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# ON THE STRUCTURE OF THE CANONICAL FUNDAMENTAL SET FOR THE 2-BRIDGE LINK ORBIFOLDS

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**Abstract**

Let $\mathcal{O}(p/q, n)$ denote the orbifold with singular set the 2-bridge knot or link $p/q$ and isotopy group cyclic of order $n$. The paper is devoted to study of the fundamental set of the orbifold $\mathcal{O}(p/q, n)$ in the hyperbolic, Euclidean and spherical three-dimensional spaces. The main goal of the paper is to construct the simplest possible fundamental polyhedron for $\mathcal{O}(p/q, n)$ in the sense that it has just two pairs of mutually congruent curvilinear faces. The fundamental polyhedron is non-convex and consists of $2p$ tetrahedra whose vertices can be defined in a canonical way from the presentation of the orbifold fundamental group.

## 1. Introduction

The aim of this paper is the study of the structure of the fundamental set for the 2-bridge link orbifolds. A 2-bridge knot or link $p/q$ is determined by a pair of relatively prime integers $(p, q)$ with $0 < q < p$. Two pairs $(p, q)$ and $(p', q')$ determine the same link if and only if $p = p'$ and $qq' \equiv \pm 1$ ( $mod\ p$ ). A hyperbolic ( resp. Euclidean, spherical ) 2-bridge link orbifold $\mathcal{O}(p/q, n)$ has as its singular set the 2-bridge link $p/q$ and as its underlying space the three-dimensional sphere. Away from the 2-bridge link the orbifold

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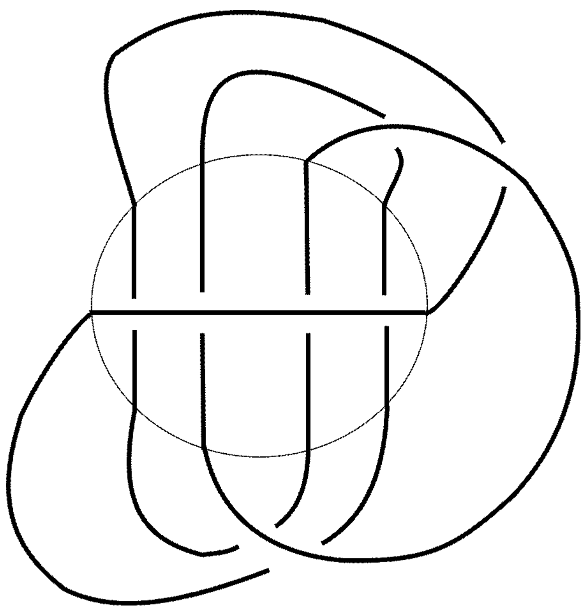

Figure 1. The figure-eight knot.

is locally isometric to hyperbolic (resp. Euclidean, spherical) three dimensional space and on the 2-bridge link it is locally the quotient space of the action of an $n$-fold rotation.

In Section 2 will be constructed the canonical fundamental set for $\mathcal{O}(p/q, 2)$-orbifold in the spherical space. Topologically this construction is well-known [15]. It is very surprising but the same construction can be realized not only in spherical space but in hyperbolic and Euclidean spaces too. It will be do in the next three sections.

In Section 3 the structure of the fundamental set of the figure-eight orbifold $\mathcal{O}(5/2, 3)$ will be constructed in Euclidean space in explicit way. More precisely, integer Cartesian coordinates of the vertices of the fundamental polyhedron for $\mathcal{O}(5/2, 3)$ will be given.

In Section 4 the geometrical parameters of fundamental set for the figure-eight orbifolds $\mathcal{O}(5/2, n)$, $n > 3$ will be computed.

In Section 5 the existence of the conic figure-eight manifold will be considered. Using the above mentioned construction of the fundamental set it will be shown that the conic figure-eight manifold will be realized in hyperbolic, Euclidean and spherical geometries.

In Section 6 an algorithm for producing of the canonical fundamental set for the $\mathcal{O}(p/q, n)$ orbifold will be given. This work was inspired by the papers [6], [9], [12].

## 2. The canonical fundamental set for orbifold $\mathcal{O}$(p/q,2) in the spherical space

Denote by $\mathbb{S}^3$ the three-dimensional spherical space. Let us consider a lens $L$ bounded

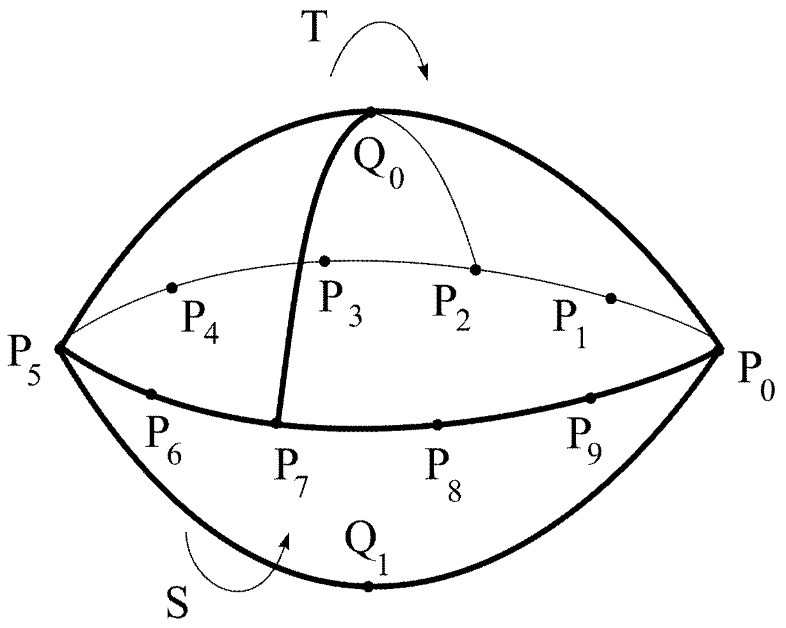


Figure 2. The fundamental set L of the figure-eight orbifold $\mathcal{O}(5/2, 2)$.

by two spherical planes in $\mathbb{S}^3$ with dihedral angle $\frac{\pi}{p}$, where $p$ is a positive integer. Choose $2p$ points on the line ( circle ) of intersection of this planes $P_i$, $i = 0, \ldots, 2p-1$ such that the spherical distance between $P_i$ and $P_{i+1}$ ( $i \; mod \; 2p$ ) is equal to $\frac{\pi}{p}$. Draw the line $P_0P_p$ on the top of the lens and the line $P_qP_{p+q}$ on the bottom of the lens ( See Figure 2 ).

**Lemma 2.1** *Let $S$ and $T$ are spherical isometries which identify faces of the lens $L$ in the following way:*

$$S : Q_1P_p \ldots P_0 \to Q_1P_0 \ldots P_p$$

$$T : Q_0P_q \ldots P_{p+q} \to Q_0P_{p+q} \ldots P_q$$

*then*

*(i) the group $< S, T >$ is a discrete subgroup of isometries of the spherical space $\mathbb{S}^3$ and has the following presentation:*

$$< \; S, T \mid S^2 = T^2 = (TS)^{2p} = 1 \; >$$

*(ii) the lens $L$ is fundamental set for the group $< S, T >$*

*(iii) the orbifolds $\mathbb{S}^3/ < S, T >$ and $\mathcal{O}(p/q, 2)$ are isomorphic*

Proof. The properties (i) and (ii) are immediate consequences of the Poincare theorem [2]. Let us verify the property (iii).

To show that orbifold $\mathbb{S}^3/ < S, T >$ coincides with the orbifold $\mathcal{O}(p/q, 2)$ consider two-fold covering $M$ of $\mathbb{S}^3/ < S, T >$ branched over union of lines $P_0P_p$ and $P_qP_{p+q}$. The

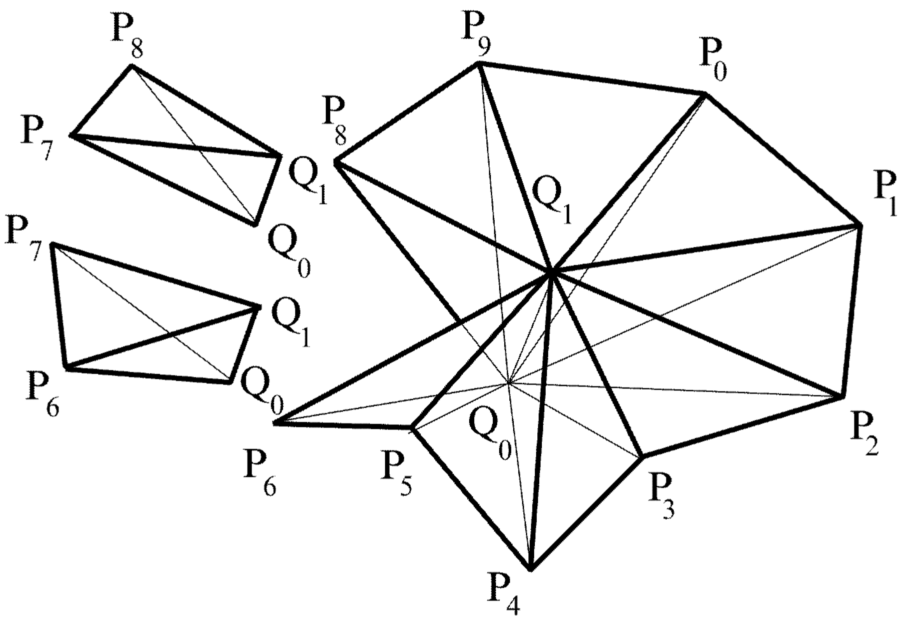


Figure 3. The fundamental set $\mathcal{P}$ of the figure-eight orbifold $\mathcal{O}(5/2, 3)$.

fundamental group of this covering is given by kernel of the epimorphism $\phi :< S, T > \to \mathbb{Z}_2 = \{0, 1\}$ defined by $\phi(S) = 1$ and $\phi(T) = 1$. Using the Reidemeister-Schreier method one can easy to check that the group $Ker\phi$ is generated by $TS$ and has just one relation $(TS)^{2p} = 1$. The fundamental set of this group formed by L and S(L) identified along the bottom hemisphere of L, where L is shown at Figure 2.

Let us denote poles of the lens $L \cup S(L)$ as $North = Q_0$, $South = S(Q_0)$, and set $V_i = P_{2i}$, $i = 0, \ldots, p$ and $V_p = V_0$. Then isometry $TS$ acts on the set $L \cup S(L)$ in the following way: the triangle $NorthV_iV_{i+1}$ is identified with the triangle $SouthV_{i+q}V_{i+q+1}$ for every $i = 0, \ldots, p-1$. Hence $M$ is the standard lens space $L(p, q)$. Orbifold $\mathbb{S}^3/ < S, T >$ is obtained as a factor space of $M = L(p, q)$ by means of involution $S$ having 1-dimensional fixed point set. It was shown in [5], the link in $\mathbb{S}^3$ whose double branched covering is a lens space $L(p, q)$ is necessarily a 2-bridge link $p/q$, and hence $\mathbb{S}^3/ < S, T >$ coincides with the orbifold $\mathcal{O}(p/q, 2)$.

