On a Cone-Manifold with the Euclidean Structure on the Whitehead Link *

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Abstract

A 3-dimensional cone-manifold $S^3(W, 2\pi/\alpha)$ with the Euclidean structure on the Whitehead link is studied in this article. A method is given to construct the canonical fundamental set $FS(\alpha)$ of the cone-manifold $S^3(W, 2\pi/\alpha)$ in the 3-dimensional Euclidean space $\mathbb{E}^3$. The desired fundamental set $FS(\alpha)$ is obtained as a 32-faced polyhedron in $\mathbb{E}^3$. This fundamental set is the simplest one in some sense.

1 Introduction

The investigation of cone-manifolds was started in [1, 2]. Then vital exploration of these manifolds was continued in [3, 4, 5, 6, 7, 8]. One of the fundamental results of this theory is the Rigidity Theorem, which is recently proved by S. Kojima.

This work was inspired by papers [7] and [9].

In the paper [7] H. Helling, A.C. Kim, J.L. Mennicke for the first time strictly proved that the orbifold $S^3(W, 2\pi/n)$ is hiperbolic for all $n = 3, 4, 5, \ldots$.

Then, in the paper [9] A. Mednykh and A. Rasskazov constructed the canonical fundamental set for 2-bridge link orbifolds.

In this paper some 3-dimensional cone-manifold $S^3(W, 2\pi/\alpha)$, where $\alpha = 2.506179\ldots$, with the Euclidean structure on the Whitehead link is studied. This 3-manifold belongs to an infinite family of cone-manifolds $S^3(W, \beta)$ with the 3-dimensional sphere $\mathbb{S}^3$ as the underlying space, the Whitehead link W as the singular set, and the angle β, $0 < \beta < 2\pi$. Using results of [3, 5] it is possible to show that $S^3(W, \beta)$ is hyperbolic if $0 < \beta < 2\pi/\alpha$ and spherical if $\beta > 2\pi/\alpha$.

Following K.N.Jones [8] we say that a *3-dimensional Euclidean cone-manifold* $X^3(\Sigma, \beta)$ is a metric space obtained as the quotient space of a disjoint union of a collection of geodesic 3-simplices in $\mathbb{E}^3$ by an isometric pairing of codimension-one faces in such a combinatorial fashion that the underlying topological spase is a manifold. *Hyperbolic* and *spherical cone-manifolds* are defined similarly.

Such a space possesses a Riemannian metric of constant sectional curvature on the union of the top-dimensional cells and the codimension-1 cells. On each codimension-2 cell, the structure is completely described by an angle β, which is the sum of the dihedral angles around all of the codimension-2 simplicial faces which are identified to give the cell. The *singular set* of a cone-manifold is the closure of all the codimension-2 cells for which this angle β, called the *cone angle* , is not 2π (the Riemannian metric may be extended smoothly over all cells whose

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angle is 2π). In this paper, we consider the case where the singular set is a link with a constant cone angle $0 < \beta < 2\pi$ on each component.

We should note here that there is a strong connection between cone-manifolds and orbifolds (see [2]), namely that *orbifolds* are cone-manifols with all cone angles of the form $2\pi/k$ for some integer k (k is the order of the isotropy of a non-vertex cone point).

Denote by $S^3(W, 2\pi/\alpha)$ the cone-manifold $X^3(\Sigma, \beta)$, whose underlying space X^3 is $\mathbb{S}^3$, the singular set Σ is homeomorphic to the Whitehead link W, and the cone angle $\beta = 2\pi/\alpha$.

The aim of this paper is to prove the following theorem.

Theorem 2.1. *Let $0 < \beta < 2\pi$ be the real root of the equation:*

$$9 + 8\cos\beta - 13\cos 2\beta + 4\cos 3\beta = 0$$

and $\alpha = 2\pi/\beta$.

Then there exists a 32-faced polyhedron in $\mathbb{E}^3$ which is the canonical fundamental set of the cone-manifold $S^3(W, 2\pi/\alpha)$. In particular, $S^3(W, 2\pi/\alpha)$ admits the Euclidean structure.

This paper is organized as follows.

In Section 1, the standard Wirtinger procedure [10] yields a presentation for the fundamental group $\pi_1(\mathbb{S}^3 \setminus W)$ of the complement of the Whitehead link W.

Let Γ be the presentation for the fundamental group $\pi_1(\mathbb{S}^3 \setminus W)$ in the group $SU(2)$, which satisfies the following conditions on generators of Γ (denoted by S and T):

$$r(S,T) = T^{-1}S^{-1}TS^{-1}T^{-1}STS^{-1}TST^{-1}STS^{-1}T^{-1}S = I. \tag{1}$$

$$\mathrm{tr}(S) = \mathrm{tr}(T) = 2\cos\frac{\pi}{\alpha}, \tag{2}$$

where tr(S) denotes the trace of the matrix S.

Using (1), we obtain the following equation:

$$9 + 8\cos\beta - 13\cos 2\beta + 4\cos 3\beta \quad = \quad 0, \tag{3}$$

where $\beta = 2\pi/\alpha$. Studing this equation we obtain that it has exactly one real root $0 < \beta < 2\pi$ and the number $\alpha = 2\pi/\beta$ is irrational. Since α is irrational, Γ is not discrete. However we can use Γ to construct the canonical fundamental set $FS(\alpha)$ of the cone-manifold $S^3(W, 2\pi/\alpha)$.

In Section 2 the canonical fundamental set $FS(\alpha)$ of the cone-manifold $S^3(W, 2\pi/\alpha)$ in the 3-dimensional Euclidean space $\mathbb{E}^3$ is constructed.

For this purpose we find images of S and T in the group $SO(3)$, using the canonical epimorphism $SU(2) \to SO(3)$.

