THE DEFECT OF WEAK APPROXIMATION FOR HOMOGENEOUS SPACES

Mikhail Borovoi

Abstract. Let $X = H\backslash G$ be a homogeneous space defined over a number field k, where G is a connected linear k-group satisfying certain conditions, and H is a connected k-subgroup of G. For a finite set S of places of k we define a finite abelian group $A_S(X)$ which is the defect of weak approximation for X with respect to S. We describe $A_S(X)$ in terms of the Brauer group of X.

Introduction

Let k be an algebraic number field, $\mathcal{V}$ the set of places of k, and $S \subset \mathcal{V}$ a finite subset. Let X be an algebraic variety over k such that $X(k) \neq \varnothing$. We say that X *satisfies the condition of weak approximation with respect to* S if

$$(WA_S) \qquad X(k) \text{ is dense in } \prod_{v\in S} X(k_v)\,,$$

k_v denoting the completion of k at v. We say that X *satisfies the condition of weak approximation* if

$$(WA) \qquad X \text{ satisfies } (WA_S) \text{ for all finite } S\,.$$

This is the same as to require that $X(k)$ is dense in $\prod_{v\in\mathcal{V}} X(k_v)$.

Let $k_S = \prod_{v\in S} X(k_v)$; then $X(k_S) = \prod_{v\in S} X(k_v)$. Let $X(k)\widehat{_S}$ denote the closure of $X(k)$ in $X(k_S)$, and let $X(k)\widehat{_\mathcal{V}}$ denote the closure of $X(k)$ in $\prod_{v\in\mathcal{V}} X(k_v)$. Clearly, (WA_S) holds for X if and only if $X(k)\widehat{_S} = X(k_S)$, and (WA) holds for X if and only if $X(k)\widehat{_\mathcal{V}} = \prod_{v\in\mathcal{V}} X(k_v)$.

Let T be an algebraic k-torus. Set $A(T) = (\prod_{v\in\mathcal{V}} T(k_v))/T(k)\widehat{_\mathcal{V}}$. The abelian group $A(T)$ is the defect of weak approximation for T; in other words, it is the measure of failure of (WA) for T. In particular, T satisfies (WA) if and only if $A(T) = 0$. The group $A(T)$ was studied by Voskresenskiĭ [Vo1], [Vo2]. He showed that $A(T)$ is finite, and related it to a certain group $H^1(k, \mathrm{Pic}(\bar{V}(T)))$. As Sansuc later showed, $A(T)$ can be computed in terms of the Brauer group of T, cf. [Sa], 8.12.

For any connected linear k-group G one can define the set $A(G) = (\prod_{v\in\mathcal{V}} G(k_v))/G(k)\widehat{_\mathcal{V}}$. As above, $A(G)$ is the defect of weak approximation for

Partially supported by the US–Israel Binational Science Foundation and by the Hermann Minkowski Center for Geometry

G. Sansuc ([Sa], 3.3), generalizing Voskresenskiĭ's results, proved that the subgroup $\widehat{G(k)}_{\mathcal{V}}$ is normal and that the quotient group $A(G)$ is finite and abelian. He showed that it is possible to compute $A(G)$ in terms of the Brauer group of G, cf. [Sa], 8.12.

In this paper we consider the case of a homogeneous space. Let $X = H\backslash G$, where G is a connected linear k-group and $H \subset G$ is a connected k-subgroup. We suppose that $A(G) = 0$ and $Ш(G) = 0$, where $Ш(G)$ is the Tate-Shafarevich group of G. These conditions hold for example when G is a simply connected group, or an adjoint group, or an absolutely simple group, or if G splits over a cyclic extension, cf. [Sa], 5.3. For such X we construct a certain finite abelian group $A_S(X)$ which is the defect of (WA_S) for X. We construct $A_S(X)$ in terms of H and G, but then we compute $A_S(X)$ in terms of the Brauer group of X. We obtain a formula for $A_S(X)$ in terms of the Brauer group of X, similar to Sansuc's formula ([Sa], 8.12) for $A(G)$ in terms of the Brauer group of G for a connected k-group G.

The plan of the paper is as follows. In Section 1 we state the main results. In Section 2 we restate the problem of weak approximation for homogeneous spaces in terms of Galois cohomology. In Section 3 we prove a theorem describing $A_S(H\backslash G)$ in terms of H and G. In Section 4 we prove a theorem describing $A_S(H\backslash G)$ in terms of the Brauer group of $H\backslash G$.

The results of this paper were announced in [Bo1]. The case of a homogeneous space $H\backslash G$ of a *simply connected* group G was treated in [Bo2].

Acknowledgements. This paper was partly written when I was staying at SFB 343 "Diskrete Strukturen in der Mathematik" at Bielefeld University. I am grateful to SFB 343 for hospitality, support, and excellent working conditions.

The paper is inspired by Sansuc's paper [Sa]. The computation of $A_S(H\backslash G)$ in terms of H and G is based on results of Kottwitz [Ko1], [Ko2] in the more functorial version of [Bo3].

I am grateful to Boris Kunyavskii for valuable remarks.

Notation

k is a field of characteristic 0, $\bar{k}$ is an algebraic closure of k, $\Gamma = \mathrm{Gal}(\bar{k}/k)$. When k is a number field, then $\mathcal{V}$, $\mathcal{V}_\infty$, $\mathcal{V}_f$ are the set of all places of k, of infinite places, of finite places, resp. For $v \in \mathcal{V}$, k_v denotes the completion of k at v. For a finite subset $S \subset \mathcal{V}$ we set $k_S = \prod_{v\in S} k_v$. We write $\mathbf{A}$ for the adèle ring of k. For a connected k-group G we set

$$H^1(k_S, G) = \prod_{v\in S} H^1(k_v, G), \quad H^1(\mathbf{A}, G) = \bigoplus_{v\in\mathcal{V}} H^1(k_v, G)\,,$$

where $\bigoplus$ denotes the subset of the direct product consisting of families (ξ_v) for which $\xi_v = 1$ for all but a finite number of places v.

