ON THE VALIDITY OF THE LOG–SOBOLEV INEQUALITY FOR SYMMETRIC FLEMING–VIOT OPERATORS

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Abstract. We prove that Fleming–Viot operators with parent–independent mutation satisfy a logarithmic Sobolev inequality if and only if the set of types is finite.

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1. Introduction

Let (X, μ) be a finite measure space and $(\mathcal{E}, D(\mathcal{E}))$ be a densely defined (not necessarily closed) quadratic form on $L^2(\mu)$. $(\mathcal{E}, D(\mathcal{E}))$ is said to determine a logarithmic Sobolev inequality with constant $c > 0$ if

$$\int f^2 \log f^2 \, d\mu \leq c\,\mathcal{E}(f,f) + \|f\|^2_{L^2(\mu)} \log \|f\|^2_{L^2(\mu)}$$

for all $f \in D(\mathcal{E})$. Such kind of inequality has been invented in the context of quantum field theory as a tool to prove hypercontractivity of semigroups associated with certain infinite–dimensional elliptic differential operators. Meanwhile, this tool has found many other applications also in finite dimensions and the logarithmic Sobolev inequality has been verified in the case of many important examples of stochastic analysis (cf. [Ba], [Gr] and references therein). Hence it is a remarkable fact that in the class of measure–valued diffusions there is up to now not one single example in which a logarithmic Sobolev inequality has been verified. The purpose of this paper now is to give an answer to the question, whether or not a logarithmic Sobolev inequality holds for generators of Fleming–Viot processes with parent–independent mutation and, if a logarithmic Sobolev inequality does not hold, whether or not we can find a reasonable substitute for this inequality. Fleming–Viot processes can

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be viewed as diffusion approximations of empirical processes associated with a certain class of discrete time Markov chains in population genetics (cf. [EK2]) and are (apart from Dawson–Watanabe processes) the best studied class of measure–valued diffusions. Before we state our main result let us first define Fleming–Viot processes: Let S be a complete separable metric space which is interpreted as a space of types of a given population. Throughout this paper we will assume that S is compact. Let $E := \mathcal{M}_1(S)$ be the space of all probability measures on S (i.e., all possible distributions of types within the given population) equipped with the weak topology. One can then introduce random mutation on the population with the help of a Feller generator A (i.e., the generator of a sub–Markovian C_0–semigroup on the space $C(S)$ of all continuous functions on S). Throughout the whole paper we will only consider bounded mutation operators of the following type

$$Af(x) = \frac{\theta}{2} \int_S (f(y) - f(x))\, \nu_0(dy)\,,\, f \in C(S)\ ,$$

where $\theta > 0$ and $\nu_0 \in E$ such that $\operatorname{supp}(\nu_0) = S$. The Fleming–Viot process associated with mutation operator A (with no recombination and no selection) is called the Fleming–Viot process with parent–independent mutation and is defined as the unique solution of the $C_E[0,\infty)$–martingale problem $(L_{\theta,\nu_0}, \mathcal{F}C^\infty)$, where

$$\mathcal{F}C^\infty := \{F = \varphi(\langle f_1, \cdot\rangle, \dots, \langle f_d, \cdot\rangle) | f_i \in C(S), \varphi \in C^\infty(\mathbb{R}^d), d \in \mathbb{N}\} \tag{1.1}$$

and

$$\begin{aligned} L_{\theta,\nu_0} F(\mu) := & \frac{1}{2} \sum_{i,j=1}^{d} (\partial_{x_i}\partial_{x_j}\varphi)(\langle f_1, \mu\rangle, \dots, \langle f_d, \mu\rangle)\operatorname{cov}_\mu(f_i, f_j) \\ & + \frac{1}{2} \sum_{i=1}^{d} (\partial_{x_i}\varphi)(\langle f_1, \mu\rangle, \dots, \langle f_d, \mu\rangle)\langle Af_i, \mu\rangle \end{aligned} \tag{1.2}$$

(cf. [EK2]). Here $\langle f, \mu\rangle := \int f\, d\mu$. It is well–known that this process has a unique stationary distribution $m_{\theta,\nu_0} \in \mathcal{M}_1(E)$ which is even symmetrizing (cf. [EK2, 8.1]). m_{θ,ν_0} can be described as follows: Let $(\rho_1, \rho_2, \dots)$ have a Poisson–Dirichlet distribution with parameter θ and let $(\xi_n)_{n\in\mathbb{N}}$ be i.i.d. with distribution ν_0 and independent of $(\rho_1, \rho_2, \dots)$. Then

$$m_{\theta,\nu_0}[A] = P\left[\sum_{i=1}^{\infty} \rho_i \delta_{\xi_i} \in A\right]\ . \tag{1.3}$$

The Dirichlet form $(\mathcal{E}_{\theta,\nu_0}, H^{1,2}(m_{\theta,\nu_0}))$ associated with the symmetric Fleming–Viot operator $(L_{\theta,\nu_0}, \mathcal{F}C^\infty)$ is obtained as the closure of the bilinear form $(-L_{\theta,\nu_0}F, G)_{L^2(m_{\theta,\nu_0})}$, $F, G \in \mathcal{F}C^\infty$, in $L^2(m_{\theta,\nu_0})$. We will prove in Proposition 3.4 that $\mathcal{E}_{\theta,\nu_0}$ determines a Poincaré inequality with constant $\frac{2}{\theta}$ (i.e., the corresponding generator has a mass gap of size $\frac{\theta}{2}$). We will show in the appendix that the existence of a mass gap can also be deduced from a result obtained by S.N.

Ethier and R.C. Griffith in [EG] concerning the convergence to equilibrium in the total variation norm of the transition semigroup of the Fleming–Viot process. However, by that method we do not obtain the exact constant $\frac{\theta}{2}$. The main result of this paper can then be formulated as follows (cf. Theorem 3.5): The bilinear form $(\mathcal{E}_{\theta,\nu_0}, H^{1,2}(m_{\theta,\nu_0}))$ determines a logarithmic Sobolev inequality if and only if $|S| < +\infty$. In this case the best (i.e., smallest) constant for which a logarithmic Sobolev inequality holds can be estimated from above by $\frac{320}{\min_{s\in S}\nu_0(\{s\})}$ (cf. Remark 2.9 concerning a further discussion of this constant). We also found that the situation is even worse: The set

$$\{F^2 | F \in H^{1,2}(m_{\theta,\nu_0}), \mathcal{E}_{\theta,\nu_0}(F,F) + \|F\|^2_{L^2(m_{\theta,\nu_0})} \leq 1\}$$

is uniformly integrable if and only if $|S| < +\infty$ which implies that there is no reasonable substitute for the logarithmic Sobolev inequality which can be formulated in terms of the bilinear form $\mathcal{E}_{\theta,\nu_0}$ and which could serve as an infinite dimensional substitute for compactness in the class of symmetric Fleming–Viot operators.

2. The finite dimensional case

We start with the case where the type space S is finite and thus S (resp. $\mathcal{M}_1(S)$) can be identified with the set $\{1, \ldots, |S|\}$ (resp. the $(|S|-1)$–dimensional simplex $\Delta_{|S|-1} = \{x \in \mathbb{R}^{|S|-1} | x_i \geq 0 \text{ and } \sum_{i=1}^{|S|-1} x_i \leq 1\}$).
Throughout the paper let $|x| := \sum_{i=1}^d x_i$ for any vector $x \in \mathbb{R}^d$ and $\mathbb{R}^d_+ := \{x \in \mathbb{R}^d | x_i > 0\,, 1 \leq i \leq d\}$. Let

$$C^\infty(\Delta_d) := \{f \in C(\Delta_d) | \exists g \in C^\infty(\mathbb{R}^d) \text{ such that } g_{|\Delta_d} = f\}\,.$$

It is then easy to see that in the finite dimensional case expression (1.2) reduces to

$$\begin{aligned} L_q f(x) &= \frac{1}{2}\sum_{i=1}^d x_i \partial^2_{x_i} f(x) - \frac{1}{2}\sum_{i,j=1}^d x_i x_j \partial_{x_i}\partial_{x_j} f(x) \\ &\quad + \frac{1}{2}\sum_{i=1}^d (q_i - |q| x_i)\, \partial_{x_i} f(x)\,, f \in C^\infty(\Delta_d)\,, \end{aligned}$$

with $q \in \mathbb{R}^{d+1}_+$, $q_i = \nu_0(\{i\})$, $1 \leq i \leq d+1$.

