# REPRESENTATIONS OF BURNSIDE RINGS

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Abstract. Motivated from equivariant topology we investigate representations of Burnside rings, i.e. modules which are finitely generated and free as abelian groups. We prove a necessary and sufficient condition for a Burnside ring to be of finite representation type. For $p$-groups whose Burnside ring is of finite representation type all the indecomposable representations are classified. Finally, we discuss a class of Burnside rings whose representation theory is particularly nice.

## Introduction

Although the objects of our discussion are of some interest by themselves, the main motivation for studying Burnside rings and their modules originates from their natural occurrence in equivariant topology. Before summarizing the purely algebraic content of this article, we roughly sketch this topological background. More details can be found in [8].

Let $G$ be a finite group. Denote by $S^V$ the one point compactification of a complex $G$-representation $V$ with base point at infinity. The equivariant stable homotopy group of pointed $G$-CW-complexes, say $X$ and $Y$, is defined to be the direct limit of the equivariant homotopy groups $[S^V \wedge X, S^V \wedge Y]_G$ over all complex representations $V$ of $G$. We denote it by the symbol $\{X, Y\}_G$. As for the non-equivariant case, these are abelian groups. On $\{S^0, S^0\}_G$ there is also a multiplicative structure given by suspension (or equivalently by composition). Again by suspension, every equivariant stable homotopy group $\{X, Y\}_G$ is a module over this ring. G. Segal observed that $\{S^0, S^0\}_G$ is isomorphic to the Burnside ring $A(G)$ of $G$. Hence, every equivariant stable homotopy group naturally is a module over the Burnside ring. More generally, one can show that each equivariant stable homology or cohomology group, or even more generally, each equivariant homology or cohomology group of a theory with Mackey structure, is a module over the respective Burnside ring. In summary, module structures over Burnside rings are ubiquitous in equivariant topology. This corresponds to the general philosophy that the most basic objects in equivariant topology are not single points but orbits; these are just the generators of the Burnside ring.

There is a theorem of Hopf stating in its classical form that the homotopy classes of maps of an $n$-dimensional, connected, closed, oriented manifold into $S^n$ are in one-to-one correspondence with $\mathbb{Z}$ by the mapping degree. It can be generalized to an equivariant situation in order to find conditions for $G$-CW-complexes $X, Y$ such that the equivariant homotopy classes of maps $X \to Y$ are characterized by a generalized mapping degree, [8, II 4.11]. This generalized mapping degree is an element of the ring $C(G)$ of integer valued functions on the set of conjugacy classes of subgroups in $G$. As a consequence, there is an embedding of the equivariant stable homotopy group $\{X, Y\}_G$ into $C(G)$. This leads to the consideration of modules which are finitely generated and free as abelian groups, so called lattices or representations. The terminology coincides with the usual one if we consider a Burnside ring as an order over the ring of integers. This point of view will be explained and used in Section 2.

We concentrate in this article on the investigation of modules of this special type. Unfortunately, general modules do not need to have a decomposition into a representation and a torsion part.

The structure and the main results of this article can be summarized as follows: In the first section we shortly recall the definition and fundamental properties of Burnside rings of finite groups. In Section 2 we prove a necessary and sufficient condition for a Burnside ring to be of finite representation type. As a consequence we see that most groups have a Burnside ring of infinite representation type. For $p$-groups the only exceptions are the cyclic groups of order $p$ and $p^2$. Section 3 contains a more detailed investigation for $p$-groups. In the cases of finite representation type the indecomposable representations are classified. Moreover, we see that for $p$-groups with more than four conjugacy classes of subgroups the representation type of the Burnside ring is wild. The last section collects some nice properties for the special case of groups with square free order.

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## 1. Burnside Rings

For a finite group $G$ the Burnside ring $A(G)$ is defined to be the Grothendieck ring of finite $G$-sets. The sum is given by disjoint union and the product by the direct product of $G$-sets with diagonal action. From this definition it is obvious that Burnside rings are commutative. The unit is represented by a point with trivial action.

Up to isomorphism every $G$-set is a disjoint union of orbits $G/H$. Two such orbits $G/H$ and $G/K$ are isomorphic as $G$-sets if and only if $H$ and $K$ are conjugate in $G$, in symbols $H \sim K$. Therefore, as an abelian group $A(G)$ is free and generated by the isomorphism classes $[G/H]$ with one representative for each conjugacy class of subgroups $H$ in $G$. In particular, Burnside rings are noetherian.

Denote by $\Phi(G)$ the set of conjugacy classes of subgroups in $G$ and by $C(G)$ the ring of integer valued functions on $\Phi(G)$. There is a well defined map

$$A(G) \longrightarrow C(G), \quad S \longmapsto ((H) \mapsto |S^H|).$$

This is an injective ring homomorphism. The image is described by Dress' congruences, [6, Chapter 1.3]. In the following we usually think of $A(G)$ as a subring of $C(G)$.

The ring extension $A(G) \subset C(G)$ is integral. As a consequence, every prime ideal of the Burnside ring is an intersection of $A(G)$ with a prime ideal in $C(G)$, i.e. of the form

$$\mathfrak{q}(H,p) := \{\, z \in C(G) \mid z(H) \equiv 0 \bmod (p) \,\} \cap A(G)$$

for a class $(H) \in \Phi(G)$ and a prime ideal $(p) \subset \mathbb{Z}$. Denote by $O^p(G)$ the unique minimal normal subgroup of $G$ such that the quotient is a $p$-group. We recall a theorem of Dress [9]:

**Proposition 1.1.** *Two prime ideals $\mathfrak{q}(H,p)$ and $\mathfrak{q}(K,r)$ are equal if and only if*

$$p = r \quad \text{and} \quad \begin{cases} (H) = (K) & \text{if } \; p = r = 0, \\ O^p(H) \sim O^p(K) & \text{if } \; p = r \text{ is prime.} \end{cases}$$

We call a prime ideal $\mathfrak{q}(H,p) \subset A(G)$ a crossing point of order $n$ if there are precisely $n$ different conjugacy classes $(H_1),\dots,(H_n)$ such that $\mathfrak{q}(H,p) = \mathfrak{q}(H_i,p)$ for $i = 1,\dots,n$.
**Example:** For $G = C_{p^n}$ the cyclic $p$-group of order $p^n$ the subgroups are

$$H_0 = 1 \subset H_1 = C_p \subset \ldots \subset H_n = G.$$

For $x \in C(G)$ denote by $x_i$ the value of $x$ at $H_i$. Then $A(G)$ is the subset of elements of $C(G)$ satisfying the congruences

$$x_i - x_{i+1} \equiv 0 \mod p^{n-i} \quad \text{for} \quad i = 0,\dots,n-1.$$

By Proposition 1.1 the prime ideal spectrum is easily determined: $O^p(H) = 1$ for every subgroup $H$ in $G$. Hence $\mathfrak{q}(H,p) = \mathfrak{q}(\{1\},p)$; the ideal $\mathfrak{q}(\{1\},p)$ is a crossing point of order $n+1$. For a prime $q \neq p$, $O^q(H) = H$. Therefore, $\mathfrak{q}(H,q) = \mathfrak{q}(K,q)$ if and only if $(H) = (K)$.