Let G be a connected linear group over a field F. Then:
G^{u} is the unipotent radical of G.
$G^{\mathrm{red}} = G/G^{\mathrm{u}}$; it is a reductive group.
$G^{\mathrm{ss}} = (G^{\mathrm{red}})^{\mathrm{derived}}$; it is a semisimple group.
$G^{\mathrm{tor}} = G^{\mathrm{red}}/G^{\mathrm{ss}}$, it is a torus.

G^{sssc} denotes the universal covering of G^{ss}; it is a semisimple simply connected group.

$\rho\colon G^{\mathrm{sssc}} \to G^{\mathrm{ss}} \to G^{\mathrm{red}}$ is the composition map.

Let A be a finite abelian group, then A^D denotes the dual group for A, $A^D = \mathrm{Hom}(A, \mathbf{Q}/\mathbf{Z})$.

1. Main results

1.1. Let k be a field of characteristic 0. Set $\Gamma = \mathrm{Gal}(\bar{k}/k)$. Let G be a connected linear k-group. We refer to [Bo3], [Bo1] for the definition of the algebraic fundamental group $\pi_1(G)$ (see also 2.1 below). It is a finitely generated abelian group endowed with a Γ-action. The Γ-module $\pi_1(G)$ depends on G functorially.

Set $B(G) = (\pi_1(G)_\Gamma)_{\mathrm{tors}}$, the torsion subgroup of the group of Γ-coinvariants of $\pi_1(G)$. The finite abelian group $B(G)$ depends on G functorially. Kottwitz [Ko1], (2.4.1) proved that $B(G) = \mathrm{Pic}(G)^D$, the dual group for the Picard group $\mathrm{Pic}(G)$ of G.

1.2. Let k be a number field and S a finite set of places of k. Let G be a connected linear k-group and $H \subset G$ a connected k-subgroup. Set $X = H\backslash G$. We are going to construct a finite abelian group $C_S(H,G)$ which is the defect of (WA_S) for X.

The embedding $H \hookrightarrow G$ defines homomorphisms $\pi_1(H) \to \pi_1(G)$ and $B(H) \to B(G)$. Set $B(H,G) = \ker[B(H) \to B(G)]$, it is a finite abelian group.

Let v be a place of k. Set $\Gamma_v = \mathrm{Gal}(\bar{k}_v/k_v)$. Fix an embedding $\bar{k} \hookrightarrow \bar{k}_v$; then we can regard Γ_v as a subgroup of Γ. Set $B_v(H,G) = B(H_{k_v}, G_{k_v})$. We obtain a homomorphism $\lambda_v\colon B_v(H,G) \to B(H,G)$. Set $B^S(H,G) = \langle \lambda_v(B_v(H,G)) \rangle_{v \notin S}$, the subgroup of $B(H,G)$ generated by the groups $\lambda_v(B_v(H,G))$ for $v \notin S$. Set $B'(H,G) = B^{\varnothing}(H,G)$. Then $B^S(H,G) \subset B'(H,G)$. We set $C_S(H,G) = B'(H,G)/B^S(H,G)$.

1.3. Theorem. *Let G, H and X be as in 1.2. Suppose that $A(G) = 0$ and $\text{Ш}(G) = 0$. Then there exists a canonical surjective map $X(k_S) \to C_S(H,G)$ with kernel $\widehat{X(k)}_S$.*

We see that the condition (WA_S) for X is equivalent to the condition $C_S(H,G) = 0$. We set $A_S(X) = C_S(H,G)$ and say that $A_S(X)$ is the defect of (WA_S) for X.

Theorem 1.3 will be proved in Section 3.

1.4. Set $B^\omega(H,G) = \bigcap_S B^S(H,G)$, where S runs over all the finite subsets of $\mathcal{V}$. We set $C_\omega(H,G) = B'(H,G)/B^\omega(H,G)$. It is clear that $C_\omega(H,G) = 0$ if and only if $C_S(H,G) = 0$ for all finite $S \subset \mathcal{V}$.

1.5. Corollary. *$H\backslash G$ satisfies (WA) if and only if $C_\omega(H,G) = 0$.*

Proof. Indeed, $C_\omega(H,G) = 0$ if and only if $C_S(H,G) = 0$ for all finite S. By Theorem 1.3 $C_S(H,G) = 0$ for all S if and only if (WA_S) holds for $H\backslash G$ for all S. Thus $C_\omega(H,G) = 0$ if and only if (WA) holds for $H\backslash G$. $\square$

1.6. Corollary. *Let L/k be a finite Galois extension of k in $\bar{k}$ such that* $\mathrm{Gal}(\bar{k}/L)$ *acts trivially on $\pi_1(H)$ and $\pi_1(G)$. Let $S_0 \subset \mathcal{V}$ be the finite set of places (nonarchimedean, ramified in L) with noncyclic decomposition groups in* $\mathrm{Gal}(L/k)$. *Then $C_S(H,G) = C_{S\cap S_0}(H,G)$. In particular, if $S \cap S_0 = \varnothing$ then $H\backslash G$ satisfies (WA_S).*

Proof. Let $v \in \mathcal{V}$, and let w be a place of L over v. Let D_w denote the decomposition group of w in $\mathrm{Gal}(L/k)$. It is easy to show that the subgroup $\lambda_v(B_v(H,G)) \subset$

$B(H,G)$ depends only on the conjugacy class of D_w in $\mathrm{Gal}(L/k)$. If D_w is cyclic and $v \in S$, then, by Chebotarev's density theorem, there exists a place $v' \notin S$ such that $\lambda_{v'}(B_{v'}(H,G)) = \lambda_v(B_v(H,G))$, and so $B^{S \smallsetminus \{v\}}(H,G) = B^S(H,G)$. Thus $B^S(H,G) = B^{S\cap S_0}(H,G)$ and $C_S(H,G) = C_{S\cap S_0}(H,G)$. In particular, if $S \cap S_0 = \varnothing$, then $C_S(H,G) = C_\varnothing(H,G) = 0$, and $H\backslash G$ satisfies (WA_S). $\square$

1.7. Corollary (Real approximation)**.** *Let G, H be as in Theorem 1.3. If $S \subset \mathcal{V}_\infty$ then $H\backslash G$ satisfies (WA_S).*

Indeed, all the archimedean places have cyclic decomposition groups, hence $S \cap S_0 = \varnothing$, and $H\backslash G$ satisfies (WA_S).

