HODGE DECOMPOSITION FOR HIGHER ORDER HOCHSCHILD HOMOLOGY

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0. Introduction

Let A be a commutative algebra and M be an A-module. Let

$$\mathcal{L}(A,M):\Gamma\to Vect$$

be a functor from finite pointed sets to the category of vector spaces given by

$$\{0,1,...,n\}\mapsto M\otimes A^{\otimes n},$$

see also section 1.7. For any based simplicial finite set $Y:\Delta^{op}\to\Gamma$ we let $H_*^Y(A,M)$ denote the homology of the chain complex associated to the simplicial vector space

$$\Delta^{op}\xrightarrow{Y}\Gamma\xrightarrow{\mathcal{L}(A,M)}Vect.$$

For $Y=S^d, d\geq 1$, one uses the notion $H_*^{[d]}(A,M)$ instead of $H_*^{S^d}(A,M)$. For $Y=S^1$ one recovers the usual Hochschild homology of A with coefficient in M (see [L2]), because of this fact, we call the groups $H_*^{[d]}(A,M)$ as *Hochschild homology of order* d of A with coefficient in M. Higher order Hochschild homology was implicitly defined in [A1]. These groups are related to the cohomology of the mapping spaces X^{S^d} in the same way as usual Hochschild homology is related to the cohomology of free loop space X^{S^1} (see section 5). The main goal of this paper is to show that the higher order Hochschild homology in charecteristic zero case have naturaral decomposition, which for $d=1$ is isomorphic to the classical Hodge decomposition ([L2]). Our methods are new even for $d=1$ and are based on homological properties of Γ-modules.

Let us recall that the Hochschild and cyclic homology of commutative algebras in characteristic zero case have natural decompositions, known as Hodge decomposition. This result for Hochschild homology was proved by Quillen ([Q]). Using sophisticated combinatorics Gerstenhaber and Schack constructed the so called Eulerian idempotents and proved that these yield a decomposition of Hochschild homology (see [GS], [L1]). It turns out that both decompositions are isomorphic to each other (see [Ro]). Moreover, Loday made in [L1] the important observation that such decompositions have a more general nature. Namely he proved the following result: Let $F : \Gamma \to Vect$ be a functor from finite pointed sets to the category of vector spaces over a characteristic zero field. Let $S^1 : \Delta^{op} \to Sets_*$ be the standard simplicial model for the circle, which has only two nondegenerate simplices. Clearly the values of S^1 are in finite pointed sets, so one can take the composition of these two functors $F(S^1) : \Delta^{op} \to Vect$. In this way one gets the simplicial vector space $F(S^1)$ and one can take the homotopy groups of it. Among other things, Loday proved that there exists a natural decomposition

$$\pi_n F(S^1) \cong \bigoplus_{i=0}^{n} H_n^{(i)}(F), \ n \geq 0,$$

where the groups $H_n^{(i)}(F)$ are defined using Eulerian idempotents. If A is a commutative algebra and M is a symmetric bimodule, then for $F = \mathcal{L}(A, M)$ Loday's result gives exactly the Hodge decomposition for Hochschild homology. Loday obtained in [L1] also the similar decomposition for cyclic homology (see section 3 for more details). Later McCarthy remarked that Loday's decomposition can be obtained using the rotations of a circle [M]. We refer to [W] for relationship between Hodge decomposition of cyclic homology and Hodge filtration in algebraic geometry and [BV] and [V] for related results.

We give an alternative approach of this subject. We do not only give a new purely homological proof of the results of Loday, but we obtain essentially more. Using homological algebra of Γ-modules we constract a spectral sequence, whose abutement is $\pi_* F(L)$. Here L is any based simplicial finite set. We show that for $L = S^d$, the spectral sequence stops at E^2 level and gives a natural decomposition for $\pi_n F(L)$. We give a simple axiomatic characterization of the decomposition of $\pi_* F(S^1)$, from which we deduce that our and Loday's decomposition are isomorphic. The striking fact is that in the decompositions of $\pi_n F(S^d)$ for odd d, the same groups $H_k^{(i)}(F)$ appear but in a different way:

$$\pi_n F(S^d) \cong \bigoplus_{i+dj=n} H_{i+j}^{(j)}(F),\ n \geq 0,\ d = 2k+1 \geq 1.$$

We define the Hochschild homology of order d of a commutative algebra A with coefficient in an A-module M to be $\pi_n F(S^d)$, where $F = \mathcal{L}(A, M)$. As a sample example we compute higher order Hochschild homology for smooth algebras (for all d) and for truncated polynomial algebras (for odd d). We prove that for all d the Hochschild homology of order d of the de Rham complex of a d-connected manifold is isomorphic to the homology of the maping space X^{S^d}.

It is worth to mention that any functor $F : \Gamma \to Vect$ gives rise to an abelian spectrum, thanks to the famous result of Segal (see [Se], notice that our Γ is the opposite of the original Γ of Segal). Let $\pi_n^{st} F$ be the homotopy groups of the corresponding spectrum. It turns out that the groups $H_n^{(1)}(F)$ (which are known as Harrison homology [L1]) and $\pi_n^{st} F$ are isomorphic in the characteristic zero case. Therefore one can thinked on the groups $\pi_n^{st} F$ for $F = \mathcal{L}(A, M)$ as a modification of the André-Quillen homology in characteristic $p > 0$ case. In [PR] is proved that it is an isomorphic to the new brave algebra version of André-Quillen homology constructed by Alan Robinson (see [RW] and [Wh]).

By similar methods B. Richter in [R2] obtained a new conceptual proof of the result of Bergeron (see [Be]).

The paper is organized as follows. In Section 1 we describe the basic properties of the category of functors from finite based sets to vector spaces. In the next Section we prove the decomposition properties for $\pi_* F(S^n)$. In Section 3 we reprove Loday's decomposition theorem for cyclic homology. We use the observation that the forgetful functor from Γ to the category of finite nonempty sets is flat in some sense. In section 4 we prove a version of the Hoschscild-Konstant-Rosenberg theorem for smooth Γ-modules, which is main technical tool for calculation of higher order Hochschild homology for smooth algebras in the last section. In the same section we also prove that the higher order Hochschild homology of Sullivan cochain algebra of d-connected space X is isomorphic to the cohomology of mapping space X^{S^d}.

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1. Properties of Γ-modules

1.1. Γ-modules. Let K be a field. In what follows all vector spaces are defined over K. Moreover $\otimes$ and Hom denote $\otimes_K$ and Hom_K respectively. Let $Vect$ be the category of all vector spaces.

Let Γ be the category of finite pointed sets. For any $n \geq 0$, we let $[n]$ be the set $\{0, 1, ..., n\}$ with basepoint 0. We assume that objects of Γ are sets $[n]$. For any nonzero element $i \in [n]$ we let $p_i : [n] \to [1]$ be the pointed map given by $p_i(i) = 1$ and $p_i(j) = 0$ if $j \neq i$. We also need the pointed map $p : [2] \to [1]$ given by $p(1) = p(2) = 1$. Let Γ-*mod* be the category of all covariant functors from Γ to $Vect$. Similarly *mod*-Γ denotes the category of contravariant functors from Γ to the category of vector spaces. We call objects of Γ-*mod* (resp. *mod*-Γ) left (resp. right) Γ-modules. We will use the term Γ-module if we do not want to distinguish between left and right Γ-modules.

The categories Γ-*mod* and *mod*-Γ are abelian categories with sufficiently many projective and injective objects. For any $n \geq 0$ one defines

$$\Gamma^n := K[\mathrm{Hom}_\Gamma([n], -)] \text{ and } \Gamma_n := K[\mathrm{Hom}_\Gamma(-, [n])].$$

Here $k[S]$ denotes the free vector space generated by a set S. It is a consequence of the Yoneda lemma that for any left Γ-module F and any right Γ-module T one has natural isomorphisms

(1.1.1)

$$\mathrm{Hom}_{\Gamma-mod}(\Gamma^n, F) \cong F([n]) \text{ and } Hom_{mod-\Gamma}(\Gamma_n, T) \cong T([n]).$$

Therefore Γ^n (resp. Γ_n), $n \geq 0$, are projective generators of the category Γ-*mod* (resp. *mod*-Γ). Clearly Γ_0 and Γ^0 are constant functors with the value K. Any pointed map $[m] \to [n]$ yields a morphisms of Γ-modules $\Gamma_m \to \Gamma_n$ and $\Gamma^n \to \Gamma^m$.

1.2. Duality and functors of finite type. For any vector space V one denotes the dual vector space by V^*. One can extend this notion for Γ-modules. For a Γ-module F one defines F^* by

$$F^*([n]) := (F([n]))^*.$$

Then F^* is a right Γ-module if F is a left Γ-module and vice-verse. A functor is called *of finite type* if it has values in finite dimensional vector spaces. Clearly $F^{**} \cong F$ if F is a functor of finite type. Let F be a left Γ-module and T be a right Γ-module. Then one has a natural isomorphism

$$\mathrm{Hom}_{\Gamma-mod}(F, T^*) \cong \mathrm{Hom}_{mod-\Gamma}(T, F^*).$$

Hence the functors Γ^{n*} and Γ_n^* are injective cogenerators of the category Γ-*mod* and *mod*-Γ respectively and one has natural isomorphisms

$$\mathrm{Hom}_{\Gamma-mod}(F, \Gamma^{n*}) \cong (F([n]))^* \text{ and } \mathrm{Hom}_{mod-\Gamma}(T, \Gamma_n^*) \cong (T([n]))^*.$$

Let us notice that the functors Γ^{n*} and Γ_n^* are of finite types.

1.3. Pointwise tensor product. For any left Γ-modules F and T we put

$$(F \otimes T)([n]) := F([n]) \otimes T([n]).$$

Clearly $F \otimes T$ is still a left Γ-module. In the same way one defines the same kind of operation for right Γ-modules. It follows by definition that

$$\Gamma^n \otimes \Gamma^m \cong \Gamma^{n+m} \text{ and } \Gamma_n \otimes \Gamma_m \cong \Gamma_{nm+n+m}.$$

This follows from the facts that $[n] \vee [m] \cong [n+m]$ and $[n] \times [m] \cong [nm+n+m]$. As a consequence we see that *if F and T are projective, then $F \otimes T$ is also projective.* Moreover $\Gamma^n \cong (\Gamma^1)^{\otimes n}$ for any $n \geq 0$.

It is also clear that $F^* \otimes T^* \cong (F \otimes T)^*$, if the values of T and F are finite dimensional. Therefore *if F and T are injective, then $F \otimes T$ is also injective* provided F and T are of finite type.

1.4. The right Γ-modules t and θ^n. Let t be a right Γ-module given by

$$t([n]) := \mathrm{Hom}_{Sets_*}([n], K).$$

Here the field K is considered as a pointed set with basepoint 0. For an element $i \in [n]$ we let χ_i be the characteristic function of i.

