SCATTERING PROBLEM FOR LOCAL PERTURBATION OF FREE QUANTUM GAS.

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Abstract. Scattering theory for perturbations of the intrinsic Dirichlet (Laplace-Beltrami) operator $H_0 = -\operatorname{div}^\Gamma \nabla^\Gamma$ on $L_2(\Gamma, \pi_z)$, *ie* the space of π_z-square integrable functions on the configuration space Γ over $\mathbb{R}^d$, is studied. Here π_z denotes Poisson measure with intensity z. We show that for an arbitrary regular non-zero potential V the standard wave operators $W^\pm(H_0, H_0 + V)$ do not exist, and propose to consider Dirichlet operators of perturbed Poisson measures instead of potential perturbations of the Hamiltonian H_0. As case studies, cylindric smooth densities and finite volume Gibbs perturbations of the Poisson measure are considered. In these cases the existence of the corresponding wave operators is proved.

KEY WORDS: Scattering theory, Poisson analysis, Dirichlet forms.
RUNNING HEAD: Scattering problem for quantum gas

Introduction

Finite as well as zero temperature states of free quantum gases are usually described in terms of the Araki–Woods representation (see, *eg*, [5]). In particular, the free Bose gas of zero density has a standard representation in Fock space, which leads to the known Schrödinger representation in the functional realization of the Fock space as a space of functions square integrable with respect to the corresponding Gaussian measure, see *eg*, [4]. In contrast to this, the free Bose gas of positive density cannot be properly described in the Gaussian space representation, and a rigorous mathematical description of the ground state for this system can only be achieved in terms of Poisson analysis (see [3] as well as the references and historical comments therein).

In this paper we study scattering theory for different types of perturbations of the intrinsic Dirichlet (Laplace–Beltrami) operator $H_0 = -\operatorname{div}^\Gamma \nabla^\Gamma$ on $L_2(\Gamma, \pi_z)$ [1], where π_z is the Poisson measure with Lebesgue intensity measure $z dx$, $z > 0$, and Γ is the corresponding configuration space over $\mathbb{R}^d$. It should be mentioned here that from the physical point of view the operator H_0 is the energy operator, or Hamiltonian, of the corresponding system of the free Bose gas [3]. In Section 2 we show that for an arbitrary regular non-zero potential perturbation V the standard wave operators $W^\pm(H_0, H_0 + V)$ do not exist. (An analogous result in the Gaussian case is well known, see *eg* [4].) Hence, to study scattering theory for the pair of operators H_0 and $H_0 + V$, we need some renormalization of the perturbed Hamiltonian $H_0 + V$. As in the case of Gaussian analysis [4], such a renormalization can be given by the Dirichlet operator of a perturbed Poisson measure (see [2, 3]). In Section 3 we consider perturbations of π_z by cylindric smooth densities and prove existence of the corresponding wave operators. Most important from the mathematical physics point of view is the case of a local perturbation of the Poisson measure by a Gibbs factor, describing a locally perturbed Bose gas in which interactions between particles take place only in a fixed bounded domain of $\mathbb{R}^d$.

But first, we need to recall the notions of configuration space, Poisson measure and the corresponding Dirichlet form [1, 2] (Section 1).

1. Poisson analysis and intrinsic differential geometry of Poisson spaces

In this section we provide a brief review of Poisson analysis and the intrinsic differential geometry of the corresponding Poisson spaces needed for this article. For a more detailed exposition of different aspects of Poisson

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analysis and the intrinsic differential geometry of Poisson spaces as well as their applications to quantum field theory and statistical mechanics see [7, 8, 1, 2, 3, 18, 20, 13, 14, 12, 16] and the references therein.

Let $\mathcal{L}(\Lambda)$, $\Lambda \subseteq \mathbb{R}^d$, denote the system of all bounded Borel subsets in Λ and define the configuration space Γ over $\mathbb{R}^d$ as the set of all locally finite configurations in $\mathbb{R}^d$, *ie*

$$\Gamma := \{\gamma \subset \mathbb{R}^d \mid \#(\gamma \cap \Lambda) < \infty \text{ for all } \Lambda \in \mathcal{L}(\mathbb{R}^d)\},$$

$\#(\gamma \cap \Lambda)$ being the cardinality of the set $\gamma \cap \Lambda$.

Let $\mathfrak{B}_\Lambda(\Gamma)$, $\Lambda \in \mathfrak{B}(\mathbb{R}^d)$, be the σ-algebra on Γ generated by all mappings of the form

$$\Gamma \ni \gamma \mapsto \#(\gamma \cap \Lambda') \in \mathbb{Z}_+ := \{0, 1, 2, \dots\},$$

where $\Lambda' \in \mathcal{L}(\Lambda)$. We set $\mathfrak{B}(\Gamma) := \mathfrak{B}_{\mathbb{R}^d}(\Gamma)$.

The Poisson measure $\pi_{\tilde{z}}$ over $\mathfrak{B}(\Gamma)$ is defined so that for all $\Lambda \in \mathcal{L}(\mathbb{R}^d)$ and any $\mathfrak{B}_\Lambda(\Gamma)$-measurable function $F \geq 0$

$$\int_\Gamma F(\gamma)\pi_{\tilde{z}}(d\gamma) = \exp\Big[-\int_\Lambda \tilde{z}(x)dx\Big] \sum_{n=0}^\infty \frac{1}{n!} \int_{\Lambda^n} F\Big(\{x_i\}_{i=1}^n\Big)(\tilde{z}dx)_n^1, \tag{1.1}$$

where $(\tilde{z}dx)_n^1 := \tilde{z}(x_1)\dots\tilde{z}(x_n)\,dx_1\dots dx_n$ and $\tilde{z}$ is some non-negative function from $L_1^{\mathrm{loc}}(\mathbb{R}^d)$. The existence of $\pi_{\tilde{z}}$ is insured by Kolmogorov's theorem.

Let us set for an arbitrary $\mathfrak{B}(\mathbb{R}^d)$-measurable function f over $\mathbb{R}^d$ with bounded essential support

$$\langle f, \gamma\rangle := \sum_{x\in\gamma} f(x).$$

This means that we identify a configuration $\gamma \in \Gamma$ with the following point measure over $\mathfrak{B}(\mathbb{R}^d)$:

$$\gamma \equiv \sum_{x\in\gamma} \delta_x,$$

where δ_x is Dirac-measure at the point x, *ie*, for all $A \in \mathfrak{B}(\mathbb{R}^d)$

$$\delta_x(A) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 1. *Let $\xi \in S(\mathbb{R}^d)$, $\Lambda \in \mathcal{L}(\mathbb{R}^d)$, and let $F : \Gamma \mapsto \mathbb{R}$ be a $\mathfrak{B}_\Lambda(\Gamma)$-measurable function and $\tilde{z} \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$. Then*

$$\int_\Gamma e^{\langle \xi,\gamma\rangle} F(\gamma)\pi_{\tilde{z}}(d\gamma) = \exp\Big[\int_{\mathbb{R}^d} (e^{\xi(x)}-1)\tilde{z}(x)dx\Big] \int_\Gamma F(\gamma)\pi_{\tilde{z}e^\xi}(d\gamma), \tag{1.2}$$

provided the integrals with respect to $\tilde{z}dx$ and the Poisson measure $\pi_{\tilde{z}e^\xi}$ in the right-hand side of (1.2) exist.

Proof. Since $\mathcal{D}(\mathbb{R}^d) := C_0^\infty(\mathbb{R}^d)$ is dense in $S(\mathbb{R}^d)$, it is enough to prove (1.2) for $\xi \in \mathcal{D}(\mathbb{R}^d)$. We may assume that $\Lambda \supset \operatorname{supp}\xi$. Then, using formula (1.1), we obtain:

$$\begin{aligned} \int_\Gamma e^{\langle \xi,\gamma\rangle} F(\gamma)\pi_{\tilde{z}}(d\gamma) &= \exp\Big[-\int_\Lambda \tilde{z}(x)dx\Big] \sum_{n=0}^\infty \frac{1}{n!} \int_{\Lambda^n} e^{\xi(x_1)+\dots+\xi(x_n)} F\Big(\{x_i\}_{i=1}^n\Big)(\tilde{z}\,dx)_n^1 = \\ &= \exp\Big[\int_\Lambda (e^{\xi(x)}-1)\tilde{z}(x)dx\Big] \exp\Big[-\int_\Lambda \tilde{z}(x)e^{\xi(x)}dx\Big] \sum_{n=0}^\infty \frac{1}{n!} \int_{\Lambda^n} F\Big(\{x_i\}_{i=1}^n\Big)(\tilde{z}e^\xi\,dx)_n^1 = \\ &= \exp\Big[\int_{\mathbb{R}^d} (e^{\xi(x)}-1)\tilde{z}(x)dx\Big] \int_\Gamma F(\gamma)\pi_{\tilde{z}e^\xi}(d\gamma). \qquad \square \end{aligned}$$

For an arbitrary measurable function f_n over $\mathbb{R}^{dn}$ with bounded essential support and $\gamma \in \Gamma$ we define a Wick monomial $\langle f_n, :\gamma^{\otimes n}:\rangle$ as a function of $\gamma \in \Gamma$ by the following formula [12]

$$\langle f_n, :\gamma^{\otimes n}:\rangle := \sum_{\{x_1,\dots,x_n\}\subset\gamma} \sum_{\pi\in\mathfrak{S}_n} f_n(x_{\pi(1)}, \dots, x_{\pi(n)}), \tag{1.3}$$

where $\mathfrak{S}_n$ denotes the group of all permutations of the set $\{1, \dots, n\}$.

Remark 1. Formula (1.3) allows us to regard $:\gamma^n:$, $n \in \mathbb{N}$, as the following point measure over $\mathfrak{B}(\mathbb{R}^{dn})$ [15, 18, 20]

$$:\prod_{i=1}^{n}\gamma(dx_i):=\prod_{i=1}^{n}(\gamma-\sum_{j=1}^{i-1}\delta_{x_j})(dx_i). \tag{1.4}$$

Lemma 2. *For any $\mathfrak{B}(\mathbb{R}^{dn})$-measurable function f_n over $\mathbb{R}^{dn}$ with bounded essential support*

$$\int\limits_{\Gamma}\langle f_n,:\gamma^{\otimes n}:\rangle\pi_{\tilde z}(d\gamma)=\int\limits_{\mathbb{R}^{dn}}f(x_1,\dots,x_n)(\tilde z\,dx)_n^1, \tag{1.5}$$

if the integral on the right-hand side of (1.5) *exists.*

Proof. Choosing $\Lambda \in \mathcal{L}(\mathbb{R}^d)$ in such a way that $\Lambda^n \supset \operatorname{ess\,supp} f_n$ and applying formulae (1.1) and (1.3), we get

$$\int\limits_{\Gamma}\langle f_n,:\gamma^{\otimes n}:\rangle\pi_{\tilde z}(d\gamma)=\exp\Big[-\int\limits_{\Lambda}\tilde z(x)dx\Big]\sum_{m=0}^{\infty}\frac{1}{m!}\int\limits_{\Lambda^m}\sum_{\substack{i_1,\dots,i_n=1\\ i_j\neq i_k,\,j\neq k}}^{m}f_n(x_{i_1},\dots,x_{i_n})(\tilde z dx)_m^1=$$

$$=\exp\Big[-\int\limits_{\Lambda}\tilde z(x)dx\Big]\sum_{m=n}^{\infty}\frac{1}{(m-n)!}\int\limits_{\Lambda^n}f_n(x_1,\dots,x_n)(\tilde z dx)_n^1\int\limits_{\Lambda^{m-n}}(\tilde z dx)_m^{n+1}=$$

$$=\int\limits_{\Lambda^n}f(x_1,\dots,x_n)(\tilde z dx)_n^1=\int\limits_{\mathbb{R}^{dn}}f(x_1,\dots,x_n)(\tilde z dx)_n^1. \qquad \square$$

In view of formula (1.5), for the calculation of averages from a product of Wick monomials with respect to Poisson measure $\pi_{\tilde z}$, it is useful to have some rule for expressing a product of Wick monomials by their linear combination. But in order to introduce such a rule we need some more definitions.

Let $\boldsymbol{\alpha}=\{\alpha_1,\dots,\alpha_k\}$ be a partition of the set $\{1,\dots,n\}$, $n\in\mathbb{N}$, into k non-empty subsets, *ie*,

$$\alpha_1\cup\alpha_2\cup\dots\cup\alpha_k=\{1,\dots,n\},\qquad \alpha_i\neq\emptyset,\quad \alpha_i\cap\alpha_j=\emptyset,\quad i\neq j,\quad i,j=1,\dots,k,$$

and a function $\zeta_{\boldsymbol{\alpha}}:\{1,\dots,n\}\mapsto\{1,\dots,k\}$, $k=\#\boldsymbol{\alpha}$, be defined in such a way that $i\in\alpha_{\zeta_{\boldsymbol{\alpha}}(i)}$, where the subsets $\alpha_1,\dots,\alpha_k$ are supposed to be ordered somehow, for example by the smallest number they contain.