## 2. Burnside Rings of Finite Representation Type

Recall that by a representation of a Burnside ring we mean a module which is finitely generated and torsion free as a $\mathbb{Z}$-module. Projective modules are examples for representations. Every representation is a direct sum of indecomposable ones. Such a decomposition is not necessarily unique. Examples can be easily constructed from the following proposition.

**Proposition 2.1.** *Let $M$ be a projective $A(G)$-module and $\varepsilon \in A(G)$ a primitive idempotent with $\varepsilon M \neq 0$. Let $n$ be the projective rank of $\varepsilon M$ over $\varepsilon A(G)$. Then there is a projective $A(G)$-module $N$ of rank 1 such that*

$$\varepsilon M \cong \varepsilon N \oplus \varepsilon A(G)^{(n-1)}.$$

*Moreover, $\varepsilon N \oplus \varepsilon A(G)^n \cong \varepsilon N' \oplus \varepsilon A(G)^n$ if and only if $\varepsilon N \cong \varepsilon N'$.*

*Proof.* The first part follows from Serre's splitting theorem [11, IV 3.18]. For every commutative ring $A$ and every projective module $N$ of rank 1 over $A$ the exterior algebra of $N \oplus A^n$ is isomorphic to $N$. This proves the second part. □

The primitive idempotent elements of Burnside rings are well understood. For a group $K$ denote by $K_s$ the minimal normal subgroup of $K$ such that $K/K_s$ is solvable. For a subgroup $K$ of $G$ let $e_K$ be the primitive idempotent of $C(G)$ corresponding to the class $(K) \in \Phi(G)$. Then for every perfect subgroup $H$ of $G$

$$\varepsilon_H := \sum_{\substack{(K) \\ (K_s)=(H)}} e_K$$

is a primitive idempotent of $A(G)$ and all primitive idempotents of the Burnside ring are of this form [8, IV.7.6].

Also, the projective modules of rank 1 are well known. For an element $x \in C(G)$ with all values $x(H)$ prime to the order of $G$ the submodule of $C(G)$

$$A(G)x + |G|C(G)$$

is projective of rank 1 and each such module is isomorphic to one of this form. Moreover, the Picard group of $A(G)$, which is built from the isomorphism classes of projective modules of rank 1, can be described [7]. In Section 3 below we present an explicit calculation of the isomorphism classes of projective modules of rank 1 for the Burnside ring of the cyclic group of order $p^2$.

We now consider general representations. By the theorem of Jordan-Zassenhaus there are for each fixed natural number $n$ only finitely many isomorphism classes of representations of rank $n$ as abelian groups [4]. The representations of $\mathbb{Z}$-rank 1 can be classified in a straightforward manner:

**Proposition 2.2.** *Every $A(G)$-representation of $\mathbb{Z}$-rank 1 is isomorphic to a module of the form $e_H C(G)$ for some subgroup $H$ of $G$. Two such modules $e_H C(G)$ and $e_K C(G)$ are isomorphic if and only if $(H) = (K)$.*

*Proof.* Every ring homomorphism $A(G) \to \mathbb{Z}$ is of the form $S \mapsto |S^H|$ for some subgroup $H \subset G$, [6, Proposition 5.7.4]. Hence, all representations of rank 1 are of the form $e_H C(G)$. If $e_H C(G)$ and $e_K C(G)$ are isomorphic, then $|S^H| = |S^K|$ for all finite $G$-sets $S$. This can only be the case if $(H) = (K)$. □

In general, there might exist indecomposable representations of arbitrarily large rank, or equivalently, there might exist infinitely many isomorphism classes of indecomposable representations. Then the ring under consideration is called of infinite representation type.

The following theorem provides a necessary and sufficient condition for a Burnside ring $A(G)$ to be of finite representation type in terms of the group $G$. A group $H$ is called $p$-perfect if $O^p(H) = H$.

**Theorem 2.3.** *A Burnside ring $A(G)$ has finite representation type if and only if there are at most triple points in its prime ideal spectrum. This is equivalent to the more technical but also more practical condition that for every prime $p$ and every $p$-perfect subgroup $H$ in $G$*

$$|\{ (K) \in \Phi(G) \mid O^p(K) \sim H \}| \leq 3.$$

Before proving the theorem we consider some applications.

**Corollary 2.4.** *The following holds for a group $G$:*

(i) *If $p^3$ divides the order of $G$, then the Burnside ring $A(G)$ has infinite representation type.*
(ii) *If for each prime $p$ dividing the order of $G$ the Sylow $p$-subgroups are cyclic of order $p$ or $p^2$, then $A(G)$ has finite representation type.*
(iii) *If $A(G)$ has finite representation type and there is a Sylow $p$-supgroup $C_p \times C_p$, then all subgroups of order $p$ are conjugate.*
(iv) *If $G$ is a $p$-group, then the Burnside ring has finite representation type if and only if $G$ is cyclic of order $p$ or $p^2$.*

*Proof.* Statements (i), (iii), and (iv) follow readily from Theorem 2.3. A proof of (ii) can be found in [12]. □

**The Proof of Theorem 2.3.** The equivalence of the two given conditions is an immediate consequence of Proposition 1.1. It follows a digression into the representation theory of orders. For the convenience of the reader we briefly recall the basic definitions: Let $R$ be a Dedekind domain with field of quotients $K$. Let $A$ be an algebra over $K$. An $R$-order $\Lambda$ in $A$ is a subring of $A$ such that the centre of $\Lambda$ contains $R$, $\Lambda$ is finitely generated as an $R$-module, and $K\Lambda = A$. For us the important examples are Burnside rings $\Lambda = A(G)$ (with $R = \mathbb{Z}, K = \mathbb{Q}, A = \mathbb{Q} \otimes_{\mathbb{Z}} A(G)$) and Burnside rings $\Lambda = A(G)_p$ localized at a prime $p$ (here $R = \mathbb{Z}_p$ and $K$ and $A$ are as above).

A $\Lambda$-module which is finitely generated and projective over $R$ is called a lattice or representation.

The proof of Theorem 2.3 relies on the following deep result about the representation type of commutative orders. Here, for a $\Lambda$-module $M$ the number $\mu_\Lambda(M)$ denotes the minimal number of generators and $\mathrm{rad}(M)$ the intersection of all maximal submodules in $M$. A survey of the proof and further references can be found in [4].

**Theorem 2.5** (Jacobinski, Drozd-Roiter '67)**.** *Let $R$ be a Dedekind domain with a quotient field $K$ which is a global field or a completion of a global field. Let $A$ be a seperable commutative and finite dimensional $K$-algebra. Let $\Lambda'$ be the (unique) maximal order in $A$. Then an order $\Lambda$ in $A$ has finite representation type if and only if*

$$\mu_\Lambda(\Lambda'/\Lambda) \leq 2 \quad \textit{and} \quad \mu_\Lambda(\mathrm{rad}(\Lambda/\Lambda')) \leq 1.$$

The theorem holds for coefficient rings $R = \mathbb{Z}$ or $\mathbb{Z}_p$. In case $\Lambda = A(G)$ (or $A(G)_p$) the maximal order is the ring $C(G)$ (respectively $C(G)_p$).

