ALTERNATING QUOTIENTS OF FUCHSIAN GROUPS

BRENT EVERITT

ABSTRACT. It is shown that any Fuchsian group has among its homomorphic images all but finitely many of the alternating groups A_n. This settles in the affirmative a long-standing conjecture of Graham Higman.

1. INTRODUCTION

It all started with a theorem of G. A. Miller. Suppose X is the 2-sphere S^2, the Euclidean plane $\mathbb{E}^2$ or the hyperbolic plane $\mathbb{H}^2$. Let G be a discrete group of orientation preserving isometries of X. By classical work of Fricke and Klein (see for instance [22]), G, if finitely generated and nonelementary, has a presentation,

generators:	$a_1, b_1, \dots, a_g, b_g$	(hyperbolic)
	$x_1, \dots, x_e$	(elliptic)
	$y_1, \dots, y_s$	(parabolic)
	$z_1, \dots, z_t$	(hyperbolic boundary elements)
relations:	$x_1^{m_1} = \dots = x_e^{m_e} = 1,$	
	$\prod_{i=1}^{e} x_i \prod_{j=1}^{s} y_j \prod_{k=1}^{t} z_k \prod_{l=1}^{g}[a_l, b_l] = 1.$	

When $X = \mathbb{H}^2$, G is called a *Fuchsian* group.

The division into spherical, Euclidean and Fuchsian is governed by the quantity,

$$\mu(G) = 2g - 2 + \sum_{i=1}^{e}\left(1 - \frac{1}{m_i}\right) + s + t, \tag{1}$$

where $\mu(G) < 0, = 0$ or > 0 as $X = S^2, \mathbb{E}^2$ or $\mathbb{H}^2$. The quotient X/G is an orientable 2-orbifold of genus g with e cone points, s punctures and t boundary components. Its geometry and the algebraic structure of G are intimately connected. As a consequence, G is determined upto isomorphism by the parameters $(g; m_1, \dots, m_e; s; t)$, $2 \leq m_1 \leq \dots \leq m_e$, its *signature*.

The spherical and Euclidean groups have an algebraic structure that is fairly transparent; the former are finite and the latter are finite abelian extensions of $\mathbb{Z} \oplus \mathbb{Z}$, hence soluble. Fuchsian groups are a different matter altogether. To get a firmer grip on this situation, one may be tempted to consider finite (even simple) homomorphic images of these groups. The first appearance of such in the literature would seem to be Miller's result [15] on the group of signature $(0; 2, 3; 1; 0)$–the classical modular group $\mathrm{PSL}_2(\mathbb{Z})$. Namely, one can find among its homomorphic

---

1991 *Mathematics Subject Classification.* Primary 20F05.

Part of this work was done while the author was a guest of Sonderforschungsbereich 343, Unversität Bielefeld. He is grateful for their financial support and hospitality.

images every alternating group, bar A_6, A_7 and A_8. There is an added geometric incentive to such results: any compact Riemann surface (or complex algebraic curve) has as its group of conformal automorphisms a finite homomorphic image of a Fuchsian group.

In the late 1960's, Graham Higman conjectured that *any* Fuchsian group has among its homomorphic images all but finitely many of the alternating groups. In [3, 4, 7, 16, 17], the case where G has signature $(0; 2, m_1, m_2; 0; 0)$ was considered and essentially settled in the affirmative. This paper considerably extends this body of work to finally settle Higman's conjecture. So, taking the phrase "essentially all" to be synonymous with "all but finitely many", and "surjects" with "has among its homomorphic images", we prove

Main Theorem. Any Fuchsian group surjects essentially all of the alternating groups.

The theorem has a 3-dimensional interpretation also. For, suppose M is a 3-dimensional Seifert fibred space, that is, an S^1-bundle over a 2-orbifold (see [9, 14]). Since the fundamental group of M surjects π_1(underlying 2-orbifold), we have

Corollary 1. If M is Seifert fibred over an orientable hyperbolic 2-orbifold, then $\pi_1(M)$ surjects essentially all of the alternating groups.

The machinery used to prove the theorem uses what are essentially Schreier coset diagrams as a vehicle. Such diagrams appear in the literature in many guises–they are equivalent to *hypermaps* (see [11]) in the case $G = (0; m_1, m_2, m_3; 0; 0)$, and *dessin d'enfants*, as Grothendieck would have it, when $G = (0; 2, m_2, m_3; 0; 0)$–see [2].

The author has benefitted from conversations with various people, including Colin Maclachlan, Alan Ried and Paul Turner. Most of all, I must record a debt of gratitude to Graham Higman, who provided encouragement and copious improvements to an earlier version of this paper. It was his suggestion to extend what was then a result about triangle groups to all Fuchsian groups. The proof of Lemma 1 was also kindly supplied by him.

2. The plan

The proof of the theorem proceeds according to various cases. The result of this sections musings will be that the general result reduces to an investigation of the cocompact groups (that is, $s = t = 0$) of genus zero, or *Dyck groups*. The most important, and the hardest of these in turn, will be those of signature $(0; m_1, m_2, m_3; 0; 0)$–the *triangle groups*.

From the presentation, a group of signature $(g; m_1, \ldots, m_e; s; t)$ is isomorphic to one of signature $(g; m_1, \ldots, m_e; s + t; 0)$. Moreover by (1), the former is Fuchsian if and only if the latter is. We may assume then, without loss of generality, that $t = 0$. Write $(g; m_1, \ldots, m_e; s)$ for $(g; m_1, \ldots, m_e; s; 0)$ from now on.

We can surject a group with signature $(g; m_1, \ldots, m_e; s)$ onto one of signature $(g'; m_1, \ldots, m'_i, \ldots, \hat{m}_j, \ldots, m_e; s')$, where $g' \leq g$, $s' \leq s$, m'_i is a divisor of m_i, and the hat denotes ommission. Here's how: map all but g' of the a_i's, and the corresponding b_i's to the identity; send x_i to $x_i^{m_i/m'_i}$, x_j to the identity and all but s' of the parabolics to the identity. All other generators map to themselves. The map extends to the desired homomorphism.

Proposition 1. For every Fuchsian group to surject essentially all the alternating groups, it is sufficient that the Dyck groups do.

Proof. Suppose G has signature $(g, m_1, \dots, m_e; s)$ with $g \geq 2$. Map G onto $\langle A, B \,|\, \text{—} \rangle$, free of rank two, by sending $a_1 \mapsto A$, $a_2 \mapsto B$, and all the other generators to the identity. Since A_n is 2-generated for $n \geq 3$ (see [5]), we are done.

A group of genus one with elliptic generators $x_1, \dots, x_e$, can be surjected onto a group of signature $(1; m_1; 0)$ for $m_1 \geq 2$, by mapping $x_2, \dots, x_e$ and $y_1, \dots, y_s$ to the identity. Now, the map $\theta : (0; 2, 2, 2, 2m_1; 0) \to S_2$ sending all generators to the permutation $(1, 2)$ has kernel isomorphic to a group of signature $(1; m_1; 0)$ by [18]. Consequently, if $(0; 2, 2, 2, 2m_1; 0)$ surjects essentially all the alternating groups, then $(1; m_1; 0)$ can be in the kernel of only finitely many of these homomorphisms. The simplicity of A_n for $n \geq 5$ thus gives the desired result for $(1; m_1; 0)$ as well. For groups of genus one with no periods, hence signature $(1; \text{—}; s)$ for $s \geq 1$, we surject onto $(1; \text{—}; 1)$. But $\langle a_1, b_1, y_1 \,|\, y_1[a_1, b_1] = 1 \rangle \cong \langle a_1, b_1 \,|\, \text{—} \rangle$, is free of rank two.

A group of genus zero with no periods must, by (1), have at least three parabolic generators, and hence surject $(0; \text{—}; 3)$. But this is free of rank two also. With a single period we have $s \geq 2$, and the group surjects $(0; m_1; 2) \cong \mathbb{Z}_{m_1} * \mathbb{Z}$, the free product of $\mathbb{Z}_{m_1}$ and $\mathbb{Z}$. This surjects $\mathbb{Z}_{m_1} * \mathbb{Z}_3$, which in turn surjects a Fuchsian triangle group $(0; 3, m_1, r; 0)$.

With two periods and one parabolic, we have $(0; m_1, m_2; 1) \cong \mathbb{Z}_{m_1} * \mathbb{Z}_{m_2}$, where $m_2 \geq 3$, so surject some Fuchsian triangle group $(0; m_1, m_2, r; 0)$. A group with more parabolics, $(0; m_1, m_2; s)$ for $s \geq 2$, surjects $(0; m_1; 2)$ done above. Finally, $(0; m_1, \dots, m_e; s)$, $e \geq 3$, surjects either $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ or $\mathbb{Z}_{m_1} * \mathbb{Z}_{m_2}$ for $m_2 \geq 3$. Surject the former onto a Fuchsian $(0; 2, 2, 2, p; 0)$. The latter has already been done. $\square$

In the sequel we abbreviate $(0; m_1, \dots, m_e; 0)$ to $(m_1, \dots, m_e)$.

Proposition 2. For every Dyck group to surject essentially all the alternating groups, it is sufficient that the following ones do.

1. The Fuchsian triangle groups (p, q, r) with $2 \leq p < q < r$ distinct primes;
2. the triangle groups $(2, 4, r)$ for $r \geq 5$ a prime;
3. the groups $(2, 2, 2, p)$ with $p \geq 3$ a prime;
4. the groups $(2, 3, 8)$, $(2, 3, 9)$, $(2, 3, 10)$, $(2, 3, 12)$, $(2, 3, 15)$, $(2, 3, 25)$, $(2, 4, 6)$, $(2, 4, 8)$, $(2, 4, 9)$, $(2, 5, 6)$, $(2, 5, 9)$ and $(3, 4, 5)$;
5. the groups $(2, 2, 2, 4)$, $(2, 2, 3, 3)$, $(2, 2, 3, 5)$, $(2, 2, 4, 4)$, $(2, 3, 3, 3)$, $(3, 3, 3, 3)$ and $(2, 2, 2, 2, 2)$.

The proof is elementary, if a little tedious, and relies on repeated application of the surjections mentioned at the beginning of the section. Also, if $G = \langle x_1, x_2, x_3 \rangle$ has signature $(2, m_1, 2m_2)$, then there is a epimorphism $\theta : G \to S_2$ with $\theta(x_1) = \theta(x_3) = (1, 2)$ and $\theta(x_2)$ the identity. By Theorem 1 of [18] the kernel of this map is isomorphic to (m_1, m_1, m_2). So, as in the proof of Proposition 1, the result for $(2, m_1, 2m_2)$ yields (m_1, m_1, m_2) also.

Proof of Proposition 2. One proceeds according to the number of elliptic generators, or periods. The idea is that any Dyck group surjects one listed in the proposition, or is a normal subgroup of one as described above. We do the first case as an illustration, and leave the rest to the reader.

The hyperbolic triangle group $(2, m_1, m_2)$ surjects $(2, q, r)$ for q, r prime. Unless $(2, q, r)$ is, after a possible reordering, one of $(2, 2, r)$ for $r \geq 2$, $(2, 3, 3)$ or $(2, 3, 5)$, it is Fuchsian, with $\frac{1}{q} + \frac{1}{r} < \frac{1}{2}$ by (1). Thus, either q and r are distinct, giving a group listed in part 1, or $(2, q, q) \cong (q, q, 2) \lhd (2, q, 4) \cong (2, 4, q)$. In this case $\frac{2}{q} < \frac{1}{2}$, so $q \geq 5$, and we have a group listed in part 2.

So we are done bar the three exceptions given. The first of these gives, after a possible relabelling, that $(2, m_1, m_2)$ has the form $(2, m_1, 2^l)$, for $m_1 \geq 3$ and $l \geq 2$. If $m_1 = 3$ or 4 then $l \geq 3$, as $(2, 3, 4)$ is spherical and $(2, 4, 4)$ Euclidean, so the group surjects $(2, 3, 8)$ or $(2, 4, 8)$, both listed in part 4. For $m_1 \geq 5$, $(2, m_1, 2^l)$ surjects $(2, m_1, 4) \cong (2, 4, m_1)$. This in turn surjects $(2, 4, r)$, r prime, and we have a group listed in part 2 unless $r = 2$ or 3. In the first case, $m_1 = 2^n \geq 8$, so $(2, 4, m_1)$ surjects $(2, 4, 8)$. In the second, $m_1 = 2^{l_1} 3^{n_1}$, and the group surjects $(2, 4, 9)$ when $l_1 = 0$, or $(2, 4, 6)$ otherwise. The other two exceptions give $(2, m_1, m_2)$ of the form $(2, 2^{l_1} 3^{n_1}, 2^{l_2} 3^{n_2})$, or $(2, 2^{l_1} 3^{n_1}, 2^{l_2} 3^{n_2} 5^{n_3})$. These are handled similarly.

The rest is much the same–the remaining cases that need to be done are the general triangle groups and the groups with four and five periods. A group with six or more periods always surjects a Fuchsian group with five. $\square$

In [3, 4, 7], the groups $(2, 3, r)$ for all $r \geq 7$ and $(2, 4, r)$ for all $r \geq 5$ were done. Theorems 1-3 of [16] take care of $(2, q, r)$ with $5 \leq q < r$ prime, with the exception of sixty sporadic cases. These sixty, and those remaining from parts 4 and 5 of Proposition 2 are handled in the Appendix. This leaves the $(2, 2, 2, p)$ and the triangle groups (p, q, r) with $3 \leq p < q < r$. See Sections 4 and 5 for these.

In later sections, various permutation groups will be constructed as homomorphic images of Fuchsian groups. We will identify these images as alternating by employing the following well known result of Jordan.

Theorem 1 ([12], refer to [20] Theorem 13.9)**.** Let G be a primitive permutation group of degree n containing a prime cycle for some prime $p \leq n - 3$. Then G is either the alternating group A_n or the symmetric group S_n.

The following lemma, well known to the cognoscenti, allows one to replace primitivity by more easily verifiable criteria. Recall that the support of a permutation $\sigma \in S_n$ consists of those elements of $\{1, 2, \ldots, n\}$ not fixed by σ.

Lemma 1. Let $G = \langle \sigma_1, \sigma_2, \ldots, \sigma_k \rangle$ be a transitive permutation group of degree n containing a prime cycle μ. For each σ_i, suppose there is a point in the support of μ whose image under σ_i is also in the support of μ. Then G is primitive.