So $\chi_i(i) = 1$ and $\chi_i(j) = 0$ if $j \neq i$. Let us remark that, if $i \neq 0$, then χ_i is a pointed map. One has the exact sequence

$$(1.4.1) \qquad \Gamma_2 \xrightarrow{\alpha} \Gamma_1 \xrightarrow{\beta} t \to 0.$$

Here α and β are given by

$$\alpha([n] \xrightarrow{f} [2]) = p_1 \circ f + p_2 \circ f - p \circ f$$

and

$$\beta([n] \xrightarrow{g} [1]) = \sum_{g(i)=1} \chi_i,$$

where p and p_i are the same as in 1.1. It is quite easy to check that the sequence (1.4.1) is indeed exact. We also refer to [P1], where a projective resolution SQ_* of t was constructed, whose beginning is just the above exact sequence. Using this projective resolution one can describe the stable homotopy groups of a functor (see Proposition 2.2) via the Eilenberg-MacLane cubical construction (see also E.13.2.2 of [L2]).

One observes that

$$t^* \cong \ Coker \ (\Gamma^0 \to \Gamma^1).$$

Indeed, the value of both functors on $[n]$ are the same, namely the vector space generated by $[n]$, modulo the vector space generated by $0 \in [n]$. Here $\Gamma^0 \to \Gamma^1$ is induced by the unique map $[1] \to [0]$, which has a section. Hence $\Gamma^1 \cong \Gamma^0 \oplus t^*$. Therefore t^* is projective and t is injective. Moreover the left Γ-modules $t^{\otimes n *}$ are projective generators, while the right Γ-modules $t^{\otimes n}$ are injective cogenerators.

The functor t played already an important role in Mac Lane homology theory (see [P1], [L2], [R1]) and we will see that it is also very important for present paper. We will need also the following right modules $\theta^n, n \geq 0$. Let $\mathcal{B}$ be the left Γ-module, whose value on $[m]$ is the free vector space generated by the subsets of $[m]$, modulo the vector space generated by subsets contained 0. If $f : [m] \to [k]$ then the action of f on $\mathcal{B}([m])$ is induced by the direct image of f. Let $\mathcal{B}_n$ be the subfunctor of $\mathcal{B}$ generated by the subsets of cardinality $\leq n$. We define the functor θ^n as the dual of the quotien $\mathcal{B}_n/\mathcal{B}_{n-1}$. Clearly $t \cong \theta^1$. One observes that $(\mathcal{B}_n/\mathcal{B}_{n-1})([m])$ is isomorphic to the vector space spanned on the n-elementian subsets of

$\{1, \cdots, m\}$. Hence as a vector space we have $\theta^n([m]) \cong \Lambda^n(t([m]))$. But as functors θ^n and $\Lambda^n \circ t$ are different as soon as $n > 1$.

1.5. The bifunctor $- \otimes_\Gamma -$. For a right Γ-module N and a left Γ-module M we let $N \otimes_\Gamma M$ be the abelian group generated by all elements $x \otimes y$, where $x \in N([n]),\ y \in M([n])$ and $n \geq 0$, modulo the relations

$$(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y, \quad x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2,$$

$$\alpha^*(x') \otimes y = x' \otimes \alpha_*(y).$$

Here $x_1, x_2 \in N([n]), y_1, y_2 \in M([n]), x' \in N([m])$ and $\alpha : [m] \to [n]$ is a morphism in Γ. It is well known (see Section 16.7 of [S]) that the bifunctor $- \otimes_\Gamma - : mod\text{-}\Gamma \times \Gamma\text{-}mod \to Vect$ is right exact with respect of each variable, preserves sums and for any left Γ-module F, any right Γ-module T and any $n \geq 0$, there exist natural isomorphisms

$$T \otimes_\Gamma \Gamma^n \cong T([n])$$

$$\Gamma_n \otimes_\Gamma F \cong F([n]).$$

Clearly $- \otimes_\Gamma -$ is a left balanced bifunctor in the sense of Cartan and Eilenberg ([CE]). Therefore the derived functors of $- \otimes_\Gamma -$ with respect to each variable are isomorphic and we will denote the common value by $Tor_*^\Gamma(-,-)$. Moreover, we will consider a hyperhomology spectral sequence for such Tor-groups.

1.6. Proposition. *Let F be a left Γ-module and C_* be a non-negative chain complex, whose components are projective right Γ-modules. Then there exist a first quadrant spectral sequence*

$$E_{pq}^2 = Tor_p^\Gamma(H_q(C_*), F) \Longrightarrow H_{p+q}(C_* \otimes_\Gamma F).$$

Suppose

$$Ext_{mod-\Gamma}^{m-n+1}(H_n(C_*), H_m(C_*)) = 0, \text{ for } n < m \tag{1.6.1}$$

then the spectral sequence is degenerate at E^2 and one has the decomposition

$$H_n(C_* \otimes_\Gamma F) \cong \bigoplus_{p+q=n} Tor_p^\Gamma(H_q(C_*), F)$$

which is natural on F.

Proof. The first statement is a standard tool of homological algebra (see for example section 2.4 of [Gr] and especially Remarque 3 on page 148 of *loc. cit.*) The rest is implicity in [D]. We give here the argument. A chain complex C_* is called of *type* (A, n) if $H_iC_* = 0$ for $i \neq n$ and $H_nC_* \cong A$. Let us denote such a complex by (A, n). A chain complex C_* is called *formal* if it has the same weak homotopy type as $\bigoplus_{n\geq 0}(H_nC_*, n)$. Dold in [D] constructed the Postnikov decomposition of C_* as a kind of tower $\cdots \to C_*^{n+1} \to C_*^n \to \cdots$. Moreover he defined the i-th invariant as an element in $H^{i+1}(C_*^{i-1}, H_iC_*)$ and proved that C_* is formal iff all invariants are zero. Clearly it is enough to show that in our circumstances C_* is formal. We will show by induction that all invariants are zero. Thanks to Satz 5.2 [D] the first invariant lies in $Ext^2_{mod-\Gamma}(H_0C_*, H_1C_*)$ and thus it is zero. Assume, now that all i-invariants are zero for $i < n$. Then C_*^{n-1} is formal. Thus it has the same weak homotopy type as

$$\bigoplus_{i=0}^{n-1}(H_iC_*, i).$$

We can use still Satz 5.2 [D] to get

$$H^{n+1}(C_*^{n-1}, H_nC_*) \cong \bigoplus_{i=0}^{n-1} Ext^{n+1-i}_{mod-\Gamma}(H_iC_*, H_nC_*) = 0.$$

Therefore the n-th invariant vanishes and the proof is finished.

1.7. (Co)algebras and Γ-modules. Let A be a commutative K-algebra with unit and let M be an A-module, considered as a symmetric A-A-bimodule. Following Loday [L1], [L2] we let $\mathcal{L}(A, M)$ be the left Γ-module given by

$$[n] \mapsto M \otimes A^{\otimes n}.$$

For a pointed map $f : [n] \to [m]$, the action of f on $\mathcal{L}(A, M)$ is given by

$$f_*(a_0 \otimes \cdots \otimes a_n) := b_0 \otimes \cdots \otimes b_m,$$

where

$$b_j = \prod_{f(i)=j} a_i, \ j = 0, \cdots, n.$$

This construction can be generalized for commuatative graded algebras (see [V]) as follows. Let A be a commuatative graded algebra and M be a A-module. The functor $\mathcal{L}(A, M) : \Gamma \to Vect$ on objects is given by the same rule (of course now product means the ordered one), while on morphisms it is given by

$$f_*(a_0 \otimes \cdots \otimes a_n) := (-1)^{\epsilon(f,a)} b_0 \otimes \cdots \otimes b_m,$$

where

$$\epsilon(f, a) = \sum_{j=1}^{n-1} \mid a_j \mid (\sum_{k \in I_j} \mid a_k \mid).$$

Here $I_j = \{k > j \mid 0 \leq f(k) \leq f(j)\}$. If A is an augmented algebra then $\mathcal{L}(A, K)$ is denoted by $\mathcal{L}(A)$ for brevity. Dually, if C is a cocommutative K-coalgebra, and N is an A-comodule, one obtains a right Γ-module $\mathcal{J}(C, N)$, which assigns $N \otimes C^{\otimes n}$ to $[n]$. Clearly, for a finite dimensional coalgebra C and a finite dimensional comodule N, one has

$$\mathcal{J}(C, N)^* \cong \mathcal{L}(C^*, N^*).$$

In the coaugmented case we write $\mathcal{J}(C)$ instead of $\mathcal{J}(C, K)$. Let us note that for graded (co)algebras one obtains functors from Γ to the category of graded vector spaces. Therefore for any integer i, the i-th component of $\mathcal{L}(A)$ and $\mathcal{J}(C)$ defines left and right Γ-modules, which are denoted by $\mathcal{L}_i(A)$ and $\mathcal{J}_i(C)$ respectively.

1.8. Example. Let $A = K[x]/(x^2)$ be the commutative graded algebra where the generator x has degree $d > 0$. We claim that there exists the following isomorphism of left Γ-modules:

$$\mathcal{L}_i(K[x]/(x^2)) \cong \Lambda^j \circ t^*, \ i = jd \text{ and } d \text{ is odd}$$

$$\mathcal{L}_i(K[x]/(x^2)) \cong (\theta^j)^*, \ i = jd \text{ and } d \text{ is even}$$

$$\mathcal{L}_i(K[x]/(x^2)) = 0, \ i \neq jd.$$

Indeed

$$\mathcal{L}(K[x]/(x^2))[n] = (K + Kx)^{\otimes n} = \bigoplus_{0 \le j \le n} (Kx^{\otimes j})^{\binom{n}{j}},$$

where $x^{\otimes(j+1)} = x^{\otimes j} \otimes x, j \ge 0$. Therefore the i-th dimensional part is zero, if $i \ne dj$ and equals to $\Lambda^j(K^n)$ if $i = dj$ and hence the claim is proved.