Using this criterion A. Raggi-Cardenas [12] proved a local version of Theorem 2.3:

**Theorem 2.6.** *For each prime $p$ the localized Burnside ring $A(G)_p$ has finite representation type if and only if for every $p$-perfect subgroup $H \subset G$*

$$|\{\,(K) \in \Phi(G) \mid O^p(K) \sim H\,\}| \leq 3.$$

Theorem 2.3 follows from this result and the observation that finite representation type is a local property with respect to localizations at prime ideals of the coefficient ring $R$. Here, once more the criterion of Jacobinski, Drozd and Roiter is used. The statement of the following proposition is known in case of completions instead of localizations [4].

**Proposition 2.7.** *Suppose $R, K, A, \Lambda'$ and $\Lambda$ are as in Theorem 2.5. Choose an $a \in R$ such that $a\Lambda' \subset \Lambda$. Then $\Lambda$ has finite representation type if and only if for each prime ideal $P \subset R$ with $aR \subset P$ the localized $R_P$-order $\Lambda_P$ has finite representation type.*

*Proof.* The $\Lambda$-module $M := \Lambda'/\Lambda$ is finitely generated and $aM = 0$. Therefore, it suffices to show that each module $M$ satisfying $aM = 0$ has the following properties:

(i) $\mu_\Lambda(M) = \max\{\,\mu_{\Lambda_P}(M_P) \mid P \subset R \text{ a prime ideal with } aR \subset P\,\}$;
(ii) $\mu_\Lambda(\mathrm{rad}_\Lambda(M)) = \max\{\,\mu_{\Lambda_P}(\mathrm{rad}_{\Lambda_P}(M_P)) \mid P \subset R \text{ a prime ideal with } aR \subset P\,\}$.

In order to prove this we use primary decompositions [4, 4.31]

$$M \xrightarrow{\cong} \bigoplus_{P \supset aR} M_P \quad \text{and} \quad \Lambda/a\Lambda \xrightarrow{\cong} \bigoplus_{P \supset aR} \Lambda_P/a\Lambda_P. \tag{1}$$

As $aM = 0$, the $\Lambda$-module $M$ is in fact a $\Lambda/a\Lambda$-module. This structure is compatible with the primary decompositions of $M$ and $\Lambda/a\Lambda$. From this equation (i) immediately follows.

The module $\mathrm{rad}_\Lambda(M)$ also has the property $a\,\mathrm{rad}_\Lambda(M) = 0$. Because of (i), we only have to show that for every prime ideal $P \subset R$ with $aR \subset P$

$$\Big(\mathrm{rad}_\Lambda(M)\Big)_P \cong \mathrm{rad}_{\Lambda_P}(M_P) \quad \text{holds.} \tag{2}$$

From the primary decomposition of $M$ we get

$$\operatorname{rad}_\Lambda(M) \cong \bigoplus_{P \supset aR} \operatorname{rad}_\Lambda(M_P). \tag{3}$$

The module $M_P$ is $P$-primary and so is its submodule $\operatorname{rad}_\Lambda(M_P)$. As a consequence

$$\Big(\operatorname{rad}_\Lambda(M_P)\Big)_P \cong \operatorname{rad}_\Lambda(M_P) \quad \text{and} \quad \Big(\operatorname{rad}_\Lambda(M_P)\Big)_Q = 0 \quad \text{for each } Q \neq P. \tag{4}$$

From the localization of the isomorphism (3) above we get for each prime ideal $P$ with $aR \subset P$

$$\Big(\operatorname{rad}_\Lambda(M)\Big)_P \cong \operatorname{rad}_\Lambda(M_P). \tag{5}$$

It remaines to show that $\operatorname{rad}_\Lambda(M_P) = \operatorname{rad}_{\Lambda_P}(M_P)$: Each submodule $N \subset M_P$ is $P$-primary. Hence $N = N_P$. Consequently, every $\Lambda$-submodule in $M_P$ is in fact a $\Lambda_P$-submodule. As a $\Lambda_P$-submodule is always a $\Lambda$-submodule, we get the identity

$$\operatorname{rad}_\Lambda(M_P) = \bigcap_{\substack{N \subset M_P \text{max.} \\ \Lambda\text{-submod.}}} N = \bigcap_{\substack{N \subset M_P \text{max.} \\ \Lambda_P\text{-submod.}}} N = \operatorname{rad}_{\Lambda_P}(M_P). \tag{6}$$

□

## 3. The Classification Problem for p-Groups

We have seen that the only $p$-groups whose Burnside rings are of finite representation type are the cyclic groups of order $p$ and $p^2$. In principle, for these groups it would be possible to find all indecomposable representations by inductive extensions. The problem then is to decide when the list is complete, because we do not know the number or the maximal rank of the indecomposable representations. Fortunately, another method can be applied for Burnside rings of $p$-groups: We construct a functor on the category of representations into another additive category which is just *not* an equivalence of categories but a functor preserving the important and forgetting the superfluous information. For our purposes this means that the functor should induce a bijection of isomorphism classes of the indecomposable objects but need not be faithful or surjective on objects. This will be described in detail in Subsection 3.2. Not only does it enable us to classify the indecomposable representations in the cases of finite representation type very efficiently, it also gives important information in cases of infinite representation type.

Originally this technique was developed for representations of algebras. M.C.R. Butler used it for the first time in integral representation theory. We suitably adapt his method from [3] to our situation. As a preliminary step we reduce to a localized situation.

**3.1. Genera of Representations.** We say that representations $M$ and $N$ of an $R$-order $\Lambda$ are in the same genus if for each prime ideal $P \subset R$ the localized $\Lambda_P$-representations $M_P$ and $N_P$ are isomorphic.

**Remark 3.1.** *If $G$ is a $p$-group two $A(G)$-representations $M, N$ are in the same genus if and only if the $A(G)_p$-representations $M_p$ and $N_p$ are isomorphic. Here the subscript $p$ denotes localization with respect to the prime ideal $(p) \subset \mathbb{Z}$.*

*Proof.* We only have to check the isomorphism at prime ideals of $\mathbb{Z}$ for which the localization of $A(G)$ is not the maximal order [4, 31.2]. For a $p$-group $G$ the only prime ideal with this property is $(p) \subset \mathbb{Z}$. □

**Proposition 3.2.** *For a $p$-group $G$ and an $A(G)$-representation $M$ the representation $M_p$ of $A(G)_p$ is indecomposable if and only if $M$ is indecomposable.*

*Proof.* If $M$ is indecomposable, then so is $M_p$. Now suppose $M_p$ is decomposable. Every $A(G)_p$-representation is a localization of an $A(G)$-representation, [4, 23.14]. So there are $A(G)$-representations $N$ and $L$ such that $M_p \cong N_p \oplus L_p$. As we have just seen, this means that $M$ and $N \oplus L$ are in the same genus. From this, one can deduce that $M$ is decomposable, too, [4, 23.14]. □

As a consequence of these two facts, our program is first to find the indecomposable $A(G)$-genera – which means the indecomposable representations of $A(G)_p$ – and then to classify the isomorphism classes of $A(G)$-representations in each of these.