Definition 2.1. If $q \in \mathbb{R}^{d+1}_+$ denote by $D(q)$ the Dirichlet distribution with parameters q_i, $1 \leq i \leq d+1$, on Δ_d. $D(q)$ is the measure given by

$$\nu(dx) := \frac{\Gamma(|q|)}{\prod_{i=1}^{d+1}\Gamma(q_i)} \prod_{i=1}^d x_i^{q_i - 1} (1 - |x|)^{q_{d+1}-1}\, dx_1 \ldots dx_d\,.$$

Denote by ϱ_q its density.

For $q \in \mathbb{R}_+^{d+1}$ the Dirichlet distribution is a symmetrizing measure for the operator L_q. The associated bilinear form $(\mathcal{E}_q, C^\infty(\Delta_d))$ is given by

$$\mathcal{E}_q(f,g) := \frac{1}{2} \sum_{i,j=1}^{d} \int x_i(\delta_{ij} - x_j)\partial_{x_i} f \partial_{x_j} g \, \varrho_q \, dx \, ; f,g \in C^\infty(\Delta_d) \; .$$

$(\mathcal{E}_q, C^\infty(\Delta_d))$ is closable in $L^2(D(q))$ (cf. [MR, I.2 and I.3]). Let $H^{1,2}(D(q))$ be the domain of the closure. It is easy to see that the generator $(L_q, D(L_q))$ associated with the closure extends the operator $(L_q, C^\infty(\Delta_d))$.

Remark 2.2. (i) It is known that L_q has a discrete spectrum with eigenvalues $n(n+|q|-1)/2$ and multiplicity $\binom{n+d-1}{n}$, $n \geq 0$ (cf. [Sh]). In particular, L_q has a mass gap of size $\frac{|q|}{2}$ (independent of the dimension), which implies L^2–ergodicity of the associated semigroup $(e^{tL_q})_{t\geq 0}$. If $q_i \geq \frac{1}{2}$ for all i, we know from [Sh] that the L^2–semigroup is ultracontractive, i.e., $\|e^{tL_q}\|_{2,\infty} < \infty$ for all $t > 0$ (more precisely, $\|e^{tL_q}\|_{2,\infty} \leq$ constant $\cdot t^{-d/2}$, $t > 0$). Consequently, a logarithmic Soboelv inequality for $\mathcal{E}_q$ could be obtained using [Da, Theorem 2.2.3] and the existence of a mass gap. We emphasize that due to the restriction $q_i \geq \frac{1}{2}$ for all i this result cannot be used to obtain a logarithmic Sobolev inequality in the general finite dimensional case.
(ii) Similar to the space $H^{1,2}(D(q))$ one can define the space $H_0^{1,2}(D(q))$ as the closure of the subspace $C_0^\infty(\overset{\circ}{\Delta}_d)$ in $H^{1,2}(D(q))$, where $\overset{\circ}{\Delta}_d$ denotes the open interior of Δ_d. It is known that the two spaces coincide if and only if $q_i \geq 1$ for all $1 \leq i \leq d+1$ (cf. [Sh, Lemma 1.1]).
(iii) By Theorem 3.4 in [EK2] the closure of $(L_q, C^\infty(\Delta_d))$ in $C(\Delta_d)$ generates a Feller semigroup. Consequently, $(\alpha - L_q)(C^\infty(\Delta_d)) \subset C(\Delta_d)$ dense for all $\alpha > 0$ (cf. [EK1, 1.2.3]). Since $C(\Delta_d) \subset L^2(D(q))$ densely and continuously we conclude that $(\alpha - L_q)(C^\infty(\Delta_q)) \subset L^2(D(q))$ dense and hence $C^\infty(\Delta_d)$ is dense in $D(L_q)$ w.r.t. the graph norm.

The next two Propositions are the main tool in the proof of the logarithmic Sobolev inequality in the finite dimensional case.

Proposition 2.3. *Let $q \in \mathbb{R}_+^{d+1}$ and assume that $(\mathcal{E}_q, C^\infty(\Delta_d))$ determines a logarithmic Sobolev inequality with constant c. Let $(k_n)_{n\leq m+1} \subset \mathbb{N}$ such that $0 = k_0 < k_1 < \ldots < k_m < k_{m+1} = d+1$ and $p_n := \sum_{l=k_{n-1}+1}^{k_n} q_l$, $1 \leq n \leq m+1$. Then $(\mathcal{E}_p, C^\infty(\Delta_m))$ too determines a logarithmic Sobolev inequality with constant c.*

Proof. Let $T : \Delta_d \to \Delta_m \, , x \mapsto (\sum_{l=k_0+1}^{k_1} x_l, \sum_{l=k_1+1}^{k_2} x_l, \ldots, \sum_{l=k_{m-1}+1}^{k_m} x_l)$. Then $T(D(q)) = D(p)$ by the amalgamation property of Dirichlet distributions (cf. [FKoN, Theorem 1.4]). Let $f \in C^\infty(\Delta_m)$. Then by the change of variables formula

$$\begin{aligned}
\mathcal{E}_q(f\circ T, f\circ T) &= \frac{1}{2}\sum_{i,j=1}^{d}\int x_i(\delta_{ij}-x_j)\partial_{x_i}(f\circ T)\,\partial_{x_j}(f\circ T)\,\varrho_q\,dx \\
&= \frac{1}{2}\sum_{i=1}^{d}\int x_i\left(\partial_{x_i}(f\circ T)\right)^2\,\varrho_q\,dx - \frac{1}{2}\sum_{i,j=1}^{d}\int x_i x_j\partial_{x_i}(f\circ T)\partial_{x_j}(f\circ T)\,\varrho_q\,dx \\
&= \frac{1}{2}\sum_{i=1}^{m}\int\left(\sum_{l=k_{j-1}+1}^{k_j} x_l\right)\left(\partial_{z_j} f\right)^2\circ T\,\varrho_q\,dx \\
&\quad - \frac{1}{2}\sum_{i,j=1}^{m}\int\left(\sum_{k=k_{i-1}+1}^{k_i} x_k\right)\left(\sum_{l=k_{j-1}+1}^{k_j} x_l\right)(\partial_{z_i} f)\circ T(\partial_{z_j} f)\circ T\,\varrho_q\,dx \\
&= \frac{1}{2}\sum_{i,j=1}^{m}\int z_i(\delta_{ij}-z_j)\partial_{z_i} f\,\partial_{z_j} f\,\varrho_p\,dz\ .
\end{aligned}$$

Since $(\mathcal{E}_q, C^\infty(\Delta_d))$ determines a logarithmic Sobolev inequality with constant c it follows from the change of variables formula again that

$$\begin{aligned}
\int f^2\log f^2\,\varrho_p\,dx &= \int (f\circ T)^2\log(f\circ T)^2\,\varrho_q\,dx \\
&\le c\,\mathcal{E}_q(f\circ T, f\circ T) + \|f\circ T\|^2_{L^2(D(q))}\log\|f\circ T\|^2_{L^2(D(q))} \\
&= c\,\mathcal{E}_p(f,f) + \|f\|^2_{L^2(D(p))}\log\|f\|^2_{L^2(D(p))}\ . \quad \square
\end{aligned}$$

Proposition 2.4. *(Additivity principle) Let* $q\in\mathbb{R}_+^{d+2}$ *and*

$$T:[0,1]\times\Delta_d\to\Delta_{d+1}, (t,z)\to(z, t(1-|z|))\ .$$

(i) Let $f\in C^\infty(\Delta_{d+1})$. *Then*

$$\begin{aligned}
&\mathcal{E}_q(f,f) \\
&\quad = \int_0^1 \mathcal{E}_{(q_1,\dots,q_d,q_{d+1}+q_{d+2})}((f\circ T)(t,\cdot),(f\circ T)(t,\cdot))\varrho_{(q_{d+1},q_{d+2})}(t)\,dt \\
&\quad\quad + \int_{\Delta_d}\frac{1}{1-|z|}\mathcal{E}_{(q_{d+1},q_{d+2})}((f\circ T)(\cdot,z),(f\circ T)(\cdot,z))\varrho_{(q_1,\dots,q_d,q_{d+1}+q_{d+2})}(z)\,dz\ .
\end{aligned}\tag{2.1}$$

(ii) If $(\mathcal{E}_{(q_1,\dots,q_d,q_{d+1}+q_{d+2})}, C^\infty(\Delta_d))$ *and* $(\mathcal{E}_{(q_{d+1},q_{d+2})}, C^\infty([0,1]))$ *determine logarithmic Sobolev inequalities with constant c then* $(\mathcal{E}_q, C^\infty(\Delta_{d+1}))$ *too determines a logarithmic Sobolev inequality with constant c.*

Proof. For the proof of (i) and (ii) let $\hat{q} := (q_1,\dots,q_{d+1})$, $\bar{q} := (q_1,\dots,q_d,q_{d+1}+q_{d+2})$, $q' = (q_{d+1},q_{d+2})$. Then $T(D(q')\otimes D(\bar{q})) = D(q)$ by [FKoN, Theorem 1.4].