1.8. For completeness we state here a result from [Bo2]. Let T be a k-torus. Set $Ч_S(T) = \mathrm{coker}\,[H^1(k,T) \to H^1(k_S,T)]$. For a connected k-group H, let H^{tor} denote the biggest quotient torus of H, see Notation.

1.9. Proposition. *Let G, H be as in 1.2. Assume that G is simply connected. Then $C_S(H,G) = Ч_S(H^{\mathrm{tor}})$.*

Proof. See [Bo2].

1.10. Let X be a k-variety. We denote by $\mathrm{Br}(X)$ the cohomological Brauer group of X. Consider the morphisms $X_{\bar k} \to X \to \mathrm{Spec}\, k$ and the induced homomorphisms $\mathrm{Br}(k) \xrightarrow{\alpha} \mathrm{Br}(X) \xrightarrow{\beta} \mathrm{Br}(X_{\bar k})$. Set $\mathrm{Br}_1(X) = \ker\beta$, $\mathrm{Br}_a(X) = \mathrm{Br}_1(X)/\mathrm{im}\,\alpha$.

Let k be a number field, S a finite set of places of k. Set $Б_S(X) = \ker[\mathrm{Br}_a(X) \to \prod_{v\notin S} \mathrm{Br}_a(X_{k_v})]$, $Б(X) = Б_\varnothing(X)$, $Б_\omega(X) = \bigcup_S Б_S(X)$.

1.11. Theorem. *Let G, H, X be as in Theorem 1.3. Then $A_S(X) = (Б_S(X)/Б(X))^D$.*

Theorem 1.11 will be proved in Section 4. This theorem shows that the group $A_S(X) = C_S(H,G)$ does not depend on the representation of X in the form $X = H\backslash G$.

1.12 Corollary. *Set $A(X) = C_\omega(H,G)$, then $A(X) = (Б_\omega(X)/Б(X))^D$.*

Note that the formula of Corollary 1.12 is similar to Sansuc's formula $A(G) = (Б_\omega(G)/Б(G))^D$ ([Sa], 8.12) for the defect of weak approximation $A(G)$ for a connected k-group G.

2. Orbits

2.1. Let k, S, G, H, X be as in 1.2, and assume that $A(G) = 0$. The group G acts on X on the right. Let $\mathcal{O}(X,G,k)$ denote the set of orbits of the group $G(k)$ in $X(k)$. Let $\mathcal{O}(X,G,k_S)$ denote the set of orbits of $G(k_S)$ in $X(k_S)$. We have a canonical map $i_S\colon \mathcal{O}(X,G,k) \to \mathcal{O}(X,G,k_S)$ induced by the embedding $X(k) \to X(k_S)$.

Let $v \in \mathcal{V}$, $x_v \in X(k_v)$, then the morphism $G_{k_v} \to X_{k_v}$, $g \mapsto x_v \cdot g$, is smooth. It follows that the map $G(k_v) \to X(k_v)$, $g \mapsto x_v \cdot g$, is open. Now let $x_S \in X(k_S)$. We see that the map $G(k_S) \to X(k_S)$, $g \mapsto x_S \cdot g$, is open. Hence the orbit $x_S \cdot G(k_S)$ is an open subset in $X(k_S)$. Since all the orbits are open, we conclude that every orbit of $G(k_S)$ in $X(k_S)$ is open and closed.

By assumption (WA) holds for G. Hence (WA_S) holds for G, i.e. $G(k)$ is dense in $G(k_S)$. Let o_S be an orbit of $G(k_S)$ in $X(k_S)$. Assume that o_S has a k-point x.

Since the map $G(k_S) \to X(k_S)$, $g \mapsto x \cdot g$, is continuous, the $G(k)$-orbit $x \cdot G(k)$ is dense in the $G(k_S)$-orbit o_S. Since o_S is closed in $X(k)$, we see that the closure of $x \cdot G(k)$ in $X(k_S)$ is o_S. We conclude that the closure $X(k)\widehat{_S}$ of $X(k)$ in $X(k_S)$ is $\bigcup i_S(o)$ where o runs over $\mathcal{O}(X, G, k)$.

2.2. Set

$$K = \ker[H^1(k, H) \to H^1(k, G)], \quad K_S = \ker[H^1(k_S, H) \to H^1(k_S, G)]\,.$$

The Galois cohomology exact sequences associated with the subgroup H of the group G (cf. [Se], I-5.4, Prop. 36) yield identifications $\mathcal{O}(X, G, k) = K$, $\mathcal{O}(X, G, k_S) = K_S$. With these identifications the map $i_S\colon \mathcal{O}(X, G, k) \to \mathcal{O}(X, G, k_S)$ becomes the restriction to K of the localization map $\mathrm{loc}_S\colon H^1(k, H) \to H^1(k_S, H)$.

2.3. We wish to construct an exact sequence $K \xrightarrow{\mathrm{loc}_S} K_S \to C_S(H, G) \to 0$. This will give us a surjective map $\mathcal{O}(X, G, k_S) \to C_S(H, G)$ with kernel $i_S(\mathcal{O}(X, G, k)$, see 2.2. This in turn will give us a surjective continuous map $X(k_S) \to C_S(H, G)$ with kernel $X(k)\widehat{_S}$, see 2.1. This will prove Theorem 1.3.

3. The defect of weak approximation

In this section we prove Theorem 1.3.

