{M}\,N)^{-4}\,\mathbf{E}\,Y_B^8Y_A^4 \ll \mathbf{E}\,Y_A^4 \ll \rho^{-2}\mathcal{M}, \\ &P_5 \ll \mathcal{M}^{-4/5}\,\mathbf{E}\,|Y_A\eta_A|^{4/5} \ll \mathcal{M}^{-4/5}\bigl(\mathbf{E}\,Y_A^4 + \mathbf{E}\,\eta_A^2\bigr) \ll \mathcal{M}^{-4/5}m^3b_4 \ll \rho^{-2}\mathcal{M}, \end{aligned}$$

Here we applied the inequality $a\,b^4 \le a^5+b^5$ to $a = |Y_A|$ and $b = |\eta_A|^{1/2}$. Finally, the inequalities

$$P_6 \ll \mathcal{M}^{-2/3}\,\mathbf{E}\,|Y_B\eta_A^3| \ll \mathcal{M}^{-2/3}\,\mathbf{E}\,\eta_A^2 \ll \mathcal{M}^{-2/3}mb_4 \ll \rho^{-1}\mathcal{M}$$

complete the proof of (4.8). Lemma is proved.

Write $\eta_B = \eta_m+\dots+\eta_N$ and $Y_B = Y_m+\dots+Y_N$,

$$\begin{aligned} &u_1(t) = \mathbf{E}_1\,\mathrm{e}\{Y_1g(1+\eta_B)\}, \;\; u_2(t) = \mathbf{E}_1\,\mathrm{e}\{Y_1g(1+\eta_B^{(N)})\}, \;\; u_3(t) = \mathbf{E}_1\,\mathrm{e}\{Y_1g(V_2)\}, \\ &r_1(t,\theta) = \mathbf{E}_1\,\mathrm{e}\bigl\{Y_1\bigl(g(1+\eta_B)+\theta(g(V)-g(1+\eta_B))\bigr)\bigr\}, \qquad V = 1+\eta_B - Y_B^2/N, \\ &r_2(t,\theta) = \mathbf{E}_1\,\mathrm{e}\bigl\{Y_1\bigl(g(V_2)+\theta(g(V)-g(V_2))\bigr)\bigr\}, \qquad V_2 = 1+\eta_B^{(N)} - (Y_B^{(N)})^2/N. \end{aligned}$$

Lemma 4.4. *Assume that the condition (A) is satisfied. Then for each $\theta \in (0,1)$ and t, $100 \le |t| \le L$ ($L = \big(4\,N\,b_3\big)^{-1}$) and every integer $k \ge m/4$ $\big(m = 400Nt^{-2}\ln|t|\big)$,*

$$|u(t)|^k \ll |t|^{-10}, \quad |u_j(t)|^k \ll |t|^{-10}, \qquad \text{and} \quad |r_i(t,\theta)|^k \ll |t|^{-10}, \tag{4.12}$$

for $j = 1,2,3$ and $i = 1,2$ and

$$|w(t)|^k \ll (1 + Y_B^{20})t^{-10}, \qquad\qquad |v(t)|^k \ll (1 + Y_B^{20})t^{-10}, \tag{4.13}$$

where the functions $u(t)$, $v(t)$ and $w(t)$ are given by (3.13).

Proof of Lemma 4.4. Let us prove the first inequality in (4.12).
The inequality

$$|\,\mathbf{E}\,\mathrm{e}\{W\}|^2 \le 1 - t^2\,\mathbf{E}\,(W - \,\mathbf{E}\,W)^2 + (4/3)|t|^3\,\mathbf{E}\,|W|^3,$$

see, e.g., Petrov (1987), applied to the random variable $Y_1 g(V_B)$ conditionally given $Y_{m+1}, \dots, Y_N$, implies

$$|u(t)|^2 \le 1 - t^2 g^2(V_B)\big(\frac{1}{N} - (\,\mathbf{E}\,Y_1)^2\big) + \frac{4}{3}\,|t|^3 b_3 |g(V_B)|^3 \le 1 - \frac{t^2}{4N}, \tag{4.14}$$

for $|t| \le H_1$, provided that c_0 is sufficiently small. Here we used (2.4) to estimate $|\,\mathbf{E}\,Y_1|$ and used the inequalities $8/9 \le g \le 8/7$. We obtain

$$|u(t)|^{m/4} \le \big(1 - \frac{t^2}{4N}\big)^{m/8} \le \exp\{-t^2 m/(32N)\} \le |t|^{-10}.$$

thus obtaining the first inequality of (4.12). Rest inequalities in (4.12) can be proved in the same way.

Let us prove the first inequality of (4.13). The second inequality will follow as a byproduct. Write $\tilde c_g = 1 + \sup_x |g'(x)|$. We may and shall assume that $\tilde c_g < 10$. Let $\mathbb{I}_{\mathcal{B}}$ denote the indicator of the event $\mathcal{B} = \{100\tilde c_g Y_B^2 \mathcal{M} \le 1\}$.

Expanding the exponent in powers of $itd_1 g'(V)$ and then in powers of $it\eta_1 Y_B g'(V)$ we get

$$w(t) = v(t) + t\,R_1 \qquad \text{and} \qquad v(t) = u(t) + t^2 R_2, \tag{4.15}$$

where

$$|R_1| \le \tilde c_g\,\mathbf{E}_{\,1}|d_1| \le 2\tilde c_g b_3, \qquad |R_2| \le 2^{-1}\tilde c_g\,\mathbf{E}_{\,1}\eta_1^2 Y_B^2 \le \tilde c_g b_4 Y_B^2.$$

Furthermore, $|t|b_3 \le (4N)^{-1}$ and $\tilde c_g \mathbb{I}_{\mathcal{B}} \mathcal{M} Y_B^2 \le 100^{-1}$. Therefore,

$$\mathbb{I}_{\mathcal{B}}\big(|t\,R_1| + t^2|R_2|\big) \le \frac{2\,\tilde c_g}{4\,N} + \frac{t^2}{100N} \le \frac{t^2}{50N}. \tag{4.16}$$

It follows from (4.14), (4.15) and (4.16) that

$$\left|\mathbb{I}_{\mathcal{B}} w(t)\right| \le |u(t)| + t^2/(50N) \le \left(1 - t^2/(4\,N)\right)^{1/2} + t^2/(50N).$$

Invoking the inequality $(1-a)^{1/2} \le 1 - a/2$, $0 \le a \le 1$, we obtain

$$\mathbb{I}_{\mathcal{B}} |w(t)| \le 1 - \frac{t^2}{8\,N} + \frac{t^2}{50N} \le 1 - \frac{t^2}{10\,N}\,.$$

For a sufficiently large $c_2 > 0$, this inequality implies

$$\mathbb{I}_{\mathcal{B}} |w(t)|^{m/4} \le \exp\bigl\{-\frac{t^2}{20}\,\frac{m}{N}\bigr\} \le |t|^{-c_2/20} \le |t|^{-10}. \tag{4.17}$$

Now write $1 = \mathbb{I}_{\mathcal{B}} + (1 - \mathbb{I}_{\mathcal{B}})$ and note that $1 - \mathbb{I}_{\mathcal{B}} \ll (Y_B^2 \mathcal{M})^{10}$. Since, $\mathcal{M} \le N\beta_3 \ll |t|^{-1}$, for $|t| \le L$, we get

$$1 - \mathbb{I}_{\mathcal{B}} \ll Y_B^{20} t^{-10}$$

and combining this inequality with (4.17) we obtain (4.13) thus completing the proof of the lemma.

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