1.9. Category of surjections. Let Ω be the small category of all finite sets and surjections. We will assume that the objects of Ω are the sets

$$\underline{n} := \{1, ..., n\}, \ n \ge 0,$$

where $\underline{0}$ denotes the empty set. Clearly $\underline{0}$ is isolated object. A covariant (resp. contravariant) functor $\Omega \to Vect$ is termed left (resp. right) Ω-module. Let M be a left $K[\Sigma_n]$-module. One denotes by $\Theta(M)$ (resp. $\Theta^{op}(M)$) the unique left (resp. right) Ω-module, which assigns M to $\underline{n}$ and 0 to $\underline{m}$, where $m \ne n$ and for which the action of $\Sigma_n = Hom_\Omega(\underline{n}, \underline{n})$ on M coincides (resp. is opposite) with the given one. Clearly if M is a simple $K[\Sigma_n]$-module, then $\Theta(M)$ and $\Theta^{op}(M)$ are simple Ω-module as well. We claim that in this way one obtains all simple Ω-modules up to isomorphism. Indeed, let T be a left (resp. right) Ω-module and n be the minimal number, for which $T(\underline{n}) \ne 0$. Since $K[\Sigma_n] = K[Hom_\Omega(\underline{n}, \underline{n})]$, we see that $M = T(\underline{n})$ is a represenatation of Σ_n. Therefore $\Theta(M)$ (resp. $\Theta^{op}(M)$) is well defined. Now one can show that there exist unique morphism Ω-modules $\Theta(M) \to T$ (resp. $T \to \Theta^{op}(M)$) which is identity on $\underline{n}$. From this observation one easily deduce the claim.

1.10. Dold-Kan type theorem for Γ-modules. For any $n \ge 1$ and any $1 \le i \le n$, one defines the pointed maps

$$r_i : [n] \to [n-1]$$

by $r_i(i) = 0$, $r_i(j) = j$ if $j < i$ and $r_i(j) = j - 1$, if $j > i$. For a left Γ-module T, let $r_{i*} : T([n]) \to T([n-1])$ be the homomorphism induced by r_i. One defines the functor

$$cr(T) : \Omega \to Vect$$

as follows. On objects one puts

$$cr(T)(\underline{n}) := \bigcap_{i=1}^{n} ker\ (r_{i*}).$$

For a surjection $f : \underline{n} \to \underline{m}$ one denotes by $f_0 : [n] \to [m]$ the unique pointed map which extends f. Then one easily shows that the image of $cr(T)(\underline{n}) \subset T([n])$ under the homomorphism $(f_0)_* : T([n]) \to T([m])$ lies in $cr(T)(\underline{m})$ and therefore $cr(T)$ is well defined. In [P3] we proved that the functor

$$cr : \Gamma - mod \to \Omega - mod$$

is an equivalence of categories. Here is the describtion of the quasi-inverse of the functor cr. Let T be a left Ω-module. For a surjection $\beta : [n] \to [m]$ in Γ one defines the homomorphism $(\beta)_* : T(\underline{n}) \to T(\underline{m})$ by

$$(\beta)_* x = 0 \text{ if } \beta^{-1}(0) \neq \{0\}$$

$$(\beta)_* x = \beta'_*(x) \text{ if } \beta^{-1}(0) = \{0\},\ x \in T(\underline{n}).$$

Here $\beta' : \underline{n} \to \underline{m}$ is the restriction of β on $\underline{n}$. We let $i_!(T)([m])$ be the vector space generated by $\alpha \otimes x$, where $\alpha : [n] \to [m]$ is a pointed map and $x \in T(\underline{n})$. The defining relations are

$$\alpha \otimes x + \alpha \otimes y = \alpha \otimes (x + y)$$

$$\alpha \otimes ax = a(\alpha \otimes x), a \in K$$

$$\alpha\beta \otimes x = \alpha \otimes (\beta)_* x, \quad \alpha \in \Gamma,\ .$$

Here β is a surjection in Γ. For a pointed map $f : [m] \to [k]$ one defines $i_!T(f) : i_!T([m]) \to i_!T([k])$ by $\alpha \otimes x \mapsto f\alpha \otimes x$. Clearly $i_!T$ is a Γ-module and by [P3] in this way one gets the functor $i_!$ from Ω-modules to Γ-modules, which is the quasi-inverse of cr.

Let us notice that the similar equivalence exists also for right modules

$$cr : mod - \Gamma \cong \Omega - mod.$$

1.11. Polynomial modules. For a left $K[\Sigma_n]$-module M one denotes by $\theta(M)$ and $\theta^{op}(M)$ the Γ-modules corresponding to $\Theta(M)$ and $\Theta^{op}(M)$ under equivalence 1.10. For $M = K$ with trivial action (resp. with sign representation) one has the isomorphism $\theta^{op}(M) \cong \Lambda^n \circ t$ (resp. $\theta^{op}(M) \cong \theta^n$). This follows from the fact that $cr(F)(\underline{m}) = 0$ if $\underline{n} \neq \underline{m}$ and $cr(F)(\underline{n}) = M$ for $F = \Lambda^n \circ t$ or $F = \theta^n$. These examples suggest to introduce the following notion. We will say that *the degree* of a left Γ-module T is less or equal to n if $cr(T)(k) = 0$ for $k > n$. In this case we write $deg(T) \leq n$. One writes $deg(T) = n$ if $deg(T) \leq n$ and $deg(T) \not\leq n-1$. Similarly for right modules. Then $deg(T) \leq n$ if and only iff $deg(T^*) \leq n$. For example $deg(t) = 1,\ deg(\Gamma^n) = n,\ deg(\theta^n) = n$. Therefore any left (resp. right) Γ-module of degree $\leq n$ admits the projective (resp. injective) resolutions whose components have degree $\leq n$. Moreover in this case if Γ-module is of finite type then one can chooce the resolutions of finite type as well.

1.12. Proposition. *If K is a field of characteristic zero, then for any right Γ-module T of degree $\leq n$ the injective dimension of T is $\leq n-1$. Thus*

$$inj.dim(T) \leq deg(T) - 1,\ T \in mod - \Gamma.$$

Similarly for any left Γ-module T one has

$$proj.dim(T) \leq deg(T) - 1,\ T \in \Gamma - mod.$$

Proof. Thanks to 1.10 one can work with Ω-modules. The functor $\Theta^{op} : K[\Sigma_n] \to mod\text{-}\Omega$ has the left adjoint, which assigns the cokernel

$$\bigoplus T(\underline{n-1}) \to T(\underline{n})$$

to right Ω-module T. Here the sum is taken over all morphisms $\underline{n} \to \underline{n-1}$. Therefore Θ^{op} takes injective objects into injective objects. Since K has characteristic zero, we see that $\Theta^{op}(T(\underline{n}))$ is injective. Clearly there is the natural map $\Theta^{op}(T(\underline{n})) \to T$, whose kernel and cokernel have less than $< n$ degrees and the result follows by induction. Similar argument based on the functor

$$\omega^n : K[\Sigma_n] \to \Omega - mod$$

shows the result for left modules. Here $\omega^n(M) = M \otimes_{K[\Sigma_n]} K[Hom_\Omega(\underline{n}, -)]$.

1.13. Exercise. Let $\omega_n : K[\Sigma_n] \to mod - \Omega$ given by $\omega_n(M) = M \otimes_{K[\Sigma_n]} K[Hom_\Omega(-, \underline{n})]$. Show that ω^n is exact and sends projective objects in projectives.

1.14. Proposition. i) *Let F be a right Γ-module of degree d, then $Hom_{mod-\Gamma}(\theta^i, F) = 0$ as soon as $i > d$.*

ii) *Let K be a field of characteristic zero. Assume $m > j - i$. Then*

$$Ext^m_{mod-\Gamma}(\theta^i, \theta^j) = 0.$$

Proof. i) Since $cr(\theta^i)(\underline{m}) \neq 0$ only for $m = i$, we see that

$$Hom_{mod-\Omega}(cr(\theta^i), cr(F)) = 0$$

if $i > d$. Hence result follows from 1.10.

ii) Let $Sym^* : Vect \to Vect$ be the functor which assignes the underline vector space of the symmetric algebra $Sym^*(V)$ to V. One defines the transformations

$$d : \Lambda^{i+1} \circ t^* \otimes Sym^j \circ t^* \to \Lambda^i \circ t^* \otimes Sym^{j+2} \circ t^*$$

and

$$d : Sym^n \circ t^* \to \mathcal{B}_n / \mathcal{B}_{n-1}$$

by

$$d(x_0 \wedge \cdots \wedge x_i \otimes y_1 \cdots y_j) :=$$

$$\sum_{k=0}^{i} (-1)^k x_0 \wedge \cdots \wedge \hat{x}_k \cdots \wedge x_i \otimes y_k^2 y_1 \cdots y_j$$

and

$$d(z_1 \cdots z_n) := \{z_1, \cdots, z_n\} mod \mathcal{B}_{n-1}.$$

Here $x_0, \cdots, x_i, y_0, \cdots, y_j, z_1, \cdots, z_n \in [m]$. We claim that for any n the following sequences are exact

$$0 \to \Lambda^n \circ t^* \to \Lambda^{n-1} \circ t^* \otimes Sym^2 \circ t^* \to \Lambda^{n-2} \circ t^* \otimes Sym^4 \circ t^* \to \cdots$$

$$\cdots \to t^* \otimes Sym^{2n-2} \circ t^* \to\to Sym^{2n} \circ t^* \to \mathcal{B}_{2n} / \mathcal{B}_{2n-1} \to 0$$

and

$$0 \to \Lambda^n \circ t^* \otimes t^* \to \Lambda^{n-1} \circ t^* \otimes Sym^3 \circ t^* \to \cdots$$
$$\cdots \to Sym^{2n+1} \circ t^* \to \mathcal{B}_{2n+1}/\mathcal{B}_{2n} \to 0.$$

This facts are true in any characteristic and can be proved as follows. Let $\mathcal{K}$ be bigraded object which in bidegree (i,j) is zero if i is odd and is $\Lambda^{i/2} \circ t^* \otimes Sym^j \circ t^*$ if i is even. This object is equip with the differential as defined above. One easily shows that $\mathcal{K}(S \vee T) \cong \mathcal{K}(S) \otimes \mathcal{K}(T)$. So everything reduces to the case, when above sequences are evaluated on [1]. In this case the assertion is obvious. In characteristic zero case, one can take the dual to obtain

$$0 \to \theta^n \to Sym^n \circ t \to \cdots \to \Lambda^{n/2} \circ t \to 0$$

if n is even and

$$0 \to \theta^n \to Sym^n \circ t \to \cdots \to t \otimes \Lambda^{(n-1)/2} \circ t \to 0$$

if n is odd. By 1.4 we know that the $t^{\otimes n}$ is injective. Since we are in characteristic zero case, it follows that $\Lambda^n \circ t$, $Sym^n \circ t$ and $\Lambda^n \circ t \otimes Sym^n \circ t$ are injective too. Therefore $Ext^m_{mod-\Gamma}(\theta^i, \theta^j)$ is a subquotient of $Hom_{mod-\Gamma}(\theta^i, \Lambda^m \circ t \otimes Sym^{j-2m} \circ t)$. Since $deg(\Lambda^m \circ t \otimes Sym^{j-2m} \circ t) = j - m$ the result follows from part i).