(i) Let us first calculate the right hand side of (2.1). For simplicity we introduce the following notations:

$$d_i f(t,z) := (\partial_i f)(T(t,z)) \ , 1 \le i \le d+1 \ .$$

Then

$$\partial_t (f \circ T)(t,z) = (1-|z|) d_{d+1} f(t,z)$$

and

$$\partial_{z_i} (f \circ T)(t,z) = d_i f(t,z) - t d_{d+1} f(t,z) \ .$$

It follows that

$$\begin{aligned} I &:= \frac{1}{2} \sum_{i,j=1}^{d} z_i(\delta_{ij} - z_j) \partial_{z_i}(f \circ T)(t,z) \partial_{z_j}(f \circ T)(t,z) \\ &= \frac{1}{2} \sum_{i,j=1}^{d} z_i(\delta_{ij} - z_j) d_i f(t,z) d_j f(t,z) - \sum_{i=1}^{d} t z_i (1-|z|) d_i f(t,z) d_{d+1} f(t,z) \\ &\quad + \frac{1}{2} t^2 |z| (1-|z|) (d_{d+1} f)^2(t,z) \end{aligned}$$

and

$$II := \frac{1}{2} \frac{1}{1-|z|} t(1-t) (\partial_t (f \circ T))^2 (t,z) = \frac{1}{2} (1-|z|) t(1-t) (d_{d+1} f)^2 (t,z) \ .$$

Adding both terms we obtain that

$$\begin{aligned} I + II = \frac{1}{2} &\sum_{i,j=1}^{d} z_i(\delta_{ij} - z_j) d_i f(t,z) d_j f(t,z) - \sum_{i=1}^{d} t z_i (1-|z|) d_i f(t,z) d_{d+1} f(t,z) \\ &+ \frac{1}{2} t (1-|z|)(1 - t(1-|z|)) (d_{d+1} f)^2 (t,z) \end{aligned}$$

and hence by the change of variables formula

$$\begin{aligned} &\int_0^1 \mathcal{E}_{\bar{q}}((f \circ T)(t,\cdot), (f \circ T)(t,\cdot)) \varrho_{q'}(t) \, dt \\ &\qquad + \int_{\Delta_d} \frac{1}{1-|z|} \mathcal{E}_{q'}((f \circ T)(\cdot, z), (f \circ T)(\cdot, z)) \varrho_{\bar{q}}(z) \, dz \\ &= \frac{1}{2} \sum_{i,j=1}^{d+1} \int_{\Delta_{d+1}} x_i (\delta_{ij} - x_j) \partial_{x_i} f(x) \partial_{x_j} f(x) \varrho_q(x) \, dx \ . \end{aligned}$$

(ii) The proof of (ii) is a small modification of Faris' additivity theorem (cf. [Gr, 2.3]). Let $f \in C^\infty(\Delta_{d+1})$. Then

$$\begin{aligned} \int f^2 \log f^2 \varrho_q \, dx &\le \int \int (f \circ T)^2(t,z) \log (f \circ T)^2 (t,z) \varrho_{\bar{q}}(z) \, dz \varrho_{q'}(t) \, dt \\ &\le c \int \mathcal{E}_{\bar{q}}((f \circ T)(t,\cdot), (f \circ T)(t,\cdot)) \varrho_{q'}(t) \, dt \qquad (2.2) \\ &\quad + \int \|(f \circ T)(t,\cdot)\|^2_{L^2(D(\bar{q}))} \log \|(f \circ T)(t,\cdot)\|^2_{L^2(D(\bar{q}))} \varrho_{q'}(t) \, dt \ , \end{aligned}$$

since $\mathcal{E}_{\bar q}$ determines a logarithmic Sobolev inequality with constant c. Since $\mathcal{E}_{q'}$ determines a logarithmic Sobolev inequality with constant c we obtain from the Semiboundedness–Theorem ([Gr, 2.1]) that

$$\begin{aligned}\int (f\circ T)^2(t,z)\log \|(f\circ T)(t,\cdot)\|^2_{L^2(D(\bar q))}\varrho_{q'}(t)\,dt \\ \le c\,\mathcal{E}_{q'}((f\circ T)(\cdot,z),(f\circ T)(\cdot,z)) \\ + \|(f\circ T)(\cdot,z)\|^2_{L^2(D(\bar q))}\log\|f\|^2_{L^2(D(\bar q))}\end{aligned}$$

for all $z\in\Delta_d$. Integrating the last inequality w.r.t. $\frac{1}{1-|z|}\varrho_{\bar q}(z)\,dz$ we conclude that

$$\begin{aligned}&\int\int (f\circ T)^2(t,z)\log\|(f\circ T)(t,\cdot)\|^2\varrho_{q'}(t)\,dt\varrho_{\bar q}(z)\,dz \\ &\le c\int\frac{1}{1-|z|}\mathcal{E}_{q'}((f\circ T)(\cdot,z),(f\circ T)(\cdot,z))\varrho_{\bar q}(z)\,dz \\ &+\|f\|^2_{L^2(D(q))}\log\|f\|^2_{L^2(D(q))}\end{aligned}\tag{2.3}$$

and combinig (2.2), (2.3) and (2.1) we obtain that

$$\int f^2\log f^2\varrho_q\,dx\le c\,\mathcal{E}_q(f,f)+\|f\|^2_{L^2(D(q))}\log\|f\|^2_{L^2(D(q))}\ .$$

This proves (ii). $\square$

The one–dimensional case.

Lemma 2.5. *Let $q\in\mathbb{R}^2_+$, $\min\{q_1,q_2\}\ge\frac12$. Then $(\mathcal{E}_q, C^\infty([0,1]))$ determines a logarithmic Sobolev inequality with constant $\frac{4}{\frac{|q|}{2}+\min\{q_1,q_2\}-1}$.*

Proof. In order to prove the assertion it is enough to show that the following inequality

$$\Gamma_2(f,f)\ge\frac12(\frac{|q|}{2}+\min\{q_1,q_2\}-1)\Gamma(f,f)\tag{2.4}$$

is satisfied for all $f\in C^\infty([0,1])$. Here $\Gamma(f,f)(x)=\frac12 x(1-x)\dot f^2(x)$ is the square field operator associated with $\mathcal{E}_q$ and

$$\Gamma_2(f,f)(x)=\frac12\{L_q\Gamma(f,f)(x)-2\Gamma(L_qf,f)(x)\}$$

is the iterated gradient. Indeed, if $(e^{tL_{\theta,\nu_0}})_{t\ge0}$ denotes the semigroup corresponding to the generator of $(\mathcal{E}_q,H^{1,2}(D(q)))$ it follows from [Sh, (6.2)] that $e^{tL_{\theta,\nu_0}}(C^\infty([0,1]))\subset C^\infty([0,1])$, $t\ge0$. It is well–known that inequality (2.4) then implies that $(\mathcal{E}_q, C^\infty([0,1]))$ determines a logarithmic Sobolev inequality with constant $\frac{4}{\frac{|q|}{2}+\min\{q_1,q_2\}-1}$ (cf. [Ba, Prop. 6.5]). To prove inequality (2.4) note that

$$\begin{aligned}\Gamma_2(f,f)(x) &= \frac{1}{4}x^2(1-x)^2\ddot{f}^2(x) + \frac{1}{4}x(1-x)(1-2x)\dot{f}(x)\ddot{f}(x) \\ &\quad + \frac{1}{4}\left((|q|-1)\,x(1-x) + \frac{1}{2}\,(q_1-|q|x)\,(1-2x)\right)\dot{f}^2(x)\ . \end{aligned} \tag{2.5}$$