**3.2. Representation Equivalent Categories.** An additive functor $F : \mathcal{A} \to \mathcal{B}$ of additive categories induces a bijection of isomorphism classes of objects if and only if $F$ is dense (i.e. for every object in $\mathcal{B}$ there is an isomorphic object in the image of the functor) and has the property that objects $A_1, A_2$ in $\mathcal{A}$ are isomorphic if and only if $F(A_1)$ and $F(A_2)$ are isomorphic in $\mathcal{B}$. Then automatically, the image of an indecomposable object is again indecomposable. Therefore, in fact $F$ induces a bijection of isomorphism classes of indecomposable objects. In this case we say that the categories $\mathcal{A}$ and $\mathcal{B}$ are representation equivalent. This generates an equivalence relation on categories.

In particular, two categories are representation equivalent if there exists an additive functor which is full, dense and detects isomorphisms (i.e. a morphism in the domain is an isomorphism if and only if its image is an isomorphism). Such a functor is classically called a representation equivalence, or in a more modern terminology an epivalence. The original terminology was changed, as the word "equivalence" seems to imply that the existence of a representation equivalence is a symmetric property which is not the case.

Our aim is to construct an epivalence. We need some notation: Fix a $p$-group $G$. Let $\Lambda$ denote the order $A(G)_p$ and $R$ its coefficient ring $\mathbb{Z}_p$. Let $\bar{\Lambda}$ be the maximal $R$-order $C(G)_p$ and $e_1, \ldots, e_n$ its primitive idempotent elements. For a $\Lambda$-representation $M$ denote $\bar{M}$ the $\bar{\Lambda}$-module $\bar{\Lambda} \otimes_\Lambda M$. Note that this module is in general not $R$-torsionfree and so not a $\bar{\Lambda}$-representation. The canonical map $M \to \bar{M}$ is an injection and in the following we consider $M$ as a $\Lambda$-submodule in $\bar{M}$.

We call a $\Lambda$-representation without a direct summand of the form $e_i\Lambda$ a reduced representation. Denote by *Rep*$(G)$ the (additive) category of reduced $\Lambda$-representations.

We write *Syst*$(G)$ for the category of tuples of finitely generated $\bar{\Lambda}$-modules

$$V_* = (V;\, V_1, \ldots, V_n) \quad \text{or for short} \quad V_* = (V;\, V_i)$$

satisfying

(a) $V_i \subset V$ is a submodule for $i = 1, \ldots, n$;
(b) $|G|\bar{\Lambda}V = 0$ (Note that $|G|\bar{\Lambda} \subset \Lambda$.);
(c) $\sum_{i \neq t} V_i = V$ for all $t = 1, \ldots, n$;
(d) For $\lambda = \sum \lambda_i e_i \in \bar{\Lambda}$ the action on $V_j$ is multiplication with the coordinate $\lambda_j$.

A morphism $f : V_* \to W_*$ in *Syst*$(G)$ is a $\bar{\Lambda}$-morphism $f : V \to W$ with $f(V_i) \subset W_i$. With the obvious sum and zero object this, too, is an additive category.

We show that for each $\Lambda$-representation $M$ the tupel $(\bar{M}/M;\, (M + e_iM)/M)$ is an object in *Syst*$(G)$. Of course it satisfies condition (a), and (b) follows from the fact that

$|G|\bar{\Lambda} \subset \Lambda$ and hence $|G|\bar{\Lambda}\bar{M} = |G|\bar{\Lambda} \otimes_\Lambda M$ is contained in $M$. Property (c) is fulfilled by the following easy calculation:

$$\sum_{i\neq t}(M + e_iM) = M + \sum_{i\neq t} e_iM = (\sum_{i=1}^{n} e_i)M + \sum_{i\neq t} e_iM = e_tM + \sum_{i\neq t} e_iM = \bar{M}.$$

Finally, (d) holds, because $(M + e_iM)/M \cong e_iM/(M \cap e_iM)$ as $\Lambda$-modules.

We extend this construction to a functor $\Delta : Rep(G) \to Syst(G)$ by defining

$$\Delta(M) = (\bar{M}/M;\ (M + e_iM)/M),$$
$$\Delta(f : M \to N) = (1 \otimes f : \bar{M} \to \bar{N}).$$

This is an additive functor, i.e. the induced maps of morphism sets

$$\Delta_{M,N} : \mathrm{Mor}_{Rep(G)}(M, N) \longrightarrow \mathrm{Mor}_{Syst(G)}(\Delta M, \Delta N)$$

are homomorphisms of abelian groups. It is not an equivalence of categories, because it is not faithful. For example, $\Delta(|G|\,\mathrm{id}_M) = 0$ for every representation $M$.

**Theorem 3.3.** *The functor $\Delta : Rep(G) \to Syst(G)$ is an epivalence. Therefore, the categories $Rep(G)$ and $Syst(G)$ are representation equivalent.*

*Proof.* We have to show that $\Delta$ is full, dense and that it detects isomorphisms. To show that $\Delta$ is full, we can use essentially the same arguments as Butler in [3, Prop. 2.4.(a)].

To show that $\Delta$ detects isomorphisms, as an intermediate step, we prove that a $\Lambda$-representation $M$ is reduced if and only if $e_iM \cap M \subset pe_iM$ for $i = 1, \dots, n$. The proof is similar to [3, 3.3(a)]: For a representation $M = e_i\Lambda$ the intersection $M \cap e_iM = M$ is not contained in $pe_iM$. This extends to the case that $e_i\Lambda$ is a summand of $M$. So for a nonreduced representation $M$ there is an $i$ such that $M \cap e_iM \not\subset pe_iM$.

Now suppose $M$ is a $\Lambda$-representation such that $M \cap e_iM \not\subset pe_iM$ for an index $i$. As an $R$-module, $e_iM$ is of the form $R^m$. An element $\lambda \in \Lambda$ acts by multiplication with the coordinate $\lambda_i$. Choose a basis $b_1, \dots, b_m$ of the $R$-module $e_iM$ and an $r \leq m$ such that $\alpha_1 b_1, \dots, \alpha_r b_r$ is a basis of the submodule $M \cap e_iM$ for suitable $\alpha_j \in R$. As $R = \mathbb{Z}_p$, all $\alpha_j$ are up to units powers of $p$. If $p$ divides all $\alpha_j$, then $M \cap e_iM$ is contained in $pe_iM$. So suppose $\alpha_1 = 1$. Then $\Lambda b_1$ is contained in $M$. On the other hand $\Lambda b_1$ is a direct summand of $e_iM$ and hence of $\bar{M}$. The splitting $\bar{M} \to \Lambda b_1$ restricts to a splitting of the inclusion $\Lambda b_1 \subset M$. So there is a direct summand $e_i\Lambda$ in $M$.

Now we can translate the proof of [3, Prop. 2.4.(c)] directly to our situation and show that $\Delta$ detects isomorphisms.

It remains to prove that $\Delta$ is dense. Let $V_*$ be an arbitrary object in $Syst(G)$. We have to find a reduced $\Lambda$-representation $M$ such that $\Delta(M)$ is isomorphic to $V_*$. Here we use a method similar to the one in [3, 3.2(a)]: Because $|G|V_i = 0$ and $\lambda \in \Lambda$ acts on $V_i$ by multiplication with the coordinate $\lambda_i$, the module $V_i$ is a quotient of a $\Lambda$-module $(e_i\Lambda)^{d_i}$ of the form $V_i \cong R/p^{r_1} \times \cdots \times R/p^{r_{d_i}}$ for some $r_1, \dots, r_{d_i}$ with $|G| \leq p^{r_k}$ for all $k$. Let $f_i : (e_i\Lambda)^{d_i} \to V_i$ be a fixed projection. Then $\ker(f_i)$ is contained in $p(e_i\Lambda)^{d_i}$. Let $F$ be the direct sum of modules $\bigoplus_{i=1}^{n}(e_i\Lambda)^{d_i}$. The sum of the projections $f_i$, $f : F \to V$, is a surjective homomorphism of $\Lambda$-modules and $M := \ker(f)$ is a $\Lambda$-representation.