2. Decomposition of $\pi_* F(S^n)$

2.1. Link with homotopy theory and the main idea. Let $Sets_*$ be the category of all pointed sets. There is a standard way to prolong a left Γ-module F to a functor from the category of simplicial sets with basepoint $s.Sets_*$ to the category of simplicial vector spaces $s.Vect$. First, one can prolong F by direct limits to a functor $Sets_* \to Vect$, then by the degreewise action one obtains a functor $s.Sets_* \to s.Vect$. By abuse of notation we will still denote this functor by F. Similarly any right Γ-module T can be prolonged to a functor from the category of simplicial sets with basepoint $s.Sets_*$ to the category of cosimplicial vector spaces $cos.Vect$. It is of great interest to understand the structure of the groups $\pi_* F(L)$ for a based simplicial set L. Of course of special interest is the case $L = S^n$. Let us remark that any Γ-set A (that is a functor

$\Gamma \to Sets_*$, with property $A([0]) = [0]$) gives rise to a binatural transformation

$$X \wedge A(Y) \to A(X \wedge Y)$$

as follows: For any $x \in X$ define $\hat{x} : Y \to X \wedge Y$ by $y \mapsto (x, y)$. Then apply A to get $A(\hat{x}) : A(Y) \to A(X \wedge Y)$. Now one defines $X \wedge A(Y) \to A(X \wedge Y)$, by $(x, z) \mapsto A(\hat{x})(z)$. Using this transformation one obtains a map $S^1 \wedge A(S^n) \to A(S^{n+1})$. Here S^1 is a simplicial model of the circle, which has only two nondegenerate simplices and $S^{n+1} = S^1 \wedge S^n$. Of course this can be done with left Γ-modules as well. Indeed if F is a left Γ-module, then $F = F_0 \oplus A$, where F_0 is a constant Γ-module with value $F([0])$. Clearly $A([0]) = 0$ and one can define

$$\pi_*^{st}(F) : = colim \pi_{*+n} A(S^n) \cong colim \pi_{*+n} F(S^n)$$

Similarly for a right Γ-module T one puts

$$\pi^i_{st} F \ := lim_n \pi^{n+i} F(S^n).$$

Here π^* denotes the "cohomotopy groups" of a cosimplicial vector space, meaning the homology of the associated cochain complex.

Mimicking Korollar 6.12 in [DP], one can prove that this limit always stabilizes and one has the isomorphism

$$\pi_i^{st}(F) \cong \pi_{i+n} F(S^n) \text{ if } n > i. \tag{2.1.2}$$

We do not give the argument here, because in [P2] a much general statement is proved.

Now we give the main idea of our approach. For simplicity we take the classical case $d = 1$. Let us recall that the n dimensional simplices of S^1 are $[n]$. Moreover $s_i : [n] \to [n+1]$ is the unique monotone injection, whose image does not contain i, while $d_i : [n] \to [n-1]$ is given by $d_i(j) = j$ if $j < i$, $d_i(i) = i$ if $i < n$, $d_n(n) = 0$ and $d_i(j) = j - 1$ if $j > i$. Thus $\pi_* F(S^1)$ is by definition the homology of the following complex

$$F(S^1) \ = \ (F([0]) \longleftarrow F([1]) \longleftarrow F([2]) \longleftarrow \cdots \ ),$$

where the boundary map is the alternating sum of the face homomorphisms. Using the tensor product of functors (see 1.5) we can write $F([n]) = \Gamma_n \otimes_\Gamma F$. Therefore $F(S^1) = \Gamma_{S^1} \otimes_\Gamma F$, where Γ_{S^1} is the simplicial right Γ-module whose n-dimensional simplices are Γ_n. Now one can use the result from 1.6 in order to construct a spectral sequence whose abutment is $\pi_* F(S^1)$. It turns out that in the characteristic zero case, this spectral sequence degenerates and we obtain the expected decomposition. In order to realize this program we start with the translation of "homotopy theory" in the language of homological algebra of Γ-modules.

2.2. Proposition. *For any left Γ-module F, one has an isomorphism*

$$Tor_*^\Gamma(t, F) \cong \pi_*^{st} F.$$

Moreover, for any right Γ-module T one has an isomorphism

$$\pi^i_{st} T \cong Ext^*_{mod-\Gamma}(t, T).$$

Proof. We use the well-known axiomatic characterisation of Tor functors. In order to show the existence of the isomorphism in dimension 0, one notes that $\pi_0^{st} F \cong \pi_1 F(S^1)$ thanks to (2.1.2). By our choice of the model of S^1 one sees that $\pi_1 F(S^1) \cong Coker(F([2]) \to F([1])$ and hence by the exact sequence (1.4.1) we get $\pi_0^{st} F \cong t \otimes_\Gamma F$, because $F([n]) \cong \Gamma^n \otimes_\Gamma F$. Obviously the functors $T \mapsto \pi_*^{st} T$ form an exact connected sequence of functors Γ-$mod \to Vect$ and it is enough to show that these are zero in positive dimensions for projective T. Therefore one only needs to consider functors like $T = \Gamma^n$. The chain complex associated to the simplicial abelian group $\Gamma^n(S^m)$ is the same as the chains of the product of n copies of S^m. Thanks to the Künneth theorem the homology of this complex in dimensions $> m$ and $< 2m$ is zero, and we are done. A similar argument works for right modules.

In what follows we give the formulation of results only for left Γ-modules. We leave the task to the interested reader to make the trivial reformulation for the corresponding results for right Γ-modules.

2.3. Fundamental spectral sequence. Let L be a based simplicial finite set. It is well-known that the homology $H_*(L)$ of L

with coefficient in K is a coaugmented coalgebra. Therefore one can consider the right Γ-module $\mathcal{J}_i(H_*L)$ for any $i \geq 0$ (see 1.7).

2.4. Theorem. *Let F be a left Γ-module and let L be a based simplicial finite set. Then there exists a spectral sequence*

$$E^2_{pq} = Tor_p^{\Gamma}(\mathcal{J}_q(H_*L), F) \Longrightarrow \pi_{p+q}(F(L)),$$

where $F(L)$ is the composition $\Delta^{op} \xrightarrow{L} \Gamma \xrightarrow{F} Vect$. Moreover any simplicial map $L \to L'$ induces an isomorphism $\pi_*(F(L)) \to \pi_*(F(L'))$ as soon as $H_*(L) \to H_*(L')$ is an isomorphism.

Proof. We let Γ_L be the following composition

$$\Delta^{op} \xrightarrow{L} \Gamma \xrightarrow{\Gamma_?} mod-\Gamma,$$

where the functor $\Gamma_?$ sends $[n]$ to Γ_n. Therefore Γ_L is a simplicial object in mod-Γ, which assigns Γ_{L_n} to $[n]$. By abuse of notation we denote by Γ_L also the chain complex associated to this simplicial right Γ-module. Let us calculate the homology of this chain complex. By definition we have $\Gamma_L([n]) = K[Hom_{\Gamma}([n], L)] \cong K[L^n]$. Therefore by the Eilenberg-Zilber and Künneth theorem we have $H_*(\Gamma_L)[n] = H_*(L)^{\otimes n}$. Hence one has an isomorphism of right Γ-modules

$$H_i(\Gamma_L) \cong \mathcal{J}_i(H_*(L)).$$

The relations from 1.5 show that $F(L)([n]) \cong \Gamma_{L_n} \otimes_{\Gamma} F$. So one can write: $F(L) \cong \Gamma_L \otimes_{\Gamma} F$. Thus the spectral sequence constructed in Proposition 1.6 for $C_* = \Gamma_L$ gives the expected result.

2.5. Corollary. *Let F be a left Γ-module. Then there exists a spectral sequence*

$$E^2_{pq} \Longrightarrow \pi_{p+q}F(S^d),\ d \geq 1,$$

with $E^2_{pq} = 0$ if $q \neq dj$ and

$$E^2_{pq} = Tor_p^{\Gamma}(\Lambda^j \circ t, F) \text{ if } q = dj \text{ and } d \text{ is odd}$$

$$E^2_{pq} = Tor_p^{\Gamma}(\theta^j, F) \text{ if } q = dj \text{ and } d \text{ is even.}$$

Moreover, if K is a field of characteristic zero, then the spectral sequence degenerates:

$$\pi_n F(S^d) \cong \bigoplus_{p+dj=n} Tor_p^{\Gamma}(\Lambda^j \circ t, F)$$

if d is odd and

$$\pi_n F(S^d) \cong \bigoplus_{p+dj=n} Tor_p^{\Gamma}(\theta^j, F)$$

for even d.

Proof. Example 1.8 shows that $\mathcal{J}_i(H_*(S^d)) = 0$ if $i \neq jd$ and for $i = jd$ $\mathcal{J}_i(H_*(S^d)) \cong \Lambda^j \circ t$ or $\mathcal{J}_i(H_*(S^d)) \cong \theta^j$ depending of parity of d. Hence the previous result implies that the spectral sequence is of the expected form. By 1.4 we know that $t^{\otimes n}$ is an injective right Γ-module. If K has characteristic zero, then Λ^q is a natural direct summand of $\otimes^q$. Therefore $\Lambda^i \circ t$ is also injective, so $Ext^i_{mod-G}(-, \Lambda^j \circ t) = 0$ for $i > 0$ and the result for odd d follows from Proposition 1.6. Assume now d is even. By Proposition 1.6 we have to show that

$$Ext^{2(j-i)+1}_{mod-\Gamma}(\theta^i, \theta^j) = 0$$

if $i < j$. But this is a consequence of Proposition 1.14 ii).

2.6. Relation with Loday's decomposition. Observe that the summand of the decomposition 2.5 corresponding to $j = 0$ is trivial if $p > 0$, because the functor $\Lambda^0 \circ t$ is the constant functor with value K and thus it is isomorphic to Γ_0 and hence it is projective. Now we consider the case when $d = 1$. Let us recall that in [L1] Loday proved that if K has characteristic zero, then for any left Γ-module F one has the following decomposition

$$\pi_n F(S^1) \cong \bigoplus_{i=0}^{n} H_n^{(i)}(F). \tag{2.6.1}$$

We will show that our decomposition in Corollary 2.5 for $d = 1$ essentially coincides with Loday's decomposition. In fact we prove more, namely we give the axiomatic characterization of the decomposition of $\pi_n F(S^1)$.

2.7. Theorem. *Let K be a field of characteritic zero. Let*

$$\pi_n F(S^1) \cong \bigoplus_{i=0}^{n} H_n^{(i)}(F)$$

be a natural decomposition with properties:

i) $H_n^{(0)}(F) = 0$ if $n > 0$

ii) *For any $i \geq 1$, the sequence of functors $H_n^{(i)}(-) : \Gamma\text{-}mod \to Vect$, $n \geq i$, form the exact connected sequence of functors.*

iii) *For any $n \geq 0$, the functor $H_n^{(n)}(-)$ is nonzero.*

Then, for any $i \geq 1$ one has a natural isomorphism

$$H_n^{(i)}(F) \cong Tor_{n-i}^{\Gamma}(\Lambda^i \circ t, F), \ \ n \geq i.$$

Moreover for the decomposition (2.6.1) *the properties* i)-iii) *hold and therefore our decomposition is isomorphic to the one given in* [L1].