Proof. Suppose on the contrary that G is imprimitive with block system $\mathfrak{B}$. For $\sigma \in G$, let $\bar{\sigma}$ be the permutation induced by σ on $\mathfrak{B}$, and $\overline{G}$ the group generated by the $\bar{\sigma}_i$. The map $\sigma \mapsto \bar{\sigma}$ is an epimorphism from G onto $\overline{G}$, and $\overline{G}$ acts transitively on $\mathfrak{B}$. All blocks $B \in \mathfrak{B}$ thus have the same size, say $|B|$. If $B \in \mathfrak{B}$ is in $\text{supp}(\bar{\mu})$, the support of $\bar{\mu}$, then B and its image under μ are distinct blocks, and so B is contained in $\text{supp}(\mu)$. Taking the union of all the blocks in $\text{supp}(\bar{\mu})$ thus gives

$$|B||\text{supp}(\bar{\mu})| \leq |\text{supp}(\mu)|. \tag{2}$$

Now μ has order p a prime, and $\bar{\mu}$ is a homomorphic image of μ. Thus, if $\bar{\mu} \neq 1$, then $\bar{\mu}$ has order p, and so $|\text{supp}(\bar{\mu})| \geq p$. Since $\mathfrak{B}$ is non-trivial, we have $|B| > 1$, and hence by (2), $|\text{supp}(\mu)| > p$. This contradicts the fact that μ is a p-cycle, so we must have $\bar{\mu} = 1$. This means that $\bar{\mu}$ fixes every block, or equivalently, any point and its image under μ lie in the same block. But μ is a single cycle, so there is a block B^* with $\mathrm{supp}(\mu) \subseteq B^*$. By the condition stated in the Lemma, B^* and its image under σ_i intersect for all i, so are equal. Since the σ_i generate G, the whole group must fix B^*, and by transitivity, $B^* = \{1, 2, \ldots, n\}$, so there is just one block. This is the desired contradiction. □

3. Coset diagrams

Suppose G has a finite presentation $\langle X \mid R \rangle$, and let $K_0 = K_0(X; R)$ be the standard 2-complex realising G, in the sense that $\pi_1(K_0) \cong G$. The 1-skeleton of K_0 consists of a single vertex incident with oriented loops or *edges*, that are in one to one correspondence with the generators X. Each edge $x \in X$ is a pair of oppositely oriented arcs, an x-arc and an x^{-1}-arc. The former coincides with the edge under its given orientation and the latter to the edge with the reverse orientation. The faces of K_0 are in one to one correspondence with the relators R, and are obtained by sewing discs onto the 1-skeleton, each with boundary label a relator word $r \in R$. The reader is referred to [10], Section 6.3 for more details.

A *Schreier coset diagram K for G* is a cellular (that is, k-cells go to k-cells) covering of K_0 (see [19], Sections 2.2.1 and 4.3.2, or [1]). The covering realises a subgroup $H \cong \pi_1(K)$ of G, the vertices of K in one to one correspondence with the cosets of H in G. Conversely, every subgroup can be realised in this way from some diagram.

Moreover, and herein lies their utility for us, each coset diagram K for G yields a homomorphism $\theta_K : G \to \mathrm{Sym}\{\text{vertices of } K\} \cong S_n$. Here, n is the sheet number of the covering, hence the number $|K|$ of vertices in K. For any $g \in G$, the image of vertex v under the permutation $\theta_K(g)$ is the terminal vertex of the path starting at v with label g. In particular, $\theta_K(G)$ is transitive if and only if K is connected.

All of which is, of course, well known. The simplicial complexes that form coset diagrams for G are characterised by two simple properties:

1. For each vertex v and generator $x \in X$, there is precisely one x-arc and one x^{-1}-arc having initial vertex v.
2. For each vertex v and relator word $r \in R$, the path starting at v with label r forms the boundary of a face, distinct from the face obtained by choosing a different v or r. Moreover, every face arises in this way.

Notice that the first condition does not preclude the possibility of an x-loop incident with v. The second condition has the unfortunate consequence that there will be several faces sharing the same set of boundary edges. To alleviate matters, we use an equivalent construct, suggested by Higman and used in [3, 4, 7, 8, 16, 17]. It is what results by identifying such "multiple" faces.

For the remainder of this section then, let $G = (m_1, \ldots, m_e)$ be some fixed but arbitrarily chosen Dyck group. By performing a series of Teitze transformations, we may get a convenient presentation for G,

$$\langle x_1, x_2, \ldots, x_{e-1} \mid x_1^{m_1} = x_2^{m_2} = \cdots = x_{e-1}^{m_{e-1}} = (x_1 x_2 \ldots x_{e-1})^{m_e} = 1 \rangle.$$

A directed graph with edges labelled $x_1, \ldots, x_{e-1}$ that satisfies property (1) above is a *G-graph*. Ordering the edges incident with every vertex of a G-graph as shown in Figure 1 yields a 2-cell embedding into some closed orientable surface (see [21] for more details on graph embeddings). Each face f of this surface complex S has

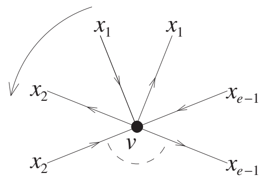


FIGURE 1

boundary label $x_i^{m_f}$ for some i, or $(x_1 \dots x_{e-1})^{m_f}$. Call S a G-*diagram* if for each face, m_f divides m_i or m_e (as the case may be).

Proposition 3. A coset diagram for G yields a unique G-diagram, and vice-versa.

Proof. In a G-diagram, a path starting at vertex v and with label $r = x_i^{m_i}$ or $(x_1 \dots x_{e-1})^{m_e}$, circumnavigates some face f precisely $\frac{m_i}{m_f}$ of $\frac{m_e}{m_f}$ times, and is closed. Take the underlying G-graph and sew on a 2-cell for each such vertex and relator pair. The result is easily seen to be a coset diagram for G. Conversely, the 1-skeleton of a coset diagram K is a G-graph in which a path from any vertex with label a relator word is closed (as it bounds a face in K). Embed the graph as above. The result is a G-diagram. □

We transfer to G-diagrams the nomenclature of coset diagrams for G: each oriented x_i-edge of the underlying G-graph is a pair consisting of an x_i-arc and x_i^{-1}-arc. Call a face an x_i-face or $(x_1 \dots x_{e-1})$-face whenever it has boundary label some power of x_i or $x_1 \dots x_e$ respectively. Since the G-graph underlying a G-diagram is just the 1-skeleton of the corresponding coset diagram K, we deduce information about θ_K from either.

A collection of G-diagrams can, in certain situations, be combined to form a single new G-diagram. This was first observed by Higman in the setting $G = (2, m_1, m_2)$. To this end, suppose a G-diagram contains a pair of vertices α and β, each incident with x_1-loops, while the path starting at α and with label $x_1 \dots x_{e-1}$ terminates at β. Call the pair a *handle*, and denote it $[\alpha, \beta]$.

Let $K_1, \dots , K_t$, $t \leq m_1$, be a collection of disjoint G-diagrams, each containing at least one of the handles $[\alpha_1, \beta_1], \dots , [\alpha_{m_1}, \beta_{m_1}]$, with these $2m_1$ vertices distinct. Take the disjoint union of all the underlying G-graphs, remove the x_1-loops at the vertices α_j and β_j, and replace them by x_1-edges from α_j to α_{j+1} and β_j to β_{j-1}, subscripts taken modulo m_1. Embed the graph in the usual way, and call the resulting surface complex $[\![K_1, \dots , K_t]\!]$ the *composition* of $K_1, \dots , K_t$.

Proposition 4. $[\![K_1, \dots , K_t]\!]$ is also a G-diagram.

Proof. The underlying graph of $[\![K_1, \dots , K_t]\!]$ is clearly a G-graph, so it remains to show that all faces have boundary labels of the required form. If the boundary of a face does not contain an x_1-edge with initial vertex one of the α_j or β_j, then all edges are contained in a single K_i and since K_i is a G-diagram, we are done.

Otherwise, we obtain the boundary label for the face by starting at an α_j or β_j and traversing a path with label some power of x_1 or some power of $x_1 \dots x_e$, until it closes (which it does by repeating an arc). The path obtained by traversing x_1-edges passes through the vertices $\alpha_{j+1}, \dots, \alpha_{m_1}, \alpha_1, \dots, \alpha_j$ or $\beta_{j-1}, \dots, \beta_1, \beta_{m_1}, \dots, \beta_j$, as the case may be, and closes with label $x_1^{m_1}$. For the other type, observe that before composition, the path starting at α_j say, with label some power of $x_1 \dots x_e$, arrived at vertex β_j after e directed edges, proceeded to traverse the x_1-loop at β_j and then an x_2-edge. After composition, the path from α_j with such a label arrives instead at β_{j+1} after e directed edges, traverses the new x_1-edge to β_j, and is then back on track (as it were) with the path before composition. So the boundary label behaves as if the composition never happened, and is of the required form. $\square$

The G-diagrams in this paper are for $G = (2,2,2,p)$ or (p,q,r) having the presentations at the beginning of Sections 4 and 5. In practice, when drawing diagrams for such G, we simplify them (purely for notational purposes). To this end, a shaded q-gon (in particular, a 2-gon in Section 4) indicates a y-face with boundary label y^q, and a shaded wedge a y-face with label y. The orientation on the y-arcs runs anticlockwise around the q-gon. A similar orientation convention reigns in Section 5 with x-faces. In Section 4 an undirected *x-edge* or *z-edge* denotes an x or z-face with label x^2 or z^2 respectively (the face has been squashed flat if you like). Finally, in all diagrams, x (or z)-faces with boundary x (or z) are removed completely, leaving only the incident vertex. In Section 5 such vertices will be called *free*. On occasion, we will talk of *attaching* x-arcs to free vertices. By this we mean attach the arcs to the underlying G-graph and re-embed.

As a consequence of all this, when $G = (2,2,2,p)$, the unshaded faces are precisely the xyz-faces, while for $G = (p,q,r)$ they are the x and xy-faces. After all the simplifications then, for an embedded G-graph to be a G-diagram, it is sufficient that the xy and xyz-faces have a number of y-arcs dividing the appropriate order in their boundaries, and the x-faces a number of x-arcs similarly. These criteria can usually be verified at a glance. That's the beauty of it.

4. The groups $(2,2,2,p)$

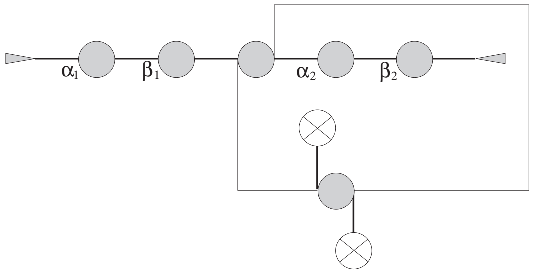


Figure 2

Suppose

$$G = \langle x, y, z \mid x^2 = y^2 = z^2 = (xyz)^p = 1 \rangle, \quad p \geq 7 \text{ prime},$$

and consider Figures 2–4. Ignoring the configurations, each is a spherical $(2,2,2,7)$-diagram. Replace each by $\frac{p-7}{2}$ 2-gons joined by z-edges into a chain, thus . The number of y-arcs in the boundary of each unshaded face increases by $p-7$, yielding three $(2,2,2,p)$-diagrams, say K_1^p, K_2^p

and K_3^p. Much of the time we deal with all p simultaneously, so by a slight abuse of notation, we just write K_1, K_2 and K_3.

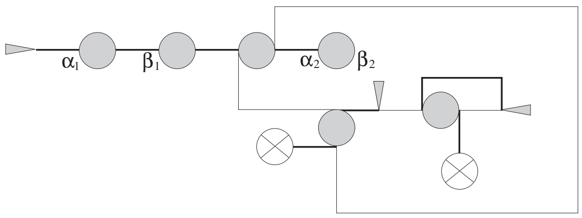


FIGURE 3

Diagrams K_1 and K_2 contain two handles $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$ while K_3 has one. Take $k_1 \geq 0$ copies of K_1 and $k_2 \geq 0$ copies of K_2, and form a sequence of compositions

$$\underbrace{[\![K_2[\![\ldots[\![K_2}_{k_2}\underbrace{[\![K_1[\![\ldots[\![K_1}_{k_1}, K_1]\!]\ldots]\!].$$

Note there will be an unused handle in one of the K_1's appearing in the first composition $[\![K_1, K_1]\!]$. Using this handle, compose with a single copy of K_3 to obtain

$$[\![K_3[\![K_2[\![\ldots[\![K_2[\![K_1[\![\ldots[\![K_1, K_1]\!]\ldots]\!]. \tag{3}$$

Call the resulting G-diagram $K(p, k_1, k_2)$, or K for short, and let $\theta_K : G \to S_n$, $n = |K|$, be the homomorphism arising from K. Since $|K| = 2pk_1 + (2p+1)k_2 + 2p$, with $2p$ and $2p+1$ relatively prime, any integer $n \geq 2p(2p-1)+2p = 4p^2$ is equal to $|K|$ for some suitable choice of k_1 and k_2. Thus, the family of diagrams $K(p, k_1, k_2)$ yield permutation representations of G for all but finitely many possible degrees.

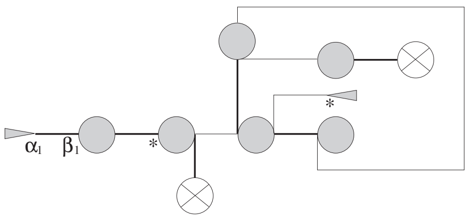


FIGURE 4

So is the image of θ_K equal to the alternating group $A_{|K|}$? In each of the $(2, 2, 2, 7)$-diagrams K_i^7, $i = 1, 2$ or 3, $\theta_{K_i^7}$ sends x, y and z to even permutations. The additions at —⊗ add $p - 7$ transpositions to the images of x and z arising from each diagram. Each composition in (3) juxtaposes the images of y and z. The images of x are juxtaposed and augmented by two new transpositions. The cumulative effect of all this is that θ_K sends x, y and z to even permutations in S_n.

Suppose J is a G-diagram and $g \in G$. The *cycle structure* of $\theta_J(g)$ is a function $\mathbf{s} : \{1, 2, 3, \ldots\} \to \{0, 1, 2, \ldots\}$, such that $\mathbf{s}(i)$ is the number of cycles of length i when $\theta_J(g)$ is written as a product of disjoint cycles. Write the structures as a sequence, $1^{\mathbf{s}(1)} \ldots i^{\mathbf{s}(i)} \ldots$, omitting those i for which $\mathbf{s}(i) = 0$. Consider $xy \in$

$(2,2,2,p)$, and let $\mathbf{s}_i^p$ be the cycle structure of $\theta_{K_i^p}(xy)$. Then $\mathbf{s}_1^7 = 1^2\,2^6$, $\mathbf{s}_2^7 = 1\,2^5\,4$ and $\mathbf{s}_3^7 = 1\,2\,5\,6$. Since K_i^p is obtained from K_i^{p-2} by adding a new z-edge and 2-gon at each $-\!\!\otimes$, $\mathbf{s}_i^p$ is merely $\mathbf{s}_i^{p-2}$ augmented by two transpositions. Thus, $\mathbf{s}_1^p = 1^2\,2^{p-1}$, $\mathbf{s}_2^p = 1\,2^{p-2}\,4$ and $\mathbf{s}_3^p = 1\,2^{p-6}\,5\,6$.