We claim that $M \cap e_iM \subset pe_iM$ for all $i$. First we show that $e_iM = e_iF$: Obviously $e_iM \subset e_iF$. Now suppose $x \in e_iF$. As $V$ is the sum of all $V_j$ with $j \neq i$ and all $f_j$ are surjective, there are elements $x_j \in V_j$ such that

$$f(x) = \sum_{j \neq i} f_j(x_j) = \sum_{j \neq i} f(x_j).$$

Thus $x - \sum_{j \neq i} x_j$ is contained in $\ker(f) = M$. Multiplication with $e_i$ shows that $e_i x = x$ is an element of $e_i M$.

From this we deduce that $M \cap e_i M = M \cap e_i F = \ker(f_i) \subset pe_i F = pe_i M$. Hence $M$ is reduced. Moreover, the morphism $f : \bar{M} = F \to V$ induces an isomorphism $\Delta M \to V_*$. □

**3.3. Wild Subcategories.** For most $p$-groups $G$ the category $\mathit{Syst}(G)$ is difficult to investigate. But it contains a well known subcategory related to the $n$-subspace problem.

Consider the full subcategory $\mathrm{D}(n)$ of $\mathit{Syst}(G)$ of the objects $V_* = (V; V_1, \ldots, V_n)$ with $pV = 0$. In particular, $V$ is a vector space over the field $\mathbb{F}_p$ with $p$ elements.

For a $p$-group $G$ the number of elements in each orbit is a power of $p$; if the orbit is not a fixed point, its order is divisible by $p$. Hence for $\lambda = \sum \lambda_i e_i \in \Lambda$ all coefficients $\lambda_i$ are equal in $\mathbb{F}_p$ and the action of $\lambda$ on an object $V_*$ of $\mathrm{D}(n)$ is multiplication with this element of $\mathbb{F}_p$. This provides no additional structure. Consequently, $\mathrm{D}(n)$ is just the category of $\mathbb{F}_p$-vector spaces with $n$ distinguished subspaces $(V; V_1, \ldots, V_n)$ subject to the relation $\sum_{i \neq t} V_i = V$ for all $t = 1, \ldots, n$.

It is known that $\mathrm{D}(2)$ and $\mathrm{D}(3)$ are of finite, while $\mathrm{D}(4)$ is of infinite but tame representation type; for $n \geq 5$ the category $\mathrm{D}(n)$ is of wild representation type, [2].

This implies that for every $p$-group with more than four conjugacy classes of subgroups the representations of the Burnside ring are not classifiable.

So far, the only $p$-group for which the Burnside ring has infinite but possibly tame representation type is $C_{p^3}$. I am indebted to Ernst Dieterich for pointing out the criterion of Drozd and Greuel [10]. For $p \neq 2$ this should imply that $A(C_{p^3})$ is of wild representation type.

**3.4. Classification in the Cases of Finite Representation Type.** For the two $p$-groups with finite representation type we use the epivalence $\Delta$ to classify the indecomposable genera.

*The cyclic group of order $p$:*
Let $(V; V_1, V_2)$ be an object in $\mathit{Syst}(C_p)$. Then $p\bar{\Lambda}V = 0$ and in particular $pV = 0$. As we have seen in the last paragraph, this implies that the $\Lambda$-action provides no additional structure. So $\mathit{Syst}(C_p)$ is just the category $\mathrm{D}(2)$ of tuples of $\mathbb{F}_p$-vector spaces $(V; V_1, V_2)$ with $V = V_1 = V_2$. Obviously, up to isomorphism there is only one indecomposable object in this category:

$$V_0 = (\mathbb{F}_p; \mathbb{F}_p, \mathbb{F}_p).$$

An easy calculation shows that $V_0 = \Delta\Lambda$. This proves that beside the two representations of $\mathbb{Z}$-rank 1, $e_1 C(C_p)$ and $e_2 C(C_p)$, the only indecomposable $A(C_p)$-representations are those in the genus of $A(C_p)$, that is the projective modules of rank 1.

*The cyclic group of order $p^2$:*
Here, $\Lambda \subset \bar{\Lambda} = R^3$ is generated by $p^2 e_1$, $pe_1 + pe_2$ and $1 = e_1 + e_2 + e_3$. Let $(V; V_1, V_2, V_3)$ be an object in $\mathit{Syst}(C_{p^2})$. As $p^2\bar{\Lambda}V = 0$, we can consider $V$ as a module over $\mathbb{Z}/p^2$ and the first generator $p^2 e_1$ of $\Lambda$ acts by multiplication with 0. The second generator $pe_1 + pe_2$

acts on $V_1$ and $V_2$ by multiplication with $p$ and on $V_3$ by multiplication with 0. On the other hand $V = V_1 + V_2$. So it acts on all of $V$ by multiplication with $p$ and $pV_3 = 0$.

This means that we can describe $Syst(C_{p^2})$ as the category of tupels of $\mathbb{Z}/p^2$-modules $(V; V_1, V_2, V_3)$ subject to the conditions

(a) $V_1, V_2$ and $V_3$ are submodules of $V$;
(b) $V = V_i + V_j$ for $i \neq j$;
(c) $pV_3 = 0$.

**Proposition 3.4.** *Every indecomposable object in* $Syst(C_{p^2})$ *is isomorphic to one of the following list:*

$$\begin{aligned}
S_1 &= \big(\mathbb{Z}/p\,;\ \mathbb{Z}/p,\, \mathbb{Z}/p,\, 0\,\big)\\
S_2 &= \big(\mathbb{Z}/p\,;\ \mathbb{Z}/p,\, 0,\, \mathbb{Z}/p\,\big)\\
S_3 &= \big(\mathbb{Z}/p\,;\ 0,\, \mathbb{Z}/p,\, \mathbb{Z}/p\,\big)\\
S_4 &= \big(\mathbb{Z}/p\,;\ \mathbb{Z}/p,\, \mathbb{Z}/p,\, \mathbb{Z}/p\,\big)\\
S_5 &= \big(\mathbb{Z}/p \oplus \mathbb{Z}/p\,;\ \langle(1,0)\rangle,\, \langle(1,1)\rangle,\, \langle(0,1)\rangle\,\big)\\
S_6 &= \big(\mathbb{Z}/p^2\,;\ \mathbb{Z}/p^2,\, \mathbb{Z}/p^2,\, 0\,\big)\\
S_7 &= \big(\mathbb{Z}/p^2\,;\ \mathbb{Z}/p^2,\, \mathbb{Z}/p^2,\, \mathbb{Z}/p\,\big)\\
S_8 &= \big(\mathbb{Z}/p^2 \oplus \mathbb{Z}/p\,;\ \langle(1,0)\rangle,\, \langle(1,1)\rangle,\, \langle(0,1)\rangle\,\big)\\
S_9 &= \big(\mathbb{Z}/p^2 \oplus \mathbb{Z}/p\,;\ \langle(1,0)\rangle,\, \langle(1,1)\rangle,\, \langle(p,0),(0,1)\rangle\,\big)
\end{aligned}$$

*Proof.* An easy consideration shows that all objects in this list are indecomposable.