Now, for an arbitrary $\mathfrak{B}(\mathbb{R}^{dn})$-measurable function f_n over $\mathbb{R}^{dn}$ we can introduce a mapping $f_n\mapsto f_n^{\boldsymbol{\alpha}}$, $k=\#\boldsymbol{\alpha}$, by the following formula

$$f_n^{\boldsymbol{\alpha}}(x_1,\dots,x_{\#\boldsymbol{\alpha}})=f_n(x_{\zeta_{\boldsymbol{\alpha}}(1)},\dots,x_{\zeta_{\boldsymbol{\alpha}}(n)}), \tag{1.6}$$

and formulate the rule as follows:

Theorem 1. *For all partitions $\boldsymbol{\alpha}$ of the set $\{1,\dots,n\}$ into k non-empty subsets $\alpha_1,\dots,\alpha_k$*

$$\langle f_n,:\gamma^{\otimes\#\alpha_1}:\otimes\dots\otimes:\gamma^{\otimes\#\alpha_k}:\rangle=\sum_{\tilde{\boldsymbol{\alpha}}:|\tilde\alpha_i\cap\alpha_j|\leq1,\ i=1,\dots,\#\tilde{\boldsymbol{\alpha}},\ j=1,\dots,k}\langle f_n^{\tilde{\boldsymbol{\alpha}}},:\gamma^{\otimes\#\tilde{\boldsymbol{\alpha}}}:\rangle. \tag{1.7}$$

The proof is just a combinatorial exercise based on formula (1.4) and definition (1.6), see [18, 20] and the references therein.

Remark 2. Using the tradition of graphical representation for complicated formulae, widely followed in quantum field theory, it is possible to restate formula (1.7) in the form of a generalized Wick theorem for Poisson fields γ, see [18, 20, 16, 12] and the references therein.

Along with the Wick monomials defined by formula (1.3), in this article we also use the following Charlier polynomials

$$Q_0=1,\qquad Q_n(f_n,\gamma):=\langle f_n,:(\gamma-\tilde z)^{\otimes n}:\rangle,\quad n\in\mathbb{N}, \tag{1.8}$$

where by definition

$$:(\gamma-\tilde z)^{\otimes n}:=\sum_{k=0}^{n}C_n^k(-1)^k\tilde z^{\otimes k}\otimes:\gamma^{\otimes(n-k)}:.$$

For example,

$$Q_1(\varphi,\gamma)=\langle\varphi,\gamma-\tilde z\rangle,\qquad Q_2(\varphi_1\otimes\varphi_2,\gamma)=\langle\varphi_1,\gamma-\tilde z\rangle\langle\varphi_2,\gamma-\tilde z\rangle-\langle\varphi_1\varphi_2,\gamma\rangle$$

$$Q_3(\varphi_1\otimes\varphi_2\otimes\varphi_3,\gamma)=\langle\varphi_1,\gamma-\tilde z\rangle\langle\varphi_2,\gamma-\tilde z\rangle\langle\varphi_3,\gamma-\tilde z\rangle-\langle\varphi_1\varphi_2,\gamma\rangle\langle\varphi_3,\gamma-\tilde z\rangle-$$

$$- \langle\varphi_1\varphi_3,\gamma\rangle\langle\varphi_2,\gamma-\tilde{z}\rangle - \langle\varphi_2\varphi_3,\gamma\rangle\langle\varphi_1,\gamma-\tilde{z}\rangle + 2\langle\varphi_1\varphi_2\varphi_3,\gamma\rangle.$$

Alternatively, Charlier polynomials can be defined through the following generating functional [6, 7, 8, 20]:

$$\exp\left[\langle\ln(1+\varphi),\gamma\rangle - \langle\varphi,\tilde{z}\rangle\right], \tag{1.9}$$

where $\varphi \in S(\mathbb{R}^d)$.

It is easy to see from (1.8) and (1.5) or directly from (1.9) that the Charlier polynomials of different order are mutually orthogonal, more precisely:

$$\int_\Gamma Q_n(\varphi_1\otimes\cdots\otimes\varphi_n,\gamma)Q_m(\phi_1\otimes\cdots\otimes\phi_m,\gamma)\pi_{\tilde{z}}(d\gamma) = \delta_{m,n}\sum_{\pi\in\mathfrak{S}_n}\langle\varphi_1,\tilde{z}\phi_{\pi(1)}\rangle\ldots\langle\varphi_n,\tilde{z}\phi_{\pi(n)}\rangle,$$

where $\delta_{m,n}$ equals 1 if $n=m$, and 0 otherwise.

In [13] a generalized Wick theorem for compensated Poisson fields was obtained which gives the rule for expressing a product of Charlier polynomials by their linear combination (see also [19, 9]). For the case of the product of two Charlier polynomials it gives the following formula:

$$\begin{aligned} Q_n(v_n,\gamma)Q_m(f_m,\gamma) = Q_{n+m}(v_n\widehat{\otimes}f_m,\gamma) + Q_{n+m-1}(v_n\widehat{\widehat{\otimes}}f_m,\gamma) + \\ + Q_{n+m-2}(v_n\widehat{\underline{\otimes}}f_m,\gamma) + Q_{n+m-3}(v_n\widehat{\widehat{\underline{\otimes}}}f_m,\gamma), \end{aligned} \tag{1.10}$$

where by definition

$$\begin{aligned}
(v_n\widehat{\otimes}f_m)(x_1,\ldots,x_{n+m}) &:= \mathcal{P}_s^{n+m}v_n(x_1,\ldots,x_n)f_m(x_{n+1},\ldots,x_{n+m}),\\
(v_n\widehat{\widehat{\otimes}}f_m)(x_1,\ldots,x_{n+m-1}) &:= mn\mathcal{P}_s^{n+m-1}v_n(x_1,\ldots,x_n)f_m(x_n,\ldots,x_{n+m-1}),\\
(v_n\widehat{\underline{\otimes}}f_m)(x_1,\ldots,x_{n+m-2}) &:= mn\mathcal{P}_s^{n+m-2}\int_{\mathbb{R}^d}v_n(x_1,\ldots,x_{n-1},x)f_m(x,x_n,\ldots,x_{n+m-2})\tilde{z}dx,\\
(v_n\widehat{\widehat{\underline{\otimes}}}f_m)(x_1,\ldots,x_{n+m-3}) &:= m(m-1)n(n-1)\mathcal{P}_s^{n+m-3}\int_{\mathbb{R}^d}v_n(x_1,\ldots,x_{n-1},x)f_m(x,x_{n-1},\ldots,x_{n+m-3})\tilde{z}dx
\end{aligned}$$

and $\mathcal{P}_s^n$ denotes the projection on the symmetric subspace of $\mathcal{D}(\mathbb{R}^{nd})$.

Any vector field $\mathbf{v}\in C_0^1(\mathbb{R}^d\mapsto\mathbb{R}^d)$ defines a one-parameter group of diffeomorphisms $\psi_{t,\mathbf{v}}$ of $\mathbb{R}^d$, $t\in\mathbb{R}$. Let for any $\gamma\in\Gamma$

$$\psi_{t,\mathbf{v}}\gamma := \{\psi_{t,\mathbf{v}}x \mid x\in\gamma\}.$$

Then, we can define the directional derivative of a functional $F:\Gamma\mapsto\mathbb{R}^d$ along the vector field $\mathbf{v}\in C_0^1(\mathbb{R}^d\mapsto\mathbb{R}^d)$ as follows

$$(\nabla_{\mathbf{v}}^\Gamma F)(\gamma) = \frac{d}{dt}F(\psi_{t,\mathbf{v}}\gamma)\big|_{t=0}, \tag{1.11}$$

provided the right-hand side of (1.11) makes sense.

For the case when F belongs to the set of smooth cylinder functionals $\mathfrak{F}C_b^\infty(\mathcal{D},\Gamma)$, *ie* has the following form

$$F(\gamma) = g_F\big(\langle\varphi_1,\gamma\rangle,\ldots,\langle\varphi_n,\gamma\rangle\big) \tag{1.12}$$

for some $n\in\mathbb{N}$, $g_F\in C_b^\infty(\mathbb{R}^n)$ and $\varphi_1,\ldots,\varphi_n\in\mathcal{D}:=C_0^\infty(\mathbb{R}^d)$, it is possible to write its directional derivative explicitly:

$$(\nabla_{\mathbf{v}}^\Gamma F)(\gamma) = \sum_{i=1}^n\partial_i g_F\big(\langle\varphi_1,\gamma\rangle,\ldots,\langle\varphi_n,\gamma\rangle\big)\langle\nabla_{\mathbf{v}}\varphi_i,\gamma\rangle, \tag{1.13}$$

where $(\nabla_{\mathbf{v}}\varphi)(x) = \langle\nabla\varphi(x),\mathbf{v}(x)\rangle_{T_x(\mathbb{R}^d)}$ is the usual directional derivative on $\mathbb{R}^d$ along the vector field $\mathbf{v}$. Note that the set of smooth cylinder functionals is dense in $L_2(\Gamma,\pi_{\tilde{z}})$.

We define the tangent space $T_\gamma(\Gamma)$ at the point $\gamma\in\Gamma$ as the Hilbert space of $\mathfrak{B}(\mathbb{R}^d)$-measurable γ-square-integrable vector fields $\mathbf{V}:\mathbb{R}^d\mapsto\mathbb{R}^d$ with the scalar product

$$\langle\mathbf{V}_1,\mathbf{V}_2\rangle_{T_\gamma(\Gamma)} := \int_{\mathbb{R}^d}\langle\mathbf{V}_1(x),\mathbf{V}_2(x)\rangle_{T_x(\mathbb{R}^d)}\gamma(dx), \qquad V_1,V_2\in T_\gamma(\Gamma).$$

The intrinsic Γ-gradient of a function $F\in\mathfrak{F}C_b^\infty(\mathcal{D},\Gamma)$ is a mapping $\Gamma\ni\gamma\mapsto(\nabla^\Gamma F)(\gamma)\in T_\gamma(\Gamma)$ such that $(\nabla_{\mathbf{v}}^\Gamma F)(\gamma) = \langle(\nabla^\Gamma F)(\gamma),\mathbf{v}\rangle_{T_\gamma(\Gamma)}$ for any $\mathbf{v}\in C_0^1(\mathbb{R}^d\mapsto\mathbb{R}^d)$.

As follows from (1.13) for smooth cylinder functions of the form (1.12) we have

$$\nabla^\Gamma F(\gamma, x) = \sum_{i=1}^{n} \partial_i g_F\big(\langle\varphi_1,\gamma\rangle,\dots,\langle\varphi_n,\gamma\rangle\big)\nabla\varphi_i(x),$$

where $\nabla = \nabla^{\mathbb{R}^d}$ is the usual gradient in $\mathbb{R}^d$.

As can be shown from the definitions of Charlier polynomials and the intrinsic Γ-gradient above we have that

$$\nabla^\Gamma Q_n(\varphi_1\otimes\cdots\otimes\varphi_n,\gamma) = \sum_{\tau\subset\{1,\dots,n\},\,\tau\neq\emptyset} (-1)^{|\tau|-1}\big(|\tau|-1\big)!\,Q_{|\bar\tau|}\Big(\bigotimes_{i\in\bar\tau}\varphi_i,\gamma\Big)\nabla\Big(\prod_{i\in\tau}\varphi_i\Big), \tag{1.14}$$

where by definition $\bar\tau := \{1,\dots,n\}\setminus\tau$.