## 3. The structure of the fundamental set of the figure-eight orbifold $\mathcal{O}$(5/2,3) in Euclidean space

The aim of this section is to show that figure-eight orbifold $\mathcal{O}(5/2, 3)$ has a fundamental set in the Euclidean space $\mathbb{E}^3$ shown at Figure 3 which is topologically similar to the fundamental set of the orbifold $\mathcal{O}(5/2, 2)$ in the spherical space shown at Figure 2.

More precisely, we shall describe an algorithm for construction of the fundamental set $\mathcal{P}$ for the figure-eight orbifold which effectively works in every space of the constant

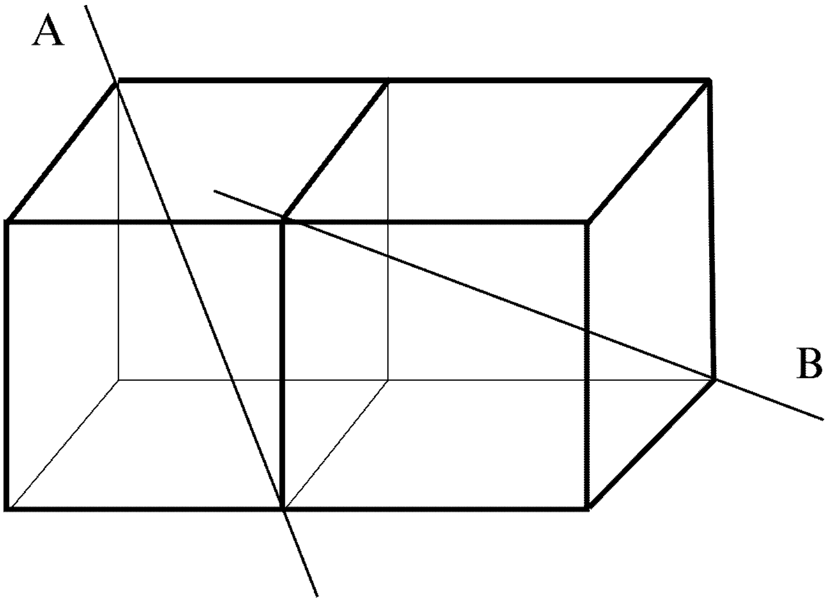


Figure 4. The axes $A$ and $B$ of the rotations $S$ and $T$ in the Euclidean space.

sectional curvature. This algorithm also can be applied for construction of fundamental set for every 2-bridge link orbifold with some inessential modifications.

We start with explicit description of the vertices of $\mathcal{P}$.

It was shown in [18] that the orbifold $\mathcal{O}(5/2, 3)$ can be obtained as a quotient space $\mathbb{E}^3/ < S, T >$, where $< S, T >$ is the group generated by two rotations of order three $S$ and $T$ whose axis shown at Figure 4. For convenience we redraw Figure 4 to Figure 5. It is easy to see that Figure 5 is obtained from Figure 4 by scale exchange. The axes of the elements $T$, $S$ are given by the lines $[(-1, 0, 0), (0, -1, 1)]$ and $[(0, 0, 0), (1, 1, 1)]$ respectively. We shall use the following properties of the isometry $S$ and $T$:

$$S(-1, 0, 0) = (0, -1, 0),\ S(0, -1, 1) = (1, 0, -1),$$

$$TS(-1, 0, 0) = (0, 0, 1),\ TS(0, -1, 1) = (-1, 1, 2).$$

Consider the half-turns $b$ and $e$ about the lines

$$(-1/2, -1/4, 0) + k(1, 0, 1), \quad (-1/4, -1/4, -1/4) + k(1, 0, -1).$$

One can easy verify the properties of the isometry $b$: $bSb^{-1} = T$, $b^2 = 1$ and e: $eS = se$, $eT = te$, $e^2 = 1$, $be = eb$, where $s = S^{-1}$ and $t = T^{-1}$. Consider the element $TSTst$. It has an axis that is the image of the axis of rotation of $T$ under transformation $TS$. So this axis coincide with the line $[TS(-1, 0, 0), TS(0, -1, 1)] = [(0, 0, 1), (-1, 1, 2)]$. Denote by $P_1$ a point of intersection of axes $b$ and $TSTst$. By direct calculation one can find that $P_1 = (1/4, -1/4, 3/4)$. All others vertices of the fundamental polyhedra are obtained by

$$P_3 = T^{-1}(P_1),\ P_5 = T^{-1}(P_9),\ P_4 = e(P_9),$$

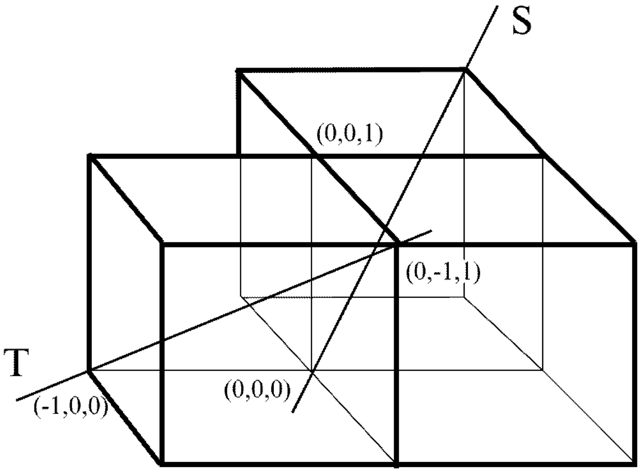


Figure 5. The axes of the rotations $S$ and $T$ in the Euclidean space.

$$P_7 = S^{-1}(P_3),\ P_6 = e(P_1),\ P_8 = e(P_3),$$
$$P_9 = S^{-1}(P_1),\ P_2 = e(P_7),\ P_0 = e(P_5).$$

Define the points $Q_0$, $Q_1$ as points of intersection of axis $e$ with axes $S$, $T$ respectively.

Hence, we have

$$\begin{array}{lll}
Q_1 = (-1/4,-1/4,-1/4) & P_2 = (-1/4,-3/4,3/4) & P_6 = (-5/4,-1/4,-3/4) \\
Q_0 = (-3/4,-1/4,1/4) & P_3 = (-1/4,-5/4,1/4) & P_7 = (-5/4,1/4,-1/4) \\
P_0 = (1/4,1/4,1/4) & P_4 = (-3/4,-5/4,-1/4) & P_8 = (-3/4,3/4,-1/4) \\
P_1 = (1/4,-1/4,3/4) & P_5 = (-3/4,-3/4,-3/4) & P_9 = (-1/4,3/4,1/4)
\end{array}$$

Define the polyhedron $\mathcal{P}$ as one formed by the vertices $P_i$, $Q_j$, the edges $P_iQ_j$, $P_iP_{i+1}$ and the faces $P_iP_{i+1}Q_j$, where $i = 0,\dots,9$, $j = 0,1$ and $P_{10} = P_0$. It is more convenient to consider non-convex polyhedron $\mathcal{P}$ as curvilinear polyhedron with the ten vertices $P_i$, $i = 0,\dots,9$, twelve edges $P_iP_{i+1}$, $i = 0,\dots,9$, $P_0P_5$, $P_2P_7$ and four curvilinear faces:

$$Q_1P_5\ \dots P_0,\quad Q_1P_0\ \dots P_5,\quad Q_0P_7\ \dots P_2,\quad Q_0P_2\ \dots P_7.$$

It is easy to see that each of these curvilinear faces consists of five triangles.

Remark that described curvilinear polyhedron is homeomorphic to the spherical polyhedron shown at Figure 2. As in the spherical case the lines $[P_0, P_5]$, $[P_2, P_7]$ are fixed by elements $S$ and $T$ respectively. Using the transformation $x \to 4x + (2,1,0)$, $x \in \mathbb{E}^3$ one can easy to find more convenient coordinates for vertices of the polyhedron $\mathcal{P}$:

$$
\begin{array}{lll}
Q_0=(-1,0,1) & P_2=(1,-2,3) & P_6=(-3,0,-3)\\
Q_1=(1,0,-1) & P_3=(1,-4,1) & P_7=(-3,2,-1)\\
P_0=(3,2,1) & P_4=(-1,-4,-1) & P_8=(-1,4,-1)\\
P_1=(3,0,3) & P_5=(-1,-2,-3) & P_9=(1,4,1)
\end{array}
$$

Now we will show that the constructed polyhedron $\mathcal{P}$ is the fundamental set for group $< S, T >$ in $\mathbb{E}^3$. Check the conditions of the Poincare theorem and define the presentation of group $< S, T >$. For this define interior dihedral angles between adjacent faces occurring in the polyhedron. We have:

$$\cos(\angle P_3P_4) = \cos(\angle P_8P_9) = \frac{7}{9}, \qquad \angle P_0P_5 = \angle P_2P_7 = \frac{2\pi}{3},$$

$$\cos(\angle P_0P_1) = \cos(\angle P_1P_2) = \cos(\angle P_2P_3) = \cos(\angle P_4P_5) = \cos(\angle P_5P_6) =$$
$$\cos(\angle P_6P_7) = \cos(\angle P_7P_8) = \cos(\angle P_9P_0) = \sqrt{\frac{2}{3}}.$$

| Edge cycle | Sum of angles | Relation in group |
|---|---|---|
| $P_2P_7$ | $\frac{2\pi}{3}$ | $T^3=1$ |
| $P_0P_5$ | $\frac{2\pi}{3}$ | $S^3=1$ |
| $P_5P_6,\ P_6P_7,\ P_7P_8,\ P_8P_9,\ P_9P_0,\ P_0P_1,\ P_1P_2,\ P_2P_3,\ P_3P_4,\ P_4P_5$ | $2\pi$ | $sTStstSTst=1$ |
| $Q_1P_6,\ Q_1P_4$ | $2\pi$ | $Ss$ |
| $Q_1P_7,\ Q_1P_3$ | $2\pi$ | $Ss$ |
| $Q_1P_8,\ Q_1P_2$ | $2\pi$ | $Ss$ |
| $Q_1P_9,\ Q_1P_1$ | $2\pi$ | $Ss$ |
| $Q_0P_6,\ Q_0P_8$ | $2\pi$ | $Tt$ |
| $Q_0P_5,\ Q_0P_9$ | $2\pi$ | $Tt$ |
| $Q_0P_3,\ Q_0P_1$ | $2\pi$ | $Tt$ |

Table 1.

In such a way the conditions of the Poincare theorem fulfilled and the following statement is true.