Proof. The fact that (2.6.1) satisfies i)-iii) is very easy to check based on the definition given in [L1]. So it is enough to show the first statement. If $n = 1$, then the result is clear because there is only one summand both decomposition of $\pi_1 F(S^1)$. In order to prove the theorem, we still use the axiomatic characterization of Tor-groups. Since the functors $H_n^{(i)}(-) : \Gamma\text{-}mod \to Vect, n \geq i$ form the exact connected sequence of functors, it is enough to show that for projective F one has the isomorphisms

$$H_i^{(i)}(F) \cong \pi_i F(S^1) \text{ and } H_n^{(i)}(F) = 0, \text{ for } n > i.$$

Let us recall that the functors $t^{*\otimes n}$, $n \geq 0$ are projective generators in the category of left Γ-modules. Hence one only needs to consider the case $F = t^{*\otimes n}$. Let us calculate $\pi_* F(S^1)$, when $F = t^{*\otimes n}$. We recall that $t^*(S)$ is the free vector space generated by S, modulo the relation $* = 0$. Thus $\pi_* t^*(S^1)$ is nothing but the reduced homology of S^1. Therefore it is clear that

2.7.1. $\pi_k t^{*\otimes m}(S^1) \cong K$, if $k = m$ and $\pi_k t^{*\otimes m}(S^1) = 0$ if $k \neq m$

It follows from the property ii) that the functor $H_i^{(i)}(-)$ is right exact for any $i \geq 0$. Now consider the case $n = 2$. The isomorphism 2.7.1 shows that $H_2^{(2)}(t^{*\otimes n}) = 0$ except the case, when $n = 2$. If this were also true for $n = 2$, then $H_2^{(2)}(F) = 0$ for all projective F, and therefore $H_2^{(2)}(-) = 0$, because $H_2^{(2)}(-)$ is right exact. This contradicts to the fact that all components are nontrivial in general. Thus $H_2^{(2)}(t^{*\otimes 2}) = K$. Comparing the Hodge decomposition for $n = 2$ with 2.7.1, we can conclude that $H_2^{(1)}(F) = 0$ for any projective F. This shows that the result is proved for $n = 2$. Now we can finish the proof by induction, based on the same argument.

2.8. Stable homotopy and Harrison homology. For any left Γ-module F, the vector space $F([n])$ is the $K[\Sigma_n]$-module, because $\Sigma_n = Aut_\Gamma([n])$. Therefore the subspace of shuffles Sh_n of $K[\Sigma_n]$ acts on $F([n])$ (see [L1]). Moreover, by [L1], the subspaces $Sh_n(F) := Sh_nF([n]) \subset F([n]), n \geq 0$, define the subcomplex of the complex associated to the simplicial module $F(S^1)$ (let us recall that in dimension n, the simplicial module $F(S^1)$ is just $F([n])$). By definition *Harrison homology* $Harr_*(F)$ of F is the homology of the corresponding quotient.

2.9. Corollary. *Assume K has characteritic zero. Then one has a natural isomorphism*

$$Harr_{*-1}(F) \cong \pi_*^{st}(F).$$

Proof. By Theorem 3.7 [L1], we know that $H_*^{(1)}(F) \cong Harr_*(F)$. Therefore our statement is a consequence of Proposition 2.2 and Theorem 2.7.

2.10. Link with Mac Lane homology. Let $P(\mathbf{Z})$ be the category of finitely generated free abelian groups. Let $\mathcal{F}(\mathbf{Z})$ be the category of all functors from $P(\mathbf{Z})$ to the category of all abelian groups. Let $T \in \mathcal{F}(\mathbf{Z})$ be a functor. We refer to [JP] (see also [L2] Chapter 13) for the definition of Mac Lane (co)homology $HML(\mathbf{Z}, T)$ of $\mathbf{Z}$ with coefficient in T. For any functor $T \in \mathcal{F}(\mathbf{Z})$ one denotes by $\bar{T}$ the precomposition of T with the functor $\bar{\mathbf{Z}} : \Gamma \to P(\mathbf{Z})$, given by $S \mapsto \bar{\mathbf{Z}}[S]$. Here S is a pointed set and $\bar{\mathbf{Z}}[S]$ is the free abelian group generated by S modulo the relation $* = 0$.

2.11. Proposition. *For any $T \in \mathcal{F}(\mathbf{Z})$ one has a natural isomorphism*

$$HML_*(\mathbf{Z}, T) \otimes \mathbf{Q} \cong Harr_{*-1}(\bar{T}) \otimes \mathbf{Q}.$$

Proof. According to Corollary 2.9 one only needs to show that

$$HML_*(\mathbf{Z},T)\otimes\mathbf{Q}\cong\pi_*^{st}(\bar{T})\otimes\mathbf{Q}.$$

This fact is probably well known to experts, but we give the argument here. Following Breen [B], we consider the chain complex $W_*(X)$ of the simplicial set $\mathbf{Z}[S^n]\otimes X$, where $X\in P(\mathbf{Z})$ and $n>0$ is a natural number. By the well known properties of Eilenberg-Mac Lane spaces we know that $H_i(W_*(X))\otimes\mathbf{Q}=0$ for $i<2n$ and $i\neq n$. Moreover $H_n(W_*(X))=X$ and each component of $W_i(X)$ is of the form $\mathbf{Z}[X^m]$ for some m. Varying X one obtains the componentwise projective complex $X\mapsto W(X^*)$ in the category $\mathcal{F}(\mathbf{Z})$, whose homology up to torsion is the functor I^* in dimension $<n$ and zero in dimensions $<2n$. Here $I^*(X)=X^*=Hom(X,\mathbf{Z})$. Therefore W_* can be used to calculate $HML_*(\mathbf{Z},T)=Tor_*^{P(\mathbf{Z})}(I^*,T)$ up to torsion and the result follows.

2.12. The groups $\pi_*F(S^d)$ and with puzzles play. Let d be a odd number. Comparing Corollary 2.5 and Theorem 2.7 one sees that in the characteristic zero case the groups

$$\pi_nF(S^d)\cong\bigoplus_{i+dj=n}H_{i+j}^{(j)}(F) \tag{2.12.1}$$

for different d differ only by the way of taking the pieces $H_n^{(i)}(F)$ in the decomposition (2.12.1). The same remark is true also for even d. The knowledge of the decomposition for $d=2$ completly determines the decomposition for all even dimensional spheres. However in even case in the decomposition of $\pi_nF(S^d)$ only the group $Harr_{n-d+1}(F)\cong H_{n-d+1}^{(1)}(F)$ is from Loday's decomposition and all other groups are new.

2.13. Stable homotopy and homology of small categories. We refer the reader to [BW] for definition of the (co)homology of small category with coefficient in a bifunctor. For any left Γ-module F and right Γ-module T, we let D^F and D_T be the bifunctors on Γ given by

$$D^F([m],[n])=t([m])\otimes F([n])\text{ and }D_T([m],[n])=T([m])\otimes t^*([n]).$$

Using Proposition 2.2 and Corollary 3.11 of [JP] we obtain the following isomorphisms

$$\pi^*_{st} T \cong H^*(\Gamma, D_T) \text{ and } \pi^{st}_* F \cong H^*(\Gamma, D^F). \tag{2.13.1}$$

2.14. A question. We do not know the answer on the following question. *Does the spectral sequence 2.4 degenerates if K is a field of characteristic zero and L is a connected based simplicial finite set?* One can prove that answer is yes if L is connected and reduced homology is nontrivial only in odd degrees.

3. Decomposition for cyclic homology

3.1. More categories. Let Fin be the category of finite nonempty sets. We assume that objects of Γ are sets $[n]$. By forgetting the base point one gets the functor $i : \Gamma \to Fin$. Let ΔC be Connes' cyclic category (see [L2]). It is well-known that the simplicial circle S^1 has a natural cyclic structure, meaning that the functor $S^1 : \Delta^{op} \to \Gamma$ fits in the commutative diagramm (see Proposition 2.11 of [L1]):

$$\begin{array}{ccc} \Delta^{op} & \to & \Delta C^{op} \\ S^1 \downarrow & & \downarrow S^1 \\ \Gamma & \to & Fin \end{array}$$

Therefore any functor $T : Fin \to Vect$ gives rise to the cyclic module $T(S^1)$. We let $HC_*(T)$ be the cyclic homology of this cyclic module. Let us remark, that for $T([n]) = A^{\otimes n+1}$ one gets the cyclic homology of A. Here A is any commutative algebra. In [L1] Loday proved that in the characteristic zero case, the groups $HC_*(T)$ have a canonical decomposition:

$$HC_n(T) \cong HC^{(1)}_n(T) \oplus \cdots \oplus HC^{(n)}_n(T), \ n \geq 1. \tag{3.1.1}$$

Moreover he proved that there is a natural isomorphism

$$H^{(1)}_n(T) \cong HC^{(1)}_n(T) \text{ for } n \geq 3.$$

We give an alternative proof of these facts based on the same ideas as were used before.

3.2. From Γ-modules to Fin-modules. A *left* (resp. *right*) Fin-*module* is a covariant (resp. contravariant) functor from Fin to the category $Vect$. The precomposition with i defines the functor from Fin-modules to Γ-modules. We let i^* denote this functor. Thus for any Fin-module F one has $i^*(F)([n]) = F([n])$. It is a well known fact that i^* has left and right adjoint functors, which are known as left and right Kan extensions of i (see section 17.1 of [S]). We let $i_!F$ be the value of the left Kan extension of i on $F \in \Gamma$-*mod*. Thus one has an isomorphism

$$\mathrm{Hom}_{Fin-mod}(i_!F, G) \cong \mathrm{Hom}_{\Gamma-mod}(F, i^*G). \tag{3.2.1}$$

A similar fact is true for right modules. We see that $i_!\Gamma^n$ and $i_!\Gamma_n$ are standard representable functors from Fin to $Vect$:

$$i_!\Gamma^n = K[\mathrm{Hom}_{Fin}([n], -)] \text{ and } i_!\Gamma_n = K[\mathrm{Hom}_{Fin}(-, [n])].$$

Therefore these are projective generators in Fin-modules. It is clear that for Fin-modules one has duality and pointwise tensor products as for Γ-modules. Moreover

$$i_!\Gamma^n \otimes i_!\Gamma^m \cong i_!\Gamma^{n+m+1} \text{ and } i_!\Gamma_n \otimes i_!\Gamma_m \cong i_!\Gamma_{nm+n+m}.$$

These are consequence of the facts that $[n]\coprod[m] \cong [n+m+1]$ and $[n] \times [m] \cong [nm + n + m]$. Therefore *if F and T are projective Fin-module, then $F \otimes T$ is also projective.* Similarly, *if F and T are injective Fin-module, then $F \otimes T$ is also injective* provided F and T have values in finite dimensional vector spaces. We have a bifunctor $-\otimes_{Fin}- : mod\text{-}Fin \times Fin\text{-}mod \to Vect$ similar to 1.5 with the same kind of properties. For example one has the isomorphism: $F([n]) \cong i_!\Gamma_n \otimes_{Fin} F$. We also observe that Proposition 1.6 is still valid for Fin-modules.