For any $F, G \in \mathfrak{F}C_b^\infty(\mathcal{D},\Gamma)$ introduce the pre-Dirichlet form $\mathcal{E}^\Gamma_{\pi_z}$ by

$$\mathcal{E}^\Gamma_{\pi_z}(F,G) = \int_\Gamma \langle\nabla^\Gamma F, \nabla^\Gamma G\rangle_{T_\gamma(\Gamma)}\pi_z(d\gamma). \tag{1.15}$$

As is shown in [1] this pre-Dirichlet form is closable and its closure $\big(\mathcal{E}^\Gamma_{\pi_z}, \mathrm{Dom}(\mathcal{E}^\Gamma_{\pi_z})\big)$ is a Dirichlet form on $L_2(\Gamma,\pi_z)$. Moreover, the corresponding operator $\big(H^\Gamma_{\pi_z}, \mathrm{Dom}(H^\Gamma_{\pi_z})\big)$ associated with this Dirichlet form, in the sense that

$$\mathcal{E}^\Gamma_{\pi_z}(F,G) = \langle H^\Gamma_{\pi_z}F, G\rangle_{L_2(\Gamma,\pi_z)}, \qquad \text{for all } F,G\in\mathrm{Dom}(H^\Gamma_{\pi_z}),$$

is the Friedrichs' extension of the operator $-\operatorname{div}^\Gamma\nabla^\Gamma$, where the Γ-divergence $\operatorname{div}^\Gamma$ is defined in such a way that

$$\int_\Gamma \langle\mathbf{V}(\gamma),\nabla^\Gamma F(\gamma)\rangle_{T_\gamma(\Gamma)}\pi_z(d\gamma) = -\int_\Gamma (\operatorname{div}^\Gamma\mathbf{V})(\gamma)F(\gamma)\pi_z(d\gamma), \qquad F\in\mathfrak{F}C_b^\infty(\mathcal{D},\Gamma)$$

for any vector field $\mathbf{V}$ of the form

$$\mathbf{V} = \sum_{i=1}^n F_i\mathbf{v}_i, \qquad \mathbf{v}_i\in C_0^1(\mathbb{R}^d\mapsto\mathbb{R}^d),\ F_i\in\mathfrak{F}C_b^\infty(\mathcal{D},\Gamma),\ i=1,\dots,n,\ n\in\mathbb{N},$$

ie,

$$\operatorname{div}^\Gamma\mathbf{V} = \operatorname{div}^\Gamma\sum_{i=1}^n\mathbf{v}_iF_i := \sum_{i=1}^n\big(\langle\nabla^\Gamma F_i,\mathbf{v}_i\rangle_{T(\Gamma)} + F_i\operatorname{div}\mathbf{v}_i\big).$$

It should be mentioned that $H^\Gamma_{\pi_z}$ is essentially self-adjoint on the set of smooth cylinder functions $\mathfrak{F}C_b^\infty(\mathcal{D},\Gamma)$ (cf. [1, Sect. 5]).

2. Potential perturbations. Non-existence of standard wave operators

Set $H_0 := H^\Gamma_{\pi_z}$. The operator H_0 has the following second quantization structure [1]: $H_0 1 = 0$, *ie* 1 is a ground state of H_0, and for any $f_n\in S(\mathbb{R}^{dn})$

$$H_0Q_n(f_n,\cdot) = Q_n(-\Delta_nf_n,\cdot), \qquad n\geq 1, \tag{2.1}$$

where $-\Delta_n$ is the Laplacian in $L_2(\mathbb{R}^{dn})$, $\Delta := \Delta_1$. Under the Wiener–Itô–Segal isomorphism between $L_2(\Gamma,\pi_z)$ and the Fock space over $L_2(\mathbb{R}^d, z dx)$ the operator H_0 transforms into the second quantization operator from the one particle operator $-\Delta$. It immediately follows that H_0 is a non-negative essentially self-adjoint operator and $\sigma(H_0) = \sigma_{\mathrm{ac}}(H_0) = [0,\infty)$, where $\sigma(H_0)$ and $\sigma_{\mathrm{ac}}(H_0)$ are the spectrum and the absolutely continuous spectrum of the operator H_0, respectively, [4, Vol. II, Chapter 6]. Moreover, the absolutely continuous subspace $\mathcal{H}_{\mathrm{ac}}$ corresponding to the operator H_0 coincides with the orthogonal complement to the ground state $\mathcal{H}_{\mathrm{ac}}(H_0) = \{1\}^\perp$. By the spectral representation theorem, (2.1) implies that for any bounded Borel function $\varphi:\mathbb{R}\mapsto\mathbb{C}$

$$\varphi(H_0)Q_n(f_n,\cdot) = Q_n\big(\varphi(-\Delta_n)f_n,\cdot\big), \qquad f_n\in L_2(\mathbb{R}^{dn}).$$

In particular,

$$e^{-itH_0}Q_n(f_n,\cdot) = Q_n\big(e^{-it\Delta_n}f_n,\cdot\big), \qquad f_n\in L_2(\mathbb{R}^{dn}). \tag{2.2}$$

Let $V\in L_2(\Gamma,\pi_z)$ be a real-valued function, and H be some self-adjoint extension of the symmetric operator H_0+V defined on $\mathfrak{F}C_b^\infty(\mathcal{D},\Gamma)$. Such a self-adjoint extension exists because H_0+V commutes on $\mathfrak{F}C_b^\infty(\mathcal{D},\Gamma)$ with the operator of complex conjugation, *ie*,

$$\overline{(H_0+V)F(\gamma)} = (H_0+V)\overline{F(\gamma)}, \qquad F\in\mathfrak{F}C_b^\infty(\mathcal{D},\Gamma).$$

For example, if V is semi-bounded from below, H can be the Friedrich's extension of the corresponding symmetric operator.

From the point of view of standard non-stationary scattering theory [10, 17] it would be natural to define wave operators $W^{\pm}(H, H_0)$ as the strong limits:

$$W^{\pm}(H, H_0) = \operatorname*{s-lim}_{t \to \mp\infty} e^{itH} e^{-itH_0} P_{\mathrm{ac}}(H_0), \tag{2.3}$$

if they exist. Here $P_{\mathrm{ac}}(H_0)$ is the projection on the absolutely continuous subspace $\mathcal{H}_{\mathrm{ac}}(H_0)$ of the operator H_0. Let us show that the wave operators $W^{\pm}(H, H_0)$ do not exist for an arbitrary non-zero potential V.

Theorem 2. *Suppose that V is a real-valued non-zero potential $V \in L_2(\Gamma, \pi_z)$ and that the operator Ve^{-H_0} is bounded in $\mathcal{H}$. Let H be a self-adjoint extension of the symmetric operator $H_0 + V$ defined on $\mathfrak{F}C_b^{\infty}(\mathcal{D}, \Gamma)$. Then the wave operators defined by (2.3) do not exist.*

Proof. Suppose that $W^{\pm}(H, H_0)$ exist. Then (see, *eg*, [10])

$$(H + i)^{-1} W^{\pm} = W^{\pm}(H_0 + i)^{-1}$$

and

$$\operatorname*{s-lim}_{t \to \pm\infty} \big((H + i)^{-1} - (H_0 + i)^{-1}\big) e^{-itH_0} P_{\mathrm{ac}}(H_0) = 0. \tag{2.4}$$

Denote the closure of a linear operator T by $\overline{T}$. Clearly, $\overline{e^{-H_0} V} = (Ve^{-H_0})^*$ is a bounded operator in $\mathcal{H}$ and

$$\begin{aligned} \overline{e^{-H_0} V}(H_0 + i)^{-1} = \overline{e^{-H_0} V (H_0 + i)^{-1}} = \overline{e^{-H_0}(H_0 + V + i)(H + i)V(H_0 + i)^{-1}} = \\ = -\big((H_0 + i)e^{-H_0} + (Ve^{-H_0})^*\big)\big((H + i)^{-1} - (H_0 + i)^{-1}\big). \end{aligned}$$

Hence, by (2.4)

$$\operatorname*{s-lim}_{t \to \pm\infty} \overline{e^{-H_0} V}(H_0 + i)^{-1} e^{-itH_0} P_{\mathrm{ac}}(H_0) = 0.$$

As the set $(H_0 + i)^{-1}\mathcal{H}_{\mathrm{ac}}(H_0)$ is dense in $\mathcal{H}_{\mathrm{ac}}(H_0)$, we have

$$\operatorname*{s-lim}_{t \to \pm\infty} \overline{e^{-H_0} V} e^{-itH_0} P_{\mathrm{ac}}(H_0) = 0. \tag{2.5}$$

We will now show that condition (2.5) is not valid, and therefore wave operators do not exist.

Define $u_{\varphi}(\gamma) = Q_1(\varphi, \gamma) \equiv \langle \varphi, \gamma \rangle - \langle \varphi, z \rangle \in P_{\mathrm{ac}}(H_0)\mathcal{H}$. Here $\varphi \in S(\mathbb{R}^d)$, where $S(\mathbb{R}^d)$ is the space of tempered test functions. Note that $e^{-itH_0} u_{\varphi} = u_{\varphi_t}$, see (2.2), where

$$\varphi_t(x) := (e^{-it\Delta}\varphi)(x) = (4\pi i t)^{-d/2} \int_{\mathbb{R}^d} e^{i|x-y|^2/4t} \varphi(y) dy. \tag{2.6}$$

It is sufficient to show that for some $\varphi \neq 0$

$$\lim_{t \to \infty} \|e^{-H_0} V u_{\varphi_t}\|_{\mathcal{H}} \neq 0.$$

Let $V = \sum_{n=0}^{\infty} Q_n(v_n, \gamma)$ be the expansion of V in a series of orthogonal Charlier polynomials [7, 8]. Then, we have

$$V u_{\varphi(t)} = \sum_{n=0}^{\infty} Q_n(v_n, \gamma) Q_1(\varphi_t, \gamma).$$

By formula (1.10)

$$Q_1(\varphi_t, \gamma) Q_n(v_n, \gamma) = Q_{n+1}(\varphi_t \hat{\otimes} v_n, \gamma) + Q_n(\varphi_t \hat{\overline{\otimes}} v_n, \gamma) + Q_{n-1}(\varphi_t \underline{\hat{\otimes}} v_n, \gamma).$$

As $\|\varphi_t\|_{L_\infty(\mathbb{R}^d)} \to 0$ when $t \to \infty$, we have that $\|\varphi_t \hat{\overline{\otimes}} v_n\|_{L_2(\mathbb{R}^{dn})} \to 0$ when $t \to \infty$. Analogously, one can prove that $\|\varphi_t \underline{\hat{\otimes}} v_n\|_{L_2(\mathbb{R}^{d(n-1)})} \to 0$ as $t \to \infty$. This is clear for $v_n \in S(\mathbb{R}^{dn})$. For an arbitrary $v_n \in L_2(\mathbb{R}^{dn})$, it follows from the estimate

$$\|\varphi_t \underline{\hat{\otimes}} v_n\|_{L_2(\mathbb{R}^{d(n-1)})} \leq z n \|\varphi\|_{L_2(\mathbb{R}^d)} \|v_n\|_{L_2(\mathbb{R}^{dn})}$$

and the fact that $S(\mathbb{R}^{dn})$ is dense in $L_2(\mathbb{R}^{dn})$, $n \in \mathbb{N}$. Let n_0 be the minimal integer for which $\|v_{n_0}\| \neq 0$ and let P_n, $n \in \mathbb{N}$, be the projector on the subspace generated by Charlier polynomials of order n. It suffices to show that

$$\|P_{n_0+1} \overline{e^{-H_0} V} u_{\varphi_t}\|_{\mathcal{H}} \not\to 0 \text{ as } t \to \infty.$$

We have

$$P_{n_0+1} V u_{\varphi_t}(\gamma) = Q_{n_0+1}(\varphi_t \hat{\otimes} v_{n_0}, \gamma) + Q_{n_0+1}(\varphi_t \hat{\overline{\otimes}} v_{n_0+1}, \gamma) + Q_{n_0+1}(\varphi_t \underline{\hat{\otimes}} v_{n_0+2}, \gamma).$$

Clearly, the norm of the second and the third terms go to zero as t tends to infinity. Hence,

$$\lim_{t\to\infty}\|P_{n_0+1}\overline{e^{-H_0}V}u_{\varphi_t}\|_{\mathcal{H}}=\lim_{t\to\infty}\|Q_{n_0+1}\big(e^{-\Delta_{n_0+1}}\varphi_t\widehat{\otimes}v_{n_0},\gamma\big)\|_{\mathcal{H}}= \tag{2.7}$$

$$((n_0+1)!)^{\frac{1}{2}}z^{\frac{n_0+1}{2}}\|e^{-\Delta}\varphi\|_{L_2(\mathbb{R}^d)}\|e^{-\Delta_{n_0}}v_{n_0}\|_{L_2(\mathbb{R}^{dn_0})}\neq 0.$$

Here we have used the following equality

$$\lim_{t\to\infty}\int\limits_{\mathbb{R}^{d(n+1)}}\hat{v}_n(x_1,\dots,x_{k-1},x_{k+1},\dots,x_{n+1})\hat{\varphi}_t(x_k)\overline{\hat{v}_n(x_1,\dots,x_{l-1},x_{l+1},\dots,x_{n+1})\hat{\varphi}_t(x_l)}(dx)_{n+1}^1=0,\quad k\neq l, \tag{2.8}$$

where $\hat{v}_n:=e^{-\Delta_n}v_n$, $\hat{\varphi}:=e^{-\Delta}\varphi$ and $(dx)_n^1:=dx_1\dots dx_n$, $n\in\mathbb{N}$. Note that (2.8) follows from the weak convergence

$$\varphi_t=e^{-it\Delta}\varphi\xrightarrow{w}0,\quad \text{as}\quad t\to\infty.$$

By (2.7), we see that (2.5) is not valid. Hence, the wave operators $W^{\pm}(H,H_0)$ do not exist. $\square$

We have proved in Theorem 2 that $Vu_{\varphi_t}\to 0$ as $t\to\infty$ iff $\varphi=0$. In Sections 3 and 4, we will need the following generalization of this fact.