**Theorem 3.1** *Let $S$ and $T$ are Euclidean isometries which identify curvilinear faces of the polyhedron $\mathcal{P}$ in the following way:*

$$S : Q_1P_5 \ \ldots \ P_0 \to Q_1P_0 \ \ldots \ P_5$$

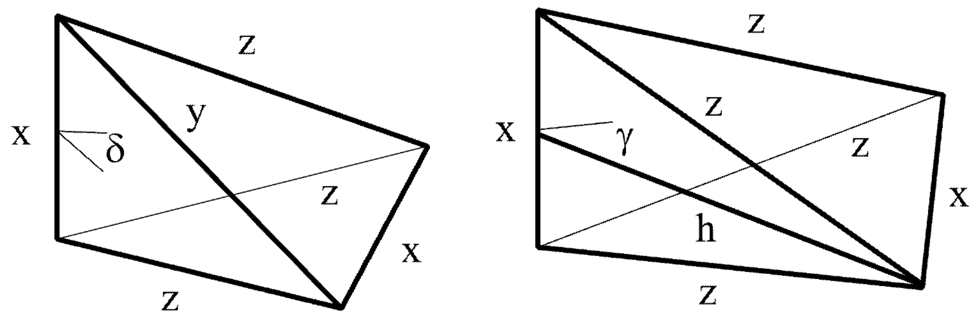


Figure 6. The tetrahedra $t_1$ ( on the left ) and $t_3$ ( on the right ).

$$T : Q_0P_2 \ \dots \ P_7 \to Q_0P_7 \ \dots \ P_2$$

*then*

*(i) the group* $< S, T >$ *is a discrete subgroup of isometries of the space* $\mathbb{E}^3$ *and has the following presentation:*

$$< \ S, T \mid S^3 = T^3 = S^{-1}TST^{-1}S^{-1}T^{-1}STS^{-1}T^{-1} = 1 \ >$$

*(ii) the polyhedron* $\mathcal{P}$ *is fundamental set for the group* $< S, T >$

*(iii) the orbifolds* $\mathbb{E}^3 / < S, T >$ *and* $\mathcal{O}(5/2, 3)$ *are isomorphic*

Another approach to the proof of the theorem 3.1 can be obtained by the following scheme. Note that the fundamental polyhedron $\mathcal{P}$ can be decomposed in ten tetrahedra $t_i = Q_0Q_1P_iP_{i+1}$, $i = 0, \dots, 9$. It is easy to verify that tetrahedra $t_1$, $t_2$, $t_4$, $t_5$, $t_6$, $t_7$, $t_9$ are congruent to each other and the remain two tetrahedra $t_3$ and $t_8$ are congruent to each other too. Let us denote the angles and lenghts of the edges of tetrahedra $t_1$, $t_3$ as it is shown at the Figure 6. Then we have the following system of equations:

$$\left\{ \begin{array}{l} \sin \frac{\gamma}{2} = \frac{x}{2h} \\ \sin \delta = \frac{x\sqrt{3}}{2h} \\ \gamma + 4\delta = \pi \\ h^2 = z^2 - (\frac{x}{2})^2 \\ \frac{x}{2y} = \frac{\tan \delta}{\sqrt{3}} \\ x^2 + y^2 = z^2 \end{array} \right. \tag{1}$$

Up to similarity the system (1) has the unique solution. The above mentioned polyhedron correspond to $x = 2\sqrt{2}$. This scheme can be realized also for construction of the fundamental set of the 2-bridge link orbifolds and cone-manifolds in the spherical and hyperbolic cases.

## 4. The structure of the fundamental set of the figure-eight orbifold $\mathcal{O}(5/2,n)$ in the hyperbolic space

We note that the topological structure of the fundamental set for the orbifold $\mathcal{O}(5/2,2)$ in spherical space and of the fundamental set for the orbifold $\mathcal{O}(5/2,3)$ in Euclidean space are the same. Now the main idea is to construct a fundamental set for orbifold $\mathcal{O}(5/2,n)$, $n>3$ of the same topological type in the hyperbolic space.

This section will be divided on three parts. In the first part we just suppose that above mentioned polyhedron $\mathcal{P}$ exists in the hyperbolic space $\mathbb{H}^3$. In such a way it is possible to obtain system of equations for dihedral angles and lenghts of edges of this polyhedron. In the second part we solve the above system of equations and show that this solutions are sufficient for constructing of polyhedron. In the third part the conditions of the Poincare theorem for constructed polyhedron will be carefully verified. It will be shown that $\mathcal{P}$ is unique fundamental set for the hyperbolic orbifold $\mathcal{O}(5/2,n)$, $n>3$.

Suppose that the corresponding fundamental set $\mathcal{P}$ is already realized in the hyperbolic space. By the rule of identification faces under isometries $S$ and $T$ we have that polyhedron $\mathcal{P}$ consists from ten tetrahedra each of them is congruent to $t_1$ or $t_3$ shown at the Figure 6. By analogy with Euclidean case we write a following system of equations for angles and lenghts of tetrahedra $t_1$ and $t_3$ in the hyperbolic case.

$$\begin{cases} \sin\frac{\gamma}{2} = \frac{\sinh\frac{x}{2}}{\sinh h} \\ \sin\delta = \frac{\sinh x \sin\frac{\pi}{n}}{\sinh h} \\ \gamma + 4\delta = \pi \\ \cosh h = \frac{\cosh z}{\cosh\frac{x}{2}} \\ \coth y \tanh\frac{x}{2} = \cot\frac{\pi}{n}\tan\delta \\ \cosh z = \cosh x \cosh y \end{cases} \tag{2}$$

System (2) is equivalent to the seven order algebraic equation respect the variable $u=\cos 2\delta$

$$(Y+4u^4p^2)((1-u)^2(2-p^2)-(1+u)Yp^2) = 4u^2(1-u)(2-p^2)(Y+2u^2p^2)^2 \tag{3}$$

where

$$Y = 1-u-4u^2p^2 \tag{4}$$

$$u = \cos 2\delta \tag{5}$$

$$p^2 = 2\sin^2\frac{\pi}{n} \tag{6}$$

The above equation is equivalent to the following:

$$(2u^2p^2+2u-1)(u+1/2)(u-1)(4u^2p^2+u-1)(u+2\sin^2\frac{\pi}{n}-1) = 0 \tag{7}$$

We are interesting in the following factor of the equation (3)

$$2u^2p^2 + 2u - 1 = 0 \tag{8}$$

In this case

$$\cosh x = \frac{1}{2up^2}, \tag{9}$$

$$\cosh h = \sqrt{1 + \frac{\cosh x - 1}{2u^2}}, \tag{10}$$

$$\cosh z = \cosh \frac{x}{2} \cosh h, \tag{11}$$

$$\cosh y = \frac{\cosh z}{\cosh x}, \tag{12}$$

$$\gamma = \pi - 4\delta. \tag{13}$$

We show it will be sufficient to construct the fundamental polyhedra for all $n > 3$. All others factors of (7) $u + 1/2 = 0,\ \ u - 1 = 0,\ 4u^2p^2 + u - 1 = 0,\ u + 2\sin^2\frac{\pi}{n} - 1 = 0$ with a few exceptions lead to degenerate geometrical situations. Solving the equation (8) we have

$$u = \cos 2\delta = \frac{-1 + \sqrt{1 + 4\sin^2\frac{\pi}{n}}}{4\sin^2\frac{\pi}{n}}, \tag{14}$$

**Remark 4.1** Formula (14) for determining of the dihedral angle $\delta$ is also valid in the Euclidean case ($n = 3$) and in the spherical case ($n = 2$).

**Remark 4.2** We note from (14) that $\delta = \frac{1}{2}\arccos u$ and

$$0 < \delta < \frac{\pi}{4}. \tag{15}$$

Hence $\gamma = \pi - 4\delta$ satisfies

$$(0 < \gamma < \pi). \tag{16}$$

Inequalities (15) and (16) are very important in our case. They both together with the condition $2\gamma + 8\delta = 2\pi$ give us that ten tetrahedra $t_0,\ldots,t_9$ forming polyhedron $\mathcal{P}$ have mutually disjoint interiors.

Check the Poincare theorem conditions for the fundamental group of orbifold $\mathcal{O}(5/2, n)$

$$\Gamma =<\ S, T \mid S^n = T^n = S^{-1}TST^{-1}S^{-1}T^{-1}STS^{-1}T^{-1} = 1\ > .$$

Let us consider $\mathcal{P}$ as curvilinear polyhedron with edges $P_0P_5$, $P_2P_7$, $P_iP_{i+1}$, $i = 0, \ldots, 9$, and with four curvilinear faces

$$Q_1P_5\ \ldots P_0, \quad Q_1P_0\ \ldots P_5, \quad Q_0P_7\ \ldots P_1, \quad Q_0P_2\ \ldots P_7.$$

Then under action of the group $< S, T >$ polyhedron $\mathcal{P}$ has exactly three edge cycles. Two of them $P_0P_5$ and $P_2P_7$ are elliptic with sum of angles $\frac{2\pi}{n}$. They correspond to relations $S^n = 1$ and $T^n = 1$ respectively. The third edge cycle $P_iP_{i+1}$, $i = 0, \ldots, 9$ has sum of angles $2\gamma + 8\delta = 2\pi$ and corresponds to relation $sTStstSTst = 1$.

And as a result we have the following theorem:

**Theorem 4.3** *Let $S$ and $T$ are hyperbolic isometries which identify curvilinear faces of the polyhedron $\mathcal{P}$ in the following way:*

$$S : Q_1P_5 \ \ldots \ P_0 \to Q_1P_0 \ \ldots \ P_5$$

$$T : Q_0P_2 \ \ldots \ P_7 \to Q_0P_7 \ \ldots \ P_2$$

*then*

*(i) the group $< S, T >$ is a discrete subgroup of isometries of the space $\mathbb{H}^3$ and has the following presentation:*

$$< \ S, T \mid S^n = T^n = S^{-1}TST^{-1}S^{-1}T^{-1}STS^{-1}T^{-1} = 1 \ >$$

*(ii) the polyhedron $\mathcal{P}$ is fundamental set for the group $< S, T >$*

*(iii) the orbifolds $\mathbb{H}^3/ < S, T >$ and $\mathcal{O}(5/2, n)$, $n \geq 4$ are isomorphic*

It is interesting to compare the values of angle $\delta$ of the polyhedra $\mathcal{P}$ for different $n$. We recall that for $n = 2$ the $\mathcal{P}$ is realized in the spherical space, for $n = 3$ in the Euclidean space and for $n > 3$ in the hyperbolic one.

| $n$ | $p^2$ | $u$ | $\delta$ |
|---|---|---|---|
| 2 | 2 | 0.30901 | $36^0$ |
| 3 | 1.5 | 0.33333 | $35.2643^0$ |
| 4 | 1 | 0.36602 | $34.2646^0$ |
| 5 | 0.690983 | 0.39915 | $33.4246^0$ |
| 6 | 0.5 | 0.41421 | $32.7652^0$ |
| 7 | 0.376510 | 0.43028 | $32.2571^0$ |
| 8 | 0.29289 | 0.44262 | $31.8644^0$ |
| 9 | 0.23395 | 0.45217 | $31.5583^0$ |
| 10 | 0.19098 | 0.45965 | $31.3117^0$ |
| $\infty$ | 0 | 0.5 | $30^0$ |

Table 2.

## 5. The fundamental set of the figure-eight conic structures

A cone-manifolds is a PL manifold [10] together with a, possibly empty, codimension two locally flat, submanifold called the singular set. ( In a 3-dimensional cone-manifold the singular set will consist of curves, but not graphs ). In this paper, a geometric cone-manifold will be modeled on some space of constant curvature [18]. Points of the singular set have neighborhood homeomorphic to neighborhoods in the model. Points on the singular set have neighborhoods homeomorphic to neighborhoods constructed as follows: take an angle $\alpha$ wedge in the model. ( A wedge is the intersection or union of two half spaces that intersect; the angle $\alpha$ is the dihedral angle where $0 < \alpha < 2\pi$ ). Then identify the two boundaries of the wedge, using the natural rotation by $\alpha$, to form a topological space $W_\alpha$. Points on the singular set have neighborhoods homeomorphic to neighborhoods in this topological space. The homeomorphism carries the singular set to the axis of rotation in the topological space. Transition functions are isometries. A singular curve $\Sigma$ whose points have only neighborhoods homeomorphic to neighborhoods in $W_\alpha$ will be called an $\alpha$-curve. Note that if we let $\alpha$ be equal to $2\pi$, then a $2\pi$-curve is actually a regular geodesic. But, of course, in general $\Sigma$ can be thought of as a singular geodesic.