3.3. Proposition. i) *The functor i^* is exact and sends injective Fin-modules to injective Γ-modules.*

ii) *The functor $i_! : \Gamma\text{-}mod \to Fin\text{-}mod$ (resp. $i_! : mod\text{-}\Gamma \to mod\text{-}Fin$) is exact and sends projective Γ-modules to projective Fin-modules.*

iii) *One has isomorphisms*

$$\mathrm{Ext}^*_{Fin-mod}(i_!F, G) \cong \mathrm{Ext}^*_{\Gamma-mod}(F, i^*G),$$

$$\mathrm{Ext}^*_{mod-Fin}(i_!F, G) \cong \mathrm{Ext}^*_{mod-\Gamma}(F, i^*G)$$

and

$$Tor_*^{Fin}(i_!F, G) \cong Tor_*^{\Gamma}(F, i^*G).$$

iv) *If F and G are right Γ-modules, then $i_!(F \otimes G) \cong i_!F \otimes i_!G$ and $i_!(\Lambda^i \circ F) \cong \Lambda^i \circ i_!(F)$.*

v) *One has the following isomorphism*

$$i_!t \cong \mathrm{Hom}_{Sets}(-, K).$$

Proof. i) Clearly i^* is exact and commutes with products. Therefore it is enough to show that $i^*(F)$ is injective, when $F = (i_!\Gamma_n)^*$ or $F = (i_!\Gamma^n)^*$. Since i^* is also compatible with duality one needs only to use the following isomorphisms:

$$i^*(i_!\Gamma^n) \cong \Gamma^{n+1} \text{ and } i^*(i_!\Gamma_n) \cong \bigoplus_{m=0}^{n} \Gamma_n,$$

which are consequences of the facts that there exists bijections

$$\mathrm{Hom}_{Fin}([n], [m]) \cong \mathrm{Hom}_{\Gamma}([n+1], [m]),$$

and

$$\coprod_{k=0}^{n} \mathrm{Hom}_{\Gamma}([m], [n]) \cong \mathrm{Hom}_{Fin}([m], [n]),$$

which are natural with respect of m. Here the restriction of the last isomorphism on the k-th component is the precomposition with the transposition $(0, k)$ of $[n]$.

ii) and iii) are consequences of i) and (3.2.1).

iv) By definition of the Kan extension one has the morphisms $F([n]) \to i_!F([n])$ and $G([n]) \to i_!G([n])$. Tensoring these morphisms one gets $F([n]) \otimes G([n]) \to (i_!F([n])) \otimes (i_!G([n]))$. By the universal property of left Kan extension there is a natural transformation $i_!(F \otimes G) \to$

$i_!F \otimes i_!G$, which is an isomorphism. Indeed, it is an isomorphism when $F = i_!\Gamma_n$ and $G = i_!\Gamma_m$, because $i_!\Gamma_n \otimes i_!\Gamma_m \cong i_!\Gamma_{nm+n+m}$ and $\Gamma_n \otimes \Gamma_m \cong \Gamma_{nm+n+m}$. The general case follows from the fact that the functor $i_!$ is right exact and commutes with sums. The last isomorphism follows from the previous one.

v). Since $i_!$ is right exact one can apply the sequence (1.4.1) to get

$$i_!(t) \cong Coker\ (i_!(\alpha) : i_!\Gamma_2 \to i_!\Gamma_1).$$

In order to identify this cokernel with the desired functor, it is enough to note that the following sequence is exact:

$$K[\mathrm{Hom}_{Fin}([n],[2])] \xrightarrow{i_!(\alpha)} K[\mathrm{Hom}_{Fin}([n],[1])] \to \mathrm{Hom}_{Sets}([n],K) \to 0.$$

Here the last homomorphism is given by $g \mapsto \sum_{g(i)=1} \chi_i$, where $g : [n] \to [1])$ is a map in Fin.

3.4. A spectral sequence for $HC_*(F)$. According to [L1] the cyclic homology $HC_*(F)$ of a left Fin-module F is defined as the homology of the bicomplex

$$\begin{array}{ccccc}
\downarrow & & \downarrow & & \downarrow \\
F([2]) & \longleftarrow & F([1]) & \longleftarrow & F([0]) \\
\downarrow & & \downarrow & & \\
F([1]) & \longleftarrow & F([0]) & & \\
\downarrow & & & & \\
F([0]), & & & &
\end{array}$$

where each column is the chain complex associated to the simplicial vector space $F(S^1)$. Let us observe that one can write $F([n]) \cong i_!\Gamma_n \otimes_{Fin} F$. Therefore, $HC_*(F) \cong H_*(L_{**} \otimes_{Fin} F)$, where L_{**} is the following bicomplex of right Fin-modules:

$$\begin{array}{ccccc}
\downarrow & & \downarrow & & \downarrow \\
i_!\Gamma_2 & \longleftarrow & i_!\Gamma_1 & \longleftarrow & i_!\Gamma_0 \\
\downarrow & & \downarrow & & \\
i_!\Gamma_1 & \longleftarrow & i_!\Gamma_0 & & \\
\downarrow & & & & \\
i_!\Gamma_0 . & & & &
\end{array}$$

Let us calculate the value of L_{**} on $[n]$. Since $i_!\Gamma_m = K[Hom_{Fin}(-,[m])]$ we see that $H_*(L_{**}([n]))$ is the cyclic homology of the cyclic module associated to the cyclic space $S^1 \times \cdots \times S^1$ ($(n+1)$-times). Now one can use the standard facts about cyclic homology theory (see Theorem 7.2.3 and Proposition 4.4.8 of [L2]) to conclude that $H_*(L_{**}([n]))$ is isomorphic to the exterior algebra $\Lambda^*(x_1, \cdots, x_n)$, where each x_i has degree 1. A similar argument as in Example 1.8 shows that $H_i(L_{**}) \cong \Lambda^i \circ \bar{t}$, where $\bar{t}$ is defined as follows: Let us recall that $i_!\Gamma_0$ is the constant functor with value K. There is a canonical transformation $i_!\Gamma_0 \to i_!t$ whose value on $[n]$ is a homomorphism which sends an element $\lambda \in K$ to $\lambda_{[n]} \in i_!t([n]) = Hom_{Sets}([n], K)$. Here $\lambda_{[n]} : S \to K$ is the constant map with value λ. By definition $\bar{t}$ fits in the exact sequence:

$$0 \to i_!\Gamma_0 \to i_!t \to \bar{t} \to 0. \tag{3.4.1}$$

Theorem 3.5. *Let F be a left Fin-module.*

i) *Then there exists a spectral sequence*

$$E^2_{pq} = Tor_p^{Fin}(\Lambda^q \circ \bar{t}, F) \Longrightarrow HC_{p+q}(F).$$

For $q = 1$ and $p \geq 2$ one has $E^2_{p1} \cong Tor_p^{\Gamma}(t, i^*F)$.

ii) *If K is a field of characteristic zero, then the spectral sequence degenerates:*

$$HC_{p+q}(F) \cong \bigoplus_{p+q=n} Tor_p^{Fin}(\Lambda^q \circ \bar{t}, F).$$

iii) *Furhermore in this case one has an isomorphism between our and Loday's decomposition (3.1.1)*

$$HC_n^{(i)}(F) \cong Tor_{n-i}^{Fin}(\Lambda^i \circ \bar{t}, F), \ n \geq i.$$

Proof. i) One can apply the Fin-version of Proposition 1.6 for $C_* = Tot(L_{**})$. By above calculation the spectral sequence is of the expected form. Since $i_!\Gamma_0$ is projective, the exact sequence (3.4.1) shows that there exist an isomorphism $E^2_{p1} \cong Tor_p^{Fin}(i_!t, F)$ provided $p \geq 2$. Therefore the last statement is a consequence of Proposition 3.3. iii).

ii) By Proposition 1.6 it is enough to show that in characteristic zero one has $Ext^i_{mod-Fin}(\Lambda^n \circ \bar{t}, \Lambda^m \circ \bar{t}) = 0$ if $n < m$ and $i \geq 2$. We will work with tensor powers instead of exterior powers, because these are retracts of tensor powers. The short exact sequence (3.4.1) shows that the complex $(i_!\Gamma_0 \to i_!t)^{\otimes n}$ is a (nonprojective) resolution of $\bar{t}^{\otimes n}$. Therefore one has a spectral sequence

$$E_1^{pq} = Ext^q_{mod-Fin}(i_!t^{\otimes p} \otimes i_!\Gamma_0^{\otimes n-p}, \bar{t}^{\otimes m}) \Longrightarrow Ext^{p+q}_{mod-Fin}(\bar{t}^{\otimes n}, \bar{t}^{\otimes m}).$$

By Proposition 3.3 iv) we know that the functor $i_!$ commutes with tensor products. Since $i_!\Gamma_0 = K$ one can delete it in the tensor product. Therefore Proposition 3.3 iii) shows that

$$E_1^{pq} \cong Ext^q_{mod-\Gamma}(t^{\otimes p}, i^*(\bar{t}^{\otimes m})).$$

One easily sees that $i^*(\bar{t}^{\otimes m}) \cong t^{\otimes m}$ and therefore it is injective. Hence $E_1^{pq} = 0$ if $q > 0$. By Lemma 3.6 we see that $E_1^{pq} = 0$ even when $q = 0$.

iii) By the argument given in the proof of Theorem 2.7 it is enough to show that $HC_m(i_!t^{*\otimes n}) = 0$ for $n \neq m$ and $HC_n(i_!t^{*\otimes n}) = K$ for all n. By Proposition 4.3.10 of [L2] one can reduce the problem to the cases $n = 0, 1$. For $n = 0$ one obtains the functor $i_!\Gamma_0$. Clearly $i_!\Gamma_0(S^1) \cong K[S^1]$. Therefore $HC_0(i_!\Gamma_0) = K$ and $HC_m(i_!\Gamma_0) = 0$ for $m > 0$, thanks to Theorem 7.2.3 of [L2]. Similarly $HC_*(i_!\Gamma_1) = HC_*(K[S^1 \times S^1]) = \Lambda^*(x_1)$, where x_1 has the dimension 1. By 1.4 we know that $t^* \oplus \Gamma_0 \cong \Gamma_1$. Therefore $HC_m(i_!t^*) = 0$ for $m \neq 1$ and $HC_1(i_!t^*) = K$ and we are done.