Let $V_* = (V;\, V_1, V_2, V_3)$ be an object in $Syst(C_{p^2})$. We show that $V_*$ decomposes into direct summands of type $S_1, \ldots, S_9$. Note that a subobject $U_* \subset V_*$ is a direct summand if and only if $U \subset V$ and all $U_i \subset V_i$ are direct summands.

We introduce some notation: For $V = (\mathbb{Z}/p^2)^m \oplus (\mathbb{Z}/p)^n$ we consider submodules $\bar{V} := (\mathbb{Z}/p^2)^m \oplus 0$ and $V' := 0 \oplus (\mathbb{Z}/p)^n$. Choose fixed generators $e_1, \ldots, e_m$ of $\bar{V}$.

There is a particular decomposition of the $\mathbb{Z}/p^2$-module $V_1$ into a free and a torsion submodule and a particular choice of generators for the free part of the form

$$V_1 = \langle e_1 + r_1, \ldots, e_m + r_m\rangle \oplus V_1' \quad \text{with } r_1, \ldots, r_m \in V' \text{ and } V_1' \subset V'.$$

To see this, choose for every $e_i$ an $r_i \in V_3$ such that $e_i + r_i$ is an element of $V_1$. This is possible because $\bar{V} \subset V = V_1 + V_3$. Futher, since $pV_3 = 0$, we have $V_3 \subset p\bar{V} \oplus V'$. We may suppose that $r_i$ has no part in $p\bar{V}$ but is contained in $V'$. The elements $e_i + r_i$ generate the free part of $V_1$. We can choose a complementary torsion part to be contained in $V'$. With the same argument we can choose for $V_2$ a decomposition of the form

$$V_2 = \langle e_1 + u_1, \ldots, e_m + u_m\rangle \oplus V_2' \quad \text{with } u_1, \ldots, u_m \in V' \text{ and } V_2' \subset V'.$$

If there is a non-zero element $x \in (V_1 \cap V_2)\backslash p\bar{V}$, it is contained in $V'$ or has to be of order $p^2$. In the first case $V_*$ has a direct summand of type $S_1$ or $S_4$, in the second of type $S_6$ or $S_7$. We split these off and call the remaining summand $V_*$ again.

Further, we can split off summands of type $S_2$ or $S_3$ until $(V_1\backslash p\bar{V}) \cap V_3 = 0 = (V_2\backslash p\bar{V}) \cap V_3$. Now suppose there is a non-zero element $x \in V_1'$. By the previous reduction steps $x$ is not contained in $V_2$ or $V_3$. But $V = V_2 + V_3$. So there are $x_2 \in V_2$ and $x_3 \in V_3$ such that $x = x_2 + x_3$. Note that $x_2 \in V_2\backslash p\bar{V}$, because otherwise $x_3 \in (V_1\backslash p\bar{V}) \cap V_3$. Hence $V_*$ has a direct summand

$$(\langle x_2, x_3\rangle;\, \langle x\rangle, \langle x_2\rangle, \langle x_3\rangle)$$

of type $S_5$. Having split off all summands of this type, in the remaining summand $V_1' = 0$ and $V_2' = 0$. It decomposes into summands

$$(\langle e_i + r_i, r_i - u_i\rangle;\, \langle e_i + r_i\rangle, \langle e_i + u_i\rangle, \langle r_i - u_i\rangle),$$

respectively

$$(\langle e_i + r_i, r_i - u_i\rangle;\ \langle e_i + r_i\rangle, \langle e_i + u_i\rangle, \langle pe_i, r_i - u_i\rangle)$$

of type $S_8$, respectively $S_9$. □

Having classified the indecomposable objects in $Syst(G)$ we now use the epivalence $\Delta$ and translate to a classification of the indecomposable genera of $A(C_{p^2})$.

**Theorem 3.5.** *The indecomposable genera of $A(C_{p^2})$ are represented by the following modules:*

$$\begin{aligned}
L_i &= e_i A(C_{p^2}), \quad \textit{for } i = 1,2,3\\
M_1 &= \langle p^2 e_1\,,\, pe_1 + pe_2\rangle\\
M_2 &= \langle p^2 e_1\,,\, e_1 + e_2\rangle\\
M_3 &= \langle p^2 e_1\,,\, pe_1 + e_3\rangle\\
M_4 &= \langle pe_2\,,\, e_2 + e_3\rangle\\
M_5 &= \langle p^2 e_1\,,\, pe_1 + pe_2\,,\, e_1 + e_2 + e_3\rangle\\
M_6 &= \langle p^2 e_1\,,\, pe_1 + pe_2\,,\, pe_1 + e_3\rangle\\
M_7 &= \langle p^2 e_1\,,\, e_1 + e_2\,,\, pe_1 + e_3\rangle\\
M_8 &= \langle p^2 e_1\,,\, pe_2\,,\, pe_1 + e_2 + e_3\rangle\\
M_9 &= \langle p^2 e_1\,,\, pe_1 + pe_2\,,\, pe_1 + e_3\,,\, e_1 + e_2 + e_3'\rangle
\end{aligned}$$

*The last representation $M_9$ is a subset of $C(C_{p^2}) \oplus e_3' C(C_{p^2})$ where $e_3' C(C_{p^2})$ is another copy of $e_3 C(C_{p^2})$.*

*Proof.* We calculate directly:

$$\begin{array}{lllll}
\Delta M_1 \cong S_1, & \Delta M_2 \cong S_6, & \Delta M_3 \cong S_2, & \Delta M_4 \cong S_3, & \\
\Delta M_5 \cong S_8, & \Delta M_6 \cong S_4, & \Delta M_7 \cong S_7, & \Delta M_8 \cong S_5, & \Delta M_9 \cong S_9.
\end{array}$$

Here we have identified the modules $M_i$ with their respective localizations. □

Only the modules in one of the indecomposable genera, namely in the genus of $M_9$, are not ideals in $A(C_{p^2})$. It seems to be a coincidence that this is a similar result as for integral group rings [4, Chap. 34].

**3.5. Isomorphism Classes in the Various Genera.** It remains to classify the isomorphism classes of modules in each genus of indecomposable $A(C_{p^2})$-representations. We will describe the modules as subsets of $C(C_{p^2})$-representations like the modules $M_1, \ldots, M_9$ in Theorem 3.5 above. Note that for every group $G$ a $C(G)$-representation is a direct sum of modules $e_H C(G)$ and has a canonical ring structure.