**Theorem 5.1** *For every $2 \le n \le \infty$ the polyhedra $\mathcal{P}$ is a fundamental set for the conic manifold $\mathcal{O}(n)$. Moreover $\mathcal{O}(n)$ is spherical for $2 \le n < 3$, Euclidean for $n = 3$ and the hyperbolic for $n > 3$. The monodromy group of $\mathcal{O}(n)$ is generated by isometries $S$, $T$, which identify following curvilinear faces of polyhedra $\mathcal{P}$*

$$S : Q_1P_5 \ \ldots \ P_0 \to Q_1P_0 \ \ldots \ P_5$$

$$T : Q_0P_2 \ \ldots \ P_7 \to Q_0P_7 \ \ldots \ P_2$$

The proof of this theorem for the Euclidean case is given in Section 3. For the hyperbolic case it is based on the fact that the system (2) has the above described solution for all ( not necessary integer ) $n > 3$. For the spherical case $2 \le n < 3$ geometrical parameters of the tetrahedra $t_1, \ldots, t_9$ can be found from the spherical analog of the system (2):

$$\left\{ \begin{array}{l} \sin\frac{\gamma}{2} = \frac{\sin\frac{x}{2}}{\sin h} \\ \sin\delta = \frac{\sin x \sin\frac{\pi}{n}}{\sin h} \\ \gamma + 4\delta = \pi \\ \cos h = \frac{\cos z}{\cos\frac{x}{2}} \\ \cot y \tan\frac{x}{2} = \cot\frac{\pi}{n}\tan\delta \\ \cos z = \cos x \cos y \end{array} \right. \qquad (17)$$

Hence, as above we have the following equation respect the variable $u = cos\ 2\delta$:

$$2u^2p^2 + 2u - 1 = 0. \tag{18}$$

Thus,

$$u = \cos 2\delta = \frac{-1 + \sqrt{1 + 4\sin^2\frac{\pi}{n}}}{4\sin^2\frac{\pi}{n}} \tag{19}$$

$$\cos x = \frac{1 - u - 2u^2p^2}{2u^2p^2} \tag{20}$$

$$\cos h = \sqrt{1 - \frac{\cos x - 1}{2u^2}} \tag{21}$$

$$\cos z = \frac{\cos\frac{x}{2}}{\cos h} \tag{22}$$

$$\cos y = \frac{\cos z}{\cos x} \tag{23}$$

## 6. Algorithm for producing of the fundamental set of orbifold $\mathcal{O}$(p/q,n) in the hyperbolic space

The aim of this section to give an algorithm of construction of the canonical fundamental set of the orbifold $\mathcal{O}(p/q, n)$ in the hyperbolic space. For the sake of simplicity we suppose that the singular set of the orbifold consist of one component only. In this case $p$ is necessary odd. The algorithm under consideration can be extended to the 2-bridge orbifolds with two component singular set and even for conic manifolds with an arbitrary 2-bridge link singular set. It will be done in the forthcoming papers of authors.

### 6.1 Points and lines in the hyperbolic space $\mathbb{H}^3$.

We consider the upper half space model of the hyperbolic space $\mathbb{H}^3$. Recall that any matrix $l \in SL(2,\mathbb{C})$ with $tr\ l = 0$ determines a half-turn about a line $L$ in $\mathbb{H}^3$. A matrix $l$ with these properties will be called a *line matrix*. According to [3] we identify the line $L$ in $\mathbb{H}^3$ with the line matrix $l$ from $SL(2,\mathbb{C})$. Moreover, if $l$ determines the line $L$ and $F$ is an isometry of $\mathbb{H}^3$ determined by matrix $f \in SL(2,\mathbb{C})$, then $flf^{-1}$ determines the line $F(L)$.

**6.2 Canonical polyhedra for given representation of the fundamental group in $PSL(2,\mathbb{C})$.**

Let $\mathcal{O}(p/q,n)$ be hyperbolic orbifold and group

$$\Gamma(n) =< S, T \mid S^n = T^n = SWT^{-\varepsilon}W^{-1} = 1 > \qquad (24)$$

where

$$W = T^{\varepsilon_1}S^{\varepsilon_2}\dots T^{\varepsilon_{p-2}}S^{\varepsilon_{p-1}}, \quad e_i \equiv iq \ (\ mod\ 2p\ ), \ -p < e_i < p, \ \varepsilon_i = sign(e_i)$$

$$\varepsilon = -1 \ if\ q\ is\ even, \ \varepsilon = +1 \ if\ q\ is\ odd$$

is *faithful discrete presentation* of the fundamental group of $\mathcal{O}(p/q,n)$ in $PSL(2,\mathbb{C})$. All representations (24) of the group $\Gamma(n)$ of such type are carefully described in [13]. Using this presentation we write from now on $\Gamma = \Gamma(n)$ for two-bridge knot orbifold. We need the following definitions:

**Definition 6.2.1** A closed subset $\mathcal{P}$ of the $\mathbb{H}^3$ is said to be *topological polyhedron* if it is homeomorphic to the 3-ball an $\partial P$ contains connected planar graph $J$. Edges and vertices of the graph $J$ are called *edges* and *vertices* of $\mathcal{P}$.

**Definition 6.2.2** A topological polyhedron $\mathcal{P}$ is set to be HL-*polyhedron* if the following conditions are satisfied:
**H0.** Every vertex of the polyhedron $\mathcal{P}$ is a point in $\mathbb{H}^3$
**H1.** Every edge of the polyhedron $\mathcal{P}$ is a geodesic segment in $\mathbb{H}^3$
**H2.** Every face of the polyhedron $\mathcal{P}$ consists of finite number of geodesic triangles in $\mathbb{H}^3$.

**Definition 6.2.3** A HL-polyhedron $\mathcal{P}$ is set to be *canonical* for the group $\Gamma$ with representation (24) if the following conditions are satisfied:
**C0.** Vertices of the polyhedron $\mathcal{P}$ are the points $P_0, \ \dots, \ P_{2p}$
**C1.** Edges of the polyhedron $\mathcal{P}$ are the geodesic segments $P_iP_{i+1}, \ P_0P_p, \ P_qP_{p+q}$
**C2.** The polyhedron $\mathcal{P}$ has four faces. More precisely, if we denote by $Q_0$ and $Q_1$ the midpoints of the segments $P_qP_{p+q}$ and $P_0P_p$ then faces of polyhedron $\mathcal{P}$ are:

$$Q_1P_0\dots P_p = \cup_{j=0}^{p-1}Q_1P_jP_{j+1}, \quad Q_1P_p\dots P_0 = \cup_{j=p}^{2p-1}Q_1P_jP_{j+1}$$

$$Q_0P_0\dots P_p = \cup_{j=0}^{p-1}Q_0P_jP_{j+1}, \quad Q_0P_p\dots P_0 = \cup_{j=p}^{2p-1}Q_0P_jP_{j+1}$$

**C3.** Faces of $\mathcal{P}$ are mutually paired by isometries $S$ and $T$:

$$S : Q_1P_p \ \dots \ P_0 \to Q_1P_0 \ \dots \ P_p,$$

$$T : Q_0P_q \ \dots \ P_{p+q} \to Q_0P_{p+q} \ \dots \ P_q.$$

### 6.3 Algorithm for the construction of the canonical polyhedron $\mathcal{P}$.

Let us consider a faithful discrete presentation of the fundamental group of $\mathcal{O}(p/q, n)$ in $PSL(2,\mathbb{C})$ given by formula (24). Besides the elements $S$,$T$ and their inverses $s = S^{-1}$, $t = T^{-1}$ the following involutions $b$, $c$ and $e$ belonging to the group $PSL(2,\mathbb{C})$ will be used:

$$bS = Tb, b^2 = 1;$$
$$cS = tc, c^2 = 1;$$
$$eS = se, eT = te, e^2 = 1;$$

The involutions $b$ and $c$ exists and are uniquely determined by $S$ and $T$ if the orders of the rotations $S$ and $T$ are equal [3]. The involution $e$ exists in all cases. Besides, the involutions $b$, $c$ and $e$ commute and the following equation take place:

$$bce = 1.$$

Let us consider the group $\Gamma_0 =<\Gamma, b>$ if $q$ is even and $\Gamma_0 =<\Gamma, c>$ if $q$ is odd. By virtue of (24) the $\Gamma_0$ have the following representation:

$$\Gamma_0 =< \ S, T, b \mid S^n = T^n = b^2 = SWTW^{-1} = bSb^{-1}T^{-1} = 1 \ >$$

or

$$\Gamma_0 =< \ S, T, c \mid S^n = T^n = c^2 = SWT^{-1}W^{-1} = cSc^{-1}T = 1 \ > .$$

From the definition of $W$, $b$ and $c$ we have $(SWb)^2 = 1$ or $(SWc)^2 = 1$. Indeed,

$$W = T^{\varepsilon_1}S^{\varepsilon_2}\dots T^{\varepsilon_{p-2}}S^{\varepsilon_{p-1}}, \quad e_i \equiv iq \ ( \ mod \ 2p \ ), \ -p < e_i < p, \ \varepsilon_i = sign(e_i),$$

then if $\varepsilon = -1$

$$bWb^{-1} = S^{-\varepsilon_{p-1}}T^{-\varepsilon_{p-2}}\dots S^{-\varepsilon_2}T^{-\varepsilon_1}, \quad e_i \equiv iq \ ( \ mod \ 2p \ ), \ -p < e_i < p, \ \varepsilon_i = sign(e_i)$$

and $SWTW^{-1} = SWbSWb = 1$. If $\varepsilon = +1$ then

$$cWc^{-1} = S^{-\varepsilon_{p-1}}T^{-\varepsilon_{p-2}}\dots S^{-\varepsilon_2}T^{-\varepsilon_1}, \quad e_i \equiv iq \ ( \ mod \ 2p \ ), \ -p < e_i < p, \ \varepsilon_i = sign(e_i)$$

and $SWT^{-1}W^{-1} = SWcSWc = 1$.

Hence representation for the group $\Gamma_0$ can be rewritten in the form

$$\Gamma_0 =< \ S, T, b \mid S^n = T^n = b^2 = (SWb)^2 = bSb^{-1}T^{-1} = 1 \ >$$

or

$$\Gamma_0 =< \ S, T, c \mid S^n = T^n = c^2 = (SWc)^2 = cSc^{-1}T = 1 \ >,$$

depending on $\varepsilon = +1$ or $\varepsilon = -1$.

The following two lemmata play the key role for construction of the canonical polyhedron $\mathcal{P}$.

**Lemma 6.3.1** *Let* $G =< A, B \mid A^n = B^2 = (AB)^2 = 1 >$ *be a dihedral subgroup of the group* $PSL(2, \mathbb{C})$ *of order* $2n$, $n \geq 2$. *Then it has a unique fixed point in* $\mathbb{H}^3$.

See proof in [1].