3.6 Lemma. If $n \neq m$ then $Hom_{mod-\Gamma}(t^{\otimes n}, t^{\otimes m}) = 0$.

Proof. By (4.1.1) it is enough to consider the case $n = 1$. Thanks to Proposition 2.2 we have to show that for any $i > 0$ one has $\pi^i t^{\otimes m}(S^i) = 0$ for $m > 1$. Since $t^{\otimes m}(X) = Hom_{Sets_*}(X^{\wedge m}, K)$ we see that $\pi^i t^{\otimes m}(X)$ for $i \geq 1$ is the same as the i-dimensional reduced cohomology of $X^{\wedge m}$ with coefficient in K and it is well known that this vanishes for $X = S^i$.

4. Hochschild-Kostant-Rosenberg theorem for functors

The goal of this section is to get a partial generalization of the Hochschild-Kostant-Rosenberg theorem (see for example 3.4.4 of [L2]) which will play the crucial role in Section 5. In this section we assume that K is a field of characteristic zero.

4.1. Inner ***Ext*** **and inner** ***Tor*****.** It is well known that in the category of left Γ-modules there exist the inner Hom (see for example [S]). Thus for any left Γ-modules T and U there exist a left Γ-module $\mathcal{H}om(T, U)$ such thet for any left Γ-module F one has the functorial isomorphism:

$$\text{(4.1.1)} \qquad \mathrm{Hom}_{\Gamma-mod}(F \otimes T, U) \cong \mathrm{Hom}_{\Gamma-mod}(F, \mathcal{H}om(T, U)).$$

Thanks to Yoneda lemma we have

$$\mathcal{H}om(T, U)([n]) \cong Hom_{\Gamma-mod}(\Gamma^n \otimes T, U).$$

Usually inner Hom is defined by this isomorphism. This is actually true in any category of functurs. Since Γ admits the sums one can describe the inner Hom in much easy way. For this purposes we need the functor

$$\bar{\Delta}_n : \Gamma - mod \to \Gamma - mod, n \geq 0,$$

defined by $(\bar{\Delta}_n U)([m]) := U([n] \vee [m])$. We claim that the natural transformation

$$\eta : Hom_{\Gamma-mod}(\Gamma^n \otimes T, U) \to Hom_{\Gamma-mod}(T, \bar{\Delta}_n U)$$

is an isomorphism and hence

$$\mathcal{H}om(T, U)([n]) \cong Hom_{\Gamma-mod}(T, \bar{\Delta}_n U).$$

The transformation η is defined as follows. Let $\alpha : \Gamma^n \otimes T \to U$ be a natural transformation. One denotes by $\eta(\alpha)([m])$ the composition of a homomorphism $T(i) : T([m]) \to T([n] \vee [m])$ with the homomorphism $\alpha' : T([n] \vee [m]) \to U([n] \vee [m])$. Here i is the canonical map $i : [m] \to [n] \vee [m]$, while $\alpha'(x) = \alpha(i \otimes x)$, $x \in T([n] \vee [m])$. Clearly η is a morphism of functors. One easily checks that it is an isomorphism when $T = \Gamma^k$ and therefore standard argument shows that it is an isomorphism for any T.

Clearly for the right derived functors of $\mathcal{H}om(T, U)$ one has the similar descroption

$$\mathcal{E}xt^*_{\Gamma-mod}(T, U)([n]) \cong Ext^*_{\Gamma-mod}(T, \bar{\Delta}_n U). \tag{4.1.2}$$

One has the spectral sequence
(4.1.3)
$$E_2^{pq} = Ext^p_{\Gamma-mod}(F, \mathcal{E}xt^q_{\Gamma-mod}(T, U)) \Longrightarrow Ext^{p+q}_{\Gamma-mod}(F \otimes T, U).$$

which is a consequence of the spectral sequence for composite functors. In the same way one defines the inner Hom and inner Ext for right Γ-modules, which has similar properties. We also need the inner Tor groups, which can be defined in the same spirit. Let T be right Γ-module and let U be a left Γ-module. One defines the left Γ-module $\mathcal{T}or^\Gamma_*(T, U)$ by

$$\mathcal{T}or^\Gamma_*(T, U)([n]) := Tor^\Gamma_*(T, \Delta_n U). \tag{4.1.4}$$

Here $\Delta_n : \Gamma - mod \to \Gamma - mod, n \geq 0$, is a functor given by $(\Delta_n U)([m]) := U([n] \times [m])$. Then for any right Γ-module F one has a spectral sequence

$$E^2_{pq} = Tor^\Gamma_p(F, \mathcal{T}or^\Gamma_q(T, U)) \Longrightarrow Tor^\Gamma_{p+q}(F \otimes T, U). \tag{4.1.5}$$

Lemma 4.2. *For the functors $F, T : \Gamma \to Vect$ one has an isomorphism*

$$\pi^{st}_*(F \otimes T) \cong \pi^{st}_*(F) \otimes T([0]) \oplus F([0]) \otimes \pi^{st}_*(T).$$

Proof. This follows from the isomorphism (2.1.2) and from the Eilenberg-Zilber and Künneth theorems.

4.3. Smooth functors. For any left Γ-module T we let $\Pi_0 T$ be the left Γ-module $[n] \mapsto \pi_0^{st}(\Delta_n T)$. Thus $\Pi_0 T \cong \mathcal{T}or_0^{\Gamma}(t, T)$. A left Γ-module F is called 0-*smooth* if $\pi_i^{st}(F) = 0$ for any $i > 0$ and it is called s- *smooth* for $s > 0$ if Γ-modules $\Delta_n F$ and $\Pi_0 F$ are $(s-1)$-smooth for any $n \geq 0$. Since $\Delta_0 F = F$, we see that any s-smooth functor is $(s-1)$-smooth as well. A left Γ-module F is called ∞-*smooth* or *smooth* for brevity if it is s-smooth for any $s \geq 0$.

Lemma 4.4. *Let $0 \leq s \leq \infty$. Then the following holds.*

i) *The direct sum of s-smooth left Γ-modules is s-smooth.*

ii) *If F and G are s-smooth left Γ-modules, then $F \otimes G$ is s-smooth.*

iii) *Any projective left Γ-module is smooth.*

iv) *If F is a s-smooth left Γ-module, then*

$$Tor_i^{\Gamma}(t^{\otimes r}, F) = 0, \text{ for } i > 0, 0 \leq r \leq s+1.$$

v) *If F is smooth left Γ-module and V be a right Γ-module of finite type and of finite degree, then*

$$Tor_i^{\Gamma}(V, F) = 0, \text{ for } i > 0.$$

Proof. i) follows from the fact that the functors $\Delta_n, n \geq 0$ and Π_0 commutes with direct sums.

ii) for $s = 0$ this follows from Lemma 4.2. For $s > 0$ one uses the induction. One observes that the functors $\Delta_n, n \geq 0$ commutes with pointwise tensor product and

$$\Pi_0(F \otimes T) \cong \Pi_0(F) \otimes T \oplus F \otimes \Pi_0(T),$$

which is a consequence of Lemma 4.2 and ii) follows.

iii) By Proposition 2.2 any projective left Γ-module is 0-smooth. Now take $s > 0$ we need to show that projective left Γ-modules are s-smooth. One can restrict ourself with $\Gamma^k, k \geq 0$, because they are projective generators. By part ii) it is enough to consider the case $k = 0, 1$. Any constant functor is projective, because Γ^0 is constant. Therefore constant functors are 0-smooth. Since the functor $\Delta^n, n \geq 0$ takes constant functors to constant and Π_0 vanishes on constant functors, one obtains by induction that constants are smooth. Since $\Delta^n(\Gamma^1) = \Gamma^1 \otimes \Gamma^1([n])$ and $\pi_0^{st}\Gamma^1 \cong K$ we obtain $\Pi_0(\Gamma^1) \cong \Gamma^1$, which shows that Γ^1 is smooth too and the proof of iii) is finished.

iv) is clear when $r = 0, 1$. For $r \geq 2$ it follows from the spectral sequence (4.1.5), because (4.1.4) shows that $\mathcal{T}or_q^{\Gamma}(t, F)$ is zero for $q > 0$ and is $(s-1)$-smooth for $q = 0$.

v) By Proposition 1.12 any such V has finite injective dimension and one can use the induction with respect of injective dimension of V. When V is injective then statement reduces to iv). Indeed the functors $t^{\otimes n}$ are injective cogenerators, therefore in our circumstances V is a direct summand of the finite sum of functors $t^{\otimes r}$. The general case follows from the embeding V in an injective of same type (see 1.11) and considering the long exact sequence for Tor-groups.

Lemma 4.5. *Let A be a smooth commutative algebra of finite type. Then $\mathcal{L}(A, A)$ is smooth.*

Proof. We need to prove that the functor $F = \mathcal{L}(A, A)$ is s-smooth for any $s \geq 0$. In the characteristic zero case the stable homotopy coincides with Harrison homology and hence with André-Quillen homology up to shifting degree. Thus they are vanish for smooth algebras (see [Q]). Therefore F is 0-smooth. One observes that one has the following isomorphism

$$\Delta_n F \cong \mathcal{L}(A^{\otimes(n+1)}, A^{\otimes(n+1)}).$$

Therefore $\pi_0^{st}\Delta_n F \cong HH_1(A^{\otimes(n+1)})$. The Küneth theorem for Hochschild homology yields the following isomorphism

$$\Pi_0(F) \cong \mathcal{L}(A, \Omega_A^1) \otimes \Gamma^1.$$

It is well-known that the tensor product of smooth algebras is still smooth and for smooth algebra A the module of differential forms Ω_A^1 is finitely generated projective A-module. Therefore $\mathcal{L}(A, \Omega_A^1)$ is a direct summand of the finite sum of Γ-modules like $\mathcal{L}(A, A)$. According to Lemma 4.4 the proof can be finish by induction.

4.6. HKR theorem for functors. Odd case. *Let $0 \leq s \leq \infty$ and F be a s-smooth left Γ-module. If $i \leq s+1$ then $H_n^{(i)}(F) = 0$*

for $i < n$. Therefore if F is smooth and d is odd then $\pi_k F(S^d) = 0$ if $k \neq jd$ and $\pi_{jd} F(S^d) \cong H_j^{(j)}(F) \cong (\Lambda^j \circ t) \otimes_\Gamma F$.