Lemma 3. *Let V be a non-zero function from $L_2(\Gamma,\pi_z)$ and suppose that all the operators VP_n, $n\in\mathbb{N}$, are bounded. Then*

$$\big\{F\in\mathcal{H}\mid\lim_{t\to\infty}\|Ve^{-itH_0}F\|_{\mathcal{H}}=0\big\}=\{0\}.$$

Proof. Let

$$V=\sum_{n=0}^{\infty}Q_n(v_n,\gamma)\qquad\text{and}\qquad F=\sum_{m=0}^{\infty}Q_m(f_n,\gamma)$$

be expansions of V and F in the series of Charlier polynomials, and set

$$n_0:=\min\{n\mid\|v_n\|_{L_2(\mathbb{R}^{dn})}\neq 0\},\qquad m_0:=\min\{m\mid\|f_m\|_{L_2(\mathbb{R}^{dm})}\neq 0\}$$

and suppose that $\|Ve^{-itH_0}F\|_{\mathcal{H}}\to 0$ as $t\to\infty$. Then

$$\|P_{n_0+m_0}Ve^{-itH_0}F\|_{\mathcal{H}}\to 0\quad\text{as}\quad t\to\infty. \tag{2.9}$$

By (1.10), we have that

$$P_{n_0+m_0}(Ve^{-itH_0}F)=Q_{n_0+m_0}(v_{n_0}\widehat{\otimes}f_{m_0,t},\gamma)+Q_{n_0+m_0}(v_{n_0+1}\widehat{\overline{\otimes}}f_{m_0,t},\gamma)+Q_{n_0+m_0}(v_{n_0}\widehat{\overline{\otimes}}f_{m_0+1,t},\gamma)+$$
$$+\sum_{\substack{n\geq n_0,\,m\geq m_0\\ n+m=n_0+m_0+2}}Q_{n_0+m_0}(v_n\widehat{\underline{\otimes}}f_{m,t},\gamma)+\sum_{\substack{n\geq n_0,\,m\geq m_0\\ n+m=n_0+m_0+3}}Q_{n_0+m_0}(v_n\widehat{\underline{\overline{\otimes}}}f_{m,t},\gamma),$$

where by definition $f_{n,t}:=e^{-it\Delta}f_n$ for any $f_n\in L_2(\mathbb{R}^{dm})$, $n\in\mathbb{N}$. Clearly (see the proof of Theorem 2), for any $f_n\in S(\mathbb{R}^{dn})$, $n\in\mathbb{N}$,

$$\begin{aligned}
&\|v_{n_0}\widehat{\otimes}f_{m_0,t}\|_{L_2(\mathbb{R}^{d(n_0+m_0)})}\to\|v_{n_0}\|_{L_2(\mathbb{R}^{dn_0})}\|f_{m_0}\|_{L_2(\mathbb{R}^{dm_0})},\quad\text{as}\quad t\to\infty,\\
&\|v_n\widehat{\overline{\otimes}}f_{m,t}\|_{L_2(\mathbb{R}^{d(n+m-1)})}\to 0,\quad\text{as}\quad t\to\infty,\\
&\|v_n\widehat{\underline{\otimes}}f_{m,t}\|_{L_2(\mathbb{R}^{d(n+m-2)})}\to 0,\quad\text{as}\quad t\to\infty,\\
&\|v_n\widehat{\underline{\overline{\otimes}}}f_{m,t}\|_{L_2(\mathbb{R}^{d(n+m-3)})}\to 0,\quad\text{as}\quad t\to\infty.
\end{aligned}$$

Hence, for all $f_n\in S(\mathbb{R}^{dn})$

$$\|P_{n_0+m_0}Ve^{-itH_0}F\|_{\mathcal{H}}\to((n_0+m_0)!)^{\frac{1}{2}}z^{\frac{n_0+m_0}{2}}\|v_{n_0}\|_{L_2(\mathbb{R}^{dn_0})}\|f_{m_0}\|_{L_2(\mathbb{R}^{dm_0})}\neq 0,\quad\text{as}\quad t\to\infty. \tag{2.10}$$

Since the operator $\overline{P_{n_0+m_0}V}=(VP_{n_0+m_0})^*$ is bounded in $\mathcal{H}$, one can extend (2.10) to any vector $F\in\mathcal{H}$. This leads to a contradiction to condition (2.9), and therefore, $f_m=0$ for all $m\geq 0$ from which implies that $F\equiv 0$. $\square$

3. Existence of wave operators for cylindric Darboux transforms

Consider the operator H associated with the Dirichlet form corresponding to Poisson measure perturbed by a smooth cylindric density

$$\mathcal{E}^{\Gamma}_{\Phi^2\pi_z}(F,G) = \int_{\Gamma} \langle \nabla^{\Gamma} F(\gamma), \nabla^{\Gamma} G(\gamma)\rangle_{T_\gamma(\Gamma)}\, \Phi^2(\gamma)\pi_z(d\gamma).$$

We shall assume that $\Phi(\gamma) = \exp\left[g_\Phi\left(\langle\phi_1,\gamma\rangle,\ldots,\langle\phi_n,\gamma\rangle\right)\right]$, where $g_\Phi \in C_b^1(\mathbb{R}^n)$

$$\phi_1,\ldots,\phi_n \in C^1(\mathbb{R}^d)\cap L_1(\mathbb{R}^d)\cap L_2(\mathbb{R}^d).$$

The action of the operator H on exponentials $F(\gamma) = \exp\left[\langle\xi,\gamma\rangle\right]$, $\xi \in S(\mathbb{R}^d)$ can be written [1] in the following form:

$$(HF)(\gamma) = (H_0F)(\gamma) + 2\sum_{i=1}^{n} \partial_i g_\Phi\left(\langle\phi_1,\gamma\rangle,\ldots,\langle\phi_n,\gamma\rangle\right)\left\langle\langle\nabla\phi_i,\nabla\xi\rangle_{\mathbb{R}^d},\gamma\right\rangle F(\gamma). \tag{3.1}$$

From now on, we will assume that H is some non-negative definite self-adjoint extension of the symmetric operator defined by (3.1) in the Hilbert space $\widetilde{\mathcal{H}} = L_2\left(\Gamma,\Phi^2(\gamma)\pi_z(d\gamma)\right)$. Then $\widetilde{\mathcal{H}} \supset \mathcal{H}$, and we can consider the wave operators defined by (2.3), as operators from $\mathcal{H}$ into $\widetilde{\mathcal{H}}$.

Theorem 3. *Let H_0 and H be self-adjoint operators associated with the Dirichlet forms $\mathcal{E}^{\Gamma}_{\pi_z}$ and $\mathcal{E}^{\Gamma}_{\Phi^2\pi_z}$ correspondingly, and assume that*

(i) $g_\Phi \in C_b^1(\mathbb{R}^n)$;

(ii) $\phi_j \in C^1(\mathbb{R}^d)$ *and for some* $c>0$, $\alpha > d/2+1$

$$|\nabla\phi_j(x)| \le \frac{c}{(1+|x|)^\alpha}, \qquad x\in\mathbb{R}^d,\ j=1,\ldots,n. \tag{3.2}$$

Then, the wave operators defined by the formula

$$W^{\pm}(H,H_0) = \operatorname*{s-lim}_{t\to\mp\infty} e^{itH}e^{-itH_0}$$

exist, and moreover, $\sigma_{ac}(H) = \sigma_{ac}(H_0) = [0,+\infty)$.

Remark 3. Since $H1 = H_01 = 0$ and $\mathcal{H}_{\mathrm{ac}}(H_0) = \{1\}^{\perp}$, we can omit the projector P_{ac} in the definition of the wave operators.

Remark 4. Assumption (3.2) can be replaced by some L_p condition. For example, it suffices to assume that for some $\varepsilon > 0$

$$|\nabla\phi_i| \in L_2\left(\mathbb{R}^d,(1+|x|)^{2+\varepsilon}dx\right), \qquad i=1,\ldots,n.$$

Proof. According to the Cook criterion, existence of the wave operators $W^{\pm}(H,H_0)$ will be proved if we show that for any function F from some dense subset of $\mathcal{H}$ there exists $t_F > 0$ such that

$$\int_{t_F}^{\infty} \left\|(H-H_0)e^{\pm itH_0}F\right\|_{\widetilde{\mathcal{H}}} dt < \infty. \tag{3.3}$$

It is convenient for our purpose to choose as such a dense subset the linear span of all functions of the form

$$F(\gamma) = \exp[\langle\log(1+\varphi),\gamma\rangle],$$

where $\varphi \in \mathcal{F}^{-1}\left(C_0^\infty(\mathbb{R}^d\setminus\{0\})\right)$, $\varphi > -1$, $\mathcal{F}^{-1}\left(C_0^\infty(\mathbb{R}^d\setminus\{0\})\right)$ denoting the image under the inverse Fourier transform $\mathcal{F}^{-1}$ in $L_2(\mathbb{R}^d)$ of the set of all infinite differentiable functions whose support is a bounded subset of $\mathbb{R}^d\setminus\{0\}$. Then [1]

$$\left(e^{\pm itH_0}F\right)(\gamma) = \exp\left[\left\langle \log(1+\varphi_{\pm t}),\gamma\right\rangle\right], \qquad \varphi_{\pm t} := e^{\pm it\Delta}\varphi,$$

and (3.3) can be rewritten as follows

$$\int_{t_F}^{\infty} \left\|(H-H_0)\exp\left[\left\langle \log(1+\varphi_{\pm t}),\cdot\right\rangle\right]\right\|_{\widetilde{\mathcal{H}}} dt < \infty. \tag{3.4}$$

By (2.6)

$$\|\varphi_{\pm t}\|_{L_\infty(\mathbb{R}^d)} \le (4\pi t)^{-d/2}\|\varphi\|_{L_1(\mathbb{R}^d)}, \tag{3.5}$$

hence we can assume that there exists $t_F > 0$ such that

$$\text{for all} \quad t \geq t_F \qquad \|\varphi_{\pm t}\|_{L_\infty(\mathbb{R}^d)} \leq \frac{1}{2}\,, \tag{3.6}$$

and so expression (3.4) makes sense.

Applying formula (3.1) and taking into account the boundedness of $\partial_i g_\Phi$, $i = 1, \dots, n$, and g_Φ, we obtain that

$$\begin{aligned}
&\Big\|(H - H_0)\exp\Big[\big\langle \log(1+\varphi_{\pm t}), \cdot\big\rangle\Big]\Big\|_{\widetilde{\mathcal{H}}} = \\
&\quad = \Big\| 2\sum_{i=1}^n \partial_i g_\Phi\big(\langle\phi_1,\cdot\rangle,\dots,\langle\phi_n,\cdot\rangle\big)\Big\langle \big\langle \nabla\phi_i, \frac{\nabla\varphi_{\pm t}}{1+\varphi_{\pm t}}\big\rangle_{\mathbb{R}^d}, \cdot\Big\rangle \exp\Big[\big\langle \log\big(1+\varphi_{\pm t}\big), \cdot\big\rangle\Big]\Big\|_{\widetilde{\mathcal{H}}} \leq \\
&\quad \leq 2\sum_{i=1}^n \sup|\partial_i g_\Phi| \,\Big\|\Big\langle \big\langle \nabla\phi_i, \frac{\nabla\varphi_{\pm t}}{1+\varphi_{\pm t}}\big\rangle_{\mathbb{R}^d}, \cdot\Big\rangle \exp\Big[\big\langle \log\big(1+\varphi_{\pm t}\big), \cdot\big\rangle\Big]\Big\|_{\widetilde{\mathcal{H}}} \leq \\
&\quad \leq 2(\sup e^{g_\Phi})^{1/2}\sum_{i=1}^n |\sup \partial_i g_\Phi| \,\Big\|\Big\langle \big\langle \nabla\phi_i, \frac{\nabla\varphi_{\pm t}}{1+\varphi_{\pm t}}\big\rangle_{\mathbb{R}^d}, \cdot\Big\rangle \exp\Big[\big\langle \log\big(1+\varphi_{\pm t}\big), \cdot\big\rangle\Big]\Big\|_{\mathcal{H}}.
\end{aligned}$$