**Lemma 6.3.2** *The subgroups* $< S, SWb >$ *and* $< eSe^{-1}, eSWbe^{-1} >$ *of the group* $\Gamma$ *are the dihedral groups of the order* $2n$.

Proof. From (24) it follows that the group $< S, SWb >$ has the following presentation:

$$< A, B \mid A^n = B^2 = (AB)^2 = 1 >,$$

where $A = S^{-1}$ and $B = SWb$. Hence the group $< S, SWb >$ is dihedral group of the order $2n$. The group $< eSe^{-1}, eSWbe^{-1} >$ coincides with the group $< S, SWb >^e = e < S, SWb > e^{-1}$.

Note that we have a lemma of a same kind in the case of odd $q$ for subgroups $< S, SWc >$ and $< eSe^{-1}, eSWce^{-1} >$.

Now we are able give an explicit description of the vertices, edges and faces of the polyhedron $\mathcal{P}$.

**a) Vertices of the polyhedron.**

Polyhedron $\mathcal{P}$ has $2p$ vertices. They are determined by the following conditions:

(i) $P_0$ is a fixed point of the dihedral group $< S, SWb >$;

(ii) $P_p$ is a fixed point of the dihedral group $< eSe^{-1}, eSWbe^{-1} >$, where $e$ is an isometry of $\mathbb{H}^3$ such that $eSe^{-1} = s,\ eTe^{-1} = t,\ e^2 = 1$;

(iii) The vertices $P_i$ with even indices $i$ are given by following formulas:
$P_{2r} = S^{-\varepsilon_{2k}} T^{-\varepsilon_{2k-1}} \ldots S^{-\varepsilon_2} T^{-\varepsilon_1}(P_0)$, where $k$ is such that $r = qk\ (\ mod\ 2p\ )$,
$\varepsilon_i = sign(e_i)\ (\ 1 \leq r < p\ )$ *and* $e_i \equiv iq\ (\ mod\ 2p\ ),\ -p < e_i < p$;

(vi) The vertices $P_i$ with odd indices $i$ are given by following formulas:
$P_{p+2r} = S^{\varepsilon_{2k}} T^{\varepsilon_{2k-1}} \ldots S^{\varepsilon_2} T^{\varepsilon_1}(P_p)$, where $k$ is such that $r = qk\ (\ mod\ 2p\ )$,
$\varepsilon_i = sign(e_i)\ (\ 1 \leq r < p\ )$ *and* $e_i \equiv iq\ (\ mod\ 2p\ ),\ -p < e_i < p$;

**b) Edges of the polyhedron.**

The edges are formed by geodesic segments $P_iP_{i+1}$, $i = 1,\ ..,\ 2p$, ( $mod\ 2p$ ) and $P_0P_p$, $P_qP_{p+q}$.

**c) Faces of the polyhedron.**

Denote by $Q_0$ and $Q_1$ the midpoints of the segments $P_qP_{p+q}$ and $P_0P_p$ respectively. Then the faces of $\mathcal{P}$ are given by

$$Q_1P_0\dots P_p = \cup_{j=0}^{p-1} Q_1P_jP_{j+1}, \quad Q_1P_p\dots P_0 = \cup_{j=p}^{2p-1} Q_1P_jP_{j+1}$$
$$Q_0P_0\dots P_p = \cup_{j=0}^{p-1} Q_0P_jP_{j+1}, \quad Q_0P_p\dots P_0 = \cup_{j=p}^{2p-1} Q_0P_jP_{j+1}$$

**Remark 6.3.1** The polyhedron $\mathcal{P}$ is well defined as non-convex polyhedron in $\mathbb{H}^3$. To see that one can represent $\mathcal{P}$ as a union of the 2p tetrahedra

$$Q_1Q_0P_0P_1, \ \dots, \ Q_1Q_0P_{2p-1}P_0$$

with mutually disjoint interiors.

**6.4 The statements of the main theorems.**

**Theorem A.** *There is unique canonical polyhedron for every given faithful discrete representation of the fundamental group of the orbifold $\mathcal{O}(p/q, n)$ in $PSL(2,\mathbb{C})$.*

**Theorem B.** *The canonical polyhedron $\mathcal{P}$ is a fundamental set for the group $\Gamma$ in $\mathbb{H}^3$.*

*Proof of the Theorem A.*
The existence of a canonical polyhedron $\mathcal{P}$ is a consequence of the algorithm described in 6.3.
The uniqueness is a consequence of the lemma 6.3.1.

*Proof of the theorem B.*
We claim that constructed polyhedron $\mathcal{P}$ is the fundamental set for the group $\Gamma$. Indeed, this polyhedron has just two pares of curvilinear faces mutually paired by $S$ and $T$:

$$S : Q_1P_p\dots P_0 \to Q_1P_0\dots P_p,$$
$$T : Q_0P_q\dots P_{p+q} \to Q_0P_{p+q}\dots P_q.$$

Let us verify the conditions of the Poincare theorem for polyhedron $\mathcal{P}$. We note that the polyhedron $\mathcal{P}$ is not necessary convex in this theorem. The edges of the polyhedron $\mathcal{P}$ form three equivalence classes. Among them there are two elliptic cycles
$P_0P_p$ ( Elliptic cycle condition of the Poincare theorem is a consequence of the relation $S^n = 1$ );

$P_q P_{p+q}$ ( Elliptic cycle condition of the Poincare theorem is a consequence of the relation $T^n = 1$ );

and one ordinary cycle

$P_0 P_1, \ldots, P_{2p-1} P_0$ ( Ordinary cycle condition of the Poincare theorem is a consequence of the main relation in group $SWT^{-1}W^{-1} = 1$ );

**Theorem C.** *Let $\mathcal{P}$ be canonical polyhedron for the group $\Gamma$ with representation (24). Then the edges $P_0P_1, \ldots, P_{2p-1}P_0$ of the polyhedron $\mathcal{P}$ forming the ordinary cycle are of the same length and the angle between any two adjacent edges in this ordinary cycle is equal to $\frac{2\pi}{n}$.*

*Proof of the Theorem C* is based on the following observation. All the edges $P_0P_1, \ldots, P_{2p-1}P_0$ are images of the edge $P_0P_1$ under the action of a isometry from the group $\Gamma$. Hence, all of them are of the same length. Consider two conjugated edges $P_{i-1}P_i$ and $P_iP_{i+1}$ and denote by $l_i$ the involution from the group $\Gamma_0 =< \Gamma, b >$ whose axes contains the edge $P_{i-1}P_i$ ( i mod 2p ). In order to proof that the angles between adjacent edges $P_{i-1}P_i$ and $P_iP_{i+1}$ is equal to $\frac{2\pi}{n}$ we need to show that the product of two involutions $l_i$ and $l_{i+1}$ is a rotation by the angle $\frac{4\pi}{n}$. We have

$$l_i l_{i+1} = r_i S^{\pm 2} r_i^{-1},$$

where $r_i$ is some element of group $\Gamma$, $i = 1, \ldots 2p$. Because $S^2$ is already rotation by the angle $\frac{4\pi}{n}$ we have that all transformations $l_il_{i+1}$ are rotations by the same angle.

**Remark 6.4.9** We note that 2-bridge knots $p/q$ and $p/(p-q)$ are equivalent. So, the orbifolds $\mathcal{O}(p/q, n)$ and $\mathcal{O}(p/(p-q), n)$ are isometric. In general case the canonical fundamental polyhedra of the orbifolds $\mathcal{O}(p/q, n)$ and $\mathcal{O}(p/(p-q), n)$ are different. But it is possible to show that they are formed from the same set of tetrahedra $Q_0Q_1P_iP_{i+1}$, $i = 0, \ldots, 2p-1$.

## 7. Examples

### 7.1 Preliminaries and remarks

The examples given below will be constructed by the following schema. All orbifolds have the knot $p/q$ labelled by n or the link $p/q$ with two components labelled by m and n respectively as singular set. Underlying space of orbifolds in all cases is the three dimensional sphere $\mathbb{S}^3$. The presentation of the fundamental group of the orbifold is given by

$$\Gamma =< \ S, T \mid S^n = T^n = SWT^{\pm 1}W^{-1} = 1 \ >,$$

where $W$ is defined from the formula (24) for knots and

$$\Gamma(m, n) =< \ S, T \mid S^m = T^n = SWS^{-1}W^{-1} \ >$$

for links, where $W$ is also defined from formula (24). Besides the elements $S$,$T$ and their inverses $s = S^{-1}$, $t = T^{-1}$ the following involutions $b$,$e$,$c$ will be used:

$$bS = Tb, b^2 = 1;$$
$$c = eb, cS = tc, c^2 = 1;$$
$$eS = se, eT = te, e^2 = 1;$$

The involutions $b$ and $c$ exist and are uniquely determined by $S$ and $T$ if the orders of the rotations $S$ and $T$ are equal [3]. The involution $e$ exists in all cases.

Let us consider the group $\Gamma_0 =< \Gamma, b >$ or $\Gamma_0 =< \Gamma, c >$ in the case of knot and $\Gamma_0 =< \Gamma, e >$ in the case of link. By virtue of (24) the $\Gamma_0$ have the following representation:

$$\Gamma_0 =< \ S, T, b \mid S^n = T^n = b^2 = SWTW^{-1} = bSb^{-1}T^{-1} = 1 \ >$$

or

$$\Gamma_0 =< \ S, T, c \mid S^n = T^n = c^2 = SWT^{-1}W^{-1} = cSc^{-1}T = 1 \ >$$

for odd p. Analogously,

$$\Gamma_0 =< \ S, T, e \mid S^m = T^n = e^2 = SWS^{-1}W^{-1} = eSe^{-1}S = 1 \ >$$

for even p. From the definition of $W$,$b$ and $c$ we have $(SWb)^2 = 1$, $(SWc)^2 = 1$ or $(SWe)^2 = 1$ depending on parity of $p$ and $q$. Hence representation for the group $\Gamma_0$ can be rewritten in the form

$$\Gamma_0 =< \ S, T, b \mid S^n = T^n = b^2 = (SWb)^2 = bSb^{-1}T^{-1} = 1 \ >$$

or

$$\Gamma_0 =< \ S, T, c \mid S^n = T^n = b^2 = (SWc)^2 = cSc^{-1}T = 1 \ >$$

for knots and

$$\Gamma_0 =< \ S, T, e \mid S^m = T^n = e^2 = (SWe)^2 = eSe^{-1}S = 1 \ >$$

for links. Thus, without lost of generality we may assume that the lifting of $\Gamma_0$ in $SL(2, \mathbb{C})$ has the following representation:

$$b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \ , \quad e = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \ , \quad c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \tag{25}$$

$$S = \begin{pmatrix} \cos\mu & i\sin\mu e^{\frac{\rho}{2}} \\ i\sin\mu e^{-\frac{\rho}{2}} & \cos\mu \end{pmatrix} \ , \quad T = \begin{pmatrix} \cos\nu & i\sin\nu e^{-\frac{\rho}{2}} \\ i\sin\nu e^{\frac{\rho}{2}} & \cos\nu \end{pmatrix} , \tag{26}$$

where $\mu = \frac{\pi}{m}$, $\nu = \frac{\pi}{n}$. The geometrical sense of the variable $\rho = r + i\phi$ is very simple. It is a complex distance between axis of elements $S$ and $T$. The matrices equalities $S^n = T^n =$

$b^2=(SWb)^2=-I$, $S^n=T^n=b^2=(SWc)^2=-I$ or $S^m=T^n=e^2=(SWe)^2=-I$ are valid, where

$$I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

We have

$$S^{-1}=\begin{pmatrix} \cos\mu & -i\sin\mu e^{\frac{\rho}{2}} \\ -i\sin\mu e^{-\frac{\rho}{2}} & \cos\mu \end{pmatrix}\ ,\quad T^{-1}=\begin{pmatrix} \cos\nu & -i\sin\nu e^{-\frac{\rho}{2}} \\ -i\sin\nu e^{\frac{\rho}{2}} & \cos\nu \end{pmatrix}. \tag{27}$$

The value of $\rho$ can be determined from the following equations:

$$tr\ SWb=0,\quad or\quad tr\ SWc=0 \tag{28}$$

if p odd and

$$tr\ SWe=0, \tag{29}$$

if p even. Note that algebraic equation for variable

$$z=trS^{-1}T=2\cos^2\mu+\sin^2\mu\cosh\rho$$

was obtained earlier in [12].