The method proposed allows to derive explicit formulas for coordinates of vertices of the fundamental set $FS(\alpha)$ in $\mathbb{E}^3$.

Note, that $FS(\alpha)$ is the simplest possible fundamental set for $S^3(W, 2\pi/\alpha)$ in the sense that it has just two pairs of mutually congruent curvelinear faces, which are pairwise identified by rotations of the Euclidean space $\mathbb{E}^3$.

Similar construction for the figure-eight knot was realized in [9]. This method can be used for investigation of Euclidean structures on arbitrary 2-bridge knotes and links.

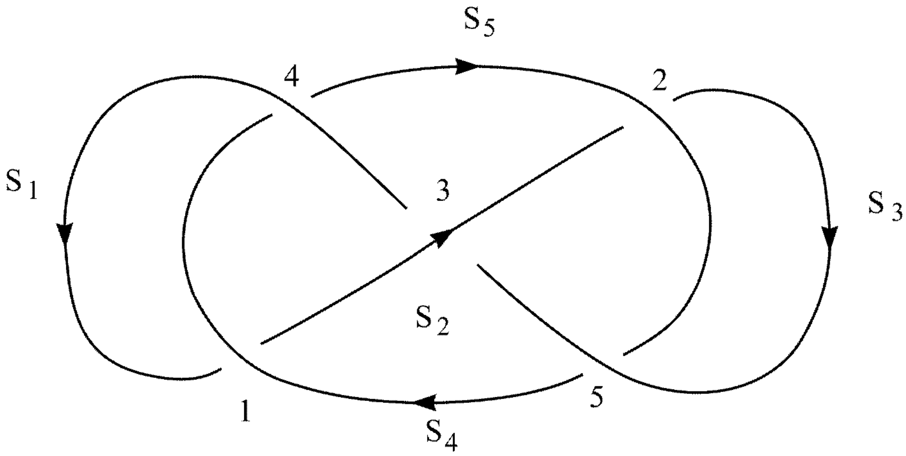

Figure 1: The Whitehead link W

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2 Geometric cone-manifolds, whose singular set is the Whitehead link

Consider a link W of 2 components called the *Whitehead link* (Fig. 1). The standard Wirtinger procedure [10] yields the presentation for the fundamental group $\pi_1(\mathbb{S}^3 \setminus W)$ for the complement of the Whitehead link:

$$\begin{aligned} S_2^{-1}S_4^{-1}S_1S_4 &= 1, \\ S_3^{-1}S_5S_2S_5^{-1} &= 1, \\ S_1^{-1}S_2S_3S_2^{-1} &= 1, \\ S_5^{-1}S_1^{-1}S_4S_1 &= 1, \\ S_4^{-1}S_3S_5S_3^{-1} &= 1. \end{aligned}$$

Here S_1, S_2, S_3, S_4, S_5 are generators of the group $\pi_1(\mathbb{S}^3 \setminus W)$ which correspond to overpasses S_1, S_2, S_3, S_4, S_5 on the Whitehead link W (see Fig. 1).

If we eliminate S_2, S_3, S_5, then we obtain the presentation

$$\pi_1(\mathbb{S}^3 \setminus W) = \langle S_1, S_4; S_4^{-1}S_1^{-1}S_4S_1^{-1}S_4^{-1}S_1S_4S_1^{-1}S_4S_1S_4^{-1}S_1S_4S_1^{-1}S_4^{-1}S_1 = 1\rangle.$$

For easier writing, we rename the generators S_1, S_4 to s, t:

$$\pi_1(\mathbb{S}^3 \setminus W) = \langle s, t; r(s,t) = t^{-1}s^{-1}ts^{-1}t^{-1}sts^{-1}tst^{-1}sts^{-1}t^{-1}s = 1\rangle.$$

Let $\chi : \pi_1(\mathbb{S}^3 \setminus W) \to PSL(2,\mathbb{C})$ be a presentation and $\pi : SL(2,\mathbb{C}) \to PSL(2,\mathbb{C})$ be the canonical projection.

Let $S, T \in SL(2,\mathbb{C})$ be such that $\chi(s) = \pi(S)$, $\chi(t) = \pi(T)$. Since the sums of the exponents of S and T in $r(S,T)$ are equal to zero, it follows that $r(S,T) = I$. Hence there is a lifting $\tilde{\chi} : \pi_1(\mathbb{S}^3 \setminus W) \to SL(2,\mathbb{C})$ (i.e. a homomorphism with the property $\chi = \pi \circ \tilde{\chi}$).

The group $\widetilde{\Gamma} = \widetilde{\chi}\left(\pi_1(\mathbb{S}^3 \setminus W)\right)$ with a fixed sistem of generators, namely $\widetilde{\Gamma} = \langle S, T\rangle$, where $S, T \in SL(2,\mathbb{C})$ will be called the *image* of the fundamental group $\pi_1(\mathbb{S}^3 \setminus W)$ under the lifting $\widetilde{\chi}$ in the group $SL(2,\mathbb{C})$.

Recall that both components of the Whitehead link have the same cone angle β. In this case the generators S and T of $\widetilde{\Gamma}$ can be represented as two elliptic elements in the group $SL(2,\mathbb{C})$, such that:

$$S = \begin{pmatrix} \cos\dfrac{\beta}{2} & F\sin\dfrac{\beta}{2} \\ -\dfrac{1}{F}\sin\dfrac{\beta}{2} & \cos\dfrac{\beta}{2} \end{pmatrix}, \qquad T = \begin{pmatrix} \cos\dfrac{\beta}{2} & \dfrac{1}{F}\sin\dfrac{\beta}{2} \\ -F\sin\dfrac{\beta}{2} & \cos\dfrac{\beta}{2} \end{pmatrix},$$

where $F = ie^{\rho/2}$, and $\rho = r + i\varphi$ is the *complex distance* in the sense of W. Fenchel [11, ch. V] from the oriented axis $\mathbf{S} = [-e^{\rho/2}, e^{\rho/2}]$ to the oriented axis $\mathbf{T} = [-e^{-\rho/2}, e^{-\rho/2}]$ of the elements S and T in the 3-dimensional hyperbolic space $\mathbb{H}^3$.