Consequently, (3.4) will be proved if we show that for some t_F so that (3.6) holds, we have

$$\int_{t_F}^{\infty} \Big\|\Big\langle \big\langle \nabla\phi_i, \frac{\nabla\varphi_{\pm t}}{1+\varphi_{\pm t}}\big\rangle_{\mathbb{R}^d}, \cdot\Big\rangle \exp\Big[\big\langle \log\big(1+\varphi_{\pm t}\big), \cdot\big\rangle\Big]\Big\|_{\mathcal{H}} dt < \infty, \qquad i = 1, \dots, n. \tag{3.7}$$

Define

$$\eta_{\pm t} := \langle \nabla\phi_j, \nabla\log(1+\varphi_{\pm t})\rangle_{\mathbb{R}^d}, \quad \text{and} \quad \xi_{\pm t} := \log(1+\varphi_{\pm t}).$$

Then

$$\begin{aligned}
&\Big\|\langle\eta_{\pm t},\cdot\rangle\exp\big[\langle\xi_{\pm t},\cdot\rangle\big]\Big\|^2_{\mathcal{H}} = \int_\Gamma \langle \eta_{\pm t}\otimes\overline{\eta_{\pm t}}, \gamma^{\otimes 2}\rangle \exp\big[2\langle \operatorname{Re}\xi_{\pm t},\gamma\rangle\big]\pi_z(d\gamma) = \\
&\quad = \exp\Big[z\int_{\mathbb{R}^d}(e^{2\operatorname{Re}\xi_{\pm t}(x)}-1)dx\Big]\int_\Gamma \langle \eta_{\pm t}\otimes\overline{\eta_{\pm t}}, \gamma^{\otimes 2}\rangle \pi_{ze^{2\operatorname{Re}\xi_{\pm t}}}(d\gamma) = \\
&\quad = \exp\Big[z\int_{\mathbb{R}^d}(e^{2\operatorname{Re}\xi_{\pm t}(x)}-1)dx\Big]\Big\{\Big|z\int_{\mathbb{R}^d} e^{2\operatorname{Re}\xi_{\pm t}(x)}\eta_{\pm t}(x)dx\Big|^2 + \int_{\mathbb{R}^d} e^{2\operatorname{Re}\xi_{\pm t}(x)}\big|\eta_{\pm t}(x)\big|^2 dx\Big\},
\end{aligned} \tag{3.8}$$

where we used formulae (1.2), (1.7) and (1.5) from Section 1.

Since

$$e^{2\operatorname{Re}\xi_{\pm t}(x)} = |e^{\xi_{\pm t}(x)}|^2 = |1+\varphi_{\pm t}|^2, \tag{3.9}$$

$\|\varphi_{\pm t}\|_{L_2(\mathbb{R}^d)} = \|\varphi\|_{L_2(\mathbb{R}^d)}$ and $\int_{\mathbb{R}^d}\varphi_{\pm t}(x)dx = (2\pi)^{d/2}(\mathcal{F}\varphi_{\pm t})(0) = 0$, we get that

$$\exp\Big[z\int_{\mathbb{R}^d}(e^{2\operatorname{Re}\xi_{\pm t}}-1)dx\Big] = \exp\Big[z\int_{\mathbb{R}^d}\big(|1+\varphi_{\pm t}|^2-1\big)\Big] = \exp\big[z\|\varphi\|^2_{L_2(\mathbb{R}^d)}\big]. \tag{3.10}$$

Moreover, taking into account (3.9), (3.6) and the equality $\nabla\varphi_{\pm t} = (\nabla\varphi)_{\pm t}$, we obtain that

$$\begin{aligned}
&\Big\{\Big|z\int_{\mathbb{R}^d} e^{2\operatorname{Re}\xi_{\pm t}(x)}\eta_{\pm t}(x)dx\Big|^2 + \int_{\mathbb{R}^d} e^{2\operatorname{Re}\xi_{\pm t}(x)}\big|\eta_{\pm t}(x)\big|^2dx\Big\}^{1/2} \leq \\
&\quad \leq \Big\{\Big(z\int_{\mathbb{R}^d}|1+\varphi_{\pm t}|^2\,\Big|\big\langle\nabla\phi_i,\frac{\nabla\varphi_{\pm t}}{1+\varphi_{\pm t}}\big\rangle_{\mathbb{R}^d}\Big|dx\Big)^2 + z\int_{\mathbb{R}^d}|1+\varphi_{\pm t}|^2\,\Big|\big\langle\nabla\phi_i,\frac{\nabla\varphi_{\pm t}}{1+\varphi_{\pm t}}\big\rangle_{\mathbb{R}^d}\Big|^2dx\Big\}^{1/2} \leq \\
&\quad \leq \Big\{\Big(z\int_{\mathbb{R}^d}\big(1+|\varphi_{\pm t}|\big)^2|\nabla\phi_i|\,\Big|\frac{\nabla\varphi_{\pm t}}{1+\varphi_{\pm t}}\Big|dx\Big)^2 + z\int_{\mathbb{R}^d}\big(1+|\varphi_{\pm t}|\big)^2|\nabla\phi_i|^2\Big|\frac{\nabla\varphi_{\pm t}}{1+\varphi_{\pm t}}\Big|^2dx\Big\}^{1/2} \leq \\
&\quad \leq \Big\{\Big(z\int_{\mathbb{R}^d}\big(1+|\varphi_{\pm t}|\big)^2|\nabla\phi_i|\,\frac{|\nabla\varphi_{\pm t}|}{\big|1-|\varphi_{\pm t}|\big|}dx\Big)^2 + z\int_{\mathbb{R}^d}\big(1+|\varphi_{\pm t}|\big)^2|\nabla\phi_i|^2\frac{|\nabla\varphi_{\pm t}|^2}{\big|1-|\varphi_{\pm t}|\big|^2}dx\Big\}^{1/2} \leq
\end{aligned}$$

$$\leq 3\Big\{\Big(\frac{3}{2}z\int\limits_{\mathbb{R}^d}|\nabla\phi_i|\,|\nabla\varphi_{\pm t}|dx\Big)^2 + z\int\limits_{\mathbb{R}^d}|\nabla\phi_i|^2\,|\nabla\varphi_{\pm t}|^2\,dx\Big\}^{1/2} \leq$$

$$\leq 3\Big\{\frac{3}{2}z\int\limits_{\mathbb{R}^d}|\nabla\phi_i|\,|\nabla\varphi_{\pm t}|dx + \Big(z\int\limits_{\mathbb{R}^d}|\nabla\phi_i|^2\,|\nabla\varphi_{\pm t}|^2dx\Big)^{1/2}\Big\}. \tag{3.11}$$

So, as one can see from (3.8), (3.10), (3.11) and the fact that all components of $\nabla\varphi$ also belong to $\mathcal{F}^{-1}\big(C_0^\infty(\mathbb{R}^d\setminus\{0\})\big)$, (3.7) will be proved if we show that

$$\int\limits_{t_F}^{\infty}\big\|\,|\nabla\phi_i|\,\varphi_{\pm t}\big\|_{L_1(\mathbb{R}^d)}dt<\infty,\quad i=1,\dots,n \tag{3.12}$$

and

$$\int\limits_{t_F}^{\infty}\big\|\,|\nabla\phi_i|\,\varphi_{\pm t}\big\|_{L_2(\mathbb{R}^d)}dt<\infty,\quad i=1,\dots,n \tag{3.13}$$

for all $\varphi\in\mathcal{F}^{-1}\big(C_0^\infty(\mathbb{R}^d\setminus\{0\})\big)$.

It is well known (see Theorem XI.24 in [17]) that (3.13) holds if

$$|\nabla\phi_i(x)|\leq\frac{c}{(1+|x|)^{1+\varepsilon}},\qquad i=1,\dots,n$$

for some c, $\varepsilon>0$. But this follows from condition (ii) of the theorem.

To prove (3.12), it suffices to show (see (3.5) and the proof of Theorem XI.16 in [17]) that for all $b>a>0$

$$I:=\int\limits_{t_F}^{\infty}t^{-d/2}\Big(\int\limits_{at\leq|x|\leq bt}\big|\nabla\phi_i(x)dx\big|\Big)dt<\infty,\qquad i=1,\dots,n.$$

By condition (ii) of the theorem, we have that

$$\int\limits_{at\leq|x|\leq bt}\big|\nabla\phi_i(x)\big|dx\leq c\int\limits_{at\leq|x|\leq bt}\frac{dx}{(1+|x|)^{\alpha}}\leq\widetilde{c}t^{d-\alpha}$$

for some constant $\widetilde{c}$. And so,

$$I\leq\widetilde{c}\int\limits_{t_F}^{\infty}t^{d/2-\alpha}dt<\infty$$

for all $\alpha>d/2+1$.

To show that $\sigma_{\mathrm{ac}}(H)=\sigma_{\mathrm{ac}}(H_0)=[0,+\infty)$ let us introduce the following subspaces $\mathcal{H}_{\mathrm{in}}^0=(\operatorname{Ker}W^+)^\perp$, $\mathcal{H}_{\mathrm{in}}=\operatorname{Ran}W^+$. Then $\mathcal{H}_{\mathrm{in}}^0$ and $\mathcal{H}_{\mathrm{in}}$ are invariant subspaces of the operators H_0 and H respectively, and the restriction $H_0\restriction_{\mathcal{H}_{\mathrm{in}}^0}$ is unitary equivalent to $H\restriction_{\mathcal{H}_{\mathrm{in}}}$ (see Proposition 4 of Section XI.3 in [17]). In our case $\mathcal{H}_{\mathrm{in}}^0=\mathcal{H}$. To prove this, let us show that $\operatorname{Ker}W^+=\{0\}$. Let $F\in\operatorname{Ker}W^+$. Then

$$\|W^+F\|=\lim_{t\to+\infty}\|\Phi^2e^{-itH_0}F\|=0$$

By Lemma 3, we see that $F=0$. Therefore, H_0 is unitary equivalent to $H\restriction_{\mathcal{H}_{\mathrm{in}}}$ and $\sigma_{\mathrm{ac}}(H)\supset\sigma_{\mathrm{ac}}(H_0)=[0,\infty)$. On the other hand $H\geq0$, and so $\sigma(H)\subset[0,\infty)$. $\square$

4. Scattering theory for local Gibbs states

Consider the operator H associated with the Dirichlet form of the Gibbs measure μ on Γ defined by

$$\mu(d\gamma)=\Xi_z^{-1}\exp\Big[-\frac{1}{2}\langle V,:\gamma^{\otimes2}:\rangle\Big]\pi_z(d\gamma),$$

where

$$V(x,y)=v(x-y)a(x)\,a(y), \tag{4.1}$$

$a\in C_0^\infty(\mathbb{R}^d)$, $a\geq0$, $v,\nabla v\in L_2^{\mathrm{loc}}(\mathbb{R}^d)$ and $v(x)=v(-x)$.

Here and below, we assume that there exists a bounded non-negative measurable function $b(\cdot)$ with bounded support such that the inequality

$$\frac{1}{2}\langle V, :\gamma^{\otimes 2}:\rangle \geq -\langle b, \gamma\rangle \tag{4.2}$$

holds for π_z-almost all $\gamma \in \Gamma$.

Remark 5. Assumption (4.2) is similar to the well-known stability condition on the potential v:

$$\exists B:\ \forall n \in \mathbb{N},\ \forall x_1, \dots, x_n \in \mathbb{R}^d \qquad \sum_{1\leq i<j\leq n} v(x_i - x_j) \geq -Bn. \tag{4.3}$$

In particular, (4.2) can be obtained from (4.3) if we take $a(\cdot) = \chi_\Lambda(\cdot)$, where χ_Λ is the characteristic function of some bounded subset Λ in $\mathbb{R}^d$.

Remark 6. Both assumptions (4.2) and (4.3) are obviously fulfilled provided $v \geq 0$ and, in this case, we can give a more simple proof of Theorem 4 below, see Appendix A.

The action of the operator H on cylinder functions $F = g\big(\langle\xi_1, \cdot\rangle, \dots, \langle\xi_n, \cdot\rangle\big) \in \mathfrak{F}C_b^\infty(S, \Gamma)$, $S := S(\mathbb{R}^d)$, is defined by the formula (see [2]):

$$(HF)(\gamma) = (H_0F)(\gamma) + \frac{1}{2}\sum_{i=1}^n \langle K_{\xi_i}, :\gamma^{\otimes 2}:\rangle \partial_i g\big(\langle\xi_1, \cdot\rangle, \dots, \langle\xi_n, \cdot\rangle\big), \tag{4.4}$$

where

$$K_\xi(x, y) := \big\langle \nabla_x V(x, y), (\nabla\xi)(x)\big\rangle_{\mathbb{R}^d} + \big\langle \nabla_y V(x, y), (\nabla\xi)(y)\big\rangle_{\mathbb{R}^d}$$

We will assume that H is some non-negative self-adjoint extension of the symmetric operator defined by (4.4) in the Hilbert space $\mathcal{H}_\mu = L_2\big(\Gamma, \mu\big)$.