We may assume that $q_1 \le q_2$, i.e., $q_1 \le \frac{|q|}{2}$. Then

$$\begin{aligned}\Gamma_2(f,f)(x) &\ge \frac{1}{4}\left(-\frac{1}{4}(1-2x)^2 + (|q|-1)\,x(1-x) + \frac{1}{2}(q_1-|q|x)(1-2x)\right)\dot{f}^2(x) \\ &= \frac{1}{4}\left(\frac{1}{2}(q_1-\frac{1}{2}) + (\frac{|q|}{2}-q_1)x\right)\dot{f}^2(x) \\ &\ge \frac{1}{4}\left((2q_1-1)x(1-x) + (\frac{|q|}{2}-q_1)x(1-x)\right)\dot{f}^2(x) \\ &= \frac{1}{2}(\frac{|q|}{2}+q_1-1)\Gamma(f,f)(x)\,, x\in[0,1]\ ,\end{aligned}$$

which implies the assertion. □

Note that in the particular case $q_1 = q_2$ inequality (2.5) shows that Γ_2 is no longer positive definite if $q_1 < \frac{1}{2}$. Consequently, the Γ_2–criterion cannot be applied in order to prove a logarithmic Sobolev inequality for Fleming–Viot operators in the general one–dimensional case.

Lemma 2.6. *Let $q \in (0,1)^2$. Then $(\mathcal{E}_q, C^\infty([0,1]))$ determines a logarithmic Sobolev inequality with constant $\frac{10}{\min\{q_1,q_2\}\min\{1-q_1,1-q_2\}}$.*

Proof. First let $f \in C^\infty([0,1])$ be such that $\int f\varrho_q\,dx = 0$. Then there exists $x_0 \in (0,1)$ with $f(x_0) = 0$. If $x \in [0,1]$ then

$$\begin{aligned}|f(x)| = |\int_{x_0}^x \dot{f}(s)\,ds| &\le \left|\int_{x_0}^x \dot{f}^2(s)s^{q_1}(1-s)^{q_2}\,ds\right|^{\frac{1}{2}} \cdot \left|\int_{x_0}^x s^{-q_1}(1-s)^{-q_2}\,ds\right|^{\frac{1}{2}} \\ &\le \sqrt{2}\,\underbrace{B(q_1,q_2)^{\frac{1}{2}}B(1-q_1,1-q_2)^{\frac{1}{2}}}_{=:\,\alpha}\,\mathcal{E}_q(f,f)^{\frac{1}{2}}\end{aligned}$$

where B denotes the Beta function. By Young's inequality, i.e., $st \le s\log s - s + e^t$ for all $s \ge 0$ and $t \in \mathbb{R}$, we conclude that

$$\int f^2\log f^2\varrho_q\,dt \le \|f\|^2\log\|f\|^2 - \|f\|^2 + 2\alpha^2\mathcal{E}_q(f,f)\ . \tag{2.6}$$

For general $f \in C^\infty([0,1])$ let $\tilde{f} := f - \int f\varrho_q dx$. By [Ba, Prop. 3.8]

$$\begin{aligned}\int f^2\log f^2\varrho_q\,dx - \|f\|^2\log\|f\|^2 &\le \int \tilde{f}^2\log\tilde{f}^2\varrho_q\,dx - \|\tilde{f}\|^2\log\|\tilde{f}\|^2 + 2\|\tilde{f}\|^2 \\ &\le 2\alpha^2\mathcal{E}_q(f,f) + \|\tilde{f}\|^2 \le 2(\alpha^2+\frac{1}{|q|})\mathcal{E}_q(f,f)\ ,\end{aligned}$$

where we used (2.6) in the last but one inequality and in the last inequality the fact that $\mathcal{E}_q$ determines a Poincaré inequality with constant $\frac{2}{|q|}$. Note that

$$\begin{aligned} B(q_1, q_2) &= \int_0^\infty \frac{t^{q_1-1}}{(1+t)^{|q|}}\, dt \le \int_0^1 t^{q_1-1}\, dt + \int_1^\infty t^{-q_2-1}\, dt \\ &\le \frac{1}{q_1} + \frac{1}{q_2} \le \frac{2}{\min\{q_1, q_2\}} \end{aligned}$$

and similarly, $B(1-q_1, 1-q_2) \le \frac{2}{\min\{1-q_1, 1-q_2\}}$, which implies that $2(\alpha^2 + \frac{1}{|q|}) \le \frac{10}{\min\{q_1,q_2\}\min\{1-q_1,1-q_2\}}$. $\square$

Lemma 2.7. *Let $q \in \mathbb{R}_+^2$. Then $(\mathcal{E}_q, C^\infty([0,1]))$ determines a logarithmic Sobolev inequality with constant $\frac{320}{\min\{q_1,q_2\}}$.*

Proof. We may assume that $q_1 \le q_2$.
1. Case $q_1 \ge \frac{2}{3}$: Then $\frac{|q|}{2} + q_1 - 1 \ge 2q_1 - 1 \ge \frac{q_1}{2}$ and 2.5 implies that $(\mathcal{E}_q, C^\infty([0,1]))$ determines a logarithmic Sobolev inequality with constant $\frac{8}{q_1}$ which implies the assertion in this case.

2. Case $q_1 < \frac{2}{3}$:
(i) $q_2 \ge \frac{5}{6}$:
(a) $|q| \ge \frac{3}{2}$: Let $p := (|q| - \frac{5}{6}, \frac{5}{6})$. Then $(\mathcal{E}_p, C^\infty([0,1]))$ determines a logarithmic Sobolev inequality with constant $\frac{48}{5}$ by 2.5. Since $(\mathcal{E}_{(\frac{5}{6}-q_1,q_1)}, C^\infty([0,1]))$ determines a logarithmic Sobolev inequality with constant $\frac{240}{q_1}$ by 2.6 it follows from 2.4 (ii) that $(\mathcal{E}_{(|q|-\frac{5}{6},\frac{5}{6}-q_1,q_1)}, C^\infty(\Delta_2))$ determines a logarithmic Sobolev inequality with constant $\frac{240}{q_1}$ and consequently, $(\mathcal{E}_q, C^\infty([0,1]))$ determines a logarithmic Sobolev inequality with constant $\frac{240}{q_1}$ by 2.3 which implies the assertion in this case.
(b) $|q| < \frac{3}{2}$: Let $p := (\frac{|q|}{2}, \frac{|q|}{2})$. Then $(\mathcal{E}_p, C^\infty([0,1]))$ determines a logarithmic Sobolev inequality with constant $\frac{40}{q_1}$ by 2.6 and since $\frac{|q|}{2} - q_1 = \frac{q_2-q_1}{2} \ge \frac{1}{12}$ it follows that $(\mathcal{E}_{(\frac{|q|}{2}-q_1,q_1)}, C^\infty([0,1]))$ determines a logarithmic Sobolev inequality with constant $\frac{320}{q_1}$ by 2.6. By 2.4 (ii) we obtain that $(\mathcal{E}_{(\frac{|q|}{2},\frac{|q|}{2}-q_1,q_1)}, C^\infty(\Delta_2))$ determines a logarithmic Sobolev inequality with constant $\frac{320}{q_1}$ and consequently $(\mathcal{E}_q, C^\infty([0,1]))$ determines a logarithmic Sobolev inequality with constant $\frac{320}{q_1}$ by 2.3. Hence the assertion is proved in this case.
(ii) $q_2 < \frac{5}{6}$: Then $(\mathcal{E}_q, C^\infty([0,1]))$ determines a logarithmic Sobolev inequality with constant $\frac{60}{q_1}$ by 2.6 which implies the assertion in this case. $\square$

The (general) finite–dimensional case.