Recall that r is the distance from $\mathbf{S}$ to $\mathbf{T}$, φ is the angle from $\mathbf{S}$ to $\mathbf{T}$.

As $r \to 0$ the hyperbolic structure on the cone-manifold $S^3(W, \beta)$ degenerates into the Euclidean one for all $0 < \beta < 2\pi$. As $r = 0$ the axes of the transformations S and T have a common fixed point (see [3, 5]).

Let $r = 0$, $\beta = 2\pi/\alpha$, then

$$S = \begin{pmatrix} \cos\dfrac{\pi}{\alpha} & ie^{i\varphi/2}\sin\dfrac{\pi}{\alpha} \\ ie^{-i\varphi/2}\sin\dfrac{\pi}{\alpha} & \cos\dfrac{\pi}{\alpha} \end{pmatrix}, \qquad T = \begin{pmatrix} \cos\dfrac{\pi}{\alpha} & -ie^{-i\varphi/2}\sin\dfrac{\pi}{\alpha} \\ -ie^{i\varphi/2}\sin\dfrac{\pi}{\alpha} & \cos\dfrac{\pi}{\alpha} \end{pmatrix},$$

and, thus $S, T \in SU(2)$.

For convinience, consider the following commutative diagram:

$$\begin{tikzcd} & SU(2) \subset SL(2,\mathbb{C}) \arrow[d, "\pi"] \\ \pi_1(\mathbb{S}^3 \setminus W) \arrow[ur, "\widetilde{\chi}"] \arrow[r, "\chi"] & PSL(2,\mathbb{C}) \end{tikzcd}$$

Conditions (1) and (2) define the presentation for the group $\widetilde{\Gamma}$ in the group $SU(2)$ in a unique manner, up to conjugation in the group $SU(2)$, and this presentation has the following form:

$$\Gamma \quad = \quad \langle S, T; r(S,T) = I, \mathrm{tr}(S) = \mathrm{tr}(T) = 2\cos\frac{\pi}{\alpha}\rangle. \tag{4}$$

Let us find values of parameters α and φ.

If we substitute matrices S and T in (1) and take the element of the left-hand side (1) then we obtain the following equation:

$$P(F,\beta) \quad = \quad P_1(F,\beta)(-1+F^4)\sin^4(\beta/2)/8F^8 = 0, \tag{5}$$

where

$$
\begin{aligned}
P_1(F,\beta) \;&=\; (3+\cos 2\beta - 4\cos\beta) + F^2(2-2\cos 2\beta) + \\
&\quad F^4(13+4\cos\beta-\cos 2\beta) + F^6(-4+4\cos 2\beta) + \\
&\quad F^8(13+4\cos\beta-\cos 2\beta) + F^{10}(2-2\cos 2\beta) + \\
&\quad F^{12}(3+\cos 2\beta - 4\cos\beta).
\end{aligned}
$$

If $(-1+F^4) = 0$ then $S = T$. This case does not correspond to geometrical situation. Similarly, $\sin(\beta/2) \neq 0$. Hence, one can suppose that the parameter β satisfies the equation:

$$P_1(F,\beta) \;=\; 0. \tag{6}$$

$P_1(F,\beta)$ is the polynomial in F with real coefficients. Let us find a root F of $P_1(F,\beta)$ of multiplicity 2. In this case, this root is also the root of the derivative:

$$
\begin{aligned}
\frac{\partial}{\partial F}P_1(F,\beta) \;&=\; 4F(1-\cos 2\beta + F^2(13+4\cos\beta-\cos 2\beta) + \\
&\quad F^4(-6+6\cos 2\beta) + F^6(26-2\cos 2\beta) + F^8(5-5\cos 2\beta) + \\
&\quad F^{10}(9-12\cos\beta+3\cos 2\beta)) = 0.
\end{aligned} \tag{7}
$$

Since F is not equal to zero, we set:

$$P_2(F,\beta) \;=\; \frac{1}{4F}\frac{\partial}{\partial F}P_1(F,\beta). \tag{8}$$

So that F is the root of $P_1(F,\beta)$ and $P_2(F,\beta)$, we have:

$$P_1(F,\beta) \;=\; P_2(F,\beta) \;=\; 0. \tag{9}$$

Eliminate F from (9), we obtain the equation (3) for the parameter β.

Now, from (3) we get the following equation:

$$11 - 2\cos\beta - 13\cos^2\beta + 8\cos^3\beta \;=\; 0. \tag{10}$$

By Cardanus formulas we obtain that the real root of (10) has the following form:

$$\cos\beta \;=\; \frac{1}{24}\left(\sqrt[3]{-6371+48\sqrt{13182}} + \sqrt[3]{-6371-48\sqrt{13182}} + 13\right). \tag{11}$$

From (11) we have $\beta = 2.507077\ldots$, hence $\alpha = 2\pi/\beta = 2.506179\ldots$.

If we substitute the value of α in (6) then we find that $\varphi = 1.162977\ldots$ (in the geometrical situation φ means the angle between axes $\mathbf{S}$ and $\mathbf{T}$).

So, we have found values of parameters α and φ, for which matrices S and T satisfy (4).

Let us prove the following lemma.

Lemma 1.1.

Let $0 < \beta < 2\pi$ be the real root of the equation:

$$9 + 8\cos\beta - 13\cos 2\beta + 4\cos 3\beta = 0$$

and $\alpha = 2\pi/\beta$.