3.1. We recall the definition of the algebaic fundamental group $\pi_1(G)$. Let G be a connected linear group over a field k of characteristic 0. Let T be a maximal torus of the reductive group G^{red} (see Notation). Consider the composed homomorphism $\rho\colon G^{\mathrm{sssc}} \to G^{\mathrm{ss}} \to G^{\mathrm{red}}$ and set $T^{(\mathrm{sssc})} = \rho^{-1}(T) \subset G^{\mathrm{sssc}}$. Set

$$\pi_1(G) = \mathrm{X}_*(T)/\rho_*\mathrm{X}_*(T^{(\mathrm{sssc})})\,,$$

where X_* denotes the cocharacter group over $\bar{k}$. Then $\pi_1(G)$ does not depend on the choice of T. It is a finitely generated abelian group with a Γ-action, where $\Gamma = \mathrm{Gal}(\bar{k}/k)$. For details see [Bo3], Ch. 1.

We have defined $B(G)$ by $B(G) = (\pi_1(G)_\Gamma)_{\mathrm{tors}}$. Both $\pi_1(G)$ and $B(G)$ depend functorially on G.

3.2. Proposition. *There is a canonical functorial isomorphism* $B(G) \simeq \mathrm{Pic}(G)^D$.

Proof. This result is essentially due to Kottwitz. Kottwitz [Ko1], 2.4.6 constructs an isomorphism $\pi_0(Z(\hat{G})^\Gamma) \simeq \mathrm{Pic}(G)$, where $Z(\hat{G})$ is the center of a connected Langlands dual group $\hat{G}$ for G. Let $\mathrm{X}^*(Z(\hat{G}))$ denote the character group $\mathrm{Hom}(Z(\hat{G}), \mathbb{G}_{m,\mathbf{C}})$ of the algebraic $\mathbf{C}$-group of multiplicative type $Z(\hat{G})$. The isomorphism $\mathrm{X}^*(Z(\hat{G})) \simeq \pi_1(G)$ (cf. [Bo3], 1.10) yields isomorphisms $\mathrm{X}^*(Z(\hat{G})^\Gamma) \simeq \pi_1(G)_\Gamma$ and $\pi_0(Z(\hat{G})^\Gamma)^D \simeq (\pi_1(G)_\Gamma)_{\mathrm{tors}} = B(G)$, whence $B(G) \simeq \mathrm{Pic}(G)^D$.

A homomorphism of connected reductive groups $\phi\colon G_1 \to G_2$ is called normal if $\phi(G_1)$ is normal in G_2. A connected Langlands dual group is functorial only with respect to normal homomorphisms, and therefore Kotwittz's $\pi_0(Z(\hat{G})^\Gamma)$ is functorial only with respect to normal homomorphisms. However our groups $\pi_1(G)$ and $B(G)$ are functorial with respect to all homomorphisms of connected linear groups, and using Lemma 2.4.5 of Kottwitz [Ko1] one can easily show that the isomorpism $B(G) \simeq \mathrm{Pic}(G)^D$ is functorial with respect to all homomorpisms. □

3.3. Let k be a number field, and let G be a connected linear k-group. Set $B_v(G) = B(G_{k_v})$ for $v \in \mathcal{V}$. We have a homomorphism $\lambda_v: B_v(G) \to B(G)$, which corresponds to the corestriction homomorphism $(\pi(G)_{\Gamma_v})_{\mathrm{tors}} \to (\pi_1(G)_\Gamma)_{\mathrm{tors}}$, where Γ_v is a decomposition group.

3.4. Let F be a local field of characteristic 0. Let G be a connected F-group. There is a functorial map $\beta_F: H^1(F,G) \to B(G)$, defined by Kottwitz [Ko2], 1.2, see also [Bo3], 3.10 and 4.1(i). The definition of [Bo3] shows that β_F is functorial with respect to all homomorphisms, not only normal. The map β_F is surjective; if F is nonarchimedean, then β_F is bijective, see [Ko2], 1.2.

Let k be a number field, $v \in \mathcal{V}$, and let G be a connected k-group. We have a map $\beta_v = \beta_{k_v}: H^1(k_v, G) \to B_v(G)$. The map β_v is surjective. If $v \in \mathcal{V}_f$ then β_v is bijective.

Set

$$\mu_v = \lambda_v \circ \beta_v: H^1(k_v, G) \to B_v(G) \to B(G)\,.$$

Since β_v is surjective, we have $\operatorname{im}\mu_v = \operatorname{im}\lambda_v$.

Consider the localization map

$$\mathrm{loc}: H^1(k,G) \to H^1(\mathbf{A},G) = \bigoplus H^1(k_v,G)$$

(we use the assumption that G is connected). We define a map $\mu = \sum \mu_v: H^1(\mathbf{A},G) \to B(G)$. This map is functorial in G.

3.5. Proposition (Kottwitz [Ko2])**.** $\ker\mu = \operatorname{im}\mathrm{loc}[H^1(k,G) \to H^1(\mathbf{A},G)]$.

Proof. See [Ko2], 2.5 and 2.6. See also [Bo3], 5.16.

3.6. Let k be a number field. Let H be a connected k-subgroup of a connected linear k-group G. Consider the sets

$$K = \ker[H^1(k,H) \to H^1(k,G)], \quad K_v = \ker[H^1(k_v,H) \to H^1(k_v,G)]\,.$$

We define

$$K_S = \prod_{v\in S} K_v, \quad K_{\mathbf{A}} = \bigoplus_{v\in\mathcal{V}} K_v\,.$$

Let $B(H,G)$ and $B_v(H,G)$ be as in 1.2. Since all our constructions are functorial, we can define maps

$$\beta_v: K_v \to B_v(H,G), \quad \lambda_v: B_v(H,G) \to B(H,G), \quad \mathrm{loc}: K \to K_{\mathbf{A}},$$
$$\mu_v = \lambda_v \circ \beta_v: K_v \to B(H,G), \quad \mu = \sum \mu_v: K_{\mathbf{A}} \to B(H,G)\,.$$

Here the symbols β_v, λ_v, μ_v and μ are used to denote the restrictions of the maps defined by the same symbols in 3.3 and 3.4. Note that $B_v(H,G)$ and $B(H,G)$ are abelian groups and $\lambda_v: B_v(H,G) \to B(H,G)$ is a homomorphism.