Theorem 2.8. *Let $q \in \mathbb{R}_+^{d+1}$. Then $(\mathcal{E}_q, C^\infty(\Delta_d))$ determines a logarithmic Sobolev inequality with constant $\frac{320}{\min\{q_1,\cdots,q_{d+1}\}}$.*

Proof. We will prove the assertion using induction. The case $d = 1$ is contained in 2.7. Suppose that the assumption is proved for all $p \in \mathbb{R}_+^{d+1}$ and let $q \in \mathbb{R}_+^{d+2}$.

Since $\mathcal{E}_{(q_1,\cdots,q_d,q_{d+1}+q_{d+2})}$ determines a logarithmic Sobolev inequality with constant $\frac{320}{\min\{q_1,\cdots,q_{d+1}+q_{d+2}\}}$ by assumption and $\mathcal{E}_{(q_{d+1},q_{d+2})}$ determines a logarithmic Sobolev inequality with constant $\frac{320}{\min\{q_{d+1},q_{d+2}\}}$ by 2.7 it follows from 2.4 (ii) that $\mathcal{E}_q$ determines a logarithmic Sobolev inequality with constant $\frac{320}{\min\{q_1,\cdots,q_{d+2}\}}$. $\square$

Remark 2.9. The Rothaus–Simon mass gap theorem (cf. e.g. [Gr, 2.5]) states that if a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on a probability space with $1 \in D(\mathcal{E})$ and $\mathcal{E}(1,1) = 0$ determines a logarithmic Sobolev inequality with constant c then it satisfies a Poincaré inequality with constant $\geq \frac{c}{2}$. Since we already know that $\mathcal{E}_q$ satisfies a Poincaré inequality with constant $\frac{2}{|q|}$ one could think that the constant specified in the theorem is not very precise. Although we did not try to find the best (i.e., smallest) constant c_q such that $\mathcal{E}_q$ determines a logarithmic Sobolev inequality with constant c_q, c_q will depend on $\min\{q_1, \ldots, q_{d+1}\}$ rather that $|q|$ as will be clear from the following example: Let $q \in \mathbb{R}_+^{d+1}$ such that $q_1 = \min\{q_1, \ldots, q_{d+1}\}$ and $f(x) = x_1$. Then by 2.3

$$\begin{aligned}
&\frac{1}{q_1}\left(\mathcal{E}_q(f,f) + \|f\|^2 \log \|f\|^2\right) \\
&\qquad = \frac{1}{q_1}\left(\frac{1}{2}\frac{q_1(|q|-q_1)}{|q|(|q|+1)} + \frac{q_1(q_1+1)}{|q|(|q|+1)}\log\left(\frac{q_1(q_1+1)}{|q|(|q|+1)}\right)\right) \\
&\qquad = \frac{1}{2}\frac{|q|-q_1}{|q|(|q|+1)} + \frac{q_1+1}{|q|(|q|+1)}\log\left(\frac{q_1(q_1+1)}{|q|(|q|+1)}\right) \to -\infty
\end{aligned}$$

if $q_1 \to 0$ and $|q|$ remains constant whereas

$$\frac{1}{q_1}\int f^2 \log f^2 \varrho_q \, dx = \frac{2}{q_1}\int f(f\log f)\varrho_q \, dx \geq -2e^{-1}\frac{1}{q_1}\int f\varrho_q \, dx = -2e^{-1}\frac{1}{|q|}$$

remains bounded from below independent of q. Hence we have found an example for which the mass gap is strictly bigger than $\frac{2}{c_q}$ (cf. [KorS] for another example in this direction which is similar to our case a differential operator with degenerating second order part).

3. The infinite dimensional case

Let S be a compact space and $E = \mathcal{M}_1(S)$ be the space of all probability measures. Fix some $\nu_0 \in E$ with $\operatorname{supp}(\nu_0) = S$, some $\theta > 0$ and let

$$Af(x) = \frac{\theta}{2}\int_S (f(y) - f(x))\,\nu_0(dy)\,,\, f \in C(S)\ . \tag{3.1}$$

Let $(L_{\theta,\nu_0}, \mathcal{F}C^\infty)$ be as in the Introduction (cf. (1.1), (1.2)) and let $m_{\theta,\nu_0} \in \mathcal{M}_1(E)$ be the unique stationary distribution of the Fleming–Viot process associated with $(L_{\theta,\nu_0}, \mathcal{F}C^\infty)$ (cf. (1.3)). Let us first note some important feature of the measure m_{θ,ν_0}:

Lemma 3.1. *Let $(A_n)_{n\le d+1}$ be a measurable partition of S such that $\nu_0(A_i)>0$ and $\Pi : E \to \Delta_d, \mu \mapsto (\mu(A_1),\dots,\mu(A_d))$. Then*

$$\Pi(m_{\theta,\nu_0}) = D(\theta(\mu(A_1),\dots,\mu(A_d))) \ .$$

Proof. (cf. [OvRS, 7.2]).

For $F : E \to \mathbb{R}$ that admits a representation $F(\mu) = \varphi(\langle f_1,\mu\rangle,\dots,\langle f_d,\mu\rangle)$, $\varphi \in C^\infty(\mathbb{R}^d)$, $f_i \in \mathcal{B}_b(S)$, $1 \le i \le d$, we can define a gradient $\nabla F : S \times E \to \mathbb{R}$ as follows: Let

$$\begin{aligned}\nabla_x F(\mu) &:= \frac{dF}{ds}(\mu + s\delta_x)|_{s=0}\\ &= \sum_{i=1}^d (\partial_{x_i}\varphi)(\langle f_1,\mu\rangle,\dots,\langle f_d,\mu\rangle) f_i(x) \ .\end{aligned}$$

If $F, G \in \mathcal{F}C^\infty$ it is then easy to see, using symmetry and invariance of L_{θ,ν_0}, that

$$\begin{aligned}-\int (L_{\theta,\nu_0}F)\, G\, dm_{\theta,\nu_0} &= -\frac{1}{2}\int (L_{\theta,\nu_0}F)\, G\, dm_{\theta,\nu_0} - \frac{1}{2}\int F(L_{\theta,\nu_0}G)\, dm_{\theta,\nu_0}\\ &= -\frac{1}{2}\int L_{\theta,\nu_0}(FG)\, dm_{\theta,\nu_0} + \frac{1}{2}\int \operatorname{cov}_\mu(\nabla F(\mu), \nabla G(\mu))\, m_{\theta,\nu_0}(d\mu)\\ &= \frac{1}{2}\int \operatorname{cov}_\mu(\nabla F(\mu), \nabla G(\mu))\, m_{\theta,\nu_0}(d\mu) \ .\end{aligned}$$

The corresponding closure $(\mathcal{E}_{\theta,\nu_0}, H^{1,2}(m_{\theta,\nu_0}))$ in $L^2(m_{\theta,\nu_0})$ is then the Dirichlet form corresponding to the Fleming–Viot process with mutation operator A as defined in (3.1).