Then α is irrational.

Proof.

Suppose, that α is rational, then $\alpha = m/n$ for some $m, n \in \mathbb{Z}$. Therefore, $\beta = 2\pi/\alpha = 2\pi n/m$. Hence, $m\beta = 2\pi n$, where

$$\cos m\beta \;=\; 1 \tag{12}$$

With u to denote $\cos\beta$, we have from (12):

$$\cos(m \arccos u) \;=\; 1 \tag{13}$$

From (13) we have $T_m(u) = 1$, where $T_m(z)$ denotes the Chebyshev polinomial. Therefore, we have, that $u = \cos\beta$ is the root of the following equations:

$$P_3(z) \;=\; 11 - 2z - 13z^2 + 8z^3 = 0 \tag{14}$$

$$S_m(z) \;=\; T_m(z) - 1 = 0, \quad m \geq 3 \tag{15}$$

Using (14), we obtain, that $P_3(z)$ is the polinomial over the field $\mathbb{Q}$ and $P_3(u) = 0$.

Since $P_3(z)$ is irreducible over $\mathbb{Q}$ then $P_3(z)$ has the minimal degree such that $P_3(u) = 0$. It is clear, that we can represent $S_m(z)$ in the following form:

$$S_m(z) \;=\; P_3(z)q(z) + r(z), \quad \deg r(z) < \deg P_3(z) \tag{16}$$

where $\deg r(z)$ denotes the degree of the polinomial $r(z)$. Since $S_m(z) = 0$, we obtain, that $r(u) = 0$. However $\deg r(z) < \deg P_3(z)$ and $P_3(z)$ is the minimal polinomial with the property $P_3(u) = 0$. Hence, $r(z) \equiv 0$.

Therefore, α is rational iff $P_3(z)$ divides $S_m(z)$ for some natural $m \geq 3$. Now we must only prove that $P_3(z)$ does not divide $S_m(z)$ for all natural $m \geq 3$. By Cardanus formulas, the equation (14) has one real root and two complex roots. We now prove that the equation (15) has only real roots for all natural m.

From (15) we get:

$$\cos(m \arccos z) = 1,$$

whence

$$m \arccos z_k = 2\pi k, \quad k = 0, 1, \ldots, m-1.$$

Hence,

$$z_k = \cos\frac{2\pi k}{m}, \quad k = 0, 1, \ldots, m-1.$$

It follows, that all z_k are real numbers for all k=0,1,...,m-1.

Thus, the equation (15) has only real roots for all natural m.

This means that $P_3(z)$ does not divide $S_m(z)$ for all natural $m \geq 3$.

Consiquently, α is irrational.

This completes the proof of the Lemma 1.1.

□

Now, by Lemma 1.1., the group Γ is not discrete. However we can use Γ to construct the canonical fundamental set $FS(\alpha)$ of the cone-manifold $S^3(W, 2\pi/\alpha)$.

3 Fundamental set of the cone-manifold $S^3(W, 2\pi/\alpha)$ in $\mathbb{E}^3$

In this section a fundamental set of the cone-manifold $S^3(W, 2\pi/\alpha)$ in $\mathbb{E}^3$ is constructed.

Let us find presentations for S and T in the group $SO(3)$, using the canonical epimorphism $SU(2) \to SO(3)$ described in [12].

As we found out in the section 1, the matrices of the rotations S and T from the group $SU(2)$ have the following form:

$$\begin{pmatrix} e^{i\left(\frac{\varphi_j+\psi_j}{2}\right)} \cos\dfrac{\theta_j}{2} & ie^{i\left(\frac{\varphi_j-\psi_j}{2}\right)} \sin\dfrac{\theta_j}{2} \\ ie^{i\left(\frac{\psi_j-\varphi_j}{2}\right)} \sin\dfrac{\theta_j}{2} & e^{-i\left(\frac{\varphi_j+\psi_j}{2}\right)} \cos\dfrac{\theta_j}{2} \end{pmatrix}, \qquad j = 1, 2$$

and for S: $\theta_1 = 2\pi/\alpha$, $\varphi_1 = \varphi/2$, $\psi_1 = -\varphi/2$; for T: $\theta_2 = -2\pi/\alpha$, $\varphi_2 = -\varphi/2$, $\psi_2 = \varphi/2$.

According to [12], S and T can be represented in the group $SO(3)$ in the form of the product of the following matrices:

$$B_{\varphi_i} C_{\theta_i} B_{\psi_i} =$$
$$= \begin{pmatrix} \cos\varphi_i & -\sin\varphi_i & 0 \\ \sin\varphi_i & \cos\varphi_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_i & -\sin\theta_i \\ 0 & \sin\theta_i & \cos\theta_i \end{pmatrix} \begin{pmatrix} \cos\psi_i & -\sin\psi_i & 0 \\ \sin\psi_i & \cos\psi_i & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for $i = 1$ and 2, respectively. Whence

$$S = \begin{pmatrix} \cos^2\dfrac{\varphi}{2} + \cos\dfrac{2\pi}{\alpha}\sin^2\dfrac{\varphi}{2} & \sin\varphi\sin^2\dfrac{\pi}{\alpha} & \sin\dfrac{\varphi}{2}\sin\dfrac{2\pi}{\alpha} \\ \sin\varphi\sin^2\dfrac{\pi}{\alpha} & \cos^2\dfrac{\varphi}{2}\cos\dfrac{2\pi}{\alpha} + \sin^2\dfrac{\varphi}{2} & -\cos\dfrac{\varphi}{2}\sin\dfrac{2\pi}{\alpha} \\ -\sin\dfrac{\varphi}{2}\sin\dfrac{2\pi}{\alpha} & \cos\dfrac{\varphi}{2}\sin\dfrac{2\pi}{\alpha} & \cos\dfrac{2\pi}{\alpha} \end{pmatrix} \qquad (17)$$