**Remark 3.6.** *For every $A(G)$-representation $M$ there is an injection into a $C(G)$-representation $C(M)$ of the same rank over $\mathbb{Z}$ such that $C(M)M = C(M)$. We call such an embedding non-degenerate. Up to isomorphism the module $C(M)$ depends only on the genus of $M$.*

*Proof.* Since $\mathbb{Q} \otimes_{\mathbb{Z}} A(G) \cong \mathbb{Q} \otimes_{\mathbb{Z}} C(G)$, we can consider $\mathbb{Q} \otimes_{\mathbb{Z}} M$ as a $C(G)$-module. It decomposes into 1-dimensional direct summands of the form $\mathbb{Q} \otimes_{\mathbb{Z}} e_H C(G)$. Define $C(M)$ to be the direct sum of the respective $C(G)$-modules $e_H C(G)$. The canonical map $C(M) \to \mathbb{Q} \otimes_{\mathbb{Z}} C(M) \cong \mathbb{Q} \otimes_{\mathbb{Z}} M$ identifies $C(M)$ with a subset of $\mathbb{Q} \otimes_{\mathbb{Z}} M$. By multiplication with suitable integers we can deform the inclusion $M \to \mathbb{Q} \otimes_{\mathbb{Z}} M$ such that the image is contained in $C(M)$. Moreover, we can choose this map such that in each $e_H$-coordinate the greatest common divisor for all elements of $M$ is equal to 1. If this is not the case just divide in each coordinate by the greatest common divisor. We have constructed a non-degenerate embedding. The module $C(M)$ depends only on $\mathbb{Q} \otimes_{\mathbb{Z}} M$ and thus on the genus of $M$. □

Now let $G$ be a $p$-group. We can quite easily determine the representations which belong to genera of modules contained in $C(G)$. The description in Theorem 3.8 below is particularly usefull, because the endomorphisms of the $A(G)$-modules $C(M)$ are very easy to handle. We require the following fact:

**Proposition 3.7.** *Let $G$ be a $p$-group and $M$ an $A(G)$-representation which is non-degenerately embedded into a direct summand $C(M)$ of $C(G)$. Then the cokernel of the embedding $M \to C(M)$ is a $p$-group.*

*Proof.* Fix an integer $k$ such that $p^k e_H$ is contained in $A(G)$ for every $(H) \in \Phi(G)$. Since the embedding of $M$ in $C(M)$ is non-degenerate, for each generator $e_H$ of $C(M)$ there are elements $m_i \in M$ and $x^i = \sum x_H^i e_H \in C(M)$ such that $e_H = \sum_i m_i x^i$. Therefore

$$p^k e_H = p^k e_H e_H = \sum_i (p^k e_H m_i) x_H^i$$

is a linear combination of elements in $M$ with integral coefficients $x_H^i$. Hence $p^k C(M)$ is a subset of $M$ and $C(M)/M$ is a quotient of $C(M)/p^k C(M)$. This proves that $C(M)/M$ is a $p$-group. □

**Theorem 3.8.** *Let $G$ be a $p$-group and $M$ an $A(G)$-representation which is non-degenerately embedded into a direct summand $C(M)$ of $C(G)$. Then each module in the genus of $M$ is isomorphic to a module of the form*

$$\varphi^{-1} M = \{x \in C(G) \mid \varphi(x) \in M\}$$

*where $\varphi : C(M) \to C(M)$ is an $A(G)$-linear map which induces an automorphism in $p$-localization. A non-degenerate submodule $N$ of $C(M)$ is isomorphic to $M$ if and only if $M = \phi^{-1} N$ for an $A(G)$-linear automorphism $\phi$ of $C(M)$.*

*Proof.* Since $\varphi$ induces a $p$-local isomorphism, it follows that every module of the form $\varphi^{-1} M$ is in the genus of $M$. Now suppose that $N$ is in the genus of $M$. We can choose $N$ to be a non-degenerate submodule of $C(M)$. An isomorphism $\varphi : N_p \to M_p$ induces an isomorphism $\mathbb{Q} \otimes_{\mathbb{Z}} N_p \to \mathbb{Q} \otimes_{\mathbb{Z}} M_p$. Since the embeddings of $N$ respectively $M$ in $C(M)$ induce isomorphisms $\mathbb{Q} \otimes_{\mathbb{Z}} N_p \cong \mathbb{Q} \otimes_{\mathbb{Z}} C(M)_p$ respectively $\mathbb{Q} \otimes_{\mathbb{Z}} M_p \cong \mathbb{Q} \otimes_{\mathbb{Z}} C(M)_p$, we get from $\varphi$ an automorphism of $\mathbb{Q} \otimes_{\mathbb{Z}} C(M)_p$, also denoted by $\varphi$.

Consider $C(M)_p$ as a subset of $\mathbb{Q} \otimes_{\mathbb{Z}} C(M)_p$. The map $\varphi$ is $A(G)$-linear, hence $C(G)$-linear and in particular $C(M)$-linear. Since $N$ in $C(M)$ is non-degenerate, we can write every $x \in C(M)_p$ as a linear combination $x = \sum x_i n_i$ of $x_i \in C(M)$ and $n_i \in N_p$. This implies that $\varphi(x) = \sum x_i \varphi(n_i)$ is an element in $C(M)_p$. We can use the same argument for the inverse of $\varphi$. This proves that $\varphi$ is an automorphism of $C(M)_p$. By multiplication of $\varphi$ with a unit in $\mathbb{Z}_p$ we can ensure that $\varphi(C(M))$ is contained in $C(M)$.

By Proposition 3.7 the quotient $C(M)/M$ is a $p$-group. Therefore, if for an element $x \in C(M)$ there is a number $s$ prime to $p$ such that $sx$ is an element of $M$, then already $x$ is contained in $M$. This implies that $M_p \cap C(M) = M$. By the same arguments it follows that $N_p \cap C(M) = N$. Therefore, $\varphi(N)$ is contained in $M$ and $\varphi^{-1}(M) \cap C(M)$ in $N$. In summary we have proved that $\varphi^{-1} M = N$ for the restriction $\varphi : C(M) \to C(M)$.

For an isomorphism $\phi : N \to M$ we can prove by the same arguments as above that it extends to an automorphism of $C(M)$. □

As an application of this theorem we show that the isomorphism classes of modules in the genus of the module $M_5 = A(C_{p^2})$, i.e. the isomorphism classes of the projective modules of rank 1, are represented by modules

$$\begin{aligned} M(a,b) :=& \{x \in C(C_{p^2}) \mid x_0 \equiv ax_1 \bmod p^2,\ \ x_1 \equiv bx_2 \bmod p\} \\ =& A(C_{p^2})\begin{pmatrix} ab \\ b \\ 1 \end{pmatrix} + p^2 C(C_{p^2}) \qquad \text{with } a, b \not\equiv 0 \bmod p. \end{aligned}$$

Two of these modules $M(a,b)$ and $M(\bar{a},\bar{b})$ are isomorphic if and only if $a \equiv \pm\bar{a} \bmod p^2$ and $b \equiv \pm\bar{b} \bmod p$. Note that the module $M(1,1)$ is just the Burnside ring $A(C_{p^2})$.

This result is well known and was already mentioned in Section 2 in less detailed form. Calculations for the genera of $M_1, \ldots, M_4, M_6, M_7, M_8$ are equally easy.