Suppose C_1 and C_2 are $(2,2,2,p)$-diagrams containing handles $[\alpha_1,\beta_1]$ and $[\alpha_2,\beta_2]$. If the diagrams are composed using these two handles, the cycles in $\theta_{C_1}(xy)$ and $\theta_{C_2}(xy)$ containing the α_i are juxtaposed in $\theta_{[\![C_1,C_2]\!]}(xy)$. Similarly, cycles containing the β_i are juxtaposed. The compositions used to construct K will therefore create some 4-cycles and some 6-cycles, and a single 3-cycle. The 5-cycle beckoning invitingly in $\mathbf{s}_3^p$ is unaffected by all these goings on.

The nett result is that $\theta_K(xy)$ contains a 5-cycle, with no other cycle of length divisible by five. So, some power of $\theta_K(xy)$ is a single 5-cycle. Among the vertices of this cycle are the two marked by an asterisk in Figure 4. One of these is fixed by $\theta_K(x)$ and the other by $\theta_K(y)$ and $\theta_K(z)$, so the conditions of Lemma 1 are fulfilled. Thus $\theta_K(G) \cong A_{|K|}$ or $S_{|K|}$ by Theorem 1, and since θ_K of x, y and z are all even, $\theta_K(G) \cong A_{|K|}$.

The cases $p = 3$ and 5 can be found in the Appendix.

5. The (p,q,r) triangle groups

Suppose

$$G = \langle x, y \mid x^p = y^q = (xy)^r = 1\rangle, \ \ 3 \le p < q < r,$$

with p, q and r prime.

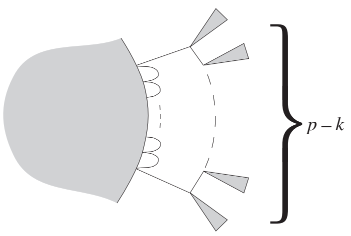


FIGURE 5. Type k pendant.

We start with some notation. An x-face is of *type* $[l_1, \dots, l_\lambda, \dots, \bar{l}_\mu, \dots, l_t]$, $\sum_{i=1}^t l_i = p$, if it has boundary label x^p, and in traversing the boundary with the orientation,

- vertices $(\sum_{i<\lambda} l_i + 1)$ through $(\sum_{i\le\lambda} l_i)$ are consecutive on some q-gon, and
- vertices $(\sum_{i<\mu} l_i + 1)$ through $(\sum_{i\le\mu} l_i)$ are incident with $\lhd$'s.

Of course the face also has type X for X any cyclic permutation of the l_i. In practice this ambiguity causes no confusion. We may say *type X x-cycle* rather than x-face of type X. In particular, Figure 5 shows a type $[k, \overline{p-k}]$ x-cycle, $1 \le k \le p$, or *type k pendant*. Suppose we have k consecutive free vertices on a q-gon, all in the boundary of the same xy-face F. Attaching a type k pendant to these vertices increases the number of y-arcs in the boundary of F by $p - 2k + 1$. The modification also produces a new x-face with boundary x^p and some y and xy-faces with label y and xy.

Let $q = lp + s$ for $l \geq 1$ and $1 \leq s \leq p-1$. Take a shaded q-gon, and attach $l-1$ type p pendants to $p(l-1)$ consecutive vertices. Attach a single type s pendant so that p consecutive vertices are left free. The result is called a *booster*.

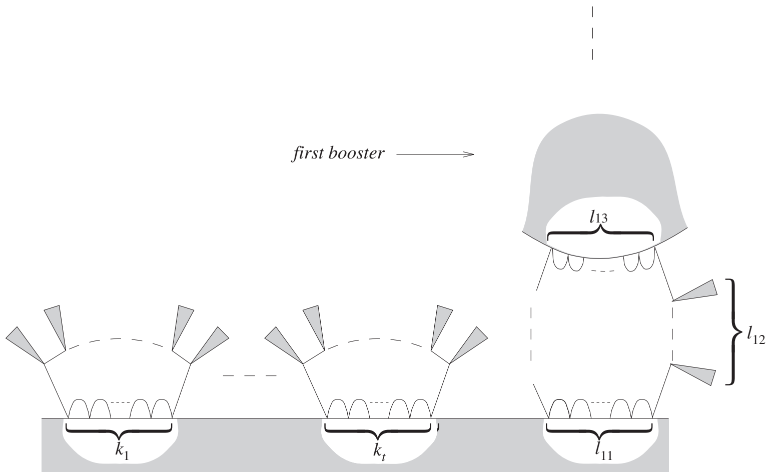


FIGURE 6. Type $\{k_1, \ldots, k_t; X_1, \ldots, X_m\}$ array.

Let $X_i = [l_{i1}, \bar{l}_{i2}, l_{i3}, \bar{l}_{i4}]$, $i = 1, \ldots, m$. Suppose also that for integers $1 \leq k_1, \ldots, k_t \leq p$, we have $l_{11} + \sum k_i$ consecutive free vertices on a q-gon and in the boundary of an xy-face F. One obtains a *type* $\{k_1, \ldots, k_t; X_1, \ldots, X_m\}$ *array*, by attaching to these free vertices,

- t pendants of types $k_1, \ldots, k_t$;
- a collection of m boosters, joined into a chain, with l_{i3} vertices of the i-th booster connected to l_{i1} vertices of the $(i-1)$-st by an x-cycle of type X_i (we take the 0-th booster to be the original q-gon). See Figure 6.

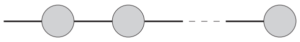
Write $\{k_1, \ldots, k_i^{\delta_i}, \ldots, k_t; X_1, \ldots, X_j^{\delta_j}, \ldots, X_m\}$ for

$$\{k_1, \ldots, \underbrace{k_i, \ldots, k_i}_{\delta_i}, \ldots, k_t; X_1, \ldots, \underbrace{X_j, \ldots, X_j}_{\delta_j}, \ldots, X_m\}.$$

Notice that a type $\{k; —\}$ array is merely a type k pendant. In attaching an array, the number of y-arcs in the boundary of xy-face F increases by

$$m(p + l + 2 - s) + \sum_{i=1}^{t}(p - 2k_i + 1) + 2 \sum_{\bar{l}_{ij} \in X_i} l_{ij}, \tag{4}$$

together with the creation of the usual (and harmless) complement of x, y and xy-faces having boundary x, x^p, y and xy. All other faces are unaffected. To see (4), start with each $X_i = [1, p-1]$, and observe that replacing it by $[1, \bar{1}, p-2]$ increases the y-arc count by two, while a change to $[2, p-2]$ has no effect.

Arrays do all the hard work in (p, q, r)-diagrams, playing a similar role to the chains  of the previous Section.

Let $\theta_K(G)$ arise from some G-diagram K, with $\mathbf{s}$ the cycle structure of $\theta_K(x^{-1}y)$ (see Section 4). Given two such structures $\mathbf{s}_1$ and $\mathbf{s}_2$, let $\mathbf{s}_1 + \mathbf{s}_2$ be their pointwise sum as functions.

Lemma 2. Suppose $K_1, \ldots, K_t$ are (p,q,r)-diagrams with $\mathbf{s}_i$ the cycle structure of $\theta_{K_i}(x^{-1}y)$. If $K = [\![K_1, \ldots, K_t]\!]$, then $\theta_K(x^{-1}y)$ has cycle structure $\sum \mathbf{s}_i$.

Proof. Only cycles in $\theta_{K_i}(x^{-1}y)$ that pass through handle points are affected by the composition. If α_j and β_j lie in such a cycle, then in $\theta_K(x^{-1}y)$ the cycle is identical, except that β_j is replaced by β_{j-1}. □

Thus, consideration of the cycle structure of $\theta_{[\![K_1,\ldots,K_t]\!]}(x^{-1}y)$ reduces to an investigation of the $\theta_{K_i}(x^{-1}y)$.

For any G-diagram K, we determine the effect on $\theta_K(x^{-1}y)$ of attaching an array to K by considering the various ingredients. From now on, when we talk of a *cycle* in K, we will mean a cycle of $\theta_K(x^{-1}y)$–the context should make clear which cycle we mean. Notice first that consecutive free vertices on a q-gon are contained in the same cycle. Attaching a type k pendant to these vertices increases the length of this cycle by $p-k$ when k is odd. When k is even, it decreases by $\frac{k}{2}$, and a new cycle of length $p-\frac{k}{2}$ is created. The vertices of a booster are organised into a single cycle of length

$$q + \begin{cases} p-s, s \text{ odd} \\ -\dfrac{s}{2}, s \text{ even.} \end{cases}$$

When contained as the i-th booster in an array, vertices may be gained or lost from this cycle (it may even be fused with cycles from neighbouring boosters) depending on whether l_{i1} and l_{i3} are even or odd. Figure 7 shows the possible orbits on the vertices, illustrated by small circles and squares.

FIGURE 7

It will become useful to have at our disposal various maneuvers, in which an array is replaced by another. Replacing an array of type $\{k_1, \ldots, k_t; X_1, \ldots, X_m\}$ by one of type $\{k_1, \ldots, k_t, \frac{p+1}{2}; X_1, \ldots, X_m\}$ is called *spoiling*. A *push-pull* substitutes $\{k_1, \ldots, k_i - 1, \ldots, k_j + 1, \ldots, k_t; X_1, \ldots, X_m\}$ instead. Replacing by $\{k_1, \ldots, k_t; X_1, \ldots, \overline{X}_i, \ldots, X_m\}$, where $\overline{X}_i = [l_{i1} \pm 1, \bar{l}_{i2}, l_{i3} \mp 1, \bar{l}_{i4}]$, will be known as *modifying a chain.*

A few brief remarks on each then. Suppose $\overline{K}$ is the result of performing such a maneuver on (p,q,r)-diagram K:

- $K \xrightarrow{\text{spoiling}} \overline{K}$: since (4) is unchanged, $\overline{K}$ is also a (p,q,r)-diagram. The modification requires $\frac{p+1}{2}$ free vertices and $|\overline{K}| = |K| + \frac{p-1}{2}$. The length of the cycle containing these free vertices changes by a non-trivial amount $< q$.
- $K \xrightarrow{\text{push-pull}} \overline{K}$: again (4) is invariant so $\overline{K}$ is a (p,q,r)-diagram. No free vertices are required and $|\overline{K}| = |K|$. The array is attached to some q-gon of

K, and the length of the cycle on this changes by

$$\sum_{\substack{k\in\{k_i,k_j\}\\ k \text{ even}}} \left(p-\frac{k}{2}\right) - \sum_{\substack{k\in\{k_i,k_j\}\\ k \text{ odd}}} \frac{k}{2}. \tag{5}$$

- $K \xrightarrow{\text{modifying chain}} \overline{K}$: again $\overline{K}$ is a (p,q,r)-diagram, with $|\overline{K}| = |K|$. The operation requires a free vertex on the $(i\mp 1)$-st booster, creating one on the $(i\pm 1)$-st. Use Figure 7 to monitor the effect on cycles in $\theta_K(x^{-1}y)$.

Enough preliminaries. We first look at:

(1). The case $p \geq 7$ and $q \geq p+6$.

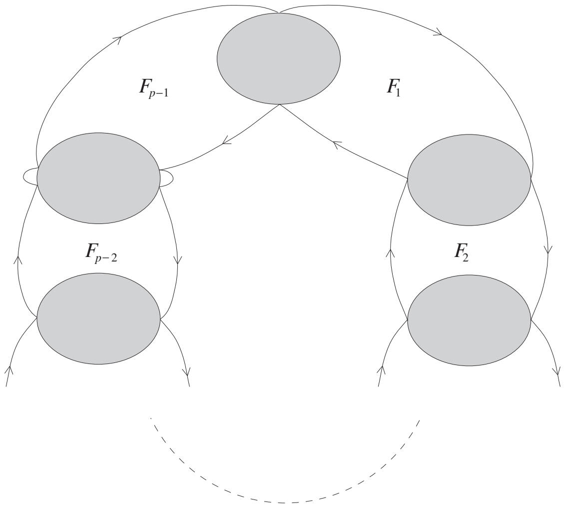


FIGURE 8

Consider Figure 8. We have q-gons, $Q_1,\ldots,Q_{p-1}$, with Q_1 at the top and the ordering going clockwise. They are connected by two type $[2,1,\ldots,1]$ x-cycles, the number of 1's being $p-2$. The connections allow Q_i to contribute one y-arc to the boundary of region F_{i-1}, subscripts taken modulo $p-1$. The usual embedding places Figure 8 on the 2-sphere, as depicted in the picture in fact. The face F_{p-1} has $q-2$ y-arcs in its boundary, faces $F_1,\ldots,F_{p-2}$ have q, and there are four other unshaded faces, two each with label xy and x^p.

Similarly for Figure 9. We have q-gons, $Q_1,\ldots,Q_p$, connected by two type $[1,\ldots,1]$ x-cycles, the number of 1's being p. The connections allow Q_i to contribute $\frac{q-1}{2}$ y-arcs to the boundary of region F_{i-1}, subscripts taken modulo p. The usual embedding places the figure on the 2-sphere also, as in the picture.

Now, let $r \geq q+2$ be prime, and m,δ and k be positive integers such that,

- m is largest with $(q+2)+m(p+l+2-s) \leq r$;
- δ is largest with $(q+2)+m(p+l+2-s)+\delta(p-3) \leq r$;
- k is determined by $p-2k+1 = r-q-m(p+l+2-s)-\delta(p-3)$.