$$T = \begin{pmatrix} \cos^2\dfrac{\varphi}{2} + \cos\dfrac{2\pi}{\alpha}\sin^2\dfrac{\varphi}{2} & -\sin\varphi\sin^2\dfrac{\pi}{\alpha} & \sin\dfrac{\pi}{\alpha}\sin\dfrac{2\pi}{\alpha} \\ -\sin\varphi\sin^2\dfrac{\pi}{\alpha} & \cos^2\dfrac{\varphi}{2}\cos\dfrac{2\pi}{\alpha} + \sin^2\dfrac{\varphi}{2} & \cos\dfrac{\varphi}{2}\sin\dfrac{2\pi}{\alpha} \\ -\sin\dfrac{\varphi}{2}\sin\dfrac{2\pi}{\alpha} & -\cos\dfrac{\varphi}{2}\sin\dfrac{2\pi}{\alpha} & \cos\dfrac{2\pi}{\alpha} \end{pmatrix} \qquad (18)$$

According to [9], consider a non-convex 32-faced polyhedron $FS(\alpha)$ formed by the vertices P_i, Q_j, the edges P_iQ_j, P_iP_{i+1} and the faces $P_iP_{i+1}Q_j$, where $i = 0, \ldots, 15$, $j = 0, 1$ and $P_{16} = P_0$. In this case it is more convinient to consider the non-convex polyhedron $FS(\alpha)$ as curvelinear polyhedron with thirty two triangular faces $Q_0P_iP_{i+1}$, $Q_1P_iP_{i+1}$, $i = 0, \ldots, 15$, sixteen tetrahedra $Q_0Q_1P_iP_{i+1}$, $i = 0, \ldots, 15$ and four curvelinear faces:

$$\begin{array}{ll} Q_0P_3P_4\ldots P_{10}P_{11}, & Q_0P_3P_2\ldots P_{12}P_{11}, \\ Q_1P_0P_1\ldots P_7P_8, & Q_1P_0P_{15}\ldots P_9P_8, \end{array}$$

where each of these curvelinear faces consists of eight triangles.

Definition 2.1. The non-convex polyhedron $FS(\alpha)$ is said to be *canonical fundamental set* for the cone-manifold $S^3(W, 2\pi/\alpha)$, if it has following properties:

(i) There are some Euclidean rotations $\mathcal{S}$ and $\mathcal{T}$ which identify curvelinear faces:

$$\begin{array}{rcl} \mathcal{S} &:& Q_0P_3P_4\dots P_{10}P_{11} \;\rightarrow\; Q_0P_3P_2\dots P_{12}P_{11}, \\ \mathcal{T} &:& Q_1P_0P_1\dots P_7P_8 \;\rightarrow\; Q_1P_0P_{15}\dots P_9P_8. \end{array}$$

(ii) The sum of interior dihedral angles at the edges of 16-gon $P_0P_1\dots P_{14}P_{15}$ is equal to 2π.

(iii) Interior dihedral angles at the edges Q_0P_3, Q_0P_{11}, Q_1P_0, Q_1P_8 are equal to $2\pi/\alpha$.

Vertices of the $FS(\alpha)$ are shown in the Fig. 3.

Theorem 2.1. *Let $0 < \beta < 2\pi$ be the real root of the equation:*

$$9 + 8\cos\beta - 13\cos 2\beta + 4\cos 3\beta = 0$$

and $\alpha = 2\pi/\beta$.

Then there exists a 32-faced polyhedron in $\mathbb{E}^3$ which is the canonical fundamental set of the cone-manifold $S^3(W, 2\pi/\alpha)$ In particular, $S^3(W, 2\pi/\alpha)$ admits the Euclidean structure.

Proof.

Let us assume that the corresponding canonical polyhedron $FS(\alpha)$ is already realized in the Euclidean space.

The aim is to find coordinates of vertices of $FS(\alpha)$ under this assumption and then to construct the polyhedron $FS(\alpha)$ in $\mathbb{E}^3$ using these coordinates.

Let N be the common normal to the axes $\mathbf{S}$ and $\mathbf{T}$, Q_0 and Q_1 be points of intersections of the N with the axes $\mathbf{S}$ and $\mathbf{T}$, respectively, P_3 and P_{11} be vertices of $FS(\alpha)$, which lie in the axis $\mathbf{S}$, P_0 and P_8 be vertices of $FS(\alpha)$, which lie in the axis $\mathbf{T}$.

Note that without loss of generality one can assume that it is possible to choose such coordinate system that points Q_0 and Q_1 are points $(0, 0, -1)$ and $(0, 0, 1)$, the common normal N has the same direction as Oz-axis, the origin $O = (0, 0, 0)$ is the centre of the axis, which joins point Q_0 to point Q_1, the plane Oxy pass through the point O paralelly with the axes $\mathbf{S}$ and $\mathbf{T}$. Further, rotate horizontally if it is necessary this coordinate system, one can suppose that the axis Ox forms angles $\varphi/2$ with projections of the axes $\mathbf{S}$ and $\mathbf{T}$ on the plane Oxy and Ox, Oy have directions as in the Fig. 2. The parameter φ as in Section 1 is the angle from $\mathbf{S}$ to $\mathbf{T}$.

Given coordinate system, two rotations $\mathcal{S}$ and $\mathcal{T}$ operate by the following rules:

$$\begin{array}{rcl} \mathcal{S}(\overline{x}) &=& (\overline{x} - (0, 0, -1)) \cdot S + (0, 0, -1), \\ && \\ \mathcal{T}(\overline{x}) &=& (\overline{x} - (0, 0, 1)) \cdot T + (0, 0, 1), \end{array} \tag{19}$$

for any point $\overline{x}$ from $\mathbb{E}^3$. Here S and T are matrices of the rotations from the group $SO(3)$, which were obtained in (17), (18).