Lemma 3.2. *Let $(A_n)_{n\le d+1}$ be a measurable partition of S such that $\nu_0(A_i)>0$ and $F(\mu) = \varphi(\langle 1_{A_1},\mu\rangle,\dots,\langle 1_{A_d},\mu\rangle)$, $\varphi \in C^\infty(\mathbb{R}^d)$. Then $F \in H^{1,2}(m_{\theta,\nu_0})$ and*

$$\mathcal{E}_{\theta,\nu_0}(F,F) = \mathcal{E}_{\theta(\nu_0(A_1),\dots,\nu_0(A_{d+1}))}(\varphi,\varphi) \ .$$

Proof. Let $q := \theta(\nu_0(A_1),\dots,\nu_0(A_{d+1}))$. It follows from [OvRS, 6.3] that $F \in H^{1,2}(m_{\theta,\nu_0})$ and

$$\mathcal{E}_{\theta,\nu_0}(F,F) = \int \operatorname{var}_\mu(\nabla F(\mu))\, m_{\theta,\nu_0}(d\mu) \ .$$

Hence,

$$\begin{aligned}\mathcal{E}_{\theta,\nu_0}(F,F) &= \int \operatorname{var}_\mu(\nabla F(\mu))\, m_{\theta,\nu_0}(d\mu)\\ &= \sum_{i,j=1}^d \int (\partial_{x_i}\varphi\partial_{x_j}\varphi)(\mu(A_1),\dots,\mu(A_d))\mu(A_i)(\delta_{ij} - \mu(A_j))\, m_{\theta,\nu_0}(d\mu)\\ &= \sum_{i,j=1}^d \int x_i(\delta_{ij} - x_j)\partial_{x_i}\varphi(x)\partial_{x_j}\varphi(x)\varrho_q(x)\, dx\\ &= \mathcal{E}_q(\varphi,\varphi) \ ,\end{aligned}$$

where we used 3.1 in the last but one equality. □

Proposition 3.3. *Let $\theta > 0$, $\nu_0 \in E$. Then $(\mathcal{E}_{\theta,\nu_0}, H^{1,2}(m_{\theta,\nu_0}))$ determines a Poincaré–inequality with constant $\frac{2}{\theta}$.*

Proof. We have to show that

$$\int (F - \langle F \rangle)^2 \, dm_{\theta,\nu_0} \leq \frac{2}{\theta} \mathcal{E}_{\theta,\nu_0}(F,F)$$

for all $F \in H^{1,2}(m_{\theta,\nu_0})$. Here we set $\langle F \rangle := \int F \, dm_{\theta,\nu_0}$. Since $\mathcal{F}C^\infty \subset H^{1,2}(m_{\theta,\nu_0})$ dense it is sufficient to prove the inequality for all $F \in \mathcal{F}C^\infty$. To this end fix $F(\mu) = \varphi(\langle f_1, \mu \rangle, \ldots, \langle f_d, \mu \rangle)$. Since each f_i can be uniformly approximated by a sequence of elementary step functions we can construct a sequence of measurable partitions $(A_n^m)_{n \leq m+1}$ and constants $c_{i,n}^m \in \mathbb{R}$ such that $f_i^m := \sum_{n=1}^{m+1} c_{i,n}^m 1_{A_n^m} \to f_i$, $m \to \infty$, uniformly for all i, $1 \leq i \leq d$.

For all m let $\varphi_m \in C^\infty(\mathbb{R}^{m+1})$ be defined by

$$\varphi_m(x) := \varphi(\sum_{n=1}^{m+1} c_{1,n}^m x_n, \ldots, \sum_{n=1}^{m+1} c_{d,n}^m x_n) \ , x \in \mathbb{R}^{m+1} \ .$$

Then

$$F_m(\mu) := \varphi_m(\langle 1_{A_1^m}, \mu \rangle, \ldots, \langle 1_{A_{m+1}^m}, \mu \rangle) = \varphi(\langle f_1^m, \mu \rangle, \ldots, \langle f_d^m, \mu \rangle) \to F(\mu)$$

for all $\mu \in E$ and in $L^p(m_{\theta,\nu_0})$ for all $p \geq 1$ by Lebesgue's Theorem. Similarly,

$$\nabla F_m(\mu) = \sum_{i=1}^{m} \partial_{x_i} \varphi(\langle f_1^m, \mu \rangle, \ldots, \langle f_d^m, \mu \rangle) f_i^m \to \nabla F(\mu)$$

for all $\mu \in E$ and in $L^p(m_{\theta,\nu_0})$ for all $p \geq 1$. Note that if $\nu_0(A_i^m) = 0$ for some i then $m_{\theta,\nu_0}\{\mu | \mu(A_i^m) = 0\} = 1$ (cf. [OvRS, 7.2]). Let $I := \{i | \nu_0(A_i^m) > 0\}$. Then $|I| > 0$ and we may assume that $I = \{1, \ldots, |I|\}$. Let $B_i := A_i^m$, $i < |I|$, and $B_{|I|} := \bigcup_{i=|I|}^{m+1} A_i^m$. Then $(B_i)_{i \leq |I|}$ is a measurable partition and $\nu_0(B_i) > 0$ for all i. Let $\psi_m(x) := \varphi_m(x, 1 - |x|, 0, \ldots, 0)$, $x \in \mathbb{R}^{|I|-1}$ and $F_m'(\mu) := \psi_m(\langle 1_{B_1}, \mu \rangle, \ldots, \langle 1_{B_{|I|_1}}, \mu \rangle)$. Then $F_m' = F_m$ m_{θ,ν_0}–a.s. If $|I| > 1$ let
$q_m := \theta(\nu_0(B_1), \ldots, \nu_0(B_{|I|}))$ and note that $|q_m| = \theta$. It follows from 3.1 and 3.2 that

$$\langle F_m \rangle = \int F_m'(\mu) \, dm_{\theta,\nu_0} = \int \psi_m \varrho_{q_m} \, dx =: \langle \psi_m \rangle$$

and

$$\begin{aligned} \int (F_m - \langle F_m \rangle)^2 dm_{\theta,\nu_0} &= \int (\psi_m - \langle \psi_m \rangle)^2 \varrho_{q_m} \, dx \\ &\leq \frac{2}{\theta} \mathcal{E}_{q_m}(\psi_m, \psi_m) = \frac{2}{\theta} \mathcal{E}_{\theta,\nu_0}(F_m, F_m) \end{aligned} \tag{3.2}$$

since $(\mathcal{E}_{q_m}, C^\infty(\Delta_{|I|-1}))$ satisfies a Poincaré–inequality with constant $\frac{2}{\theta}$. If $|I| = 1$ then $F_m \equiv \psi_m(1)$ m_{θ,ν_0}–a.s. and thus

$$\int (F_m - \langle F_m \rangle)^2 m_{\theta,\nu_0}(d\mu) = 0 = \frac{2}{\theta} \mathcal{E}_{\theta,\nu_0}(F_m, F_m) \ .$$

Consequently, (3.2) holds in this case too. Passing to the limit $m \to \infty$ in the last inequality we obtain the assertion. $\square$

Remark 3.4. We will show in the appendix that the existence of a mass gap can be deduced also from the following result obtained by S.N. Ethier and R.C. Griffith in [EG]: Let $(p_t^{\theta,\nu_0})_{t\geq 0}$ be the transition semigroup of the Fleming–Viot process corresponding to the generator $(L_{\theta,\nu_0}, \mathcal{F}C^\infty)$. Then

$$
\|p_t^{\theta,\nu_0}(\mu,\cdot) - m_{\theta,\nu_0}\|_{\mathrm{var}} \leq 1 - d_0^\theta(t)\,, t > 0\,, \mu \in E\,, \tag{3.3}
$$

where $\|\cdot\|_{\mathrm{var}}$ denotes the total variation norm, $d_0^\theta(t) = P[D_t = 0]$, $t > 0$, and $(D_t)_{t\geq 0}$ is a pure death process in $\mathbb{Z}_+ \cup \{+\infty\}$ starting at $+\infty$ with death rates $n(n+\theta-1)/2$, $n \geq 0$. Moreover, it has been shown in [T] that $e^{-\frac{\theta}{2}t} \leq 1 - d_0^\theta(t) \leq (1+\theta)e^{-\frac{\theta}{2}t}$, $t > 0$. Note that (3.3) implies that the lower bound in the last inequality is the exact exponential rate of convergence in the corresponding L^2–space, i.e.,

$$
\left(\int (p_t^{\theta,\nu_0}F - \langle F\rangle)^2 \, dm_{\theta,\nu_0}\right)^{\frac{1}{2}} \leq e^{-\frac{\theta}{2}t}\left(\int F^2 \, dm_{\theta,\nu_0}\right)^{\frac{1}{2}}, t > 0\,.
$$

Although $(\mathcal{E}_{\theta,\nu_0}, H^{1,2}(m_{\theta,\nu_0}))$ satisfies a Poincaré–inequality we will see in the following theorem that the bilinear form does not determine a logarithmic Sobolev inequality.