The vertices of the canonical fundamental polyhedron $\mathcal{P}$ of the group $\Gamma$ in $\mathbb{H}^3$ are given by dihedral subgroups of the group $<S,T,b>$, $<S,T,c>$ or $<S,T,e>$.

The edges of $\mathcal{P}$ are given by cyclic groups generated by elliptic elements of order 2 and can be calculated as intersection of the dihedral groups of their end points.

**Remark 7.1.1** In the simplest case of the figure-eight knot orbifold ( example 7.2, below ) the point $P_0=Fix<S,SWb>$ is unique fixed point of the group

$$<\ S,SWb\mid S^n=(SWb)^2=(Wb)^2=1\ >,$$

$P_1=Fix<s,sW^eb>^{ST}$ is unique fixed point of the group

$$<s,sW^eb>^{ST}=ST\ \ <s,sW^eb>\ \ (ST)^{-1}=$$

$$=<ST\ s\ (ST)^{-1},ST\ s\ eWe^{-1}\ b\ (ST)^{-1}>,$$

$P_0P_1=Fix<STStsb>$ is a segment with endpoints $P_0$ and $P_1$ fixed by the cyclic group

$$<\ STStsb\mid(STStsb)^2=1\ >.$$

We have in this case

$$<S,SWb>\ \cap\ <s,sW^eb>^{ST}=<STStsb>.$$

Indeed, $STStsb=SWb=(s^{ST})^{-2}(sW^eb)^{ST}$.

Thus, $STStsb \in < S, SWb >$, $STStsb \in < s, sW^e b >^{ST}$ and $STStsb$ is unique elliptic element of the groups $< S, SWb >$, $< s, sW^e b >^{ST}$ which axis is passing through points $P_0$, $P_1$. For simplicity we prefer write $P_0P_1 =< STStsb >$ instead of $P_0P_1 = Fix < STStsb >$.

**Lemma 7.1.2** *The edge $P_0P_1$ of the polyhedron $\mathcal{P}$ is defined by the following line matrix:*

$$l(P_0, P_1) = SWb,$$
$$W = b^{\varepsilon_1} S^{\varepsilon_2} \dots S^{\varepsilon_{2p-2}} b^{\varepsilon_{2p-1}}, \quad where$$
$$\varepsilon_i = sign(e_i) \; and \; e_i \equiv iq \; ( \; mod \; 4p \; ), \; -2p < e_i < 2p;$$
$$bS = Tb, \quad b^2 = 1$$

*Proof* is based on the fact that the group $< S, b \mid S^n = b^2 = (SWb)^2 = 1 >$ is the fundamental group of the orbifold $\mathcal{O}(2p/q, n, 2)$ with singular set 2-bridge link $2p/q$ with two components labeled by $n$ and 2. Note that we have a lemma of a same kind in the case of odd $q$ for

$$l(P_0, P_1) = SWc,$$

**Remark 7.1.3** Edges $P_iP_{i+1}$, $i = 1, \dots, 2p$ ( $mod$ $2p$ ) of the polyhedron $\mathcal{P}$ are equivalent to $P_0P_1$ and so can be defined by line matrices $l(P_j, P_{j+1}) = U_j l(P_0P_1) U_j^{-1}$ for suitable matrices $U_j$, $j = 1, \dots, 2p$ ( $mod$ $2p$ ) from group $\Gamma$. The edges $P_0P_p$ and $P_qP_{p+q}$ are given by line matrices $norm(S - S^{-1})$ and $norm(T - T^{-1})$ respectively, where $norm(M) = \frac{1}{\sqrt{det \; M}} M$ [3].

We will use the following simple observation.

Let $D$ be the dihedral group of the isometries $\mathbb{H}^3$ of the order $2n$, $n > 1$ and $h$ is an arbitrary isometry of $\mathbb{H}^3$. Denote by $D^h$ the dihedral group $hDh^{-1}$. Then the image of the unique fixed point of the group D under isometry $h$ is the ( unique ) fixed point of the group $D^h$.

The complex distances between axes $S$ ant $T$ and lengths of the edges of the polyhedron $\mathcal{P}$ are given in the tables for different values of the singular index.

Each example contains the picture of the knot or link $p/q$ which is a singular set of the orbifold $\mathcal{O}(p/q, n)$. The explicit construction of the canonical fundamental set of orbifold is also given.

In the following table the equivalence between our notations and Rolfsen notations [16] is given.

| our notations | Rolfsen notations |
|---|---|
| 5/2, 5/3 | $4_1$ |
| 7/2, 7/3 | $5_2$ |
| 8/3, 8/5 | $5_1^2$ |

Table 3.

### 7.2 The figure-eight knot orbifold $\mathcal{O}(5/2, n)$

From (24) we have the following presentation of the fundamental group of the orbifold $\mathcal{O}(5/2, n)$.

$$< S, T | S^n = T^n = SWTW^{-1} = 1 >, \; W = TSts$$

The complex distance equation in this case is

$$4u^2 - 2(M^2 + 1)u + M^4 + 4M^2 - 1 = 0,$$

where $u = cosh\ \rho$ and $M = cot\ \frac{\pi}{n}$.

In the following table $u$ is the solution of the equation above, $\rho = r + i\phi$ is the complex distance between axes of elements $S$ and $T$, $d$ is the length of edges $P_0P_5 = P_2P_7$ and $a$ is the common length of the edges $P_iP_{i+1}$, ( $i\ mod\ 10$ ).

Due to Fenchel [3], the complex distance $\rho(g, h)$ between two lines determined by involutions $g$ and $h$ can be find from the following equation: $cosh\ \rho(g, h) = \frac{1}{2}tr(gh)$.

Note that axis of element $S$ is passing through point $P_0$ and axis of element $S^{ST}$ is passing through point $P_1$. From Remark 7.1.1 we get

$$< S, SWb > \cap < s, sW^e b >^{ST} = < STSrsb > .$$

Geometrically it means that the axes of order $n$ elements $S$ and $S^{ST}$ are orthogonal to the order two element $STStsb$, which axis is passing through points $P_0$ and $P_1$. Thus we have that $a$ is a real part of the complex distance between axes of elements $S$ and $S^{ST}$. We find involutions determining axes of elements $S$ and $S^{ST}$ as

$$g = norm(S - s) = norm(S - S^{-1})$$

and

$$h = norm(S^{ST} - s^{ST}) = norm(STSts - STsts) =$$
$$= norm(STST^{-1}S^{-1} - STS^{-1}T^{-1}S^{-1})$$

respectively. We get $cosh\ \rho = cosh\ \rho(S, S^{ST}) = \frac{1}{2}(g, h)$ and

$$cosh\ (2a) = cosh\ (\rho + \overline{\rho}) = cosh\ \rho\ cosh\ \overline{\rho} + sinh\ \rho\ sinh\ \overline{\rho} = |cosh\ \rho|^2 + |sinh\ \rho|^2.$$

Now we are able to find $cosh\ a$ from the product of two involutions $g$ and $h$.

Analogously, we can find $cosh\ d$ using involution $SWb$ and $(SWb)^e$.

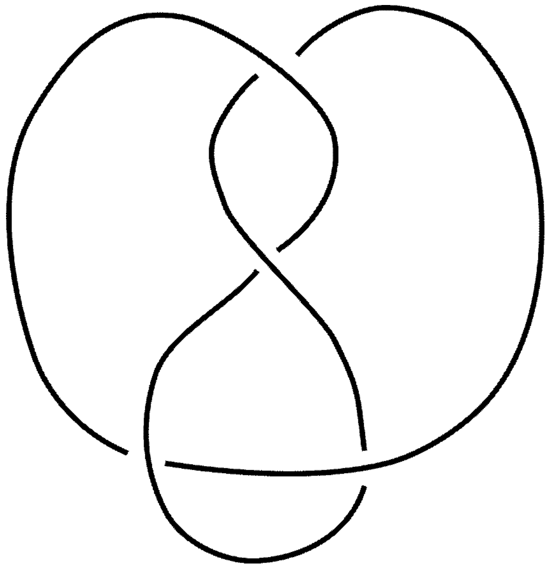

Figure 7. The $5/2 \equiv 5/3 \equiv 4_1$ knot ( The figure-eight knot ).

| $n$ | $u$ | $cosh\ r$ | $\phi$ | $cosh\ d$ | $cosh\ a$ |
|---|---|---|---|---|---|
| 4 | $0.5 - 0.866025i$ | 1.36603 | 68.5292° | 2 | 1.36603 |
| 5 | $0.72360 - 1.42058i$ | 1.84039 | 66.8492° | 2.11803 | 1.84039 |
| 6 | $1 - 2i$ | 2.41421 | 65.5304° | 2 | 2.41421 |
| 7 | $1.32799 - 2.63564i$ | 3.08626 | 64.5142° | 1.84601 | 3.08626 |
| 8 | $1.70711 - 3.34019i$ | 3.85683 | 63.7288° | 1.70711 | 3.85683 |
| 9 | $2.13716 - 4.12027i$ | 4.72648 | 63.1146° | 1.59240 | 4.72648 |

Table 4.