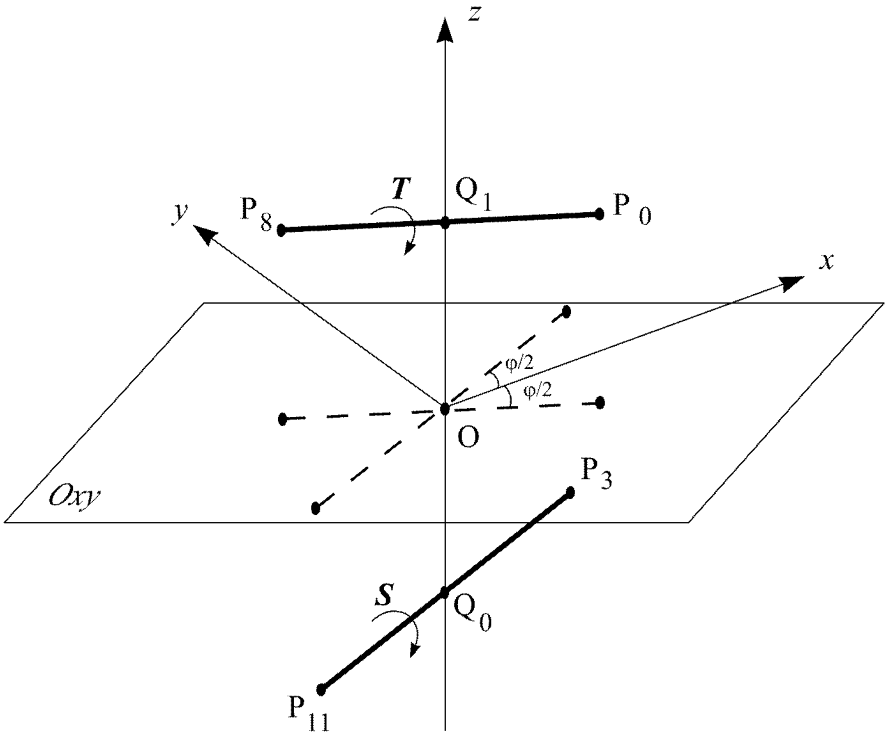

Figure 2: The location of the axes **S** and **T** in $\mathbb{E}^3$

Inverse rotations for $\mathcal{S}$ and $\mathcal{T}$ are given by formulas:

$$\begin{aligned} \mathcal{S}^{-1}(\overline{x}) &= (\overline{x} - (0,0,-1)) \cdot S^{-1} + (0,0,-1), \\ \mathcal{T}^{-1}(\overline{x}) &= (\overline{x} - (0,0,1)) \cdot T^{-1} + (0,0,1). \end{aligned} \tag{20}$$

respectively.

Using geometrical properties (i), (ii), (iii) of $FS(\alpha)$, we find coordinates of points P_0 and P_3 in the given coordinate system:

$$P_0 = \left(y \sin\frac{\pi}{\alpha}\sin\varphi\cos\frac{\varphi}{2}, -y\sin\frac{\pi}{\alpha}\sin\varphi\sin\frac{\varphi}{2}, 1\right), \tag{21}$$

$$P_3 = \left(y \sin\frac{\pi}{\alpha}\sin\varphi\cos\frac{\varphi}{2}, y\sin\frac{\pi}{\alpha}\sin\varphi\sin\frac{\varphi}{2}, -1\right), \tag{22}$$

where

$$y = \frac{4\cos(\pi/\alpha)}{\cos^2(\pi/\alpha) + \sin^2(\pi/\alpha)\cos^2\varphi}.$$

The coordinates of the vertices of $FS(\alpha)$ can be obtained from coordinates of vertices P_0 and P_3 with a help of rotations $\mathcal{S}$, $\mathcal{T}$, $\mathcal{S}^{-1}$, $\mathcal{T}^{-1}$ in the following way:

$$\begin{array}{lllllll} P_6 &=& \mathcal{S}^{-1}(P_0) & \quad & P_{13} &=& \mathcal{T}(P_3) \\ P_{10} &=& \mathcal{T}(P_6) & & P_9 &=& \mathcal{S}^{-1}(P_{13}) \\ P_{12} &=& \mathcal{S}(P_{10}) & & P_7 &=& \mathcal{T}^{-1}(P_9) \\ P_4 &=& \mathcal{T}^{-1}(P_{12}) & & P_{15} &=& \mathcal{S}(P_7) \\ P_2 &=& \mathcal{S}(P_4) & & P_1 &=& \mathcal{T}^{-1}(P_{15}) \\ P_{14} &=& \mathcal{T}(P_2) & & P_5 &=& \mathcal{S}^{-1}(P_1) \\ P_8 &=& \mathcal{S}^{-1}(P_{14}) & & P_{11} &=& \mathcal{T}(P_5) \end{array} \tag{23}$$