3.7. Proposition. *For $v \in \mathcal{V}_f$ the maps $\beta_v\colon K_v \to B_v(H,G)$ are bijections.*

Proof. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_v & \xrightarrow{\alpha} & H^1(k_v,H) & \longrightarrow & H^1(k_v,G)\\
 & & \downarrow & & \downarrow & & \downarrow\\
0 & \longrightarrow & B_v(H,G) & \xrightarrow{\alpha'} & B_v(H) & \longrightarrow & B_v(G)
\end{array}$$

where the maps α and α' are injective. The right and the middle vertical arrows are bijective because $v \in \mathcal{V}_f$, hence the left vertical arrow is bijective. $\square$

3.8. Proposition. *If* $Ш(G) = 0$ *then*

$$\ker[\mu\colon K_{\mathbf{A}} \to B(H,G)] = \mathrm{im}\,[\mathrm{loc}\colon K \to K_{\mathbf{A}}]\,.$$

Proof. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccc}
 & & & & & & Ш(G)=0\\
 & & & & & & \downarrow\\
0 & \longrightarrow & K & \longrightarrow & H^1(k,H) & \xrightarrow{i} & H^1(k,G)\\
 & & \downarrow{\scriptstyle \mathrm{loc}} & & \downarrow{\scriptstyle \mathrm{loc}} & & \downarrow{\scriptstyle \mathrm{loc}}\\
0 & \longrightarrow & K_{\mathbf{A}} & \longrightarrow & H^1(\mathbf{A},H) & \longrightarrow & H^1(\mathbf{A},G)\\
 & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu_H} & & \downarrow{\scriptstyle \mu_G}\\
0 & \longrightarrow & B(H,G) & \longrightarrow & B(H) & \longrightarrow & B(G)
\end{array}$$

The middle and the right columns of the diagram are exact by Proposition 3.5. By diagram chasing one can prove that the left column is exact.

We write down the diagram chasing. It is easy to see that the composition $K \to K_{\mathbf{A}} \to B(H,G)$ is zero. Now let $\xi_{\mathbf{A}} \in K_{\mathbf{A}}$, $\mu(\xi_{\mathbf{A}}) = 0$. Let $\eta_{\mathbf{A}}$ denote the image of $\xi_{\mathbf{A}}$ in $H^1(\mathbf{A},H)$. Then $\mu_H(\eta_{\mathbf{A}}) = 0$, hence $\eta_{\mathbf{A}} = \mathrm{loc}(\eta)$ for some $\eta \in H^1(k,H)$. Since $\eta_{\mathbf{A}}$ comes from $K_{\mathbf{A}}$, the image of $\eta_{\mathbf{A}}$ in $H^1(\mathbf{A},G)$ is zero, hence $\mathrm{loc}(i(\eta)) = 0$. Since $Ш(G) = 0$, we see that $i(\eta) = 0$, hence η is the image of some $\xi \in K$. Since $\mathrm{loc}(\eta) = \eta_{\mathbf{A}}$, the image of $\mathrm{loc}(\xi)$ in $H^1(\mathbf{A},H)$ is $\eta_{\mathbf{A}}$. The map $K_{\mathbf{A}} \to H^1(\mathbf{A},H)$ is an embedding, hence $\mathrm{loc}(\xi) = \xi_{\mathbf{A}}$. Thus $\xi_{\mathbf{A}} \in \mathrm{im}\,[\mathrm{loc}\colon K \to K_{\mathbf{A}}]$. $\square$

3.9. Let $B'(H,G)$, $B^S(H,G)$ and $C_S(H,G)$ be as in 1.2. We set $\mu_S = \sum \mu_v\colon K_S \to B(H,G)$, $\nu_S = \mu_S \mod B^S(H,G)\colon K_S \to C_S(H,G)$.

3.10. Theorem. *If* $Ш(G) = 0$ *then the sequence*

$$K \xrightarrow{\mathrm{loc}_S} K_S \xrightarrow{\nu_S} C_S(H,G) \to 0$$

is exact.

Theorem 1.3 follows from Theorem 3.10, see 2.3.

Proof. 3.10.1. We prove that ν_S is surjective.

Let $B_S(H,G)$ denote the subgroup of $B(H,G)$ generated by the groups $\lambda_v(B_v(H,G))$ for $v \in S$. Then $B'(H,G) = B^S(H,G) + B_S(H,G)$.

Let $v \in \mathcal{V}_\infty \cap S$. Using Chebotarev's density theorem, we see that there exists $v' \notin S$ such that $\lambda_v(B_v(H,G)) = \lambda_{v'}(B_{v'}(H,G))$, and so $B^S(H,G) = B^{S\smallsetminus\{v\}}(H,G)$. It follows that $B^S(H,G) = B^{S\cap\mathcal{V}_f}(H,G)$ and $C_S(H,G) = C_{S\cap\mathcal{V}_f}(H,G)$.

Let $v \in \mathcal{V}_f$. By Proposition 3.7 the map $\beta_v \colon K_v \to B_v(H,G)$ is bijective. Hence $\mu_v(K_v) = \lambda_v(B_v(H,G)$. But $\lambda_v(B_v(H,G))$ is a subgroup of $B(H,G)$ (because λ_v is a homomorphism), hence $\mu_v(K_v)$ is a subgroup of $B(H,G)$.

To prove that ν_S is surjective, we first assume that $S \subset \mathcal{V}_f$. Then for any $v \in S$, $\mu_v(K_v)$ is a subgroup of $B(H,G)$, and we see that the image $\mu_S(K_S)$ of K_S in $B(H,G)$ is a subgroup and equals $B_S(H,G)$. But $B'(H,G) = B_S(H,G)+B^S(H,G)$ and

$$C_S(H,G) = B'(H,G)/B^S(H,G) = (B_S(H,G) + B^S(H,G))/B^S(H,G)\ .$$

Since $\mu_S(K_S) = B_S(H,G)$, we see that the map $\nu_S \colon K_S \to C_S(H,G)$ is surjective.