The vertices of the canonical fundamental polyhedron $\mathcal{P}(n)$ are given by

$$P_0 = Fix < S, SWb >, \quad P_1 = Fix < s, sW^e b >^{ST},$$
$$P_2 = Fix < S, SWb >^{stST}, \quad P_3 = Fix < s, sW^e b >^{tST},$$
$$P_4 = Fix < S, SWb >^{T}, \quad P_5 = Fix < s, sW^e b >,$$
$$P_6 = Fix < S, SWb >^{ST}, \quad P_7 = Fix < s, sW^e b >^{stST},$$
$$P_8 = Fix < S, SWb >^{tST}, \quad P_9 = Fix < s, sW^e b >^{T}$$

The edges of the canonical fundamental polyhedron $\mathcal{P}(n)$ are given by

$$P_0P_1 =< STStsb >, \ P_1P_2 =< TSTstb >, \ P_2P_3 =< STstSb >,$$
$$P_3P_4 =< tSTStb >, \ P_4P_5 =< StsTSb >, \ P_5P_6 =< tsTSTb >,$$

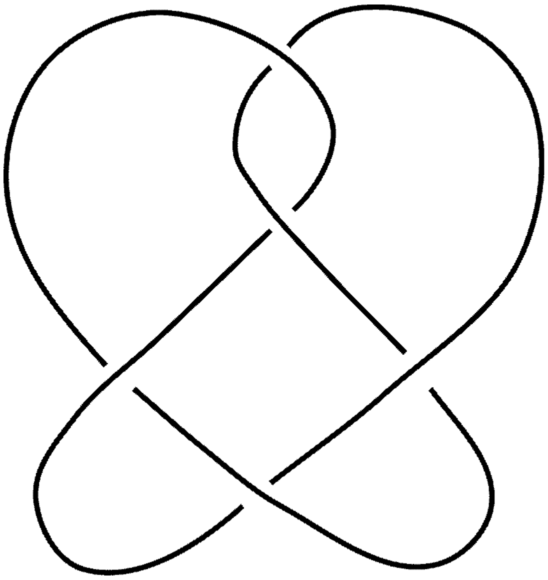

Figure 8. The $7/2 \equiv 7/3 \equiv 5_2$ knot.

$$P_6P_7 =< stSTSb >,\ P_7P_8 =< TstSTb >,\ P_8P_9 =< sTSTsb >,$$
$$P_9P_0 =< TStsTb >$$

The figure-eight orbifold is spherical if $n = 2$, Euclidean if $n = 3$ and hyperbolic if $n > 3$. Besides that, all geometrical parameters of the polyhedron $\mathcal{P}(n)$ can be calculated from formulas given in the Sections 2-4.

### 7.3 The orbifold $\mathcal{O}(7/2, n)$

From (24) we have the following presentation of the fundamental group of the orbifold $\mathcal{O}(7/2, n)$.

$$< S, T | S^n = T^n = SWTW^{-1} = 1 >,\ W = TSTsts$$

The vertices of the canonical fundamental polyhedron $\mathcal{P}(n)$ are given by

$$P_0 = Fix < S, SWb >, \quad P_{13} = Fix < s, eSWbe^{-1} >^{sTST},$$
$$P_{12} = Fix < S, SWb >^{TStst}, \quad P_{11} = Fix < s, eSWbe^{-1} >^{T},$$
$$P_{10} = Fix < S, SWb >^{st}, \quad P_9 = Fix < s, eSWbe^{-1} >^{stsTST},$$
$$P_8 = Fix < S, SWb >^{tst}, \quad P_7 = Fix < s, eSWbe^{-1} >,$$
$$P_6 = Fix < S, SWb >^{Stst}, \quad P_5 = Fix < s, eSWbe^{-1} >^{tsTST},$$
$$P_4 = Fix < S, SWb >^{t}, \quad P_3 = Fix < s, eSWbe^{-1} >^{ST},$$

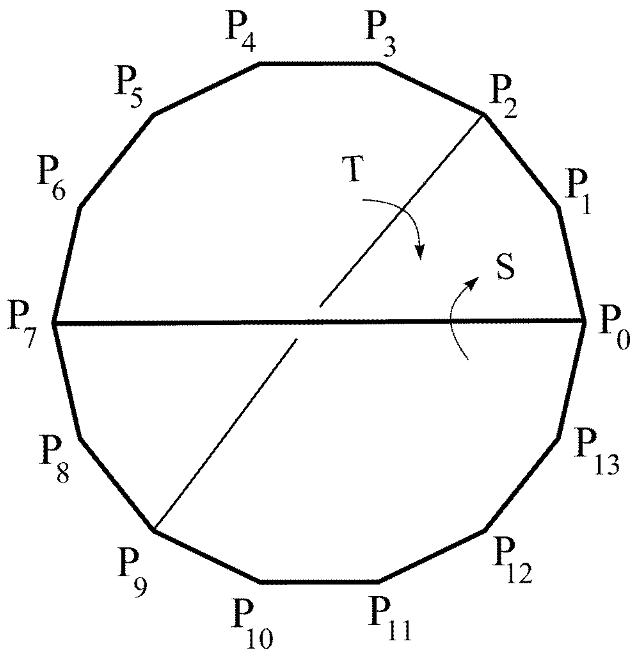


Figure 9. Schema of the fundamental set of the orbifold $\mathcal{O}(7/2, n)$

$$P_2 = Fix < S, SWb >^{STStst}, \quad P_1 = Fix < s, eSWbe^{-1} >^{TST}$$

The edges of the canonical fundamental polyhedron $\mathcal{P}(n)$ are given by

$$P_0P_{13} =< TSTstsTb >, \ P_{13}P_{12} =< sTSTStsb >, \ P_{12}P_{11} =< TStstSTb >,$$

$$P_{11}P_{10} =< stSTSTsb >, \ P_{10}P_9 =< TstsTSTb >, \ P_9P_8 =< stsTSTSb >,$$

$$P_8P_7 =< tstSTSTb >, \ P_7P_6 =< StstSTSb >, \ P_6P_5 =< tsTSTStb >,$$

$$P_5P_4 =< STstsTSb >, \ P_4P_3 =< tSTSTstb >, \ P_3P_2 =< STStstSb >,$$

$$P_2P_1 =< TSTStstb >, \ P_1P_0 =< STSTstsb >$$

The trace equation is

$$8u^3 - 4(3M^2 + 1)u^2 + (4M^4 + 8M^2 - 4)u + M^6 - M^4 + 7M^2 + 1 = 0,$$

where $u = cosh\ \rho$ and $M = cot\ \frac{\pi}{n}$.

In the following table $u$ is the solution of the equation above, $\rho = r + i\phi$ is the complex distance between axes of elements $S$ and $T$, $d$ is the length of edges $P_0P_7 = P_2P_9$ and $a$ is the length of edges $P_iP_{i+1}$, ( $i\ mod\ 14$ ). Due to Fenchel [3], the complex distance $\rho(g, h)$ between two involutions $g$ and $h$ can be find from the following equation: $cosh\ \rho(g, h) = \frac{1}{2}tr\ (gh)$. We can calculate parameter $a$ as a distance between axes of elements $T$ and $S^{ts}$ and parameter $d$ as a distance between axes of elements $SWb$ and $(SWb)^e$. Thus, $cosh\ a$ is determined by involutions $norm(T - t)$ and $norm(S^{ts} - s^{ts})$, $cosh\ d$ is determined by involutions $SWb$ and $(SWb)^e$.

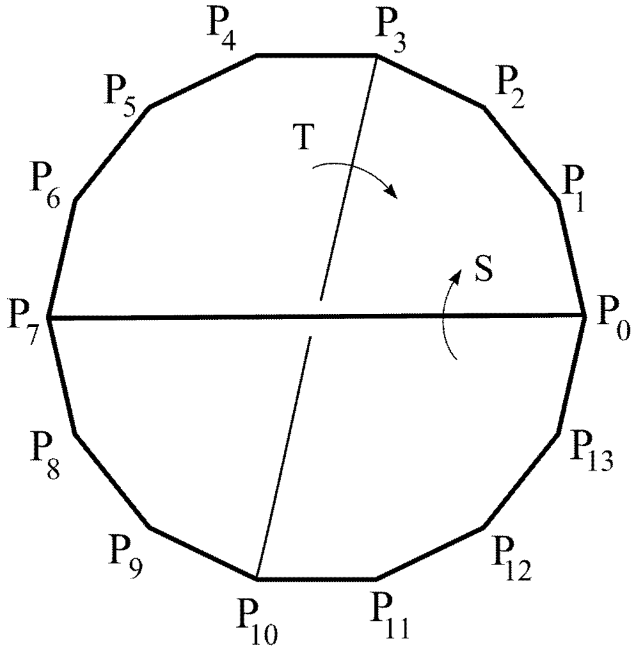


Figure 10. Schema of the fundamental set of the orbifold $\mathcal{O}(7/3, n)$

| $n$ | $u$ | $cosh\ r$ | $\phi$ | $cosh\ d$ | $cosh\ a$ |
|---|---|---|---|---|---|
| 3 | $0.716929 - 0.307986i$ | 1.12565 | 46.4958° | 2.20669 | 1.12234 |
| 4 | $0.558127 - 0.200389i$ | 1.59766 | 39.5002° | 2.98482 | 1.55480 |
| 5 | $0.460076 - 0.139596i$ | 2.27861 | 33.7576° | 2.61805 | 2.13187 |
| 6 | $0.388854 - 0.103016i$ | 3.17076 | 29.6761° | 2.20667 | 2.84944 |
| 7 | $0.335233 - 0.080183i$ | 4.26781 | 26.9034° | 1.90591 | 3.70684 |
| 8 | $0.293936 - 0.065191i$ | 5.56116 | 25.0101° | 1.69754 | 4.70273 |
| 9 | $0.261410 - 0.054814i$ | 7.04437 | 23.6851° | 1.55134 | 5.83583 |

Table 5.

### 7.4 The orbifold $\mathcal{O}(7/3, n)$

From (24) we have the following presentation of the fundamental group of the orbifold $\mathcal{O}(7/3, n)$.

$$< S, T | S^n = T^n = SWT^{-1}W^{-1} = 1 >, \ W = TStsTS$$

The vertices of the canonical fundamental polyhedron $\mathcal{P}(n)$ are given by

$$P_0 = Fix < S, SWc >, \quad P_{13} = Fix < s, eSWce^{-1} >^T,$$
$$P_{12} = Fix < S, SWc >^{Tst}, \quad P_{11} = Fix < s, eSWce^{-1} >^{TstST},$$

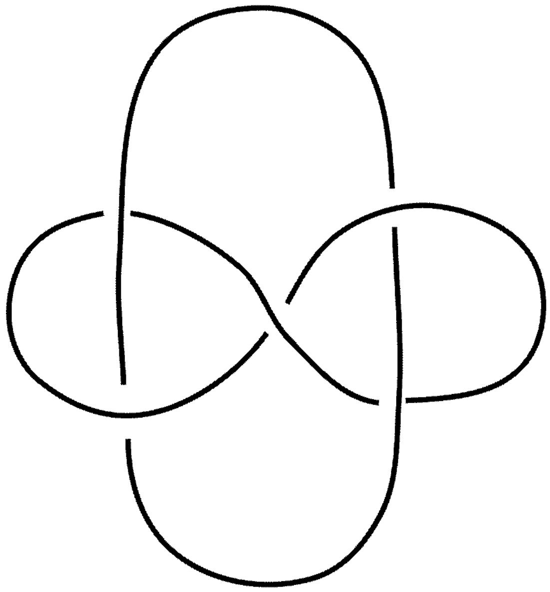

Figure 11. The $8/3 \equiv 8/5 \equiv 5_1^2$ link ( The Whitehead link ).

$$P_{10} = Fix < S, SWc >^{stSTst}, \quad P_9 = Fix < s, eSWce^{-1} >^{stST},$$
$$P_8 = Fix < S, SWc >^{st}, \quad P_7 = Fix < s, eSWce^{-1} >,$$
$$P_6 = Fix < S, SWc >^{t}, \quad P_5 = Fix < s, eSWce^{-1} >^{tST},$$
$$P_4 = Fix < S, SWc >^{tSTst}, \quad P_3 = Fix < s, eSWce^{-1} >^{STstST},$$
$$P_2 = Fix < S, SWc >^{STst}, \quad P_1 = Fix < s, eSWce^{-1} >^{ST}$$

The edges of the canonical fundamental polyhedron $\mathcal{P}(n)$ are given by

$$P_0P_1 =< STStsTSc >, \ P_1P_2 =< TStstSTc >, \ P_2P_3 =< STstSTSc >,$$
$$P_3P_4 =< tSTstSTc >, \ P_4P_5 =< StstSTsc >, \ P_5P_6 =< tSTStsTc >,$$
$$P_6P_7 =< StsTStsc >, \ P_7P_8 =< tsTStstc >, \ P_8P_9 =< stSTStsc >,$$
$$P_9P_{10} =< tstSTstc >, \ P_{10}P_{11} =< stSTstSc >, \ P_{11}P_{12} =< TstSTStc >,$$
$$P_{12}P_{13} =< sTStstSc >, \ P_{13}P_0 =< TStsTStc >$$

The trace equation is

$$8u^3 - 4(M^2 - 1)u^2 + (4M^4 + 8M^2 - 4)u - M^6 + M^4 + 9M^2 - 1 = 0,$$

where $u = cosh\ \rho$ and $M = cot\ \frac{\pi}{n}$.