Recall that P_n is the orthogonal projector on the subspace generated by all Charlier polynomials of order n in the Hilbert space $\mathcal{H}$ and consider the operator J_n acting from $\mathcal{H}$ into $\mathcal{H}_\mu$ by the formula: $J_nF = P_nF$, $n \geq 1$.

As we will show below (Lemma 4), the operators J_n, $n \geq 1$, are bounded from $\mathcal{H}$ into $\mathcal{H}_\mu$, and therefore, one can define wave operators (from $\mathcal{H}$ into $\mathcal{H}_\mu$)

$$W^\pm(H, H_0, J_n) := \operatorname*{s-lim}_{t\to\mp\infty} e^{itH} J_n e^{-itH_0} \equiv \operatorname*{s-lim}_{t\to\mp\infty} e^{itH} e^{-itH_0} P_n, \tag{4.5}$$

provided the limit exists. We will show that such wave operators indeed exist for all n, and hence the standard wave operators

$$W^\pm(H, H_0)F = \lim_{t\to\mp\infty} e^{itH} e^{-itH_0} F, \tag{4.6}$$

exist for all polynomials F.

Lemma 4. *For any $n \geq 1$, J_n are bounded operators from $\mathcal{H}$ to $\mathcal{H}_\mu$.*

Proof. By (4.2)

$$\big\|Q_n(f_n, \cdot)\big\|^2_{\mathcal{H}_\mu} \leq \Xi_z^{-1} \int_\Gamma \big|Q_n(f_n, \gamma)\big|^2 \exp\Big[-\frac{1}{2}\langle V, :\gamma^{\otimes 2}:\rangle\Big]\pi_z(d\gamma) \leq \Xi_z^{-1} \int_\Gamma \big|Q_n(f_n, \gamma)\big|^2 \exp\big[\langle b, \gamma\rangle\big]\pi_z(d\gamma).$$

Since

$$\big\|Q_n(f_n, \cdot)\big\|^2_{\mathcal{H}} = z^n\, n! \|f_n\|^2_{L_2(\mathbb{R}^d)},$$

to prove the lemma it is sufficient to show that for some $c_n > 0$ and any symmetric function $f_n \in L_2(\mathbb{R}^{dn})$

$$\int_\Gamma \big|Q_n(f_n, \gamma)\big|^2 \exp\big[\langle b, \gamma\rangle\big]\pi_z(d\gamma) \leq c_n \|f_n\|^2_{L_2(\mathbb{R}^d)}.$$

Denote $\tilde{b} := e^b - 1$,

$$e(\varphi, \gamma) := \exp\Big[\big\langle \log(1 + \varphi), \gamma\big\rangle - z\langle\varphi\rangle\Big] = \sum_{n=0}^\infty \frac{1}{n!} Q_n(\varphi^{\otimes n}, \gamma), \tag{4.7}$$

and consider for $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned}\int_\Gamma e(t\varphi, \gamma)\overline{Q_n(f_n, \gamma)} \exp\big[\langle b, \gamma\rangle\big]\pi_z(d\gamma) &= \exp\big[z\langle\tilde{b} + t\varphi\tilde{b}\rangle\big] \int_\Gamma e\big(\tilde{b} + t\varphi(1 + \tilde{b}), \gamma\big)\overline{Q_n(f_n, \gamma)}\pi_z(d\gamma) = \\ &= z^n \exp\big[z\langle\tilde{b} + t\varphi\tilde{b}\rangle\big] \Big\langle \big(\tilde{b} + t\varphi(1 + \tilde{b})\big)^{\otimes n}, \overline{f_n}\Big\rangle_{L_2(\mathbb{R}^{dn})}.\end{aligned}$$

By (4.7), we have

$$\int\limits_{\Gamma} Q_n(\varphi^{\otimes n},\gamma)\overline{Q_n(f_n,\gamma)}\exp\big[\langle b,\gamma\rangle\big]\pi_z(d\gamma)=$$

$$= n!\,z^n\exp\big[z\langle\tilde b\rangle\big]\sum_{k=0}^{n}C_n^k\frac{\big(z\langle\varphi,\tilde b\rangle\big)^k}{k!}\Big\langle \tilde b^{\otimes k}\otimes(\varphi+\varphi\tilde b)^{\otimes(n-k)},\overline{f_n}\Big\rangle_{L_2(\mathbb{R}^{dn})}.$$

Therefore,

$$\int\limits_{\Gamma}\big|Q_n(f_n,\gamma)\big|^2\exp\big[\langle b,\gamma\rangle\big]\pi_z(d\gamma)=z^n\exp\big[z\langle\tilde b\rangle\big]\sum_{k=0}^{n}\frac{(n!)^2}{(k!)^2(n-k)!}\times$$

$$\times\int\limits_{\mathbb{R}^{d(n+k)}} f_n(x_1,\dots,x_k,x_{k+1},\dots,x_n)\overline{f_n(x_1',\dots,x_k',x_{k+1},\dots,x_n)}\times$$

$$\times\prod_{i=1}^{k}\big(\tilde b(x_i)\tilde b(x_i')\big)\prod_{j=k+1}^{n}\big(1+\tilde b(x_j)\big)\,dx_1\dots dx_n\,dx_1'\dots dx_k'\le c_n\|f_n\|^2_{L_2(\mathbb{R}^{dn})}.\qquad\square$$

Theorem 4. *Let H_0 and H be self-adjoint operators associated with the Dirichlet forms corresponding to the measures π_z and μ. Suppose also that v and $|\nabla v|$ are from $L_2^{\mathrm{loc}}(\mathbb{R}^d)$, $v(x)=v(-x)$, and the potential V defined by formula (4.1) with $a\in C_0^\infty(\mathbb{R}^d)$ satisfying assumption (4.2). Then for all $n\in\mathbb{N}$ the wave operators $W^{\pm}(H,H_0,J_n)$ exist and, in particular, the wave operators $W^{\pm}(H,H_0)$ exist on all polynomials. Moreover, $\sigma_{ac}(H)=\sigma_{ac}(H_0)$.*

Remark 7. By a modification of the following proof it is possible to weaken the condition $v,\nabla v\in L_2^{\mathrm{loc}}(\mathbb{R}^d)$ allowing a non integrable positive singularity of the potential v, see *eg* [2].

Proof. Let us show that for all $n\in\mathbb{N}$ the wave operators (4.5) exist. To do this, it suffices to show that for any F from some total subset of $P_n\mathcal{H}$ there exists $t_F>0$ such that

$$\int\limits_{t_F}^{\infty}\big\|(H-H_0)e^{\pm itH_0}F\big\|_{\mathcal{H}_\mu}dt<\infty. \tag{4.8}$$

We may take $F(\gamma)=Q_n(f_n,\cdot)$ with kernels $f_n=\varphi_1\otimes\dots\otimes\varphi_n$, where as before

$$\varphi_1,\dots,\varphi_n\in\mathcal{F}^{-1}\big(C_0^\infty(\mathbb{R}^d\setminus\{0\})\big),$$

and denote $\varphi_{i,\pm t}:=e^{\pm it\Delta}\varphi_i$, $i=1,\dots,n$. Then, see (2.2),

$$e^{\pm itH_0}Q_n(f_n,\gamma)=Q_n(\varphi_{1,\pm t}\otimes\dots\otimes\varphi_{n,\pm t},\gamma),$$

and using formulae (4.4) and (1.14), we get that

$$(H-H_0)e^{\pm itH_0}Q_n(f_n,\gamma)=\frac{1}{2}\sum_{\substack{\tau\subset\{1,\dots,n\}\\ \tau\neq\emptyset}}(-1)^{|\tau|-1}\big(|\tau|-1\big)!\langle K_{\varphi_{\tau,\pm t}},:\gamma^{\otimes 2}:\rangle Q_{|\bar\tau|}\big(\bigotimes_{i\in\bar\tau}\varphi_{i,\pm t},\gamma\big),$$

where, by definition, $\varphi_{\tau,\pm t}:=\prod_{i\in\tau}\varphi_{i,\pm t}$ and $\bar\tau:=\{1,\dots,n\}\setminus\tau$.

So,

$$\big\|(H-H_0)e^{\pm itH_0}Q_n(f_n,\gamma)\big\|_{\mathcal{H}_\mu}\le\frac{1}{2}\sum_{\substack{\tau\subset\{1,\dots,n\}\\ \tau\neq\emptyset}}\big(|\tau|-1\big)!\big\|\langle K_{\varphi_{\tau,\pm t}},:\gamma^{\otimes 2}:\rangle Q_{|\bar\tau|}\big(\bigotimes_{i\in\bar\tau}\varphi_{i,\pm t},\gamma\big)\big\|_{\mathcal{H}_\mu},$$

and because of the finite number of terms in the sum over τ, (4.8) can be reduced to

$$\int\limits_{t_F}^{\infty}\big\|\langle K_{\varphi_{\tau,\pm t}},:\gamma^{\otimes 2}:\rangle Q_{|\bar\tau|}\big(\bigotimes_{i\in\bar\tau}\varphi_{i,\pm t},\gamma\big)\big\|_{\mathcal{H}_\mu}dt<\infty\qquad\forall\tau\subset\{1,\dots,n\},\,\tau\neq\emptyset. \tag{4.9}$$

We may assume that $\tau=\{1,\dots,k\}$, $k=1,\dots,n$, $n\in\mathbb{N}$.

Taking into account formula (1.8) and our assumption about the functions $\varphi_1,\dots,\varphi_n$:

$$\langle\varphi_{i,\pm t}\rangle=(2\pi)^{d/2}(\mathcal{F}\varphi_i)(0)=0,$$

one can see that

$$Q_{n-k}\Big(\bigotimes_{i=k+1}^{n}\varphi_{i,\pm t},\gamma\Big)=\Big\langle\bigotimes_{i=k+1}^{n}\varphi_{i,\pm t},:(\gamma-z)^{\otimes(n-k)}:\Big\rangle=\Big\langle\bigotimes_{i=k+1}^{n}\varphi_{i,\pm t},:\gamma^{\otimes(n-k)}:\Big\rangle,\qquad k\in\mathbb{N},$$

and so (4.9) is equivalent to

$$\int_{t_F}^{\infty}\big\|\langle K_{\varphi_{1,\pm t}\cdots\varphi_{k,\pm t}},:\gamma^{\otimes 2}:\rangle\langle\varphi_{k+1,\pm t}\otimes\cdots\otimes\varphi_{n,\pm t},:\gamma^{\otimes(n-k)}:\rangle\big\|_{\mathcal{H}_\mu}dt<\infty\qquad\forall k\in\{1,\dots,n\},\ n\in\mathbb{N}. \tag{4.10}$$

As follows from the definition of measure μ, formulae (4.2), (1.2) and Lemma 5 (see Appendix B),

$$\begin{aligned}
&\big\|\langle K_{\varphi_{1,\pm t}\cdots\varphi_{k,\pm t}},:\gamma^{\otimes 2}:\rangle\langle\varphi_{k+1,\pm t}\otimes\cdots\otimes\varphi_{n,\pm t},:\gamma^{\otimes(n-k)}:\rangle\big\|^2_{\mathcal{H}_\mu}=\\
&\quad=\Xi_z^{-1}\int_\Gamma\big|\langle K_{\varphi_{1,\pm t}\cdots\varphi_{k,\pm t}},:\gamma^{\otimes 2}:\rangle\langle\varphi_{k+1,\pm t}\otimes\cdots\otimes\varphi_{n,\pm t},:\gamma^{\otimes(n-k)}:\rangle\big|^2\exp\Big[-\frac{1}{2}\langle V,:\gamma^{\otimes 2}:\rangle\Big]\pi_z(d\gamma)\le\\
&\quad\le\Xi_z^{-1}\int_\Gamma\big|\langle K_{\varphi_{1,\pm t}\cdots\varphi_{k,\pm t}},:\gamma^{\otimes 2}:\rangle\langle\varphi_{k+1,\pm t}\otimes\cdots\otimes\varphi_{n,\pm t},:\gamma^{\otimes(n-k)}:\rangle\big|^2\exp\Big[\langle b,\gamma\rangle\Big]\pi_z(d\gamma)=\\
&\quad\le\Xi_z^{-1}e^{\langle\widetilde{b},z\rangle}\int_\Gamma\big|\langle K_{\varphi_{1,\pm t}\cdots\varphi_{k,\pm t}},:\gamma^{\otimes 2}:\rangle\langle\varphi_{k+1,\pm t}\otimes\cdots\otimes\varphi_{n,\pm t},:\gamma^{\otimes(n-k)}:\rangle\big|^2\pi_{ze^b}(d\gamma)\le\\
&\quad\le\Xi_z^{-1}e^{\langle\widetilde{b},z\rangle}P\big(ze^B,|\operatorname{supp}a|,z\|\widetilde{b}\|_{L_2(\mathbb{R}^d)},\|\varphi_i\|_{L_2(\mathbb{R}^d)},\|\varphi_{i,\pm t}\|_{L_\infty(\mathbb{R}^d)},i=k+1,\dots,n\big)\|K_{\varphi_{1,\pm t}\cdots\varphi_{k,\pm t}}\|^2_{L_2(\mathbb{R}^d)},
\end{aligned}$$

where, as in the proof of Lemma 4, $\widetilde{b}:=e^b-1$, $B=\operatorname{ess\,sup}b$, and P is a non-negative polynomial of $2n+3$ real variables ze^B, $|\operatorname{supp}a|$, $z\|\widetilde{b}\|_{L_2(\mathbb{R}^d)}$, $\|\varphi_{k+1}\|_{L_2(\mathbb{R}^d)}$, $\dots$, $\|\varphi_n\|_{L_2(\mathbb{R}^d)}$, $\|\varphi_{k+1,\pm t}\|_{L_\infty(\mathbb{R}^d)}$, $\dots$, $\|\varphi_{n,\pm t}\|_{L_\infty(\mathbb{R}^d)}$.