Theorem 3.5. *Let $\theta > 0$, $\nu_0 \in E$ such that supp$(\nu_0) = S$. Let $(\mathcal{E}_{\theta,\nu_0}, H^{1,2}(m_{\theta,\nu_0}))$ be the Dirichlet form corresponding to the Fleming–Viot process with mutation operator A as defined in (3.1). Then:*

(i)

$$
D_0 := \{F^2 | F \in H^{1,2}(m_{\theta,\nu_0}), \mathcal{E}_{\theta,\nu_0}(F,F) + \|F\|^2_{L^2(m_{\theta,\nu_0})} \leq 1\}
$$

is uniformly integrable if and only if $|S| < +\infty$. In particular, $(\mathcal{E}_{\theta,\nu_0}, H^{1,2}(m_{\theta,\nu_0}))$ determines a logarithmic Sobolev inequality if and only if $|S| < +\infty$.

(ii) If $|S| < +\infty$ then $(\mathcal{E}_{\theta,\nu_0}, H^{1,2}(m_{\theta,\nu_0}))$ determines a logarithmic Sobolev inequality with constant $\frac{320}{\min_{s\in S}\nu_0(\{s\})}$.

Proof. If $|S| < +\infty$ then by Theorem 2.8 $(\mathcal{E}_{\theta,\nu_0}, H^{1,2}(m_{\theta,\nu_0}))$ determines a logarithmic Sobolev inequality (with constant $\frac{320}{\min_{s\in S}\nu_0(\{s\})}$). In particular, D_0 is uniformly integrable since

$$
\begin{aligned}
\sup_{F^2\in D_0}\int_{\{F^2\geq c\}} F^2\, dm_{\theta,\nu_0} &\leq \frac{1}{\log c}\sup_{F^2\in D_0}\int_{\{F^2\geq c\}} F^2\log F^2\, dm_{\theta,\nu_0} \\
&\leq \frac{1}{\log c}\,\frac{320}{\min_{s\in S}\nu_0(\{s\})} \to 0\ , c\to+\infty\ .
\end{aligned}
$$

If $|S| = +\infty$ then we can find a decreasing sequence $(A_n)_{n\geq 1}$ of measurable subsets with $p_n := \nu_0(A_n) > 0$ and $\lim_{n\to\infty} p_n = 0$. Let

$$
F_n(\mu) := \left(\frac{1}{p_n(\theta p_n + 1)}\right)^{\frac{1}{2}} \mu(A_n)\ .
$$

Clearly, $F_n \in \mathcal{F}C^\infty$, $\int F_n^2\, dm_{\theta,\nu_0} = \frac{1}{p_n(\theta p_n+1)} \int t^2 \varrho_{\theta(p_n,1-p_n)}(t)\, dt = \frac{1}{\theta+1}$, and $\mathcal{E}_{\theta,\nu_0}(F_n, F_n) = \frac{1}{p_n(\theta p_n+1)} \int t(1-t)\varrho_{\theta(p_n,1-p_n)}(t)\, dt = \frac{\theta(1-p_n)}{(\theta p_n+1)(\theta+1)}$, hence

$$\mathcal{E}_{\theta,\nu_0}(F_n, F_n) + \|F_n\|^2_{L^2(m_{\theta,\nu_0})} = \frac{\theta(1-p_n)+\theta p_n+1}{(\theta p_n+1)(\theta+1)} \le 1 \ ,$$

i.e., $(F_n^2) \subset D_0$. However,

$$\begin{aligned}\int_{\{F_n^2 \ge c\}} F_n^2\, dm_{\theta,\nu_0} &= \frac{1}{p_n(\theta p_n+1)} \int_{\{\mu|\mu(A_n)\ge\sqrt{c\,p_n(\theta p_n+1)}\}} \mu(A_n)^2\, m_{\theta,\nu_0}(d\mu) \\ &= \frac{\Gamma(\theta+1)}{\Gamma(\theta p_n+2)\Gamma(\theta(1-p_n))} \int_{\sqrt{c\,p_n(\theta p_n+1)}}^1 t^{\theta p_n+1}(1-t)^{\theta(1-p_n)-1}\, dt \\ &\underset{n\to\infty}{\to} \frac{\Gamma(\theta+1)}{\Gamma(2)\Gamma(\theta)} \int_0^1 t(1-t)^{\theta-1}\, dt = \frac{1}{\theta+1} \ \forall c > 0 \ ,\end{aligned}$$

which implies that (F_n^2), hence D_0 too, is not uniformly integrable. $\square$

Corollary 3.6. *The L^2–semigroup $(e^{tL_{\theta,\nu_0}})_{t\ge0}$ associated with the symmetric Fleming–Viot operator L_{θ,ν_0} is hypercontractive (i.e., $\|e^{tL_{\theta,\nu_0}}\|_{2,4} < +\infty$ for some $t > 0$) if and only if $|S| < +\infty$.*

Proof. If $|S| < +\infty$ it follows from 3.5 and [Ba, 3.4] that $(e^{tL_{\theta,\nu_0}})_{t\ge0}$ is hypercontractive. Next suppose that $|S| = +\infty$ and that $(e^{tL_{\theta,\nu_0}})_{t\ge0}$ is hypercontractive. By [Ba, 3.6] it follows that there exist positive constants c, m such that

$$\int F^2 \log F^2\, dm_{\theta,\nu_0} \le c\,\mathcal{E}_{\theta,\nu_0}(F,F) + m\,\|F\|^2_{L^2(m_{\theta,\nu_0})} + \|F\|^2_{L^2(m_{\theta,\nu_0})} \log \|F\|^2_{L^2(m_{\theta,\nu_0})}$$

for all $F \in H^{1,2}(m_{\theta,\nu_0})$. This would imply that the set

$$\{F^2 | F \in H^{1,2}(m_{\theta,\nu_0}), \mathcal{E}_{\theta,\nu_0}(F,F) + \|F\|^2_{L^2(m_{\theta,\nu_0})} \le 1\}$$

is uniformly integrable, which is a contradiction. Hence $(e^{tL_{\theta,\nu_0}})_{t\ge0}$ cannot be hypercontractive. $\square$

Appendix A

The purpose of this appendix is to show that (3.3) can be used *directly* in order to show that $(\mathcal{E}_{\theta,\nu_0}, H^{1,2}(m_{\theta,\nu_0}))$ determines a Poincaré–inequality with constant $2\frac{(1+\theta)^2}{\theta}$. To this end note that $(p_t^{\theta,\nu_0})_{t\ge0}$ induces a sub–Markovian C_0–semigroup of contractions on $L^1(m_{\theta,\nu_0})$, again denoted by $(p_t^{\theta,\nu_0})_{t\ge0}$. By the Riesz–Thorin Interpolation Theorem $(p_t^{\theta,\nu_0})_{t\ge0}$ can be restricted to a C_0–semigroup of contractions on $L^p(m_{\theta,\nu_0})$ for all $p \in [1,\infty)$. Then (3.3) implies the following result:

Lemma A.1. *Let $p \in [1, \infty]$ and $F \in L^p(m_{\theta,\nu_0})$. Then*

$$\|p_t^{\theta,\nu_0} F - \langle F \rangle\|_p \leq (1 - d_0^\theta(t))\|F\|_p \,, t > 0 \;.$$

Proof. Consider the linear operator $U_t F := p_t^{\theta,\nu_0} F - \langle F \rangle \cdot 1_E$, $F \in L^p(m_{\theta,\nu_0})$, $t \geq 0$. We will show that

(a) $\|U_t\|_\infty \leq 1 - d_0^\theta(t)$, $t > 0$,

(b) $\|U_t\|_1 \leq 1 - d_0^\theta(t)$, $t > 0$.

Proof of (a): Let $F \in \mathcal{B}_b(E)$. Then (3.3) implies

$$|p_t^{\theta,\nu_0} F(\mu) - \langle F \rangle| \leq (1 - d_0^\theta(t))\|F\|_\infty \,, t > 0 \,, \mu \in E \;.$$

Consequently,

$$\|p_t^{\theta,\nu_0} F - \langle F \rangle \cdot 1_E\|_\infty \leq (1 - d_0^\theta(t))\|F\|_\infty \,, t > 0 \,, \tag{1}$$

which implies (a).