Notice that $2 \leq k \leq \frac{p-1}{2}$. Each q-gon Q_i of Figure 8 has a number of consecutive free vertices laying in the boundary of face F_i. Assuming for now that this number

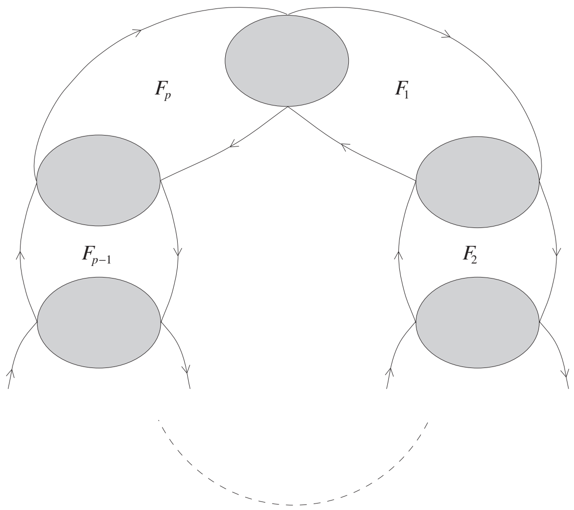


FIGURE 9

is sufficient to do so, attach to $Q_1, \ldots, Q_{p-2}$ arrays of type $\{2^\delta, k; [2, p-2]^m\}$, and one of type $\{2^\delta, k-1; [2, p-2]^m\}$ to Q_{p-1}. By (4) and the definitions of m, δ and k, each face F_i now has r y-arcs in its boundary. We thus have a spherical (p, q, r)-diagram. Call it K_1^r, or as in Section 4, just K_1 when there is no need to be more specific.

Take a single q-gon, attach to it a type $\{2^\delta, k; [2, p-2]^m\}$ array and embed. Call the resulting spherical (p, q, r)-diagram K_2^r, or just K_2. Our third diagram is slightly more complicated. Attach type $\{k; —\}$ arrays to $Q_1, \ldots, Q_{p-3}$ and Q_{p-1}, using free vertices in the boundary of $F_1, \ldots, F_{p-3}$ and F_{p-1}. To $Q_2, \ldots, Q_{p-2}$ and Q_p, attach type $\{2^\delta; [2, p-2]^m\}$'s, adjacent to $F_1, \ldots, F_{p-3}$ and F_{p-1}, while to Q_1 and Q_{p-2}, connect $\{2^\delta, k; [2, p-2]^m\}$'s adjacent to F_p and F_{p-2} (the reader should sketch the positions of the various attachments as a guide). Again, assume for now that there is sufficient space to do all these things. Each F_i receives $r-q$ new y-arcs. Call the resulting (p, q, r)-diagram K_3^r, or just K_3.

Let N be the number of new vertices introduced by an array of type $\{2^\delta, k; [2, p-2]^m\}$. Then $|K_1| = (p-1)q + (p-2)N + N + 1$ and $|K_2| = q + N$. Thus, any common divisor of $|K_1|$ and $|K_2|$ also divides

$$|K_1| - (p-1)|K_2| = 1, \tag{6}$$

so that $|K_1|$ and $|K_2|$ are relatively prime.

Next up, let $\mathbf{s}_i$ be the cycle structure of $\theta_{K_i}(x^{-1}y)$ for $i = 1, 2$ and 3. In K_3, the $\frac{q+1}{2}$ free vertices of Q_{p-1} adjacent to F_{p-2}, and the $\frac{q-1}{2}$ free vertices of Q_p adjacent to F_p, form a q-cycle in $\mathbf{s}_3$. Call any other cycle in $\mathbf{s}_1, \mathbf{s}_2$ or $\mathbf{s}_3$ with length divisible by q a *bad* cycle. As in Section 4, copies of K_1, K_2 and K_3 will form a large composite diagram K, with just one copy of K_3 used. If there are no bad cycles, then by Lemma 2, $\theta_K(x^{-1}y)$ will contain a single q-cycle, with no other cycle of length divisible by q.

We can always arrange things so that bad cycles dissappear. The philosophy is this: the vertices of Figures 8–9 and the q-gon that forms the nucleus of K_2 are organised into various cycles. In fact, there are $p-3$ q-cycles, a $(q-2)$-cycle and a $(q+2)$-cycle in Figure 8; p q-cycles in Figure 9, and a q-cycle in the q-gon of K_2. A crucial observation is that in K_1 and K_2, each of these cycles has exactly one array attached. Things are more complicated with K_3: one q-cycle has $\{k;\text{—}\}$ and $\{2^\delta, k; [2, p-2]^m\}$ arrays attached, another has $\{2^\delta; [2, p-2]^m\}$ and $\{2^\delta, k; [2, p-2]^m\}$ arrays, while $p-3$ of them have $\{k;\text{—}\}$ and $\{2^\delta; [2, p-2]^m\}$. The single q-cycle not mentioned is our precious prime cycle.

We now monitor the effect on these cycles of the attached arrays. First, using the observations following Lemma 2, one can check that the boosters in a type $\{k_1, \dots, k_t; [2, p-2]^m\}$ array contribute bad cycles only when $s=2$. In this case, the m-th booster contains a q-cycle. No problem, just modify the chain, and replace X_m by $\overline{X}_m = [1, p-1]$.

Next, the effect of the pendants in an array. Consider one of the q-cycles in K_1 or K_2. If $m=0$, so that a $\{2^\delta, k;\text{—}\}$ array is attached to the cycle, its length becomes

$$q - \delta + \begin{cases} p-k, & k \text{ odd} \\ -\dfrac{k}{2}, & k \text{ even.} \end{cases}$$

Since $k \leq \frac{p-1}{2}$, we have $p-k \geq \frac{p+1}{2}$, and so the cycle is bad only if $\delta \geq \frac{p+1}{2}$. The definitions of m, δ and k give $\delta(p-3)+p-3 \leq p+l+2-s$, so the cycle is bad only if $l-s \geq 7$, that is, $q \geq 8p+1$. In fact, $q \geq 2p+1$ will do. By an identical argument, the $(q-2)$-cycle in K_1 becomes bad only if $q \geq 2p+1$, and the $(q+2)$-cycle suffers the same fate under the addition of a type $\{4;\text{—}\}$ array, or only if $q \geq 2p+1$.

Similarly for the q-cycles in K_3. When $m=0$, we must have $q \geq 3p$ before any turn bad, and when $m=1$, we must have $q \geq 2p+1$.

What do we do with these bad cycles? When $m \geq 1$ it is simple. Take one of the $\{k_1, \dots, k_t; [2, p-2]^m\}$ arrays attached to the cycle and perform a simultaneous volley of chain modifications: either replace all $X_i = [2, p-2]$ by $\overline{X}_i = [1, p-1]$, or all X_i by $\overline{X}_i = [3, p-3]$, whichever does not create a bad cycle on the m-th booster (they both can't). When $s=2$ and $m \geq 2$, change all X_i to $[1, p-1]$. If $s=2$ and $m=1$, change $X_1 = [1, p-1]$ to $\overline{X_1} = [3, p-3]$. In any case the bad cycle is obliterated and no new bad cycles are created. Remember that when $\overline{X_1} = [3, p-3]$, we are assuming there are two free vertices where the array is attached, but more on this later.

If $m=0$ and a bad cycles arises in K_3, spoil one of the attached arrays, assuming for now that there is enough room to do so. If the bad cycle is in K_1 or K_2, it would be nice to be rid of it by spoiling the attached array. Unfortunately, spoiling changes the number of vertices, and (6) would no longer be valid. So, except for when a $\{4;\text{—}\}$ is attached to the $(q+2)$-cycle, spoil *every* array in these two diagrams (again assuming there is enough room). This certainly removes the bad cycle. The danger is that it may have created a new one elsewhere. If so, remove it by performing a push-pull on the attached array: replace $\{2^\delta, k \text{ or } k-1, \frac{p+1}{2}; [2, p-2]^m\}$ by $\{2^\delta, k-1 \text{ or } k-2, \frac{p+3}{2}; [2, p-2]^m\}$, or $\{2^\delta, 1, \frac{p+1}{2}; [2, p-2]^m\}$ by $\{2^\delta, 2, \frac{p-1}{2}; [2, p-2]^m\}$. In all the cases that bad cycles arise, $q \geq 2p+1$, so the effect (5) of these push-pulls in both non-trivial and $< q$. The new bad cycle is thus removed.

The bad cycle arising when a $\{4;\text{—}\}$ array is attached to the $(q+2)$-cycle in K_1 is removed by similarly spoiling every array in K_1 and K_2. It can be checked that this creates no new bad cycles elsewhere. This accounts for all situations where bad cycles arise.

Our final task is to see that there are sufficient free vertices in the appropriate places for all the above to happen. Fix p, and for a given q, let Δ be the maximum value obtained by δ. When $m = 0$ the largest number of consecutive free vertices needed anywhere is $2\delta + k + \frac{p+1}{2}$: room for a type $\{2^\delta, k; —\}$ array and a possible spoil. Similarly, when $m \geq 1$ we need $2(\delta+1)+k+1$: room for a $\{2^\delta, k; [2, p-2]^m\}$ array and a potential volley of chain modifications. The $m \geq 1$ needs are less than the $m = 0$ needs, and since $k \leq \frac{p-1}{2}$, these in turn are less than $2\Delta + p$.

Take four consecutive vertices on the q-gon of K_2 and two on each of $Q_1, \dots, Q_{p-2}$ of K_1. These are the handles for K_1 and K_2. Thus, before any arrays are added, the q-gons of K_1 and K_2 are left with $q - 4$ consecutive free vertices. When $p + 6 \leq q \leq 2p + 1$, we have $\Delta = 1$, so $2\Delta + p \leq q - 4$, and we are happy.

Now Δ is the largest multiple of $p - 3$ less than $p + l + 2 - s$. Thus for a fixed l, Δ and hence $2\Delta + p$ is biggest, and $q - 4$ smallest, when $s = 1$. It therefore suffices to show that $2\Delta + p \leq q - 4$ for $q = lp + 1$. We already have this for $l = 2$. If the inequality is valid for a given l, and we increase it by one, then $p + l + 2 - s$, and hence Δ, increases by at most one, and so $2\Delta + p$ by at most two. But $q - 4$ increases by $p \geq 7$, and we are home.

In K_3 the vertex requirements are greatest and the availability least, on the side of Q_{p-2} adjacent to F_{p-2}. The reader should sketch the positions of the various arrays to see why. By considering the possible values of Δ for q in the range $p + 6 \leq q \leq 4p - 1$, one can show, using the discussion of when bad cycles arise, that the $\frac{q-3}{2}$ consecutive free vertices that are available suffice. For $q \geq 4p + 1$, argue as for K_1 and K_2.

Finally, place a handle on K_3 using two vertices of the precious q-cycle.

(2). The case $p \geq 7$ and $q = p + 2$ or $p + 4$.

Diagrams K_1 and K_2 are the same as in the previous case. That there is sufficient room on K_1 and K_2 is a slightly more delicate matter, but the argument is essentially the same. Our methods can be of no help to $(11, 13, 17)$ however, which can be found in the Appendix.

Unfortunately, there are not enough free vertices on our existing K_3 once q is this close to p. Instead, consider Figure 10. When $q = p + 2$ the large xy-face has $q_0 = p + 10$ y-arcs in its boundary, while the minimum r of interest is $r_0 = p + 6$. For $r > r_0$ prime, let m and k be positive integers such that: m is largest with $r_0 + m(p + 1) \leq r$; and k is determined by $p - 2k + 1 = r - q_0 - m(p + 1)$. Add a type $\{k; [1, p - 1]^m\}$ array to the top q-gon. The resulting (p, q, r)-diagram will be our K_3 for $q = p + 2$.

Since $k \leq \frac{p+5}{2}$, there is sufficient room on the top q-gon for the array with at least three vertices to spare. Put a handle on the bottom q-gon, which also has at least three vertices to spare. The middle q-gon supplies us with a q-cycle. Bad cycles can only arise on the q-gon to which the array is attached. In such a situation, change the two $[1, p - 1]$ cycles in Figure 10 to type $[2, p - 2]$'s. This removes the bad cycle.

With $q = p+4$, a K_3 diagram for $(7, 11, 13)$ can be found in Section 6. Otherwise the argument is identical with $q_0 = p + 16$, and $r_0 = p + 6$ when $p \geq 13$, or $r_0 = 17$ when $p = 7$.

(3). The case $p = 5$ and $q \geq 17$.

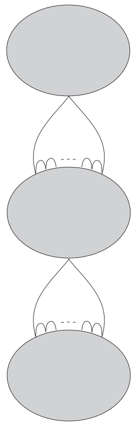


FIGURE 10

Except for the arrays, diagrams K_1, K_2 and K_3 are the same as in case 1. For $r \geq q+2$ prime, let m be largest with $(q+2)+m(5+l+2-s) \leq r$; δ_1 largest with $(q+2)+m(5+l+2-s)+4\delta_1 \leq r$; δ_2 largest with $(q+2)+m(5+l+2-s)+4\delta_1+2\delta_2 \leq r$; and k determined by $p+2k-1 = r-q-m(5+l+2-s)-4\delta_1-2\delta_2$. Add arrays in the same places as case 1, except replace each 2^δ in an array there by $1^{\delta_1}, 2^{\delta_2}$. The remainder of the argument is the same.

(4). The case $p=5$ and $q=11,13$.

Diagrams K_1 and K_2 are as in case 3. For K_3, let $r_0 = 13$ and $q_0 = 15$ when $q=11$, or $r_0 = 17$ and $q_0 = 21$ when $q = 13$. Given $r \geq r_0$ prime, take m largest with $r_0 + m(9-s) \leq r$, and k determined by $p-2k+1 = r-q_0-m(9-s)$. Add a type $\{5, k; [1, p-1]^m\}$ array to the top q-gon of Figure 10, and type $\{5; —\}$'s to the other two. The resulting $(5, q, r)$-diagram is our K_3. Proceed as in case 2.

(5). The case $p=5$ and $q=7$.

We do $(5,7,11)$ and $(5,7,13)$ in the Appendix. Diagrams K_1 and K_2 are the same as in case 1, bar the arrays. Instead, for $r \geq 17$ prime, take m largest with $9+6m \leq r$; δ largest with $9+6m+2\delta \leq r$, and k given by $5+2k-1 = r-7-6m-2\delta$. Somewhat unusually, add type $\{k; [1,4]^{m-1}, [1, \bar{\delta}, 4-\delta]\}$'s and a single type $\{k-1; [1,4]^{m-1}, [1, \bar{\delta}, 4-\delta]\}$ in all the usual places. When $m \geq 2$ and $\delta = 2$, a bad cycle arises in the chain of boosters. Remove it by modifying, $\overline{X}_m$ being $[2, \bar{2}, 1]$, and by replacing the type 2 pendant on each of the last two boosters by types 1 and 3. For K_3, follow the construction of case 2.

(6). The case $p=3$ and $q \geq 17$.