By (3.5) $\|\varphi_{i,\pm t}\|_{L_\infty(\mathbb{R}^d)}$, $i=k+1,\dots,n$, are bounded for all $t>t_F$. So, consider $\|K_{\varphi_{1,\pm t}\cdots\varphi_{k,\pm t}}\|_{L_2(\mathbb{R}^d)}$. We have that

$$\begin{aligned}
&\|K_{\varphi_{1,\pm t}\cdots\varphi_{k,\pm t}}\|_{L_2(\mathbb{R}^{2d})}=\\
&\quad=\Big\|\big\langle\nabla_xV(x,y),\nabla(\varphi_{1,\pm t}\dots\varphi_{k,\pm t})(x)\big\rangle_{\mathbb{R}^d}+\big\langle\nabla_yV(x,y),\nabla(\varphi_{1,\pm t}\dots\varphi_{k,\pm t})(y)\big\rangle_{\mathbb{R}^d}\Big\|_{L_2(\mathbb{R}^{2d})}\le\\
&\quad\le\Big\|\big\langle\nabla_xV(x,y),\nabla(\varphi_{1,\pm t}\dots\varphi_{k,\pm t})(x)\big\rangle_{\mathbb{R}^d}\Big\|_{L_2(\mathbb{R}^{2d})}+\Big\|\big\langle\nabla_yV(x,y),\nabla(\varphi_{1,\pm t}\dots\varphi_{k,\pm t})(y)\big\rangle_{\mathbb{R}^d}\Big\|_{L_2(\mathbb{R}^{2d})}=\\
&\quad\le 2\Big\|\big\langle\nabla_xV(x,y),\nabla(\varphi_{1,\pm t}\dots\varphi_{k,\pm t})(x)\big\rangle_{\mathbb{R}^d}\Big\|_{L_2(\mathbb{R}^{2d})}=\\
&\quad\le 2\Big\|\sum_{i=1}^{k}\big\langle\nabla_xV(x,y),\nabla\varphi_{i,\pm t}(x)\big\rangle_{\mathbb{R}^d}\prod_{\substack{j=1\\ j\ne i}}^{k}\varphi_{j,\pm t}(x)\Big\|_{L_2(\mathbb{R}^{2d})}\le\\
&\quad\le 2\sum_{i=1}^{k}\Big\|\big\langle\nabla_xV(x,y),\nabla\varphi_{i,\pm t}(x)\big\rangle_{\mathbb{R}^d}\prod_{\substack{j=1\\ j\ne i}}^{k}\varphi_{j,\pm t}(x)\Big\|_{L_2(\mathbb{R}^{2d})}\le\\
&\quad\le 2\sum_{i=1}^{k}\Big(\prod_{\substack{j=1\\ j\ne i}}^{k}\big\|\varphi_{j,\pm t}(x)\big\|_{L_\infty(\mathbb{R}^d)}\Big)\Big\|\big\langle\nabla_xV(x,y),\nabla\varphi_{i,\pm t}(x)\big\rangle_{\mathbb{R}^d}\Big\|_{L_2(\mathbb{R}^{2d})}.
\end{aligned} \tag{4.11}$$

By (3.5), (4.10) will follow from the estimate

$$\int_{t_F}^{\infty}\big\|\langle\nabla_xV(x,y),\nabla\varphi_{\pm t}(x)\rangle_{\mathbb{R}^d}\big\|_{L_2(\mathbb{R}^{2d})}dt<\infty\qquad\forall\varphi\in\mathcal{F}^{-1}\big(C_0^\infty(\mathbb{R}^d\setminus\{0\})\big). \tag{4.12}$$

But because of the fact that our potential V has a bounded support, from the method of stationary phase (see the corollary after Theorem XI.14 in [17]), it follows that $\big\|\langle\nabla_xV(x,y),\nabla\varphi_{\pm t}(x)\rangle_{\mathbb{R}^d}\big\|_{L_2(\mathbb{R}^d)}$ vanishes faster than any power of t, and therefore (4.12) holds.

Since e^{itH_0} and e^{itH} are contractions on $\mathcal{H}$ and $\mathcal{H}_\mu$, respectively, for all $t\in\mathbb{R}$, and $J_n:\mathcal{H}\mapsto\mathcal{H}_\mu$ is bounded (Lemma 4), the above implies the existence of the wave operators $W^\pm(H,H_0,J_n)$ for all $n\in\mathbb{N}$.

The proof of the equality $\sigma_{\rm ac}(H)=\sigma_{\rm ac}(H_0)$ is analogous to that in the proof of Theorem 3. $\square$

Corollary 1. Suppose, in addition to the assumptions of Theorem 4, that $V \geq 0$. Then, the wave operators $W^{\pm}(H, H_0)$ defined by (4.6) exist for all $F \in \mathcal{H}$.

Proof. Since $\|\cdot\|_{\mathcal{H}_\mu} \leq \|\cdot\|_{\mathcal{H}}$, the operators $e^{itH}e^{-itH}$, $t \in \mathbb{R}$, are uniformly bounded from $\mathcal{H}$ into $\mathcal{H}_\mu$. Hence, from the existence of the limits in (4.6) on the set of all polynomials (which is dense in $\mathcal{H}$) the existence of the wave operators $W^{\pm}(H, H_0)$ on the whole space $\mathcal{H}$ follows. $\square$

Appendix A.

Here, just as in Section 4, we consider a local Gibbs perturbation of the Poisson measure π_z and give another, more simple, proof of Theorem 4 for the case when the potential V is a non-negative function[1]. Namely, we verify the Cook criterion on the same set of exponents as in the proof of Theorem 3, *ie* for any $\varphi \in \mathcal{F}^{-1}\big(C_0^\infty(\mathbb{R}^d \setminus \{0\})\big)$, $\varphi > -1$, there exists such $t_\varphi > 0$ that

$$\int_{t_\varphi}^{\infty} \Big\|(H - H_0)\exp\Big[\big\langle \log(1 + \varphi_{\pm t}), \cdot\big\rangle\Big]\Big\|_{\mathcal{H}_\mu} dt < \infty, \tag{A.1}$$

where $\varphi_{\pm t} := e^{\pm it\Delta}\varphi$.

Indeed, using the definition of the measure μ in formula (4.4), the assumption about the positivity of V, and formulae (1.2), (3.10), one obtains that

$$\begin{aligned}
\Big\|(H - H_0)\exp\Big[\big\langle \log(1 + \varphi_{\pm t}), \cdot\big\rangle\Big]\Big\|_{\mathcal{H}_\mu} &= \frac{1}{4}\Xi_z^{-1}\int_\Gamma \big|\langle K_{\xi_{\pm t}}, :\gamma^{\otimes 2}:\rangle e^{\langle \xi_{\pm t}, \gamma\rangle}\big|^2 \exp\Big[-\frac{1}{2}\langle V, :\gamma^{\otimes 2}:\rangle\Big]\pi_z(d\gamma) \leq \\
&\leq \frac{1}{4}\Xi_z^{-1}\int_\Gamma \big|\langle K_{\xi_{\pm t}}, :\gamma^{\otimes 2}:\rangle e^{\langle \xi_{\pm t}, \gamma\rangle}\big|^2 \pi_z(d\gamma) = \\
&\leq \frac{1}{4}\Xi_z^{-1}\int_\Gamma \langle K_{\xi_{\pm t}} \otimes \overline{K_{\xi_{\pm t}}}, :\gamma^{\otimes 2}: \otimes :\gamma^{\otimes 2}:\rangle \exp\big[2\langle \operatorname{Re}\xi_{\pm t}, \gamma\rangle\big]\pi_z(d\gamma) = \\
&\leq \frac{1}{4}\Xi_z^{-1}\exp\Big[z\int_{\mathbb{R}^d}(e^{2\operatorname{Re}\xi_{\pm t}(x)} - 1)dx\Big]\int_\Gamma \langle K_{\xi_{\pm t}} \otimes \overline{K_{\xi_{\pm t}}}, :\gamma^{\otimes 2}: \otimes :\gamma^{\otimes 2}:\rangle \pi_{ze^{2\operatorname{Re}\xi_{\pm t}}}(d\gamma) = \\
&\leq \frac{1}{4}\Xi_z^{-1}\exp\big[z\|\varphi\|_{L_2(\mathbb{R}^d)}\big]\int_\Gamma \langle K_{\xi_{\pm t}} \otimes \overline{K_{\xi_{\pm t}}}, :\gamma^{\otimes 2}: \otimes :\gamma^{\otimes 2}:\rangle \pi_{ze^{2\operatorname{Re}\xi_{\pm t}}}(d\gamma),
\end{aligned}$$

where just as in the proof of Theorem 2, $\xi_{\pm t} := \log(1 + \varphi_{\pm t})$.

Next, using formulae (1.7), (1.5), (3.9), (3.6) and Schwarz' inequality we get

$$\begin{aligned}
\int_\Gamma \langle K_{\xi_{\pm t}} \otimes \overline{K_{\xi_{\pm t}}}, :\gamma^{\otimes 2}: \otimes :\gamma^{\otimes 2}:\rangle \pi_{ze^{2\operatorname{Re}\xi_{\pm t}}}(d\gamma) &= \Big|\int_{\mathbb{R}^{2d}} K_{\xi_{\pm t}}(x_1, x_2) ze^{\operatorname{Re}\xi_{\pm t}(x_1)} ze^{\operatorname{Re}\xi_{\pm t}(x_2)} dx_1 dx_2\Big|^2 + \\
&+ 4\int_{\mathbb{R}^{3d}} K_{\xi_{\pm t}}(x_1, x_2)\overline{K_{\xi_{\pm t}}(x_1, \tilde{x}_2)} ze^{\operatorname{Re}\xi_{\pm t}(x_1)} ze^{\operatorname{Re}\xi_{\pm t}(x_2)} ze^{\operatorname{Re}\xi_{\pm t}(\tilde{x}_2)} dx_1 dx_2 d\tilde{x}_2 + \\
&+ 2\int_{\mathbb{R}^{2d}} \big|K_{\xi_{\pm t}}(x_1, x_2)\big| ze^{\operatorname{Re}\xi_{\pm t}(x_1)} ze^{\operatorname{Re}\xi_{\pm t}(x_2)} dx_1 dx_2 \leq \\
&\leq \Big(\frac{3}{2}z\Big)^4 \Big|\int_{\mathbb{R}^{2d}} K_{\xi_{\pm t}}(x_1, x_2) dx_1 dx_2\Big|^2 + 4\Big(\frac{3}{2}z\Big)^3 \int_{\mathbb{R}^{3d}} \big|K_{\xi_{\pm t}}(x_1, x_2)\big| \big|K_{\xi_{\pm t}}(x_1, \tilde{x}_2)\big| dx_1 dx_2 d\tilde{x}_2 + \\
&+ 2\Big(\frac{3}{2}z\Big)^2 \int_{\mathbb{R}^{2d}} \big|K_{\xi_{\pm t}}(x_1, x_2)\big| dx_1 dx_2 \leq \\
&\leq \Big\{\Big(\frac{3}{2}z\Big)^4 |\operatorname{supp} a|^2 + 4\Big(\frac{3}{2}z\Big)^3 |\operatorname{supp} a| + 2\Big(\frac{3}{2}z\Big)^2\Big\} \int_{\mathbb{R}^{2d}} \big|K_{\xi_{\pm t}}(x, y)\big| dx dy.
\end{aligned}$$

[1] In this case, $\mathcal{H}_\mu \supset \mathcal{H}$ and the wave operators $W^{\pm}(H, H_0)$ exist on the whole space $\mathcal{H}$, see Corollary 1.