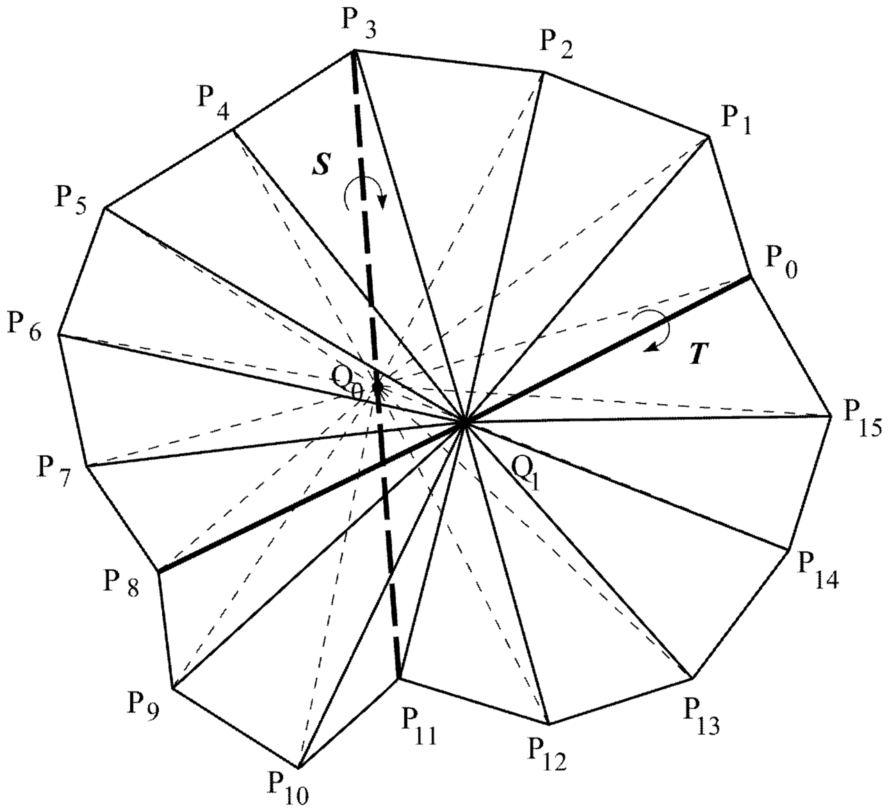

Figure 3: The fundamental set $FS(\alpha)$

First, verify that points P_0, P_1, $\ldots$, P_{14}, P_{15} obtained in (21), (22), (23), Q_0, Q_1 form 32-faced polyhedron $FS(\alpha)$ in $\mathbb{E}^3$.

Consider all tetrahedra $Q_0Q_1P_iP_{i+1}$, $i = 0, 1, \ldots, 15$. Inspecting coordinates of normals to triangular faces $Q_0P_iP_{i+1}$, $Q_1P_iP_{i+1}$ we are convinced that all tetrahedra $T_i = Q_0Q_1P_iP_{i+1}$, $i = 0, 1, \ldots, 15$ are not degenerate.

Using (21), (22), (23), we obtain, that there are four types of interior dihedral angles of tetrahedra T_i, $i = 0, 1, \ldots, 15$ at the edge Q_0Q_1, namely γ_0 (tetrahedra T_1, T_5, T_9, T_{13}), γ_1 (tetrahedra T_3, T_7, T_{11}, T_{15}), γ_2 (tetrahedra T_4, T_6, T_{12}, T_{14}), γ_3 (tetrahedra T_0, T_2, T_8, T_{10}) and γ_i, $i = 0, 1, 2, 3$, satisfy the following system of equations:

$$\left\{\begin{array}{lcl} 4\gamma_0 + 4\gamma_1 + 4\gamma_2 + 4\gamma_3 & = & 2\pi \\ \gamma_0 + 2\gamma_3 & = & \varphi \\ \gamma_0 + \gamma_1 + 2\gamma_3 & = & \pi/2 \\ \gamma_0 - \gamma_1 - \gamma_2 + \gamma_3 & = & 2\pi \end{array}\right. \tag{24}$$

where φ is the angle from $\mathbf{S}$ to $\mathbf{T}$.

From (24) we have

$$\left\{\begin{array}{lclcl} \gamma_0 & = & \gamma_1 & = & \dfrac{\pi}{2} - \varphi \\ \gamma_2 & = & \gamma_3 & = & \varphi - \dfrac{\pi}{4} \\ \gamma_0 + \gamma_2 & = & \dfrac{\pi}{4} & & \end{array}\right. \tag{25}$$

Recall that $\varphi = 1.162977\ldots$, hence $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ are positive.

For easier writing, we rename the angles γ_0, γ_2 to γ, δ, respectively.

Therefore, from the first equation of (24) we find out that the sum of positive interior dihedral angles of T_i, $i = 0, 1, \dots, 15$ at the edge Q_0Q_1 is equal to 2π.

Hence, interiors of tetrahedra T_i, $i = 0, 1, \dots, 15$ are disjoint and the union of these tetrahedra is non-convex 32-faced polyhedron in $\mathbb{E}^3$.

By (25) we get all interior dihedral angles of T_i, $i = 0, 1, \dots, 15$ at the edge Q_0Q_1:

$$
\begin{array}{lcllcl}
\angle_0 & = & \frac{\pi}{2} - \varphi & \angle_8 & = & \frac{\pi}{2} - \varphi \\
\angle_1 & = & \varphi - \frac{\pi}{4} & \angle_9 & = & \varphi - \frac{\pi}{4} \\
\angle_2 & = & \frac{\pi}{2} - \varphi & \angle_{10} & = & \frac{\pi}{2} - \varphi \\
\angle_3 & = & \varphi - \frac{\pi}{4} & \angle_{11} & = & \varphi - \frac{\pi}{4} \\
\angle_4 & = & \frac{\pi}{2} - \varphi & \angle_{12} & = & \frac{\pi}{2} - \varphi \\
\angle_5 & = & \varphi - \frac{\pi}{4} & \angle_{13} & = & \varphi - \frac{\pi}{4} \\
\angle_6 & = & \frac{\pi}{2} - \varphi & \angle_{14} & = & \frac{\pi}{2} - \varphi \\
\angle_7 & = & \varphi - \frac{\pi}{4} & \angle_{15} & = & \varphi - \frac{\pi}{4}
\end{array}
\tag{26}
$$

where $\angle_i$ denotes the dihedral angle of T_i at the edge Q_0Q_1.