Now we do not assume that $S \subset \mathcal{V}_f$. The map $K_{S\cap\mathcal{V}_f} \to C_{S\cap\mathcal{V}_f}(H,G) = C_S(H,G)$ is surjective, hence the map $\nu_S \colon K_S \to C_S(H,G)$ is surjective.

3.10.2. We prove that $\nu_S(\mathrm{loc}_S(K)) = 0$.

Let $x \in K$. Set $x_S = \mathrm{loc}_S(x) \in K_S$, $x^S = \mathrm{loc}^S(x) \in K^S$, where $K^S = \bigoplus_{v\notin S} K_v$ and loc_S, loc^S are the localization maps. Let $\mu^S \colon K^S \to B(H,G)$ be the map defined by $(x_v) \mapsto \sum \mu_v(x_v)$. By Proposition 3.5 $\mu(\mathrm{loc}(x)) = 0$ for all $x \in K$. Thus $\mu_S(\mathrm{loc}_S(x)) + \mu^S(\mathrm{loc}^S(x)) = 0$. But $\mu^S(\mathrm{loc}^S(x)) \in B^S(H,G)$. Thus $\nu_S(\mathrm{loc}_S(x)) = 0$.

3.10.3. We prove that $\ker[\nu_S \colon K_S \to C_S(H,G)] \subset \mathrm{loc}_S(K)$.

Consider the map $\mu^S \colon K^S \to B(H,G)$. We prove that $\mu^S(K^S) = B^S(H,G)$. Using Chebotarev's density theorem we can show that for every $v \in \mathcal{V}_\infty \smallsetminus S$ there exists $v' \notin S \cup \mathcal{V}_\infty$ such that $\lambda_v(B_v(H,G)) = \lambda_{v'}(B_{v'}(H,G))$. Thus $B^S(H,G) = B^{S\cup\mathcal{V}_\infty}(H,G)$. But for $v \notin S\cup\mathcal{V}_\infty$ the map $\beta_v \colon K_v \to B_v(H,G)$ is bijective (Proposition 3.7), hence $\mu_v(K_v) = \lambda_v(B_v(H,G))$. We see that for such v, $\mu_v(K_v)$ is a subgroup. Hence

$$\begin{aligned}\mu^{S\cup\mathcal{V}_\infty}(K^{S\cup\mathcal{V}_\infty}) &= \sum_{v\notin S\cup\mathcal{V}_\infty} \mu_v(K_v) = \sum_{v\notin S\cup\mathcal{V}_\infty} \lambda_v(B_v(H,G) \\ &= B^{S\cup\mathcal{V}_\infty}(H,G) = B^S(H,G)\,.\end{aligned}$$

But $\mu^{S\cup\mathcal{V}_\infty}(K^{S\cup\mathcal{V}_\infty}) \subset \mu^S(K^S)$. It follows that $\mu^S(K^S) = B^S(H,G)$.

Now let $x_S \in K_S$. Assume that $\nu_S(x_S) = 0$. Then $\mu_S(x_S) \in B^S(H,G)$. We have already proved that $B^S(H,G) = \mu^S(K^S)$, hence $\mu^S(K^S)$ is a group and there exists $x^S \in K^S$ such that $\mu^S(x^S) = -\mu_S(x_S)$. Set $x_{\mathbf{A}} = (x_S, x^S) \in K_{\mathbf{A}}$, then $\mu(x_{\mathbf{A}}) = 0$. By Proposition 3.5 $x_{\mathbf{A}} = \mathrm{loc}(x)$ for some $x \in K$. We see that $x_S = \mathrm{loc}_S(x)$. Thus $\ker \nu_S \subset \mathrm{loc}_S(K)$. This completes the proof of Theorem 3.10 and Theorem 1.3. □

4. Relation with the Brauer group

In this section we prove Theorem 1.11.

4.1. Consider the canonical map $G \to H\backslash G = X$. Set $\mathrm{Br}(X,G) = \ker[\mathrm{Br}(X) \to \mathrm{Br}(G)]$, $\mathrm{Br}_a(X,G) = \ker[\mathrm{Br}_a(X) \to \mathrm{Br}_a(G)]$, where $\mathrm{Br}_a(X)$ is defined in 1.10. The torsor $G \to X$ under H gives rise to the exact sequence

$$\mathrm{Pic}(G) \to \mathrm{Pic}(H) \to \mathrm{Br}(X) \to \mathrm{Br}(G),$$

cf. [Sa], (6.10.1). Write $\mathrm{Pic}(G,H) = \operatorname{coker}[\mathrm{Pic}(G) \to \mathrm{Pic}(H)]$, $\mathrm{Pic}_1(G,H) = \ker[\mathrm{Pic}(G,H) \to \mathrm{Pic}(G_{\bar k}, H_{\bar k})]$. From the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Pic}(G,H) & \longrightarrow & \mathrm{Br}(X) & \longrightarrow & \mathrm{Br}(G) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Pic}(G_{\bar k}, H_{\bar k}) & \longrightarrow & \mathrm{Br}(X_{\bar k}) & \longrightarrow & \mathrm{Br}(G_{\bar k})
\end{array}$$

we obtain the exact sequence

$$0 \to \mathrm{Pic}_1(G,H) \to \mathrm{Br}_1(X) \to \mathrm{Br}_1(G)\,, \tag{4.1.1}$$

where Br_1 is defined in 1.10.