Use figure 9, and allow Q_i to contribute a single y-arc to region F_{i-1}. For $r \geq q$ prime, take $m \geq 0$ largest with $q+m(5+l-s) \leq r$ and $\delta \geq 0$ largest with $q+m(5+l-s)+2\delta \leq r$. Add type $\{1^\delta; [2,1]^m\}$ arrays to each Q_i, using the free vertices adjacent to region F_i. The resulting $(3,q,r)$-diagram is our K_1.

Spoil the array on Q_1, that is, replace by one of type $\{1^\delta, 2; [2,1]^m\}$. This gives another $(3,q,r)$-diagram, K_2. Notice that $|K_1| - |K_2| = 1$, so $|K_1|$ and $|K_2|$ are relatively prime. Place a handle on Q_2 and Q_3 in each diagram. We can remove bad cycles from the chains of boosters by the methods of case 1. It is easy to show that none arise elsewhere in K_1. A bad cycle will arise on Q_1 in K_2 precisely when $m \geq 1$ and $\delta = 1$, but replacing the $\{1^\delta, 2; [2,1]^m\}$ by $\{1^\delta, 3; [1,1,\bar{1}], [2,1]^{m-1}\}$ removes it. The argument of case 1 shows that there are sufficient free vertices for all the arrays and subsequent modifications.

Take Figure 9 with the connecting type $[1,1,1]$ x-cycles allowing Q_i to contribute $\frac{q-1}{2}$ to F_{i-1}. Attach type $\{1^\delta; [2,1]^m\}$ arrays to Q_1 adjacent to F_1 and F_3, and also to Q_2 adjacent to F_2. The result is K_3. By the usual argument, there is sufficient room for the arrays as well as to spoil any array incident with a bad cycle. A q-cycle occupies the untouched vertices of Q_3 adjacent to F_3 and Q_2 adjacent to F_1, and a handle for K_3 can be safely placed here.

(7). The case $p = 3$ and $q = 13$.

You can find $(3,13,17)$ and $(3,13,19)$ in the Appendix. For $r \geq 23$ prime, use the K_1 and K_2 of case 6. For K_3 attach $\{3^3; —\}$ arrays to the bottom two q-gons of Figure 10, and place a handle on the bottom one as well. Place a type $\{3; —\}$ on the top q-gon. In addition, we need a type $\{1^\delta; [2,1]^m\}$ on the top q-gon, with δ and m chosen as in case 6, and this can be spoiled if necessary to remove bad cycles.

(8). The case $p = 3$ and $q = 11$.

We do $(3,11,13)$ in the Appendix. For $r \geq 17$ prime, diagrams K_1 and K_2 are as in case 6. For K_3 attach type $\{3^2; —\}$ arrays to the top two q-gons in Figure 10, and a $\{3^3; —\}$ array to the bottom. Place a handle on the middle q-gon (which contains our q-cycle) and a type $\{1^\delta; [2,1]^m\}$ array on the top one. Chose m and δ according to the usual scheme. Spoil the array to remove any bad cycles.

(9). The case $p = 3$ and $q = 7$.

Look in the Appendix for $(3,7,11)$. For $r \geq 13$ prime, variations on Figure 10 yield all three diagrams. For consider just the top two q-gons and the type $[1,2]$ x-cycle connecting them. Place a type $\{1^\delta; [2,1]^m\}$ array on the top one as usual and handle on each of the top two. The resulting $(3,7,r)$-diagram is K_1. Attach a type $\{2; —\}$ array to the bottom q-gon. The result is K_2. For K_3, start from scratch with Figure 10, and attach to the bottom two q-gons arrays of type $\{3; —\}$, while to the top, attach a type $\{1^\delta, 3; [1,2]^m\}$. Place a handle on the bottom q-gon.

(10). The case $p = 3$ and $q = 5$.

You can find $(3,5,7)$ and $(3,5,11)$ in the Appendix. Otherwise, for K_1 take Figure 10 with type $\{—; [2,1]^m\}$ and $\{1^\delta; —\}$ arrays attached to the second and third q-gons respectively, and with two handles on the top. For K_2, place type $\{2; —\}$ and $\{1^\delta; [1,2]^m\}$ arrays on the second and third q-gons instead. To get K_3, attach a $\{1^\delta; [2,1]^m\}$ to the top q-gon and a handle on the bottom one.

So, for every (p,q,r) triangle group under consideration we have a K_1, K_2 and K_3. Moreover, $|K_1|$ and $|K_2|$ are relatively prime. Let $p_1, p_2 > p$ be distinct primes not dividing $|K_1|$ and $|K_2|$. For k_1 and k_2 arbitrary non-negative integers we construct a sequence of diagrams $C_0, C_1, \dots, C_k, \dots, C_{k_1+k_2} = K$:

- Step 0: if k_1 or $k_2 = 0$, take $C_0 = K_3$. Otherwise, $C_0 = K_2$.
- Step i for $1 \leq i \leq k_1$: take p_1 identical copies of K_1 and let C_i be the diagram
$$[\![[\![\cdots [\![[\![C_{i-1}, \underbrace{K_1, \dots, K_1}_{p-1}]\!], \underbrace{K_1, \dots, K_1}_{p-1}]\!], \dots]\!], \underbrace{K_1, \dots, K_1}_{p-1}]\!], \underbrace{K_1, \dots, K_1}_{\leq p-2}]\!].$$
A quick sketch may help the reader to see what is going on. In particular, the two handles on each K_1 allow us to perform this composition, and moreover, C_i has at least two unused handles.
- Step i for $k_1 + 1 \leq i \leq k_1 + k_2 - 1$: take p_2 identical copies of K_2 and let C_i be a composite diagram as in the previous step, but with p_2 copies of K_2' instead of p_1 copies of K_1.
- Step $k_1 + k_2$: if k_1 or $k_2 = 0$, let $K = C_{k_1+k_2}$ be as in the previous step. Otherwise, take a diagram as in the previous step and replace one of the K_2's by a K_3.

Now $|K| = k_1p_1|K_1| + k_2p_2|K_2| + |K_3|$ Since $|K_1|$ and $|K_2|$ are relatively prime, so too are $p_1|K_1|$ and $p_2|K_2|$. By choosing k_1 and k_2 suitably, $|K|$ can thus be made to equal any integer greater than $(p_1|K_1| - 1)(p_2|K_2| - 1) + |K_3|$. Thus, if $\theta_K : (p,q,r) \to S_{|K|}$ is the homomorphism arising from K, we have permutation representations of (p,q,r) for all but finitely many possible degrees. The permutation $\theta_K(x^{-1}y)$ contains a single q-cycle with no other cycle of length divisible by q. That the resulting q-cycle in $\theta_K(G)$ contains points for the purposes of Lemma 1 is easy to check. Hence $\theta_K(G) \cong A_{|K|}$ or $S_{|K|}$. Since the generators of $\theta_K(G)$ are even permutations (p and q being odd), $\theta_K(G) \cong A_{|K|}$.

6. Appendix

And so to the remaining sporadic cases. For each, we list the images of the generators, from which of course, the diagrams can be reconstructed.

1. There are two $(2,q,r)$ groups left from part 4 of Proposition 2. Using the notation of [7], one shows that the "extended" triangle group
$$(2,q,r)^* = \langle x, y, t \,|\, x^2 = y^q = (xy)^r = t^2 = (xt)^2 = (yt)^2 = 1 \rangle,$$
surjects essentially all the symmetric groups S_n. See [7] for more details.
   (a) $G = (2,5,6)$:
   $\theta_{K_1}(x) = (2,4)(3,6)(8,11)(12,13)$,
   $\theta_{K_1}(y) = (1,2,5,7,3)(4,8,6,10,9)(11,12,14,15,13)$,
   $\theta_{K_1}(t) = (2,3)(4,6)(5,7)(9,10)(12,13)(14,15)$,
   handles: $[14,15], [5,7], [10,9]$
   $\theta_{K_2}(x) = (2,3)(4,6)(5,7)(8,10)(9,12)(11,15)$,
   $\theta_{K_2}(y) = (1,2,4,5,3)(6,8,11,9,7)(10,13,12,16,14)$,
   $\theta_{K_2}(t) = (2,3)(4,5)(6,7)(8,9)(10,12)(14,16)$,
   handles: $[16,14]$
   $\theta_{K_3}(x) = (2,4)(3,6)(8,10)(9,12)(11,14)(13,15)$,
   $\theta_{K_3}(y) = (1,2,5,7,3)(4,6,9,11,8)(10,13,14,15,12)$,
   $\theta_{K_3}(t) = (2,3)(4,6)(5,7)(8,9)(10,12)(13,15)$,
   handles: $[5,7]$
   (b) $G = (2,5,9)$:

$\theta_{K_1}(x) = (2,4)(3,6)(8,10)(9,12)(11,15)(14,17)(20,21)(22,23),$
$\theta_{K_1}(y) = (1,2,5,7,3)(4,6,9,11,8)(10,13,16,12,14)(15,18,22,23,19)$
$(17,20,24,25,21),$
$\theta_{K_1}(t) = (2,3)(4,6)(5,7)(8,9)(10,12)(13,16)(18,19)(20,21)(22,23)(24,25),$
handles: $[13,16],[24,25],[5,7]$
$\theta_{K_2}(x) = (5,6)(7,9)(8,11)(10,12)(13,14)(15,16),$
$\theta_{K_2}(y) = (1,2,4,5,3)(6,7,10,12,8)(9,13,15,14,11),$
$\theta_{K_2}(t) = (1,2)(3,4)(7,8)(9,11)(10,12)(13,14),$
handles: $[1,2]$
$\theta_{K_3}(x) = (5,6)(7,9)(8,11)(10,15)(12,17)(13,16)(14,18)(19,20),$
$\theta_{K_3}(y) = (1,2,4,5,3)(6,7,10,12,8)(9,13,16,11,14)(15,19,20,17,18),$
$\theta_{K_3}(t) = (1,2)(3,4)(7,8)(9,11)(10,12)(13,16)(15,17)(19,20),$
handles: $[1,2]$

2. As mentioned after the proof of Proposition 2, Theorems 1–3 of [16] leave behind sixty of the $(2,q,r)$ with $5 \leq q < r$ prime. By adding caravans and combs (to use the terminology of [**?**]) to their Figures 6–10, one can obtain brother-sister pairs, and so account for all but fifteen of this sixty (see [16]). For these fifteen we proceed as in number 1 of this section.

First, let A be the following $(2,q,q)^*$-diagram:
$\theta_A(x) = (3,q+3)(q,2q),$
$\theta_A(y) = (1,2,\dots,q)(q+1,q+2,\dots,2q),$
$\theta_A(t) = (1,2)(3,q)(4,q-1)\dots(\frac{q+1}{2},\frac{q+5}{2})$
$(q+1,q+2)(q+3,2q)(q+4,2q-1)\dots(q+\frac{q+1}{2},q+\frac{q+5}{2}),$
handles: $[1,2],[q+\frac{q+1}{2},q+\frac{q+5}{2}].$

Let B be the $(2,q,q)^*$-diagram determined by the above permutations, with $\theta_B(x)$ supplemented by $(q+1,q+2)(\frac{q+3}{2},2q+1)$. Similarly, C is the $(2,q,2q)^*$-diagram given by,
$\theta_C(x) = (3,2q+1)(q,3q+1)(q+3,2q+1+\frac{q-1}{2})(2q,3q+1-\frac{q-1}{2}),$
$\theta_C(y) = (1,2,\dots,q)(q+1,q+2,\dots,2q)\dots(3q+1,3q+2,\dots,4q),$
$\theta_C(t) = (1,2)(3,q)(4,q-1)\dots(\frac{q+1}{2},\frac{q+5}{2})$
$(q+1,q+2)(q+3,2q)(q+4,2q-1)\dots(q+\frac{q+1}{2},q+\frac{q+5}{2}),$
$(2q+1,3q+1)(2q+2,4q)(2q+3,4q-1)\dots(3q-1,3q+2)$

Let D be the $(2,q,2q)^*$-diagram determined by the above permutations, with $\theta_D(x)$ supplemented by $(q+1,q+2)(\frac{q+3}{2},4q+1)$.

(a) $G = (2,5,7)$:
$\theta_{K_1}(x) = (4,6)(5,7)(9,11)(14,15),$
$\theta_{K_1}(y) = (1,2,4,5,3)(6,8,10,7,9)(11,12,14,15,13),$
$\theta_{K_1}(t) = (1,2,4,8,10,7,3)(5,9,12,14,13,11,6),$
handles: $[3,2],[8,10]$
$\theta_{K_2}(x) = (4,6)(5,7)(9,11)(12,14)(13,16)(15,17)(19,21)(20,22),$
$\theta_{K_2}(y) = (1,2,4,5,3)(6,8,10,7,9)(11,12,15,17,13)(14,18,16,20,19),$
$\theta_{K_2}(t) = (1,2,4,8,10,7,3)(5,9,12,18,16,11,6)(13,20,22,19,21,14,15),$
handles: $[3,2],[8,10]$
$\theta_{K_3}(x) = (2,4)(3,6)(5,7)(8,11)(9,12)(10,13),$
$\theta_{K_3}(y) = (1,2,5,7,3)(4,8,10,6,9)(11,14,15,13,12),$
$\theta_{K_3}(t) = (1,2,8,14,15,13,6)(3,9,11,10,12,4,5),$
handles: $[14,15]$

(b) $G = (2,5,11)$:
$\theta_{K_1}(x) = (4,6)(5,7)(12,16)(13,17)(14,18)(15,19),$
$\theta_{K_1}(y) = (1,2,4,5,3)(6,8,12,13,9)(7,10,14,15,11)(16,20,22,19,21),$
$\theta_{K_1}(t) = (1,2,4,8,12,20,22,19,11,7,3)(5,10,14,18,15,21,16,13,17,9,6),$
handles: $[3,2],[20,22]$

$\theta_{K_2}(x) = (4,\ 6)(5,\ 7)(9,\ 13)(10,\ 14)(12,\ 16)(15,\ 17)(19,\ 21)(24,\ 25),$
$\theta_{K_2}(y) = (1,\ 2,\ 4,\ 5,\ 3)(6,\ 8,\ 12,\ 13,\ 9)(7,\ 10,\ 14,\ 15,\ 11)(16,\ 18,\ 20,\ 17,\ 19)$
$(21,\ 22,\ 24,\ 25,\ 23),$
$\theta_{K_2}(t) = (1,\ 2,\ 4,\ 8,\ 12,\ 18,\ 20,\ 17,\ 11,\ 7,\ 3)(5,\ 10,\ 15,\ 19,\ 22,\ 24,\ 23,\ 21,\ 16,\ 13,\ 6),$
handles: [3, 2], [18, 20]
$\theta_{K_3}(x) = (4,\ 6)(5,\ 7)(8,\ 12)(9,\ 13)(10,\ 14)(11,\ 15),$
$\theta_{K_3}(y) = (1,\ 2,\ 4,\ 5,\ 3)(6,\ 8,\ 12,\ 13,\ 9)(7,\ 10,\ 14,\ 15,\ 11),$
$\theta_{K_3}(t) = (1,\ 2,\ 4,\ 8,\ 13,\ 6,\ 5,\ 10,\ 15,\ 7,\ 3),$
handles: [3, 2]