Using (26) it is easy to verify that the sum of all $\angle_i$, $i = 0, 1, \dots, 15$ is equal to

$$
8\left(\frac{\pi}{2} - \varphi\right) + 8\left(\varphi - \frac{\pi}{4}\right) \quad = \quad 2\pi. \tag{27}
$$

Now, verify that $FS(\alpha)$ is fundamental set of the cone-manifold $S^3(W, 2\pi/\alpha)$ in $\mathbb{E}^3$, i.e. that $FS(\alpha)$ satisfies (i), (ii), (iii).

From (23) we have that $FS(\alpha)$ satisfies (i).

Using (21), (22), (23), we have, that the interior dihedral angle of T_i at the edge Q_0Q_1 is equal to the interior digedral angle of T_i at the edge P_iP_{i+1} for all $i = 0, 1, \dots, 15$.

Therefore, from (24) the sum of all dihedral angles on edges of the 16-gon $P_0P_1 \dots P_{14}P_{15}$ equals $2\pi = 360^{\circ}$.

So, $FS(\alpha)$ satisfies (ii).

By immediate calculations we find all interior dihedral angles of $FS(\alpha)$ (see table 1). (In the table 1 the equality "$\angle P_iP_{i+1} = \varkappa^{\circ}$" means that the interior dihedral angle which is formed by two faces of $FS(\alpha)$ at the edge P_iP_{i+1} equals $\varkappa^{\circ}$).

Table 1

$\angle_0$	$=$	$\angle P_0P_1$	$=$	$21.6338^o...$	$\angle_8$	$=$	$\angle P_8P_9$	$=$	$21.6338^o...$
$\angle_1$	$=$	$\angle P_1P_2$	$=$	$23.3663^o...$	$\angle_9$	$=$	$\angle P_9P_{10}$	$=$	$23.3663^o...$
$\angle_2$	$=$	$\angle P_2P_3$	$=$	$21.6338^o...$	$\angle_{10}$	$=$	$\angle P_{10}P_{11}$	$=$	$21.6338^o...$
$\angle_3$	$=$	$\angle P_3P_4$	$=$	$23.3664^o...$	$\angle_{11}$	$=$	$\angle P_{11}P_{12}$	$=$	$23.3664^o...$
$\angle_4$	$=$	$\angle P_4P_5$	$=$	$21.6338^o...$	$\angle_{12}$	$=$	$\angle P_{12}P_{13}$	$=$	$21.6338^o...$
$\angle_5$	$=$	$\angle P_5P_6$	$=$	$23.3663^o...$	$\angle_{13}$	$=$	$\angle P_{13}P_{14}$	$=$	$23.3663^o...$
$\angle_6$	$=$	$\angle P_6P_7$	$=$	$21.6337^o...$	$\angle_{14}$	$=$	$\angle P_{14}P_{15}$	$=$	$21.6337^o...$
$\angle_7$	$=$	$\angle P_7P_8$	$=$	$23.3663^o...$	$\angle_{15}$	$=$	$\angle P_{15}P_0$	$=$	$23.3663^o...$

From the explicit formulas (21), (22), (23) we have that interior dihedral angles at the edges Q_0P_3, Q_0P_{11}, Q_1P_0, Q_1P_8 are equal to $2\pi/\alpha$.

Hence, $FS(\alpha)$ satisfies (iii).

Consiquently, $FS(\alpha)$ is fundamental set of the manifold $S^3(W, 2\pi/\alpha)$ in $\mathbb{E}^3$.

This completes the proof of the Theorem 2.1.

$\square$

References

[1] Hodgson, C.: *Degeneration and regeneration of geometric structures on three-manifolds.* Ph.D.Thesis. Princeton University (1986).

[2] Thurston, W.: *Three-dimentional Geometry and Topology.* Princeton University Press. Princeton, New Jersey, 1997.

[3] Gonzales-Akuña, F. and J. M. Montesinos-Amilibia: *On the character variety of group representations in $SL(2,\mathbb{C})$ and $PSL(2,\mathbb{C})$.* Math.Z. **214** (1993), 627–652.

[4] Hilden, H.M., Lozano, M.T. and J. M. Montesinos-Amilibia: *A Characterization of Arithmetic Subgroups of $SL(2,\mathbb{C})$ and $PSL(2,\mathbb{C})$.* Math.Nachr. **159** (1992), 245–270.

[5] Hilden, H.M., Lozano, M.T. and J. M. Montesinos-Amilibia: *On a remarkable polyhedron geometrizing the figure eight knot cone manifolds.* J.Math.Sci.Univ. Tokyo **2** (1995), 501–561.

[6] Hilden, H.M., Lozano, M.T. and J. M. Montesinos-Amilibia: *On Volumes and Chern-Simons Invariants of Geometric 3-Manifolds.* J.Math.Sci.Univ. Tokyo **3** (1996), 723–744.

[7] Helling, H., Kim, A.C., Mennicke, J.L.: *Some Honey-combs in Hyperbolic 3-space.* Communications in Algebra **23** (1995), 5221–5242.

[8] Jones, K.N.: *Geometric Structures on Branched Covers over Universal Links.* Contemporary Mathematics **164** (1994), 47–58.

[9] Mednykh, A., Rasskazov, A.: *On the structure of the canonical fundamental set for the 2-bridge link orbifolds.* Preprint 1997.

[10] Crowell, R.H., Fox, R.H.: *Introduction to knot theory.* Springer-Verlag. New York-Heidelberg-Berlin, 1977.

[11] Fenchel, W.: *Elementary Geometry in Hyperbolic Space.* Walter de Gruyter. Berlin-New York, 1989.

[12] Kostrikin, A.I.: *Introduction to Algebra.* Nauka 1977 (in Russian).