In the following table $u$ is the solution of the equation above, $\rho = r + i\phi$ is the complex distance between axes of elements $S$ and $T$, $d$ is the length of edges $P_0P_7 = P_3P_{10}$ and $a$ is the length of edges $P_iP_{i+1}$, ( $i\ mod\ 14$ ).

| $n$ | $u$ | $cosh\ r$ | $\phi$ | $cosh\ d$ | $cosh\ a$ |
|---|---|---|---|---|---|
| 3 | $0.612444 - 0.487547i$ | 1.12234 | 77.0445° | 2.20669 | 1.12565 |
| 4 | $0.471262 - 0.377044i$ | 1.55480 | 77.3247° | 2.98482 | 1.59766 |
| 5 | $0.388163 - 0.313716i$ | 2.13187 | 77.8907° | 2.61805 | 2.27861 |
| 6 | $0.329760 - 0.269252i$ | 2.84944 | 78.4637° | 2.20667 | 3.17076 |
| 7 | $0.286201 - 0.235634i$ | 3.70684 | 78.9301° | 1.90591 | 4.26781 |
| 8 | $0.252537 - 0.209228i$ | 4.70273 | 79.2836° | 1.69754 | 5.56116 |
| 9 | $0.229790 - 0.187957i$ | 5.83583 | 79.5472° | 1.55134 | 7.04437 |

Table 6.

We can calculate parameters $a$ and $d$ from a product of two involutions. For example, $cosh\ a$ is determined by involutions $norm(S-s)$ and $norm(S^t-s^t)$, $cosh\ d$ is determined by involutions $SWc$ and $(SWc)^e$.

We note that 2-bridge knots 7/2 and 7/3 are equivalent. So, the orbifolds $\mathcal{O}(7/2,n)$ and $\mathcal{O}(7/3,n)$ are isometric. It is easy see from examples 7.3 and 7.4 that the canonical fundamental polyhedra of the orbifolds $\mathcal{O}(7/2,n)$ and $\mathcal{O}(7/3,n)$ are different. But it is possible to show that they are formed from the same set of tetrahedra $Q_0Q_1P_iP_{i+1}$, $i=0,\ldots,13$.

### 7.5 The orbifold $\mathcal{O}(8/3,m,n)$

From (24) we have the following presentation of the fundamental group of the orbifold $\mathcal{O}(8/3,m,n)$.

$$< S,T | S^m = T^n = SWS^{-1}W^{-1} = 1 >,\ W = TStstST$$

The vertices of the canonical fundamental polyhedron $\mathcal{P}(m,n)$ are given by

$$P_0 = Fix < S, SWe >, \quad P_1 = Fix < T, TVe >^{STSts},$$
$$P_2 = Fix < S, SWe >^{TSTst}, \quad P_3 = Fix < T, TVe >,$$
$$P_4 = Fix < S, SWe >^{STst}, \quad P_5 = Fix < T, TVe >^{tSTSts},$$
$$P_6 = Fix < S, SWe >^{t}, \quad P_7 = Fix < T, TVe >^{Sts},$$
$$P_8 = Fix < S, SWe >^{tsTSTst}, \quad P_9 = Fix < T, TVe >^{ts},$$
$$P_{10} = Fix < S, SWe >^{st}, \quad P_{11} = Fix < T, TVe >^{stSTSts},$$
$$P_{12} = Fix < S, SWe >^{Tst}, \quad P_{13} = Fix < T, TVe >^{s},$$
$$P_{14} = Fix < S, SWe >^{sTSTst}, \quad P_{15} = Fix < T, TVe >^{TSts},$$

where $V = STstsTS$.

The edges of the canonical fundamental polyhedron $\mathcal{P}(m,n)$ are given by

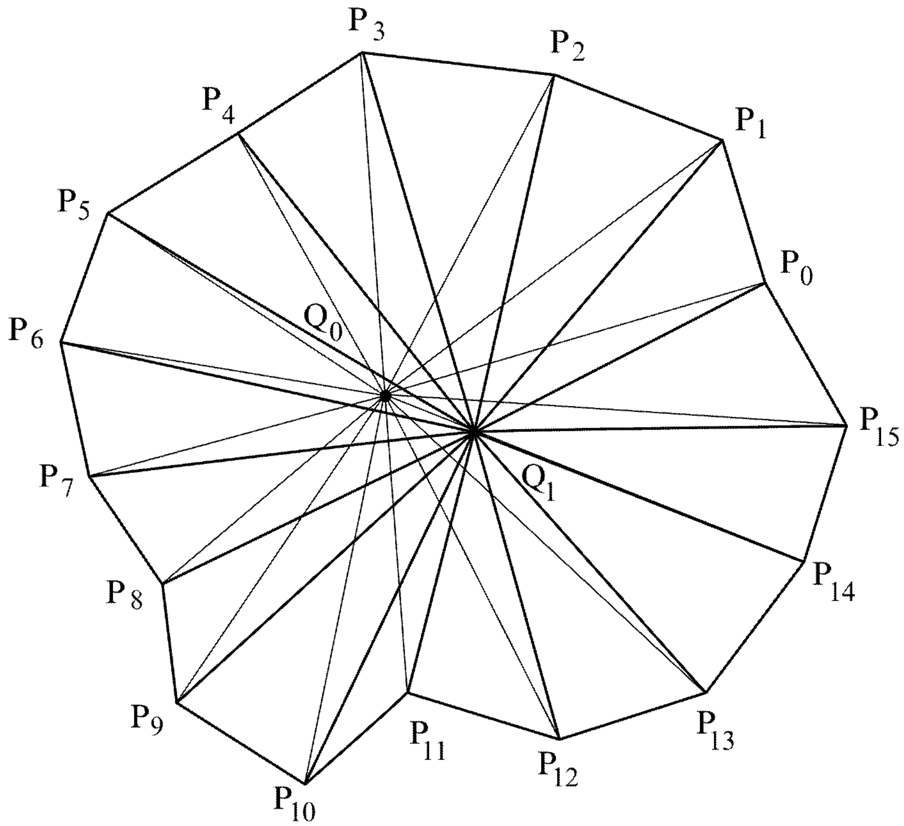


Figure 12. The fundamental set of the Euclidean cone-manifold $\mathcal{O}(8/3, \alpha, \alpha)$, $\alpha = 2.506179...$

$$P_0P_1 =< STStstSTe >,\ P_1P_2 =< TSTstSTSe >,\ P_2P_3 =< STstsTSTe >,$$

$$P_3P_4 =< tSTstsTSe >,\ P_4P_5 =< STstSTSte >,\ P_5P_6 =< tSTStstSe >,$$

$$P_6P_7 =< StstSTste >,\ P_7P_8 =< tsTSTstSe >,\ P_8P_9 =< stsTSTste >,$$

$$P_9P_{10} =< tstSTstse >,\ P_{10}P_{11} =< stSTStste >,\ P_{11}P_{12} =< TstSTStse >,$$

$$P_{12}P_{13} =< stSTstsTe >,\ P_{13}P_{14} =< TstsTSTse >,\ P_{14}P_{15} =< sTSTstSTe >,$$

$$P_{15}P_0 =< TStstSTse > .$$

The trace equation is

$$u^3 - MNu^2 + \frac{1}{2}(M^2N^2 + M^2 + N^2 - 1)u + MN = 0,$$

where $u = cosh\ \rho$ and $M = cot\ \frac{\pi}{m}$, $N = cot\ \frac{\pi}{n}$.

In the following table $u$ is the solution of the equation above, $\rho = r + i\phi$ is the complex distance between axes of elements $S$ and $T$, $d_1$ and $d_2$ are the lengths of the edges $P_0P_8, P_3P_{11}$ and $a$ is the length of edges $P_iP_{i+1}$, ( $i\ mod\ 16$ ).

| $m$ | $n$ | $u$ | $cosh\ r$ | $\phi$ | $cosh\ d_1$ | $cosh\ d_2$ | $cosh\ a$ |
|---|---|---|---|---|---|---|---|
| 3 | 3 | $0.407487 - 0.639200i$ | 1.15745 | 114.965° | 3.11347 | 3.11347 | 1.15745 |
| 3 | 4 | $0.356301 - 0.563225i$ | 1.34777 | 115.364° | 5 | 3 | 1.34777 |
| 4 | 4 | $0.300867 - 0.506038i$ | 1.61591 | 118.532° | 4.22226 | 4.22226 | 1.61591 |
| 3 | 5 | $0.326027 - 0.508943i$ | 1.55133 | 114.713° | 6.35443 | 2.46106 | 1.55134 |
| 4 | 5 | $0.269194 - 0.462720i$ | 1.88802 | 119.622° | 4.95050 | 3.29514 | 1.88804 |
| 5 | 5 | $0.237137 - 0.425330i$ | 2.22704 | 121.717° | 3.73607 | 3.73607 | 2.22703 |
| 3 | 6 | $0.303355 - 0.466707i$ | 1.76866 | 113.953° | 7.37898 | 2.04681 | 1.76867 |
| 4 | 6 | $0.247048 - 0.428196i$ | 2.16814 | 120.034° | 5.42711 | 2.65338 | 2.16815 |
| 5 | 6 | $0.215313 - 0.394886i$ | 2.57279 | 122.797° | 3.99780 | 2.94984 | 2.57280 |
| 6 | 6 | $0.194002 - 0.367234i$ | 2.98483 | 124.307° | 3.11347 | 3.11347 | 2.98483 |

Table 7.

We can calculate $cosh\ a$ using involutions $norm(T-t)$ and $norm(S^{ts}-s^{ts})$, $cosh\ d_1$ using involutions $SWe$ and $(SWe)^e$, $cosh\ d_2$ using involutions $SVe$ and $(SVe)^e$.

The canonical fundamental set for cone-manifolds $\mathcal{O}(8/3, m, n)$ can be constructed in all spaces of constant curvature. In particular, the approach described in Section 5 for the Euclidean figure-eight knot orbifold was realized in [17] for the Euclidean Whitehead link cone-manifold.

At the Figure 12 the fundamental polyhedron $\mathcal{P}$ of the Euclidean cone-manifold $\mathcal{O}(8/3, \alpha, \alpha)$ with singular index $\alpha = 2.506179\ldots$ on each component is shown.

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