(c) $G = (2, 5, 13)$:
$\theta_{K_1}(x) = (5,\ 6)(7,\ 9)(8,\ 11)(10,\ 13)(12,\ 14)(15,\ 18)(16,\ 20)(17,\ 22)(21,\ 25)(23,\ 26),$
$\theta_{K_1}(y) = (1,\ 2,\ 4,\ 5,\ 3)(6,\ 7,\ 10,\ 12,\ 8)(13,\ 15,\ 19,\ 21,\ 16)(14,\ 17,\ 23,\ 24,\ 18),$
$\theta_{K_1}(t) = (1,\ 2,\ 4,\ 5,\ 7,\ 9,\ 10,\ 15,\ 14,\ 8,\ 11,\ 6,\ 3)(12,\ 17,\ 22,\ 23,\ 26,\ 24,\ 18,\ 19,$
$21,\ 25,\ 16,\ 20,\ 13),$
handles: [1, 2], [24, 19],
$\theta_{K_2}(x) = (4,\ 6)(5,\ 7)(13,\ 16)(14,\ 17)(19,\ 21)(22,\ 24)(23,\ 26)(25,\ 27),$
$\theta_{K_2}(y) = (1,\ 2,\ 4,\ 5,\ 3)(6,\ 8,\ 12,\ 13,\ 9)(7,\ 10,\ 14,\ 15,\ 11)(16,\ 18,\ 20,\ 17,\ 19)$
$(21,\ 22,\ 25,\ 27,\ 23),$
$\theta_{K_2}(t) = (1,\ 2,\ 4,\ 8,\ 12,\ 13,\ 18,\ 20,\ 17,\ 15,\ 11,\ 7,\ 3)(5,\ 10,\ 14,\ 19,\ 22,\ 24,\ 25,$
$23,\ 26,\ 21,\ 16,\ 9,\ 6),$
handles: [3, 2], [18, 20],
$\theta_{K_3}(x) = (5,\ 6)(7,\ 9)(8,\ 11)(10,\ 15)(12,\ 18)(13,\ 19)(17,\ 23)(20,\ 24)(21,\ 25)(22,\ 26),$
$\theta_{K_3}(y) = (1,\ 2,\ 4,\ 5,\ 3)(6,\ 7,\ 10,\ 12,\ 8)(9,\ 13,\ 20,\ 21,\ 14)(11,\ 16,\ 22,\ 24,\ 17),$
$\theta_{K_3}(t) = (1,\ 2,\ 4,\ 5,\ 7,\ 13,\ 19,\ 20,\ 17,\ 23,\ 11,\ 6,\ 3)(8,\ 16,\ 22,\ 26,\ 24,\ 21,\ 25,\ 14,$
$9,\ 10,\ 15,\ 12,\ 18),$
handles: [1, 2]

(d) $G = (2, 7, 11)$:
$\theta_{K_1}(x) = (3,\ 5)(4,\ 6)(7,\ 8)(13,\ 14),$
$\theta_{K_1}(y) = (1,\ 2,\ 4,\ 6,\ 7,\ 5,\ 3)(8,\ 9,\ 11,\ 13,\ 14,\ 12,\ 10),$
$\theta_{K_1}(t) = (1,\ 2,\ 4,\ 7,\ 9,\ 11,\ 13,\ 12,\ 10,\ 8,\ 5),$
handles: [11, 12], [1, 2]
$\theta_{K_2}(x) = (6,\ 8)(7,\ 9)(11,\ 14)(18,\ 20)(19,\ 22)(21,\ 23),$
$\theta_{K_2}(y) = (1,\ 2,\ 4,\ 6,\ 7,\ 5,\ 3)(8,\ 10,\ 13,\ 15,\ 12,\ 9,\ 11)(14,\ 16,\ 18,\ 21,\ 23,\ 19,\ 17),$
$\theta_{K_2}(t) = (1,\ 2,\ 4,\ 6,\ 10,\ 13,\ 15,\ 12,\ 9,\ 5,\ 3)(7,\ 11,\ 16,\ 18,\ 20,\ 21,\ 19,\ 22,\ 17,\ 14,\ 8),$
handles: [13, 15], [3, 2,]
$\theta_{K_3}(x) = (3,\ 5)(4,\ 6)(7,\ 8)(9,\ 11)(10,\ 12)(13,\ 14),$
$\theta_{K_3}(y) = (1,\ 2,\ 4,\ 6,\ 7,\ 5,\ 3)(8,\ 9,\ 12,\ 14,\ 13,\ 11,\ 10),$
$\theta_{K_3}(t) = (1,\ 2,\ 4,\ 7,\ 9,\ 10,\ 14,\ 11,\ 12,\ 8,\ 5),$
handles: [1, 2]

(e) $G = (2, 7, 13)$:
$\theta_{K_1}(x) = \theta_C(x) + (13, 14)(23, 24),$ $\theta_{K_2}(x) = \theta_D(x) + (13, 14)(23, 24),$
$\theta_{K_1}(y) = \theta_C(y),$ $\theta_{K_2}(y) = \theta_D(y),$
$\theta_{K_1}(t) = \theta_C(t),$ $\theta_{K_2}(t) = \theta_D(t),$
handles: [1, 2], [11, 13] handles: [1, 2], [11, 13]
$\theta_{K_3}(x) = (7,\ 8)(9,\ 11)(10,\ 12)(13,\ 14),$
$\theta_{K_3}(y) = (1,\ 2,\ 4,\ 6,\ 7,\ 5,\ 3)(8,\ 9,\ 12,\ 14,\ 13,\ 11,\ 10),$
$\theta_{K_3}(t) = (1,\ 2,\ 4,\ 6,\ 7,\ 9,\ 10,\ 14,\ 11,\ 12,\ 8,\ 5,\ 3),$
handles: [1, 2]

(f) $G = (2, 7, 17)$:
$\theta_{K_1}(x) =$ as for $(2, 7, 13) + (4, 29)(6, 30)(11, 31)(13, 32)(16, 33)(28, 34),$
$\theta_{K_1}(y) =$ as for $(2, 7, 13),$
$\theta_{K_1}(t) =$ as for $(2, 7, 13) + (29, 30)(31, 32)(33, 34),$
handles: [1, 2], [17, 27]
$\theta_{K_2}(x) =$ as for $(2, 7, 13) + (4, 30)(6, 31)(11, 32)(13, 33)(16, 34)(28, 35),$
$\theta_{K_2}(y) =$ as for $(2, 7, 13),$
$\theta_{K_2}(t) =$ as for $(2, 7, 13) + (30, 31)(32, 33)(34, 35),$
handles: [1, 2], [17, 27]

$\theta_{K_3}(x) = (7,\ 8)(9,\ 11)(10,\ 13)(12,\ 15)(14,\ 17)(16,\ 18)$,
$\theta_{K_3}(y) = (1,\ 2,\ 4,\ 6,\ 7,\ 5,\ 3)(8,\ 9,\ 12,\ 16,\ 18,\ 14,\ 10)$,
$\theta_{K_3}(t) = (1,\ 2,\ 4,\ 6,\ 7,\ 9,\ 11,\ 12,\ 15,\ 16,\ 14,\ 17,\ 10,\ 13,\ 8,\ 5,\ 3)$,
handles: [1, 2],

(g) $G = (2, 7, 19)$:

$\theta_{K_1}(x) = \theta_C(x) + (11, 29)(13, 30)(16, 31)(17, 32)(20, 33)(21, 34)(23, 35)(24, 36)$
$(27, 37)(28, 38)$,
$\theta_{K_1}(y) = \theta_C(y)$,
$\theta_{K_1}(t) = \theta_C(t) + (29, 30)(31, 28)(32, 37)(33, 36)(34, 35)$,
handles: [8, 9],[19, 25]

$\theta_{K_2}(x) = \theta_{K_1}(x) + (1, 2)(12, 38)$,
$\theta_{K_2}(y) = \theta_{K_1}(y)$,
$\theta_{K_2}(t) = \theta_{K_1}(t)$,
handles: [8, 9],[19, 25]

$\theta_{K_3}(x) = (5,\ 7)(6,\ 9)(8,\ 10)(11,\ 13)(12,\ 15)(14,\ 17)(16,\ 19)(18,\ 20)$,
$\theta_{K_3}(y) = (1,\ 2,\ 4,\ 6,\ 8,\ 5,\ 3)(10,\ 11,\ 14,\ 18,\ 20,\ 16,\ 12)$,
$\theta_{K_3}(t) = (1,\ 2,\ 4,\ 6,\ 9,\ 8,\ 11,\ 13,\ 14,\ 17,\ 18,\ 16,\ 19,\ 12,\ 15,\ 10,\ 5,\ 7,\ 3)$,
handles: [1, 2]

(h) $G = (2, 11, 13)$:

$\theta_{K_1}(x) = \theta_A(x) + (5, 23)(9, 24)(16, 25)(20, 26)$,
$\theta_{K_1}(y) = \theta_A(y)$,
$\theta_{K_1}(t) = \theta_A(t) + (23, 24)(25, 26)$,
handles: as A

$\theta_{K_2}(x) = \theta_B(x) + (5, 24)(9, 25)$
$(16, 26)(20, 27)$,
$\theta_{K_2}(y) = \theta_B(y)$,
$\theta_{K_2}(t) = \theta_B(t) + (24, 25)(26, 27)$,
handles: as B

$\theta_{K_3}(x) = (6,\ 8)(7,\ 9)(10,\ 12)(11,\ 13)$,
$\theta_{K_3}(y) = (1,\ 2,\ 4,\ 6,\ 9,\ 11,\ 10,\ 8,\ 7,\ 5,\ 3)$,
$\theta_{K_3}(t) = (1,\ 2,\ 4,\ 6,\ 7,\ 11,\ 13,\ 10,\ 12,\ 8,\ 9,\ 5,\ 3)$,
handles: [3, 2]

(i) $G = (2, 11, 17)$:

$\theta_{K_1}(x) = \theta_C(x) + (4, 5)(9, 10)$,
$\theta_{K_1}(y) = \theta_C(y)$,
$\theta_{K_1}(t) = \theta_C(t)$,
handles: [1, 2], [17, 19]

$\theta_{K_2}(x) = \theta_D(x) + (4, 5)(9, 10)$,
$\theta_{K_2}(y) = \theta_D(y)$,
$\theta_{K_2}(t) = \theta_D(t)$,
handles: [1, 2], [17, 19]

$\theta_{K_3}(x) = (6,\ 8)(7,\ 10)(9,\ 12)(11,\ 14)(13,\ 16)(15,\ 17)$,
$\theta_{K_3}(y) = (1,\ 2,\ 4,\ 6,\ 9,\ 13,\ 15,\ 11,\ 7,\ 5,\ 3)$,
$\theta_{K_3}(t) = (1,\ 2,\ 4,\ 6,\ 8,\ 9,\ 12,\ 13,\ 16,\ 15,\ 17,\ 11,\ 14,\ 7,\ 10,\ 5,\ 3)$,
handles: [3, 2]

(j) $G = (2, 13, 17)$:

$\theta_{K_1}(x) = \theta_A(x) + (5, 27)(6, 28)(10, 30)(11, 29)(18, 31)(19, 32)(23, 33)(24, 34)$,
$\theta_{K_1}(y) = \theta_A(y)$,
$\theta_{K_1}(t) = \theta_A(t) + (27, 29)(28, 30)(31, 34)(32, 33)$,
handles: as A

$\theta_{K_2}(x) = \theta_B(x) + (5, 28)(6, 29)(10, 31)(11, 30)(18, 32)(19, 33)(23, 34)(24, 35)$,
$\theta_{K_2}(y) = \theta_B(y)$,
$\theta_{K_2}(t) = \theta_B(t) + (28, 30)(29, 31)(32, 35)(33, 34)$,
handles: as B

$\theta_{K_3}(x) = (6,\ 8)(7,\ 9)(10,\ 12)(11,\ 14)(13,\ 16)(15,\ 17)$,
$\theta_{K_3}(y) = (1,\ 2,\ 4,\ 6,\ 9,\ 11,\ 15,\ 13,\ 10,\ 8,\ 7,\ 5,\ 3)$,
$\theta_{K_3}(t) = (1,\ 2,\ 4,\ 6,\ 7,\ 11,\ 14,\ 15,\ 17,\ 13,\ 16,\ 10,\ 12,\ 8,\ 9,\ 5,\ 3)$,
handles: [3, 2]

(k) $G = (2, 13, 19)$:

$\theta_{K_1}(x) =$ as for $(2, 13, 17) + (4, 35)(12, 36)(17, 37)(25, 38)$,
$\theta_{K_1}(y) =$ as for $(2, 13, 17)$,
$\theta_{K_1}(t) =$ as for $(2, 13, 17) + (35, 36)(37, 38)$,
handles: as A

$\theta_{K_2}(x) =$ as for $(2,13,17) + (4,36)(12,37)(17,38)(25,39)$,
$\theta_{K_2}(y) =$ as for $(2,13,17)$,
$\theta_{K_2}(t) =$ as for $(2,13,17) + (36,37)(38,39)$,
handles: as B

$\theta_{K_3}(x) = (3,29)(4,17)(5,53)(6,54)(10,55)(11,56)(12,25)(13,39)(16,57)(18,58)$
$(19,59)(20,60)(22,61)(23,62)(24,63)(26,64)(30,43)(31,65)(32,66)(34,67)$
$(36,68)(37,69)(38,51)(40,41)(42,70)(44,71)(45,72)(46,73)(48,74)(49,75)$
$(50,76)(52,77)$,
$\theta_{K_3}(y) = (1,2,\dots,13)(14,15,\dots,26)\dots(40,41,\dots,52)$,
$\theta_{K_3}(t) = (1,2)(3,13)\dots(7,9)(14,15)(16,26)\dots(20,22)(27,28)(29,39)\dots(33,35)$
$(40,41)(42,52)\dots(46,47)(53,56)(54,55)(57,64)(58,63)(59,62)(60,61)$
$(65,69)(66,68)(70,77)(71,76)(72,75)(73,74)$,
handles: [1, 2]