Therefore, (A.1) is equivalent to

$$\int_{t_\varphi}^{\infty} \|K_{\xi_{\pm t}}\|_{L_2(\mathbb{R}^{2d})} dt < \infty \tag{A.2}$$

for all $\varphi \in \mathcal{F}^{-1}\big(C_0^\infty(\mathbb{R}^d \setminus \{0\})\big)$, $\varphi > -1$.

But because of (3.5) we can always choose t_φ so that for all $t > t_\varphi$

$$\|\varphi_{\pm t}\|_{L_\infty(\mathbb{R}^d)} < \frac{1}{2},$$

and so

$$\|K_{\xi_{\pm t}}\|_{L_2(\mathbb{R}^{2d})} \leq 2\Big\|\Big\langle \nabla_x V(x,y), \frac{\nabla\varphi_{\pm t}(x)}{1+\varphi_{\pm t}(x)}\Big\rangle_{\mathbb{R}^d}\Big\|_{L_2(\mathbb{R}^{2d})} \leq 4\Big\|\big\langle \nabla_x V(x,y), \nabla\varphi_{\pm t}(x)\big\rangle_{\mathbb{R}^d}\Big\|_{L_2(\mathbb{R}^{2d})},$$

which reduces (A.2) to (4.12). $\square$

Appendix B.

This appendix is devoted to the proof of the following

Lemma 5. *Let $K \in L_2(\mathbb{R}^d \times \mathbb{R}^d)$ be supported in $\Lambda \times \Lambda \subset \mathbb{R}^d \times \mathbb{R}^d$, $|\Lambda| < \infty$. Then, for all $n \in \mathbb{N}$, $\varphi_1, \dots, \varphi_n \in \mathcal{F}^{-1}\big(C_0^\infty(\mathbb{R}^d \setminus \{0\})\big)$, and any non-negative bounded measurable function $\tilde z$ on $\mathbb{R}^d$ such that $(\tilde z - z) \in L_2(\mathbb{R}^d)$ for some $z \geq 0$*

$$\int_\Gamma \big|\langle K, :\gamma^{\otimes 2}:\rangle\langle \varphi_1 \otimes \dots \otimes \varphi_n, :\gamma^{\otimes n}:\rangle\big|^2 \pi_{\tilde z}(d\gamma) \leq$$

$$\leq P\big(\sup \tilde z, |\Lambda|, \|\tilde z - z\|_{L_2(\mathbb{R}^d)}, \|\varphi_i\|_{L_2(\mathbb{R}^d)}, \|\varphi_i\|_{L_\infty(\mathbb{R}^d)}, i = 1, \dots, n\big)\|K\|^2_{L_2(\mathbb{R}^d)}, \tag{B.1}$$

where P is a non-negative polynomial of $2n+3$ real variables.

Proof. The main steps of the proof are the same as those leading to estimating the integral

$$\int_\Gamma \langle K, :\gamma^{\otimes 2}:\rangle \pi_{\tilde z}(d\gamma)$$

in Appendix A, with the only difference that now we cannot write down explicitly all the terms arising after applying formulae (1.7) and (1.5) to

$$\int_\Gamma \langle K \otimes \overline{K} \otimes \varphi_1 \otimes \dots \otimes \varphi_n \otimes \overline{\varphi_1} \otimes \dots \otimes \overline{\varphi_n}, :\gamma^{\otimes 2}: \otimes :\gamma^{\otimes 2}: \otimes :\gamma^{\otimes n}: \otimes :\gamma^{\otimes n}:\rangle \pi_{\tilde z}(d\gamma).$$

Nevertheless, we can divide them into three groups and give an explicit rule for estimating the terms from each of the group, leading to formula (B.1).

Let us ascribe to the first group the terms corresponding to such partitions $\widetilde{\boldsymbol{\alpha}}$ in formula (1.7), every element of which contains no more than one number from the set $\{1,2,3,4\}$. For example, as in

$$\widetilde{\boldsymbol{\alpha}} = \big\{\{1,5,6+n\},\{2\},\{3\},\{4\},\{6,7+n\},\{7\},\{8\},\dots,\{4+2n\}\big\}.$$

The contribution of the corresponding term to the integral on the left-hand side of (B.1) can be written as follows

$$\begin{aligned}
\int_{\mathbb{R}^{d(2n+1)}} & K(x_1,x_2)\overline{K(\tilde x_1,\tilde x_2)}\varphi_1(x_1)\varphi_2(y_2)\varphi_3(y_3)\varphi_4(y_4)\dots\varphi_n(y_n) \times \\
& \times \overline{\varphi_1(\tilde y_1)}\,\overline{\varphi_2(x_1)}\,\overline{\varphi_3(y_2)}\,\overline{\varphi_4(\tilde y_4)}\dots\overline{\varphi_n(\tilde y_n)}(\tilde z dx)_2^1(\tilde z d\tilde x)_2^1(\tilde z dy)_n^2(\tilde z d\tilde y_1)(\tilde z d\tilde y)_n^4 = \\
=& \langle \varphi_3, \tilde z\rangle\langle\varphi_4,\tilde z\rangle\dots\langle\varphi_n,\tilde z\rangle\langle\overline{\varphi_1},\tilde z\rangle\langle\overline{\varphi_4},\tilde z\rangle\dots\langle\overline{\varphi_n},\tilde z\rangle \times \\
& \times \int_{\mathbb{R}^d}\varphi_2(y_2)\overline{\varphi_3(y_2)}\tilde z(y_2)dy_2 \int_{\Lambda^4} K(x_1,x_2)\overline{K(\tilde x_1,\tilde x_2)}\varphi_1(x_1)\overline{\varphi_2(x_1)}(\tilde z dx)_2^1(\tilde z d\tilde x)_2^1 \leq \\
\leq & |\langle\varphi_3,\tilde z\rangle|\,|\langle\varphi_4,\tilde z\rangle|\dots|\langle\varphi_n,\tilde z\rangle|\,|\langle\overline{\varphi_1},\tilde z\rangle|\,|\langle\overline{\varphi_4},\tilde z\rangle|\dots|\langle\overline{\varphi_n},\tilde z\rangle| \times \\
& \times \|\varphi_2\|_{L_2(\mathbb{R}^d,\tilde z dx)}\|\overline{\varphi_3}\|_{L_2(\mathbb{R}^d,\tilde z dx)}\|\varphi_1\|_{L_\infty(\mathbb{R}^d)}\|\overline{\varphi_2}\|_{L_\infty(\mathbb{R}^d)}(\sup\tilde z|\Lambda|)^2\|K(x,y)\|^2_{L_2(\mathbb{R}^{2d},\tilde z dx\,\tilde z dy)}.
\end{aligned}$$

We note that the sign of complex conjugation inside the modulo and norm signs in the above formula is useful for inferring the general rule.

It is easy to see that in the general situation the norm of φ_k or $\overline{\varphi_k}$ is in $L_\infty(\mathbb{R}^d)$, if the number $4+k$ or $4+n+k$ is contained in some $\widetilde{\alpha}_i$, $i=1,\dots,\#\widetilde{\boldsymbol{\alpha}}$, together with one of the numbers 1, 2, 3 or 4, and is in $L_2(\mathbb{R}^d,\tilde{z}dx)$ if it is contained in some $\widetilde{\alpha}_i$, $i=1,\dots,\#\widetilde{\boldsymbol{\alpha}}$, together with any other number from 5 to $4+2n$. Otherwise, *ie*, if the number $4+k$ or $4+n+k$ is contained in some $\widetilde{\alpha}_i$, $i=1,\dots,\#\widetilde{\boldsymbol{\alpha}}$, alone, we get $|\langle\varphi_i,\tilde{z}\rangle|$ or $|\langle\overline{\varphi_i},\tilde{z}\rangle|$ correspondingly.

Next, we ascribe to the second group the terms corresponding to the partitions $\widetilde{\boldsymbol{\alpha}}$ with only one element containing more than one number from the set $\{1,2,3,4\}$. For example, as in

$$\widetilde{\boldsymbol{\alpha}}\big\{\{1,3,5,6+n\},\{2\},\{4\},\{6,7+n\},\{7\},\{8\},\dots,\{4+2n\}\big\},$$

which gives the following contribution to the integral on the left-hand side of (B.1):

$$\begin{aligned}
&\langle\varphi_3,\tilde{z}\rangle\langle\varphi_4,\tilde{z}\rangle\dots\langle\varphi_n,\tilde{z}\rangle\langle\overline{\varphi_1},\tilde{z}\rangle\langle\overline{\varphi_4},\tilde{z}\rangle\dots\langle\overline{\varphi_n},\tilde{z}\rangle\times\\
&\qquad\times\int_{\mathbb{R}^d}\varphi_2(y_2)\overline{\varphi_3(y_2)}\tilde{z}(y_2)dy_2\int_{\Lambda^4}K(x_1,x_2)\overline{K(x_1,\tilde{x}_2)}\varphi_1(x_1)\overline{\varphi_2(x_1)}(\tilde{z}dx)_2^1(\tilde{z}d\tilde{x}_2)\le\\
&\le|\langle\varphi_3,\tilde{z}\rangle|\,|\langle\varphi_4,\tilde{z}\rangle|\dots|\langle\varphi_n,\tilde{z}\rangle|\,|\langle\overline{\varphi_1},\tilde{z}\rangle|\,|\langle\overline{\varphi_4},\tilde{z}\rangle|\dots|\langle\overline{\varphi_n},\tilde{z}\rangle|\times\\
&\qquad\times\|\varphi_2\|_{L_2(\mathbb{R}^d,\tilde{z}dx)}\|\overline{\varphi_3}\|_{L_2(\mathbb{R}^d,\tilde{z}dx)}\|\varphi_1\|_{L_\infty(\mathbb{R}^d)}\|\overline{\varphi_2}\|_{L_\infty(\mathbb{R}^d)}\sup\tilde{z}|\Lambda|\,\|K(x,y)\|^2_{L_2(\mathbb{R}^{2d},\tilde{z}dx\,\tilde{z}dy)}.
\end{aligned}$$

Finally, to the third group, we ascribe the terms corresponding to the partition $\widetilde{\boldsymbol{\alpha}}$ in formula (1.7) with two different elements containing more than one number from the set $\{1,2,3,4\}$ (or to be precise, exactly by two of them), as in

$$\widetilde{\boldsymbol{\alpha}}=\big\{\{1,3,5,6+n\},\{2,4\},\{6,7+n\},\{7\},\{8\},\dots,\{4+2n\}\big\}.$$

The corresponding contribution to the integral in the left-hand side of (B.1) can be estimated by

$$\begin{aligned}
&|\langle\varphi_3,\tilde{z}\rangle|\,|\langle\varphi_4,\tilde{z}\rangle|\dots|\langle\varphi_n,\tilde{z}\rangle|\,|\langle\overline{\varphi_1},\tilde{z}\rangle|\,|\langle\overline{\varphi_4},\tilde{z}\rangle|\dots|\langle\overline{\varphi_n},\tilde{z}\rangle|\times\\
&\qquad\times\|\varphi_2\|_{L_2(\mathbb{R}^d,\tilde{z}dx)}\|\overline{\varphi_3}\|_{L_2(\mathbb{R}^d,\tilde{z}dx)}\|\varphi_1\|_{L_\infty(\mathbb{R}^d)}\|\overline{\varphi_2}\|_{L_\infty(\mathbb{R}^d)}\|K(x,y)\|^2_{L_2(\mathbb{R}^{2d},\tilde{z}dx\,\tilde{z}dy)}.
\end{aligned}$$

We then see that the only difference between the estimate of the terms from the different groups lies in the power of $\sup\tilde{z}|\Lambda|$.

To conclude the proof of the lemma it is sufficient to note that $\|\varphi_i\|_{L_2(\mathbb{R}^d,\tilde{z}dx)}\le(\sup\tilde{z})^{1/2}\|\varphi_i\|_{L_2(\mathbb{R}^d,dx)}$, $i=1,\dots,n$, $\|K(x,y)\|^2_{L_2(\mathbb{R}^{2d},\tilde{z}dx\,\tilde{z}dy)}\le(\sup\tilde{z})^2\|K(x,y)\|^2_{L_2(\mathbb{R}^{2d})}$, and

$$|\langle\varphi,\tilde{z}\rangle|=\Big|\int_{\mathbb{R}^d}\varphi(x)\tilde{z}(x)dx\Big|=\Big|\int_{\mathbb{R}^d}\varphi(x)\big(\tilde{z}(x)-z\big)dx\Big|\le\|\varphi\|_{L_2(\mathbb{R}^d)}\,\|\tilde{z}-z\|_{L_2(\mathbb{R}^d)}$$

for all $\varphi\in\mathcal{F}^{-1}\big(S(\mathbb{R}^d\setminus\{0\})\big)$. $\square$

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