Proof. By Theorem 2.7 we know that $H_n^{(i)}(F) \cong Tor_{n-i}^\Gamma(\Lambda^i \circ t, F)$. Since Λ^i is a direct summand of $\otimes^i$ the first statement follows from Lemma 4.4 iv). In order to obtain the last isomorphisms one needs to use Corollary 2.5.

4.7. HKR theorem for functors. Even case. *If d is a even and F is smooth left Γ-module then $\pi_k F(S^d) = 0$ if $k \neq jd$ and $\pi_k F(S^d) \cong \theta^j \otimes_\Gamma F$ if $k = jd$.*

Proof. By 1.11 we know that θ^j is of finite degree, it is also of finite type. Hence by Lemma 4.4 v) we have $Tor_i^\Gamma(\theta^n, F) = 0$ for $i > 0$ and result follows from the spectral sequence constructed in Corollary 2.5.

5. Higher order Hochschild homology

5.1. Definition of higher order Hochschild homology. Let A be a commutative algebra and M be an A-module. For any based simplicial finite set Y one defines *Y-homology of A with coefficient in M* by

$$H_*^Y(A, M) := \pi_*(\mathcal{L}(A, M)(Y)).$$

Here $\mathcal{L}(A, M)$ is the left Γ-module introduced in 1.7. Theorem 2.4 shows that this definition depends only on homotopy type of Y. In case, when $Y = S^d$, we call this objects as *Hochschild homology of order d* and denote by

$$H_*^{[d]}(A, M), \ d \geq 1.$$

Clearly, for $d = 1$ one recovers usual Hochschild homology. In the particular case $M = A$ we write $HH_*^Y(A)$ and $HH_*^{[d]}(A)$ instead of $H_*^Y(A, M)$ and $H_*^{[d]}(A, A)$. It is well-known that the homotopy of a simplicial commutative algebra has the natural structure of a graded commutative algebra. Since $\mathcal{L}(A, A)$ is a functor with values in the category of commutative algebras, we see that $HH_*^Y(A)$ carries the structure of commutative graded algebra.

The standard model of S^n has only one 0-simplex and no non-degenerate simplexes in dimensions > 0 and $< n$. Therefore

$$H_0^{[d]}(A, M) \cong M \text{ and } H_i^{[d]}(A, M) = 0 \text{ for } 0 < i < n.$$

The isomorphism (2.1.2) shows that

$$H_d^{[d]}(A, M) \cong H_1^{[1]}(A, M) \cong \Omega_A^1 \otimes_A M$$

Because of the multiplicative structure one obtains the natural maps

$$\Omega_A^n \to HH_{nd}^{[d]}(A), n \geq 1 \text{ for odd } d \tag{5.1.1}$$

$$Sym_A^n(\Omega_A^1) \to HH_{nd}^{[d]}(A), n \geq 1 \text{ for even } d. \tag{5.1.2}$$

Here $Sym_A^*(E)$ denotes the symmetric A-algebra generated by an A-module E.

Let A and B be commuatative algebras. Furtermore, let M be an A- module and N be a B-module. Since $\mathcal{L}(A \otimes B, M \otimes N) \cong \mathcal{L}(A, M) \otimes \mathcal{L}(B, N)$ we see that

$$H_*^Y(A \otimes B, M \otimes N) \cong H_*^Y(A, M) \otimes H_*^Y(B, N). \tag{5.1.3}$$

5.2. Proposition. *Let $Y = S^1 \vee S^1 \cdots \vee S^1$, (n-times) and A be a smooth algebra, then*

$$HH_*^Y(A) \cong \Omega_A^* \otimes_A \Omega_A^* \cdots \otimes_A \Omega_A^* \ (n - \text{times}).$$

Proof. For simplicity we will consider the case $n = 2$. The same argument works for $n > 2$ as well. Clearly $S^1 \vee S^1$ is the diagonal of the bisimplicial set $([n], [m]) \mapsto [n] \vee [m]$. Therefore the simplicial vector space $\mathcal{L}(A, A)(S^1 \vee S^1)$ is the diagonal of the following bisimplicial vector space $([n], [m]) \mapsto A \otimes A^{\otimes n} \otimes A^{\otimes m}$ and by Eilenberg-Zilber-Cartier theorem it is isomorphic to the total homology of the bicomplex associated to this bisimplicial vector space. Consider now the bicomplex of the form $\Omega_A^* \otimes_A \Omega_A^*$ with trivial differential. Then the map from Lemma 1.3.14 of [L2] defines the morphism from the first bicomplex to the second one. By the Hochschild-Kostant-Rosenberg theorem (see [L2]) it induces the isomorphism on E^2 term of the corresponding spectral sequences and hence we are done.

5.3. Proposition. *Let K be a field of characteristic zero. If d is odd, then for any commutative algebra A and any A-module M one has the natural decomposition.*

$$H_n^{[d]}(A,M) \cong \bigoplus_{i+dj=n} H_{i+j}^{(j)}(A,M).$$

For even d one has the natural decomposition.

$$H_n^{[d]}(A,M) \cong \bigoplus_{i+dj=n} Tor_i^{\Gamma}(\theta^j, \mathcal{L}(A,M)).$$

Moreover the summand corresponding to $j=1$ is isomorphic to $H_{i+1}^{(1)}(A,M)$.

Proof. This is a consequence of (2.12).

5.4. Proposition. *If A is a free commutative algebra over a characteristic zero field, then the maps (5.1.1) and (5.1.2) are isomorphisms. Moreover for odd d the same is true for any smooth algebra.*

Proof. For odd d this is a consequence of the Hochschild-Kostant-Rosenberg theorem 4.6, together with the well-known fact $H_n^{(n)}(A,M) \cong \Omega_A^n \otimes_A M$ ([L2]).

Assume now that d is even. According to Theorem 4.7 it is enough to consider the case $d=2$. Moreover we restrict ourself with the case, when A is free commutative algebra. Since the homology in question commutes with direct limits, the isomorphism 5.1.3 shows that, one can restrict ourself with case $A = K[x]$. Obviously the diagonal of the following bisimplicial set

$$[n] \mapsto \bigvee_{i=1}^{n} S^1$$

is a model of S^2. Here the horizontal simplicial structure is induced from the simplicial structure of S^1:

$$[n] \mapsto \bigvee_{i=1}^{n} [1].$$

This shows that for calculation of $HH^{[2]}(A)$ one can use the corresponding bicomplex. Thus one has the spectrals sequence

$$E^1_{p*} = HH^{Y_p}_*(A) \Longrightarrow HH^{[2]}(A).$$

Here $Y_p = S^1 \vee S^1 \vee \cdots \vee S^1$, ($p$-times). By Proposition 5.2 we know that $E^1_{p*} \cong \Omega^*_A \otimes_A \cdots \otimes_A \Omega^*_A$ (p-times). Therefore E^2 is isomorphic to the Hochschild homology (over A) of the graded algebra Ω^*_A with coefficient in A. Since $A = K[x]$ we see that $\Omega^*_A \cong A[\epsilon]$, where $\epsilon^2 = 0$ and ϵ has degree 1. This means that Ω^*_A is the free algebra with one generator in dimension one. Of course free in the graded sence. Thus Hochschild homology in question is isomorphic to $Sym^n_A(\Omega^1_A)$, where Ω^1_A has degree 2. By the degree reason the spectral sequence degenerates and the result is proved.

5.5. Higher order Hochschild homology of truncated polynomial algebras. Let r be a positive integer and let $A = K[x]/(x^{r+1})$. According to Proposition 5.4.15 of [L2] one knows that $HH^{(q)}_n(A) = 0$ except when $n = 2q$ or $n = 2q - 1$ for any $q \geq 1$ and in this case it is isomorphic to $A/(x^r) \cong k^r$. Therefore one can use Proposition 5.3 to get $HH^{[d]}_m(A) = 0$ if $m \neq n(d+1)$ or $m \neq n(d+1) - 1$ and

$$HH^{[d]}_m(A) \cong k^r \text{ if } m = n(d+1) \text{ or } m = n(d+1) - 1.$$

Here d is odd.

5.6. Higher order Hochschild homology and homology of mapping spaces X^{S^d}. We use some ideas from [A1] and [A2] to relate the groups $HH^{[d]}_*(A)$ with the cohomology of the mapping spaces X^{S^d}. One can define $HH^{[d]}_*(A)$ for any commuatative differential graded algebras A to be the homology of the total complex associated to the simplicial chain complex $\mathcal{L}(A, A)(S^d)$.

5.7. Theorem. *Let X be a finite dimensional d-connected CW-complex. Let $A^*(X)$ be commutative cochain algebra of X in the sense of Sullivan. Then*

$$HH^{[d]}_*(A^*(X)) \cong H^*(X^{S^d}, K).$$

For $d = 1$ this is the well-known isomorphism proved in [J].

5.8. The minimal model of X^{S^d}. Let X be a d-connected space and let $\mathcal{X} = (\Lambda^*(V), d)$ be a minimal model of X. By our assumtions the graded vector space V do not haves components in dimension $\leq d$. One defines the CDGA $\mathcal{M} = (\Lambda^*(V \oplus \bar{V}), \bar{d})$ as follows. Take $\bar{V}$ to be $\Sigma^{-d}V$. For each element $x \in V$, we let $\bar{x}$ to be the corresponding element in $\bar{V}$. Then $x \mapsto \bar{x}$ is an isomorphism $V \to \bar{V}$ of degree $-d$. Let $i : \mathcal{X} \to \mathcal{M}$ be the unique derivation of degree $-d$ extending $x \mapsto \bar{x}$. Then $\bar{d}$ is given by

$$\bar{d}(x) = dx \text{ and } \bar{d}(\bar{x}) = (-1)^d idx, x \in V.$$

One easily shows that $\mathcal{M}$ is a minimal model of X^{S^d} (mimiking to [VS], or using more general result of [H]).

5.9. Proof of theorem 5.7. The higher order Hochschild homology of commutative differential graded algebras sends weak equivalences in isomorphisms. Therefore one can use the minimal model $\mathcal{X}$ of X insread of $A^*(X)$. The minimal model $\mathcal{M}$ of X^{S^d} is isomorphic to the differential graded algebra version of $\Lambda^n_{\mathcal{X}}(\Omega^1_{\mathcal{X}}[-d])$ or $Sym^n_{\mathcal{X}}(\Omega^1_{\mathcal{X}}[-d])$ depending to parity of d (see 5.4.3 of [L2]). Since underline graded algebra is free, it follows from the differential version of Proposition of 5.4 that the homology of $\Lambda^n_{\mathcal{X}}(\Omega^1_{\mathcal{X}}[-d])$ and $Sym^n_{\mathcal{X}}(\Omega^1_{\mathcal{X}}[-d])$ are isomorphic to the higher order Hochschild homology for odd and even d respectively and result follows.

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