(l) $G = (2,17,19)$:

$\theta_{K_1}(x) = \theta_A(x) + (5,35)(15,36)(22,37)(32,38)$,
$\theta_{K_1}(y) = \theta_A(y)$,
$\theta_{K_1}(t) = \theta_A(t) + (35,36)(37,38)$,
handles: as A

$\theta_{K_2}(x) = \theta_B(x) + (5,36)(15,37)(22,38)(32,39)$,
$\theta_{K_2}(y) = \theta_B(y)$,
$\theta_{K_2}(t) = \theta_B(t) + (36,37)(38,39)$,
handles: as B

$\theta_{K_3}(x) = (10,\ 12)(11,\ 13)(14,\ 16)(15,\ 18)(17,\ 20)(19,\ 21)$,
$\theta_{K_3}(y) = (1,\ 2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 14,\ 17,\ 19,\ 15,\ 13,\ 11,\ 9,\ 7,\ 5,\ 3)$,
$\theta_{K_3}(t) = (1,\ 2,\ 4,\ 6,\ 8,\ 10,\ 14,\ 16,\ 17,\ 20,\ 19,\ 21,\ 15,\ 18,\ 13,\ 9,\ 7,\ 5,\ 3)$,
handles: [3, 2]

(m) $G = (2,17,23)$:

$\theta_{K_1}(x) =$ as for $(2,17,19) + (6,39)(7,40)(14,41)(13,42)(23,43)(24,44)(31,45)(30,46)$,
$\theta_{K_1}(y) =$ as for $(2,17,19)$,
$\theta_{K_1}(t) =$ as for $(2,17,19) + (39,41)(40,42)(43,45)(44,46)$,
handles: a A

$\theta_{K_2}(x) =$ as for $(2,17,19) + (6,40)(7,41)(14,42)(13,43)(23,44)(24,45)(31,46)(30,47)$,
$\theta_{K_2}(y) =$ as for $(2,17,19)$
$\theta_{K_2}(t) =$ as for $(2,17,19) + (40,42)(41,43)(44,46)(45,47)$,
handles: as B

$\theta_{K_3}(x) = (4,\ 6)(5,\ 7)(10,\ 12)(11,\ 14)(13,\ 16)(15,\ 18)(17,\ 20)(19,\ 22)(21,\ 24)(23,\ 25)$,
$\theta_{K_3}(y) = (1,\ 2,\ 4,\ 6,\ 8,\ 10,\ 13,\ 17,\ 21,\ 23,\ 19,\ 15,\ 11,\ 9,\ 7,\ 5,\ 3)$,
$\theta_{K_3}(t) = (1,\ 2,\ 4,\ 8,\ 10,\ 12,\ 13,\ 16,\ 17,\ 20,\ 21,\ 24,\ 23,\ 25,\ 19,\ 22,\ 15,\ 18,\ 11,\ 14,\ 9,\ 7,\ 3)$,
handles: [3, 2]

(n) $G = (2,19,23)$:

$\theta_{K_1}(x) = \theta_A(x) + (5,39)(17,40)(6,41)(16,42)(24,43)(36,44)(25,45)(35,46)$,
$\theta_{K_1}(y) = \theta_A(y)$,
$\theta_{K_1}(t) = \theta_A(t) + (39,40)(41,42)(43,44)(45,46)$,
handles: as A

$\theta_{K_2}(x) = \theta_B(x) + (5,40)(17,41)(6,42)(16,43)(24,44)(36,45)(25,46)(35,47)$,
$\theta_{K_2}(y) = \theta_B(y)$,
$\theta_{K_2}(t) = \theta_B(t) + (40,41)(42,43)(44,45)(46,47)$,
handles: as B

$\theta_{K_3}(x) = (10,\ 12)(11,\ 13)(14,\ 16)(15,\ 18)(17,\ 20)(19,\ 22)(21,\ 24)(23,\ 25)$,
$\theta_{K_3}(y) = (1,\ 2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 14,\ 17,\ 21,\ 23,\ 19,\ 15,\ 13,\ 11,\ 9,\ 7,\ 5,\ 3)$,
$\theta_{K_3}(t) = (1,\ 2,\ 4,\ 6,\ 8,\ 10,\ 14,\ 16,\ 17,\ 20,\ 21,\ 24,\ 23,\ 25,\ 19,\ 22,\ 15,\ 18,\ 13,\ 9,\ 7,\ 5,\ 3)$,
handles: [3, 2]

(o) $G = (2,23,29)$:

$\theta_{K_1}(x) = \theta_A(x) + (5,47)(21,48)(6,49)(20,50)(7,51)(19,52)(28,53)(44,54)$
$(29,55)(43,56)(30,57)(42,58)$,
$\theta_{K_1}(y) = \theta_A(y)$,
$\theta_{K_1}(t) = \theta_A(t) + (47,48)(49,50)(51,52)(53,54)(55,56)(57,58)$,
handles: as A

$\theta_{K_2}(x) = \theta_B(x) + (5,48)(21,49)(6,50)(20,51)(7,52)(19,53)(28,54)(44,55)$
$(29,56)(43,57)(30,58)(42,59)$,
$\theta_{K_2}(y) = \theta_B(y)$,
$\theta_{K_2}(t) = \theta_B(t) + (48,49)(50,51)(52,53)(54,55)(56,57)(58,59)$,
handles: as B
$\theta_{K_3}(x) = (14, 16)(15, 18)(17, 20)(19, 22)(21, 24)(23, 26)(25, 28)(27, 29)$,
$\theta_{K_3}(y) = (1, 2, 4, 6, 8, 10, 12, 14, 17, 21, 25, 26, 24, 27, 23, 19, 15, 13, 11, 9, 7, 5, 3)$,
$\theta_{K_3}(t) = (1, 2, 4, 6, 8, 10, 12, 14, 16, 17, 20, 21, 27, 29, 23, 24, 25, 28, 26, 19, 22, 15, 18, 13, 11, 9, 7, 5, 3)$,
handles: [3, 2]

3. The remaining triangle groups are below. The technique is as described in Section 5.
   (a) $G = (3,4,5)$:
   $\theta_{K_1}(x) = (4, 5, 6)(7, 9, 10)(11, 13, 15)$,
   $\theta_{K_1}(y) = (1, 2, 4, 3)(5, 6, 8, 7)(9, 11, 14, 12)(10, 13)$,
   handles: [1, 2], [14, 12]
   $\theta_{K_2}(x) = (4, 5, 6)(8, 9, 10)(11, 12, 13)$,
   $\theta_{K_2}(y) = (1, 2, 4, 3)(5, 6, 8, 7)(9, 11, 12, 10)(13, 14, 16, 15)$,
   handles: [1, 2], [14, 16]
   $\theta_{K_3}(x) = (4, 5, 6)$,
   $\theta_{K_3}(y) = (1, 2, 4, 3)(5, 6)$,
   handles: [1, 2]
   (b) $G = (3,5,7)$:
   $\theta_{K_1}(x) = (3, 5, 6)(4, 8, 9)(7, 12, 13)$,
   $\theta_{K_1}(y) = (1, 2, 4, 7, 3)(6, 10, 9, 14, 11)$,
   handles: [1, 2], [14, 11]
   $\theta_{K_2}(x) = (5, 6, 7)(8, 9, 11)$,
   $\theta_{K_2}(y) = (1, 2, 4, 5, 3)(6, 8, 10, 9, 7)(11, 12, 14, 15, 13)$,
   handles: [1, 2], [12, 14]
   $\theta_{K_3}(x) = (3, 5, 6)(4, 7, 8)(9, 10, 11)$,
   $\theta_{K_3}(y) = (1, 2, 4, 5, 3)(7, 9, 11, 10, 8)$,
   handles: [1, 2]
   (c) $G = (3,5,11)$:
   $\theta_{K_1}(x) = (5, 6, 7)$,
   $\theta_{K_1}(y) = (1, 2, 4, 5, 3)(7, 8, 10, 11, 9)$,
   handles: [1, 2], [8, 10]
   $\theta_{K_2}(x) = (5, 6, 7)(9, 11, 12)$,
   $\theta_{K_2}(y) = (1, 2, 4, 5, 3)(7, 8, 10, 11, 9)$,
   handles: [1, 2], [8, 10]
   $\theta_{K_3}(x) = (4, 6, 7)(5, 8, 9)$,
   $\theta_{K_3}(y) = (1, 2, 4, 5, 3)(7, 10, 11, 8, 9)$,
   handles: [1, 2]
   (d) $G = (3,7,11)$:
   $\theta_{K_1}(x) = (6, 8, 9)(7, 10, 11)$,
   $\theta_{K_1}(y) = (1, 2, 4, 6, 7, 5, 3)$,
   handles: [1, 2], [5, 3]
   $\theta_{K_2}(x) = (7, 8, 9)(12, 13, 14)$,
   $\theta_{K_2}(y) = (1, 2, 4, 6, 7, 5, 3)(8, 10, 12, 14, 13, 11, 9)$,
   handles: [1, 2], [4, 6]
   $\theta_{K_3}(x) = (3, 4, 5)(6, 8, 9)(7, 10, 11)$,
   $\theta_{K_3}(y) = (1, 2, 4, 6, 7, 5, 3)$,
   handles: [1, 2]
   (e) $G = (3,11,13)$:
   $\theta_{K_1}(x) = (11, 12, 13)$,
   $\theta_{K_1}(y) = (1, 2, 4, 6, 8, 10, 11, 9, 7, 5, 3)$,
   handles: [1, 2], [4, 6]
   $\theta_{K_2}(x) = (9, 11, 12)(10, 13, 14)$,
   $\theta_{K_2}(y) = (1, 2, 4, 6, 8, 10, 11, 9, 7, 5, 3)$,
   handles: [1, 2], [4, 6]
   $\theta_{K_3}(x) = (8, 10, 11)(9, 12, 13)$,
   $\theta_{K_3}(y) = (1, 2, 4, 6, 8, 10, 11, 9, 7, 5, 3)$,
   handles: [1, 2]
   (f) $G = (3,13,17)$:

$\theta_{K_1}(x) = (12,\ 14,\ 15)(13,\ 16,\ 17)$, $\quad \theta_{K_2}(x) = (10,\ 12,\ 13)(11,\ 14,\ 15)(16,\ 17,\ 18)$,
$\theta_{K_1}(y) = (1,\ 2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 13,\ 11,\ 9,\ 7,\ 5,\ 3)$ $\quad \theta_{K_2}(y) = (1,\ 2,\ 4,\ 6,\ 8,\ 10,\ 13,\ 16,\ 11,\ 9,\ 7,\ 5,\ 3)$,
handles: [11, 9], [1, 2] $\quad$ handles: [1, 2], [4, 6]
$\theta_{K_3}(x) = (9,\ 10,\ 11)(12,\ 14,\ 15)(13,\ 16,\ 17)$,
$\theta_{K_3}(y) = (1,\ 2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 13,\ 11,\ 9,\ 7,\ 5,\ 3)$,
handles: [1, 2]

(g) $G = (3, 13, 19)$:
$\theta_{K_1}(x) = (11,\ 13,\ 14)(12,\ 16,\ 17)(15,\ 18,\ 19)$, $\quad \theta_{K_2}(x) = (8,\ 10,\ 11)(9,\ 12,\ 13)(15,\ 17,\ 18)(16,\ 19,\ 20)$,
$\theta_{K_1}(y) = (1,\ 2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 15,\ 11,\ 9,\ 7,\ 5,\ 3)$, $\quad \theta_{K_2}(y) = (1,\ 2,\ 4,\ 6,\ 8,\ 11,\ 15,\ 16,\ 14,\ 9,\ 7,\ 5,\ 3)$
handles: [1, 2], [4, 6] $\quad$ handles: [1, 2], [4, 6]
$\theta_{K_3}(x) = (8,\ 10,\ 11)(9,\ 12,\ 13)(14,\ 16,\ 17)(15,\ 18,\ 19)$,
$\theta_{K_3}(y) = (1,\ 2,\ 4,\ 6,\ 8,\ 10,\ 14,\ 15,\ 11,\ 9,\ 7,\ 5,\ 3)$,
handles: [1, 2]

(h) $G = (5, 7, 11)$:
$\theta_{K_1}(x) = (7,\ 8,\ 10,\ 11,\ 9)$, $\quad \theta_{K_2}(x) = (6,\ 8,\ 10,\ 9,\ 7)$,
$\theta_{K_1}(y) = (1,\ 2,\ 4,\ 6,\ 7,\ 5,\ 3)$, $\quad \theta_{K_2}(y) = (1,\ 2,\ 4,\ 6,\ 7,\ 5,\ 3)(8,\ 11,\ 13,\ 14,\ 12,\ 9,\ 10)$,
handles: [1, 2], [4, 6] $\quad$ handles: [11, 13], [1, 2]
$\theta_{K_3}(x) = (3,\ 5,\ 9,\ 10,\ 6)(4,\ 7,\ 11,\ 12,\ 8)$,
$\theta_{K_3}(y) = (1,\ 2,\ 4,\ 6,\ 7,\ 5,\ 3)$,
handles: [1, 2]

(i) $G = (5, 7, 13)$:
$\theta_{K_1}(x) = (7,\ 8,\ 10,\ 11,\ 9)$, $\quad \theta_{K_2}(x) = (7,\ 8,\ 10,\ 11,\ 9)$,
$\theta_{K_1}(y) = (1,\ 2,\ 4,\ 6,\ 7,\ 5,\ 3)(8,\ 10,\ 13,\ 14,\ 12,\ 9,\ 11)$, $\quad \theta_{K_2}(y) = (1,\ 2,\ 4,\ 6,\ 7,\ 5,\ 3)(9,\ 11,\ 10,\ 13,\ 15,\ 14,\ 12)$,
handles: [1, 2], [13, 14] $\quad$ handles: [1, 2], [13, 15]
$\theta_{K_3}(x) = (4,\ 6,\ 8,\ 5,\ 7)(9,\ 10,\ 12,\ 13,\ 11)$,
$\theta_{K_3}(y) = (1,\ 2,\ 4,\ 6,\ 9,\ 5,\ 3)$,
handles: [1, 2]

(j) $G =$(7,11,13):
$\theta_{K_1}(x) =$ case 2, section 5, $\quad \theta_{K_2}(x) =$ case 2, section 5,
$\theta_{K_1}(y) =$ case 2, section 5, $\quad \theta_{K_2}(y) =$ case 2, section 5,
$\theta_{K_3}(x) = (7,\ 9,\ 12,\ 13,\ 11,\ 8,\ 10)$,
$\theta_{K_3}(y) = (1,\ 2,\ 4,\ 6,\ 8,\ 9,\ 12,\ 10,\ 7,\ 5,\ 3)$,
handles: [1, 2]

(k) $G = (11, 13, 17)$:
$\theta_{K_1}(x) = (7,\ 8,\ 10,\ 12,\ 14,\ 16,\ 17,\ 15,\ 13,\ 11,\ 9)$, $\quad \theta_{K_2}(x) = (8,\ 10,\ 12,\ 15,\ 17,\ 18,\ 16,\ 14,\ 13,\ 11,\ 9)$,
$\theta_{K_1}(y) = (1,\ 2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 13,\ 11,\ 9,\ 7,\ 5,\ 3)$, $\quad \theta_{K_2}(y) = (1,\ 2,\ 4,\ 6,\ 8,\ 10,\ 13,\ 14,\ 11,\ 9,\ 7,\ 5,\ 3)$,
handles: [1, 2], [4, 6] $\quad$ handles: [1, 2], [4, 6]
$\theta_{K_3}(x) =$ case 2, section 5,
$\theta_{K_3}(y) =$ case 2, section 5,

4. Diagrams for the remaining Dyck groups with four periods are below. All these are as in Section 4, except that in finding a prime cycle for the purposes of Theorem 1, one considers the images of yz rather than xy.