Proof of (b): Let $F \in \mathcal{B}_b(E)$ and $l_F := 1_{\{p_t^{\theta,\nu_0} F > \langle F \rangle\}} - 1_{\{p_t^{\theta,\nu_0} F < \langle F \rangle\}}$. Then, using that m_{θ,ν_0} is a symmetrizing measure for p_t^{θ,ν_0},

$$\begin{aligned}
\int |p_t^{\theta,\nu_0} F - \langle F \rangle \cdot 1_E| \, dm_{\theta,\nu_0} &= \int (p_t^{\theta,\nu_0} F - \langle F \rangle \cdot 1_E) l_F \, dm_{\theta,\nu_0} \\
&= \int (p_t^{\theta,\nu_0} F) l_F \, dm_{\theta,\nu_0} - \langle F \rangle \langle l_F \rangle \\
&= \int F (p_t^{\theta,\nu_0} l_F) \, dm_{\theta,\nu_0} - \langle F \rangle \langle l_F \rangle \\
&= \int F ((p_t^{\theta,\nu_0} l_F) - \langle l_F \rangle \cdot 1_E) \, dm_{\theta,\nu_0} \\
&\leq \|F\|_1 \|p_t^{\theta,\nu_0} l_F - \langle l_F \rangle \cdot 1_E\|_\infty \\
&\leq (1 - d_0^\theta(t))\|F\|_1 \,, t > 0 \,,
\end{aligned}$$

where we used (1) in the last inequality. Consequently, $\|p_t^{\theta,\nu_0} F - \langle F \rangle \cdot 1_E\|_1 \leq (1 - d_0^\theta(t))\|F\|_1$, $t > 0$, which implies (b).

The linear operator U_t is therefore continuous on $L^p(m_{\theta,\nu_0})$ for $p = 1$ and $p = \infty$ with operator norm less than $1 - d_0^\theta(t)$, $t > 0$. Hence by the Riesz–Thorin Interpolation Theorem it follows that U_t is continuous on $L^p(m_{\theta,\nu_0})$ for all $p \in [1, \infty]$ with operator norm less than $1 - d_0^\theta(t)$, $t > 0$, which implies the assertion. □

Proposition A.2. *Let $\theta > 0$, $\nu_0 \in E$. Then $(\mathcal{E}_{\theta,\nu_0}, H^{1,2}(m_{\theta,\nu_0}))$ determines a Poincaré–inequality with constant $\frac{2(1+\theta)^2}{\theta}$.*

Remark A.3. Note that we do not obtain the exact constant $\frac{2}{\theta}$ in the last Proposition.

Proof. Let $F \in L^2(m_{\theta,\nu_0})$ be such that $\langle F\rangle = 0$. Then $G_0F := \int_0^\infty p_t^{\theta,\nu_0} F\,dt$ exists in $L^2(m_{\theta,\nu_0})$ since by A.1 and the fact that $1 - d_0^\theta(t) \leq (1+\theta)e^{-\frac{\theta}{2}t}$, $t > 0$, $\int_0^\infty \|p_t^{\theta,\nu_0} F\|_2\,dt \leq (1+\theta)\int_0^\infty e^{-\frac{\theta}{2}t}\|F\|_2\,dt < +\infty$. Moreover, if in addition $F \in D(L_{\theta,\nu_0})$ then $\langle L_{\theta,\nu_0}F\rangle = 0$, $G_0L_{\theta,\nu_0}F = \int_0^\infty \frac{d}{dt}\left(p_t^{\theta,\nu_0}F\right)\,dt = -F$ and thus

$$\begin{aligned}(-L_{\theta,\nu_0}F, F)_2 = (L_{\theta,\nu_0}F, G_0L_{\theta,\nu_0}F)_2 &= \int_0^\infty (L_{\theta,\nu_0}F, p_t^{\theta,\nu_0}L_{\theta,\nu_0}F)_2\,dt \\ &= \int_0^\infty \|p_{\frac{t}{2}}^{\theta,\nu_0}L_{\theta,\nu_0}F\|_2^2\,dt\ .\end{aligned}$$

Since $\langle p_{\frac{t}{2}}^{\theta,\nu_0}L_{\theta,\nu_0}F\rangle = 0$ A.1 implies that $\|p_t^{\theta,\nu_0}L_{\theta,\nu_0}F\|_2 = \|p_{\frac{t}{2}}^{\theta,\nu_0}p_{\frac{t}{2}}^{\theta,\nu_0}L_{\theta,\nu_0}F\|_2 \leq (1+\theta)e^{-\frac{\theta}{4}t}\|p_{\frac{t}{2}}^{\theta,\nu_0}L_{\theta,\nu_0}F\|_2$ and thus

$$\begin{aligned}(-L_{\theta,\nu_0}F, F)_2 &= \int_0^\infty \|p_{\frac{t}{2}}^{\theta,\nu_0}L_{\theta,\nu_0}F\|_2^2\,dt \\ &\geq \frac{1}{(1+\theta)^2}\int_0^\infty e^{\frac{\theta}{2}t}\|p_t^{\theta,\nu_0}L_{\theta,\nu_0}F\|_2^2\,dt\ .\end{aligned} \tag{2}$$

By Hölder's inequality

$$\begin{aligned}\int_0^\infty \|p_t^{\theta,\nu_0}L_{\theta,\nu_0}F\|_2\,dt &= \int_0^\infty e^{-\frac{\theta}{4}t}e^{\frac{\theta}{4}t}\|p_t^{\theta,\nu_0}L_{\theta,\nu_0}F\|_2\,dt \\ &\leq \left(\int_0^\infty e^{-\frac{\theta}{2}t}\,dt\right)^{\frac{1}{2}}\left(\int e^{\frac{\theta}{2}t}\|p_t^{\theta,\nu_0}L_{\theta,\nu_0}F\|_2^2\,dt\right)^{\frac{1}{2}} \\ &= \left(\frac{2}{\theta}\right)^{\frac{1}{2}}\left(\int_0^\infty e^{\frac{\theta}{2}t}\|p_t^{\theta,\nu_0}L_{\theta,\nu_0}F\|_2^2\,dt\right)^{\frac{1}{2}}\ ,\end{aligned}$$

and thus

$$\begin{aligned}\int_0^\infty e^{\frac{\theta}{2}t}\|p_t^{\theta,\nu_0}L_{\theta,\nu_0}F\|_2^2\,dt &\geq \frac{\theta}{2}\left(\int_0^\infty \|p_t^{\theta,\nu_0}L_{\theta,\nu_0}F\|_2\,dt\right)^2 \\ &\geq \frac{\theta}{2}\|\int_0^\infty p_t^{\theta,\nu_0}L_{\theta,\nu_0}F\,dt\|_2^2 \\ &= \frac{\theta}{2}\|G_0L_{\theta,\nu_0}F\|_2^2 = \frac{\theta}{2}\|F\|_2^2\ .\end{aligned} \tag{3}$$

Combining (2) and (3) we conclude that

$$(-L_{\theta,\nu_0}F, F) \geq \frac{\theta}{2(1+\theta)^2}\|F\|_2^2\ . \tag{4}$$

Finally, let $F \in H^{1,2}(m_{\theta,\nu_0})$ be arbitrary and $\tilde{F} := F - \langle F \rangle \cdot 1_E$. Then $p_t^{\theta,\nu_0}\tilde{F} \in D(L_{\theta,\nu_0})$ if $t > 0$, $\langle p_t^{\theta,\nu_0}\tilde{F}\rangle = 0$ and (4) implies that

$$\mathcal{E}_{\theta,\nu_0}(p_t^{\theta,\nu_0}\tilde{F}, p_t^{\theta,\nu_0}\tilde{F}) \geq \frac{\theta}{2(1+\theta)^2}\|p_t^{\theta,\nu_0}\tilde{F}\|_2^2\ , t > 0\,. \tag{5}$$

Since $\lim_{t\to 0} p_t^{\theta,\nu_0}\tilde{F} = \tilde{F}$ in $H^{1,2}(m_{\theta,\nu_0})$ and $\mathcal{E}_{\theta,\nu_0}(\tilde{F},\tilde{F}) = \mathcal{E}_{\theta,\nu_0}(F,F)$ taking the limit $t \to 0$ in (5) we obtain the assertion. □

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