4.2. We have

$$\mathrm{Pic}(G,H)^D = \operatorname{coker}[\mathrm{Pic}(G) \to \mathrm{Pic}(H)]^D = \ker[B(H) \to B(G)] = B(H,G)\,.$$

Set

$$\begin{aligned}
B_1(H,G) &= \mathrm{Pic}_1(G,H)^D = \ker[\mathrm{Pic}(G,H) \to \mathrm{Pic}(G_{\bar k}, H_{\bar k})]^D \\
&= \operatorname{coker}[\mathrm{Pic}(G_{\bar k}, H_{\bar k})^D \to \mathrm{Pic}(G,H)^D] = \operatorname{coker}[B(H_{\bar k}, G_{\bar k}) \to B(H,G)]\,.
\end{aligned}$$

We define $B_{1v}(H,G)$, $B_1^S(H,G)$, $B_1'(H,G)$ and $C_{1S}(H,G)$ like $B_v(H,G), \dots, C_S(H,G)$ but with $B_1(H,G)$ instead of $B(H,G)$, i.e. $B_{1v}(H,G) = B_1(H_{k_v}, G_{k_v})$, $B_1^S(H,G) = \langle \lambda_v(B_{1v}(H,G))\rangle_{v\notin S}$, $B_1'(H,G) = B_1^{\varnothing}(H,G)$, $C_{1S}(H,G) = B_1'(H,G)/B_1^S(H,G)$.

4.3. Lemma. *$C_{1S}(H,G) = C_S(H,G)$.*

Proof. Let $\bar\lambda\colon B(H_{\bar k}, G_{\bar k}) \to B(H,G)$ be the homomorphism induced by the morphism of pairs $(H_{\bar k}, G_{\bar k}) \to (H,G)$. We have $B(H_{\bar k}, G_{\bar k}) = B(H_{\bar k_v}, G_{\bar k_v})$. From the commutative diagram of pairs

$$\begin{array}{ccc}
(H_{\bar k_v}, G_{\bar k_v}) & \longrightarrow & (H_{k_v}, G_{k_v}) \\
\downarrow & & \downarrow \\
(H_{\bar k}, G_{\bar k}) & \longrightarrow & (H,G)
\end{array}$$

we obtain the commutative diagram

$$\begin{array}{ccc}
B(H_{\bar k_v}, G_{\bar k_v}) & \longrightarrow & B(H_{k_v}, G_{k_v}) \\
\Vert & & \downarrow{\scriptstyle \lambda_v} \\
B(H_{\bar k}, G_{\bar k}) & \xrightarrow{\ \bar\lambda\ } & B(H,G)
\end{array}$$

We see that $\operatorname{im}[\lambda_v\colon B_v(H,G) \to B(H,G)] \supset \operatorname{im}\bar\lambda$. Hence $B^S(H,G) \supset \operatorname{im}\bar\lambda$ and $B'(H,G) \supset \operatorname{im}\bar\lambda$. We have $B_1(H,G) = B(H,G)/\operatorname{im}\bar\lambda$. Hence

$$\begin{aligned}
C_{1S}(H,G) &= B_1'(H,G)/B_1^S(H,G) = (B'(H,G)/\operatorname{im}\bar\lambda)/(B^S(H,G)/\operatorname{im}\bar\lambda) \\
&= B'(H,G)/B^S(H,G) = C_S(H,G)\,.
\end{aligned}$$

□

4.4. Let X be a k-variety. A k-point $x \in X(k)$, $x\colon \mathrm{Spec}(k) \to X$, defines a homomorphism $x^*\colon \mathrm{Br}_1(X) \to \mathrm{Br}_1(k) = \mathrm{Br}(k)$. Define $\mathrm{Br}_x(X) = \ker[x^*\colon \mathrm{Br}_1(X) \to \mathrm{Br}(k)]$. There is a structure morphism $s\colon X \to \mathrm{Spec}(k)$ defining a homomorphism $s^*\colon \mathrm{Br}(k) \to \mathrm{Br}_1(X)$. We have $x^* \circ s^* = 1_{\mathrm{Br}(k)}$. Using these two homomorphisms we can prove that $\mathrm{Br}_1(X) = \mathrm{Br}_x(X) \oplus \mathrm{Br}(k)$ and that $\mathrm{Br}_x(X) \simeq \mathrm{Br}_a(X)$.

Now let $X = H\backslash G$ as before. We set $\mathrm{Br}_1(X,G) = \ker[\mathrm{Br}_1(X) \to \mathrm{Br}_1(G)]$. We have proved in (4.1.1) that $\mathrm{Pic}_1(G,H) = \mathrm{Br}_1(X,G)$. We have

$$\mathrm{Br}_1 X = \mathrm{Br}_{x_0}(X) \oplus \mathrm{Br}(k), \quad \mathrm{Br}_1(G) = \mathrm{Br}_e(X) \oplus \mathrm{Br}(k)$$

where x_0 denotes the image in $X(k)$ of the unit element $e \in G(k)$. We have isomorphisms $\mathrm{Br}_a(X) = \mathrm{Br}_{x_0}(X)$, $\mathrm{Br}_a(G) = \mathrm{Br}_e(G)$, whence

$$\begin{aligned}\mathrm{Pic}_1(G,H) &= \mathrm{Br}_1(X,G) = \ker[\mathrm{Br}_{x_0}(X) \oplus \mathrm{Br}(k) \to \mathrm{Br}_e(G) \oplus \mathrm{Br}(k)] \\ &= \ker[\mathrm{Br}_{x_0}(X) \to \mathrm{Br}_e(G)] = \ker[\mathrm{Br}_a(X) \to \mathrm{Br}_a(G)] = \mathrm{Br}_a(X,G)\,.\end{aligned}$$

We define the groups $Б_S(X,G)$ and $Б(X,G)$ in terms of $\mathrm{Br}_a(X,G)$ in the same way as $Б_S(X)$ and $Б(X)$ were defined in terms of $\mathrm{Br}_a(X)$, i.e. $Б_S(X,G) = \ker[\mathrm{Br}_a(X,G) \to \prod_{v\notin S} \mathrm{Br}_a(X_{k_v}, G_{k_v})]$, $Б(X,G) = Б_\varnothing(X,G)$.