(a) $G = (2, 2, 2, 3)$:
$\theta_{K_1}(x) = (5,\ 7)(6,\ 8)$, $\quad \theta_{K_2}(x) = (3,\ 4)(6,\ 7)$,
$\theta_{K_1}(y) = (1,\ 2)(3,\ 5)(4,\ 6)(7,\ 8)$, $\quad \theta_{K_2}(y) = (2,\ 3)(4,\ 5)(6,\ 7)(8,\ 9)$,
$\theta_{K_1}(z) = (1,\ 3)(2,\ 4)(5,\ 8)(6,\ 7)$, $\quad \theta_{K_2}(z) = (1,\ 2)(3,\ 4)(5,\ 6)(7,\ 8)$,
handles: [1, 4], [2, 3] $\quad$ handles: [1, 2], [8, 9]

$\theta_{K_3}(x) = (3,\ 4)(5,\ 6)(7,\ 8)(9,\ 10)$,
$\theta_{K_3}(y) = (2,\ 3)(4,\ 5)(6,\ 7)(9,\ 10)$,
$\theta_{K_3}(z) = (1,\ 2)(3,\ 4)(6,\ 7)(8,\ 9)$,
handles: [1, 2]

(b) $G = (2, 2, 2, 4)$:

$\theta_{K_1}(x) = (5,\ 6)(7,\ 8)$,
$\theta_{K_1}(y) = (1,\ 2)(3,\ 5)(4,\ 6)(7,\ 8)$,
$\theta_{K_1}(z) = (1,\ 3)(2,\ 4)(5,\ 7)(6,\ 8)$,
handles: [1, 4], [2, 3]

$\theta_{K_2}(x) = (2,\ 3)(6,\ 9)$,
$\theta_{K_2}(y) = (2,\ 4)(3,\ 5)(6,\ 9)(7,\ 8)$,
$\theta_{K_2}(z) = (1,\ 2)(3,\ 6)(4,\ 7)(5,\ 8)$,
handles: [4, 1], [7, 5]

$\theta_{K_3}(x) = (4,\ 5)(7,\ 8)$,
$\theta_{K_3}(y) = (2,\ 3)(4,\ 5)(6,\ 7)(8,\ 9)$,
$\theta_{K_3}(z) = (1,\ 2)(3,\ 4)(5,\ 6)(8,\ 9)$,
handles: [1, 2]

(c) $G = (2, 2, 2, 5)$:

$\theta_{K_1}(x) = (5,\ 6)(8,\ 9)$,
$\theta_{K_1}(y) = (2,\ 3)(4,\ 5)(6,\ 7)(8,\ 10)$,
$\theta_{K_1}(z) = (1,\ 2)(3,\ 4)(5,\ 6)(7,\ 8)$,
handles: [1, 2], [10, 7]

$\theta_{K_2}(x) = (4,\ 5)(7,\ 8)(9,\ 10)(11,\ 12)$,
$\theta_{K_2}(y) = (2,\ 3)(4,\ 6)(5,\ 7)(8,\ 9)(10,\ 11)(12,\ 13)$,
$\theta_{K_2}(z) = (1,\ 2)(3,\ 4)(6,\ 7)(8,\ 10)(9,\ 11)(12,\ 13)$,
handles: [1, 2], [6, 3]

$\theta_{K_3}(x) = (3,\ 4)(5,\ 6)(7,\ 8)(9,\ 10)$,
$\theta_{K_3}(y) = (2,\ 3)(5,\ 6)(7,\ 9)(8,\ 10)$,
$\theta_{K_3}(z) = (1,\ 2)(4,\ 5)(6,\ 7)(8,\ 10)$,
handles: [1, 2]

(d) $G = (2, 2, 3, 3)$:

$\theta_{K_1}(x) = (2,\ 3)(7,\ 8)$,
$\theta_{K_1}(y) = (1,\ 2)(3,\ 6)(4,\ 7)(5,\ 8)$,
$\theta_{K_1}(z) = (1,\ 3,\ 4)(2,\ 5,\ 6)$,
handles: [1, 5], [6, 4]

$\theta_{K_2}(x) = (3,\ 4)(6,\ 7)$,
$\theta_{K_2}(y) = (3,\ 4)(6,\ 7)$,
$\theta_{K_2}(z) = (1,\ 2,\ 3)(4,\ 5,\ 6)(7,\ 8,\ 9)$,
handles: [1, 2], [8, 9]

$\theta_{K_3}(x) = (3,\ 5)(6,\ 7)$,
$\theta_{K_3}(y) = (1,\ 2)(3,\ 6)(4,\ 5)(7,\ 8)$,
$\theta_{K_3}(z) = (2,\ 3,\ 4)(6,\ 7,\ 8)$,
handles: [2, 1]

(e) $G = (2, 2, 3, 5)$:

$\theta_{K_1}(x) = (4,\ 5)(7,\ 8)$,
$\theta_{K_1}(y) = (1,\ 2)(3,\ 5)(4,\ 6)(7,\ 8)$,
$\theta_{K_1}(z) = (2,\ 3,\ 4)(5,\ 7,\ 6)$,
handles: [1, 3], [6, 2]

$\theta_{K_2}(x) = (3,\ 4)(6,\ 7)$,
$\theta_{K_2}(y) = (2,\ 4)(3,\ 5)(6,\ 8)(7,\ 9)$,
$\theta_{K_2}(z) = (1,\ 2,\ 3)(4,\ 6,\ 5)(7,\ 9,\ 8)$,
handles: [1, 2], [8, 5]

$\theta_{K_3}(x) = (4,\ 5)(6,\ 7)$,
$\theta_{K_3}(y) = (1,\ 2)(3,\ 5)(4,\ 6)(7,\ 8)$,
$\theta_{K_3}(z) = (2,\ 3,\ 4)(6,\ 7,\ 8)$,
handles: [1, 3]

(f) $G = (2, 2, 4, 4)$:

$\theta_{K_1}(x) = (5,\ 7)(6,\ 8)$,
$\theta_{K_1}(y) = (2,\ 3)(4,\ 5)$,
$\theta_{K_1}(z) = (1,\ 2)(3,\ 4,\ 6,\ 5)$,
handles: [1, 2], [4, 3]

$\theta_{K_2}(x) = (4,\ 5)(7,\ 8)$,
$\theta_{K_2}(y) = (2,\ 3)(4,\ 5)(6,\ 7)(8,\ 9)$,
$\theta_{K_2}(z) = (1,\ 2)(3,\ 4)(5,\ 6)(7,\ 9)$,
handles: [1, 2], [6, 9]

$\theta_{K_3}(x) = (4,\ 5)(7,\ 8)$,
$\theta_{K_3}(y) = (2,\ 3)(6,\ 7)$,
$\theta_{K_3}(z) = (1,\ 2)(3,\ 4,\ 6,\ 5)$,
handles: [1, 2]

(g) $G = (2, 3, 3, 3)$:

$\theta_{K_1}(x) = (3,\ 4)(7,\ 8)$,
$\theta_{K_1}(y) = (3,\ 5,\ 4)(6,\ 7,\ 8)$,
$\theta_{K_1}(z) = (1,\ 2,\ 3)(5,\ 6,\ 7)$,
handles: [1, 2], [6, 5]

$\theta_{K_2}(x) = (3,\ 4)(8,\ 9)$,
$\theta_{K_2}(y) = (3,\ 5,\ 4)(6,\ 7,\ 8)$,
$\theta_{K_2}(z) = (1,\ 2,\ 3)(5,\ 6,\ 7)$,
handles: [1, 2], [6, 5]

$\theta_{K_3}(x) = (3,\ 4)(6,\ 8)$,
$\theta_{K_3}(y) = (3,\ 5,\ 4)(6,\ 8,\ 7)$,
$\theta_{K_3}(z) = (1,\ 2,\ 3)(5,\ 6,\ 7)$,
handles: [1, 2]

(h) $G = (3, 3, 3, 3)$:

$\theta_{K_1}(x) = (3,\ 4,\ 5)$,
$\theta_{K_1}(y) = (3,\ 5,\ 4)$,
$\theta_{K_1}(z) = (1,\ 2,\ 3)(4,\ 6,\ 7)(5,\ 8,\ 9)$,
handles: [1, 2], [6, 7], [8, 9]

$\theta_{K_2}(x) = (3,\ 4,\ 5)$,
$\theta_{K_2}(y) = (3,\ 5,\ 4)(8,\ 10,\ 9)$,
$\theta_{K_2}(z) = (1,\ 2,\ 3)(4,\ 6,\ 7)(5,\ 8,\ 9)$,
handles: [1, 2], [6, 7], [8, 9]

$\theta_{K_3}(x) = (3,\ 4,\ 5)$,
$\theta_{K_3}(y) = (3,\ 5,\ 4)(6,\ 8,\ 7)$,
$\theta_{K_3}(z) = (1,\ 2,\ 3)(5,\ 6,\ 7)$,
handles: [1, 2]

5. Last but not least, the group $(2,2,2,2,2) = \langle x, y, z, w\rangle$. The details are as in Section 4, with the image of yz providing the witness prime cycle. No doubt the reader can fill in the details.

$G = (2,2,2,2,2)$:

$\theta_{K_1}(x) =$ identity,	$\theta_{K_2}(x) = (5,6)(7,8)$,	$\theta_{K_3}(x) =$ identity,
$\theta_{K_1}(y) = (2,\ 3)(4,\ 5)$,	$\theta_{K_2}(y) = (2,3)(4,5)$,	$\theta_{K_3}(y) = (2,\ 3)(4,\ 5)$,
$\theta_{K_1}(z) = (2,\ 3)(4,\ 5)$,	$\theta_{K_2}(z) = (2,3)(4,5)$,	$\theta_{K_3}(z) = (1,\ 2)(3,\ 4)$,
$\theta_{K_1}(w) = (1,\ 2)(3,\ 4)$,	$\theta_{K_2}(w) = (1,2)(3,4)(5,7)(6,8)$,	$\theta_{K_3}(w) = (2,\ 3)(4,\ 5)$,
handles: [1, 2], [3, 4]	handles: [1, 2], [3,4]	handles: [1, 3]

References

[1] D E Cohen. *Combinatorial Group Theory: a topological approach.* London Mathematical Society Student Texts 14, Cambridge University Press, 1989.

[2] P B Cohen, C Itzykson and J Wolfart. Fuchsian triangle groups and Grothendieck Dessins. *Comm. Math. Physics*, 163:605–627, 1994.

[3] M D E Conder. Generators for alternating and symmetric groups. *J. London Math. Soc.*, 22:75–86, 1980.

[4] M D E Conder. More on generators for alternating and symmetric groups. *Quart. J. Math. Oxford*, 32(2):137–163, 1981.

[5] H S M Coxeter and W O J Moser. *Generators and Relations for Discrete Groups.* Fourth Edition, Springer-Verlag, Berlin, 1980.

[6] I M S Dey and J Wiegold. Generators for alternating and symmetric groups. *J. Austral. Math. Soc.*, 12:63-68, 1971.

[7] B Everitt. Permutation Representations of the $(2,4,r)$ Triangle Groups. *Bull. Austral. Math. Soc.*, 49:499–511, 1994.

[8] B Everitt. Alternating quotients of the $(3,q,r)$ triangle groups. *Comm. Algebra*, 25(6):1817–1832, 1997.

[9] J Hempel. *3-Manifolds.* Annals of Mathematics Studies, Princeton, 1976.

[10] P Hilton and S Wylie. *Homology Theory.* Cambridge University Press, 1962.

[11] G Jones and D Singerman. Belyi functions, hypermaps and Galois groups. *Bull. London Math. Soc.*, 28:561-590, 1996.

[12] C Jordan. Sur la limite de transitivité des groups non alternés. *Bull. Soc. Math. France*, 1:40–71, 1873.

[13] R C Lyndon and P E Schupp. *Combinatorial Group Theory.* A Series of Modern Surveys in Mathematics, Springer-Verlag, 1977.

[14] J M Montesinos. *Classical Tessellations and Three–Manifolds.* Universitext, Springer-Verlag, 1987.

[15] G A Miller. On the groups generated by two operators. *Bull. Amer. Math. Soc.*, 7:424-426, 1901.

[16] Q Mushtaq and H Servatius. Permutation Representations of the Symmetry Groups of Regular Hyperbolic Tessellations. *J. London Math. Soc*, 48(1):77-86, 1993.

[17] Q Mushtaq and Gian-Carlo Rota. Alternating Groups as quotients of two generator groups. *Adv. in Math.*, 96:113–121, 1992.

[18] D Singerman. Subgroups of Fuchsian groups and finite permutation groups. *Bull. London Math. Soc.*, 2 (1970) 319–323.

[19] J Stillwell. *Classical Topology and Combinatorial Group Theory.* Graduate Texts in Mathematics, Springer-Verlag, 1980.

[20] H Wielandt. *Finite Permutation Groups.* Academic Press, 1964.

[21] A T White. *Graphs, Groups and Surfaces.* North Holland, 1973.

[22] H Zieschang, E Vogt and H-D Coldeway. *Surfaces and Planar Discontinuous Groups.* Springer Lecture Notes number 835, 1980.

Mathematics Institute, University of Warwick, Coventry CV4 7AL, England

Current address: Department of Mathematical Sciences, University of Durham, Durham DH1 3LE, England

E-mail address: `brent.everitt@durham.ac.uk`