4.5. Proposition. *$C_S(H,G) = (Б_S(X,G)/Б(X,G))^D$.*

Proof. We have

$$B_1(H,G) = \mathrm{Pic}_1(H,G)^D = \mathrm{Br}_a(X,G)^D, \quad B_{1v}(H,G) = \mathrm{Br}_a(X_{k_v}, G_{k_v})^D\,.$$

For a homomorphism $\varphi\colon A \to B$ of torsion abelian groups we have

$$\mathrm{im}\,[\varphi\colon A \to B]^D = B^D/(\mathrm{coker}\,\varphi)^D = B^D/\ker[\varphi^D\colon B^D \to A^D].$$

In our case take

$$A = \bigoplus_{v\notin S} B_{1v}(H,G), \quad B = B_1(H,G),$$

then

$$A^D = \prod_{v\notin S} \mathrm{Br}_a(X_{k_v}, G_{k_v}), \quad B^D = \mathrm{Br}_a(X,G),$$

and so

$$\begin{aligned}B_1^S(H,G)^D &= \mathrm{im}\left[\bigoplus_{v\notin S} B_{1v}(H,G) \to B_1(H,G)\right]^D \\ &= \mathrm{im}\,[A \to B]^D = B^D/\ker[B^D \to A^D] \\ &= \left(\mathrm{Br}_a(X,G)/\ker\left[\mathrm{Br}_a(X,G) \to \prod_{v\notin S} \mathrm{Br}_a(X_{k_v}, G_{k_v})\right]\right) \\ &= (\mathrm{Br}_a(X,G)/Б_S(X,G)).\end{aligned}$$

Thus

$$\begin{aligned} B_1^S(H,G) &= (\mathrm{Br}_a(X,G)/\text{Б}_S(X,G))^D, \\ B_1'(H,G) = B_1^{\varnothing}(H,G) &= (\mathrm{Br}_a(X,G)/\text{Б}(X,G))^D. \end{aligned}$$

Using this we obtain

$$\begin{aligned} C_S(H,G) = C_{1S}(H,G) &= B_1'(H,G)/B_1^S(H,G) \\ &= (\mathrm{Br}_a(X,G)/\text{Б}(X,G))^D/(\mathrm{Br}_a(X,G)/\text{Б}_S(X,G))^D \\ &= (\text{Б}_S(X,G)/\text{Б}(X,G))^D\,. \end{aligned}$$

□

4.6. Proposition. *If* $\text{Ш}(G)=0$ *and* $A(G)=0$*, then* $\text{Б}(G)=0$ *and* $\text{Б}_S(G)=0$*.*

Proof. By [Sa], 8.14 there exists an exact sequence

$$0 \to A(G) \to \text{Б}_\omega(G) \to \text{Ш}(G) \to 0\,.$$

Since $A(G)=0$ and $\text{Ш}(G)=0$, we see that $\text{Б}_\omega(G)=0$. Since $\text{Б}_S(G)\subset \text{Б}_\omega(G)$, we conclude that $\text{Б}_S(G)=0$. Take $S=\varnothing$, we obtain that $\text{Б}(G)=0$. □

4.7. Lemma. *If* $\text{Б}_S(G)=0$ *then* $\text{Б}_S(X,G)=\text{Б}_S(X)$*.*

Proof. Consider the commutative diagram with exact columns

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Б}_S(X,G) & \longrightarrow & \text{Б}_S(X) & \longrightarrow & \text{Б}_S(G) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Br}_a(X,G) & \longrightarrow & \mathrm{Br}_a(X) & \longrightarrow & \mathrm{Br}_a(G) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \prod_{v\notin S}\mathrm{Br}_a(X_{k_v},G_{k_v}) & \longrightarrow & \prod_{v\notin S}\mathrm{Br}_a X_{k_v} & \longrightarrow & \prod_{v\notin S}\mathrm{Br}_a G_{k_v} \end{array}$$

From the fact that the middle row and the lower row are exact, one can easily deduce that the upper row is exact. Since $\text{Б}_S(G)=0$, we see from the diagram that $\text{Б}_S(X,G)=\text{Б}_S(X)$. □

4.8. It follows from Proposition 4.6, Lemma 4.7 and Proposition 4.5 that $C_S(H,G)=(\text{Б}_S(X)/\text{Б}(X))^D$. This proves Theorem 1.11.

References

[**Bo1**] M. Borovoi, *On weak approximation in homogeneous spaces of algebraic groups*, Soviet Math. Dokl. **42** (1991), 247–251.

[**Bo2**] M. Borovoi, *On weak approximation in homogeneous spaces of simply connected algebraic groups*, Proc. Internat. Conf. "Automorphic Functions and Their Applications, Khabarovsk, June 27 – July 4, 1988" (N. Kuznetsov and V. Bykovsky, eds.), Khabarovsk, 1990, pp. 64–81.

[**Bo3**] M. Borovoi, *Abelian Galois cohomology of reductive group*, Memoirs of Amer. Math. Soc. **132** (1998), no. 626.

[**Ko1**] R. E. Kottwitz, *Stable trace formula: cuspidal tempered terms*, Duke Math. J. **51** (1984), 611–650.

[**Ko2**] R. E. Kottwitz, *Stable trace formula: elliptic singular terms*, Math. Ann. **275** (1986), 365–399.

[**Sa**] J.-J. Sansuc, *Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres*, J. für die reine und angew. Math. **327** (1981), 12–80.

[**Vo1**] V. E. Voskresenskiĭ, *Birational properties of linear algebraic groups*, Izv. Akad. Nauk SSSR **34** (1970), 3–19; English transl. in Math. USSR Izv. **4** (1970), 1–17.

[**Vo2**] V. E. Voskresenskiĭ, *Algebraic tori*, Nauka, Moscow, 1977. (Russian)

Raymond and Beverly Sackler School of Mathematical Sciences, Tel Aviv University, 69978 Tel Aviv, Israel

E-mail address: borovoi@math.tau.ac.il