Observe that an endomorphism $\varphi$ of $C(C_{p^2})$ which becomes an automorphism after $p$-localization is described by a diagonal matrix with respect to the usual basis

$$\varphi = \begin{pmatrix} s & & 0 \\ & t & \\ 0 & & u \end{pmatrix} \qquad \text{with} \quad s, t, u \not\equiv 0 \ \bmod p.$$

It is an automorphism if and only if $s, t, u \in \{+1, -1\}$. Every module in the genus of $A(C_{p^2})$ has the form $\varphi^{-1}A(C_{p^2})$. We can easily see that $\varphi^{-1}A(C_{p^2}) = \psi^{-1}A(C_{p^2})$ for

$$\psi = \begin{pmatrix} 1 & & 0 \\ & a & \\ 0 & & ab \end{pmatrix} \qquad \text{such that} \quad \begin{matrix} as \equiv t \ \bmod p^2, \\ bt \equiv u \ \bmod p. \end{matrix}$$

Moreover, $\psi^{-1}A(C_{p^2}) = M(a,b)$ is equal to $M(\bar{a},\bar{b})$ if and only if $a \equiv \bar{a} \ \bmod p^2$ and $b \equiv \bar{b} \ \bmod p$. By the second part of Theorem 3.8 we get our result.

There is another possibility to get all modules in the genus of a submodule in $C(G)$. Note that the isomorphism classes of projective modules of rank 1 form by tensor product an abelian group, the Picard group $\operatorname{Pic} A(G)$. For every $A(G)$-representation $M$ and every projective module $P$ of rank 1 the representation $P \otimes_{A(G)} M$ is in the same genus as $M$. This defines an action of the Picard group on each genus of representations.

**Proposition 3.9.** *Let $G$ be a $p$-group. Then* $\operatorname{Pic} A(G)$ *acts transitively on the genus of each submodule $M \subset C(G)$.*

*Proof.* Suppose, $C(M)$ is a direct summand of $C(G)$ and $M \subset C(M)$ is non-degenerate. Every $N$ in the genus of $M$ is isomorphic to a module $\varphi^{-1}M$, where $\varphi$ is an $A(G)$-linear endomorphism of $C(M)$ which is a $p$-local automorphism. We extend $\varphi$ to an endomorphism of $C(G)$ by choosing the identity on the direct complement of $C(M)$.

By Theorem 3.8 the module $P = \varphi^{-1}A(G)$ is projective of rank 1. With respect to the standard basis, $\varphi$ is described by a diagonal matrix with entries prime to $p$. There is a homomorphism $P \otimes_{A(G)} M \longrightarrow N \subset C(G)$ defined by multiplication in each coordinate. In order to prove that this is an isomorphism, we construct an inverse map: Choose a diagonal matrix $\bar{\varphi}$ such that $\varphi\bar{\varphi} = 1 + |G|^2 D$ for another diagonal matrix $D$. For every $x \in N$ and $r \in \bar{\varphi}^{-1}A(G)$ the product

$$rx = (\varphi\bar{\varphi} - |G|^2 D)rx = \bar{\varphi}r\,\varphi x - |G|^2 Drx$$

is an element in $M$. Since $P \cdot \bar{\varphi}^{-1}A(G) = A(G)$, there is a decomposition $1 = \sum_i t_i r_i$ for some $t_i \in P$ and $r_i \in \bar{\varphi}^{-1}A(G)$. The map $x \mapsto \sum_i t_i \otimes r_i x$ is the desired inverse of $\varphi$. $\square$

Now we turn to the investigation of the modules in the genus of $M_9$. These are submodules of $C(M_9) = C(\mathbb{Z}/p^2) \oplus e_3' C(\mathbb{Z}/p^2)$ of maximal rank. We show that we can choose the embedding in a particularly nice form:

**Remark 3.10.** *Every submodule of maximal rank in $C(M_9)$ is isomorphic to a non-degenerate one of the form*

$$\langle\, p^n e_1,\, p^m(ye_1 + e_2),\, z_1e_1 + z_2e_2 + e_3,\, v_1e_1 + v_2e_2 + e_3' \,\rangle$$

*with integers $y, z_1, z_2, v_1, v_2$ and $0 \le n \le 2,\ 0 \le m \le 1$.*

*Proof.* For every submodule of maximal rank in $C(M_9)$ there is a basis of the form

$$xe_1, \quad y_1e_1 + y_2e_2, \quad z_1e_1 + z_2e_2 + z_3e_3, \quad v_1e_1 + v_2e_2 + v_3e_3 + v_4e_3'$$

with $x, y_2, z_3, v_4 \neq 0$. As a module over $A(C_{p^2})$ this is isomorphic to the module with basis

$$xe_1, \quad y_1e_1 + y_2e_2, \quad z_1e_1 + z_2e_2 + e_3, \quad v_1e_1 + v_2e_2 + e_3'.$$

By dividing out common divisors in each coordinate, we can choose this to be non-degenerate. Finally, the conditions for $x, y_1, y_2$ are consequences of the action of $p^2e_1$ and $p(e_1 + e_2) \in A(C_{p^2})$. □

There is an analogue to Theorem 3.8:

**Theorem 3.11.** *Every $A(C_{p^2})$-representation in the genus of $M_9$ is isomorphic to a module of the form*

$$\varphi^{-1}M_9 = \{\, x \in C(M_9) \mid \varphi(x) \in M_9 \,\}$$

*where $\varphi$ is an $A(C_{p^2})$-linear endomorphism of $C(M_9)$ which is $p$-locally an automorphism. Submodules $N$ and $\bar{N}$ in $C(M_9)$ as in Remark 3.10 are isomorphic if and only if $N = \phi^{-1}\bar{N}$ for an $A(C_{p^2})$-linear automorphism of $C(M_9)$.*

*Proof.* We proceed as in the proof of Theorem 3.8: For a module $N$ in the genus of $M := M_9$ choose an embedding in $C(M)$ as in the remark above. An isomorphism $\varphi : N_p \to M_p$ induces an automorphism of $\mathbb{Q} \otimes_{\mathbb{Z}} C(M)_p$, also denoted by $\varphi$.

We show that $\varphi(C(M)_p)$ is contained in $C(M)_p$. Here we need a little variation of the arguments in the proof of Theorem 3.8: Since $p^2e_1$ and $p^2e_2 \in A(C_{p^2})$ are acting on $C(M)$ by multiplication in the first, respective second, coordinate, it follows that $\varphi$ commutes with multiplications by $e_1$ and $e_2$ in $C(M)$. Because $N$ is non-degenerate, there are numbers $x_i \in \mathbb{Z}$ and elements $n_i \in N$ such that $e_1 = \sum_i x_ie_1\, n_i$. Hence $\varphi(e_1) = \sum_i x_ie_1\varphi(n_i)$ is an element in $C(M)_p$. For the same reason, $\varphi(e_2)$ is an element in $C(M)_p$. Since the embedding of $N$ in $C(M)$ is of the form of Remark 3.10, there are numbers $a, b \in \mathbb{Z}$ and an element $n \in N$ such that $e_3 = n + ae_1 + be_2$. From this follows that $\varphi(e_3)$ is contained in $C(M)_p$. The same argument applies to $e_3'$.

For the rest of the proof we can use the same arguments as in the proof of Theorem 3.8. □

This theorem enables us to determine all the representations in the genus of $M_9$. The crucial point is that the $A(C_{p^2})$-linear endomorphisms of $C(M_9)$ which are $p$-local automorphisms are of a particularly nice form with respect to the standard basis:

$$\varphi = \begin{pmatrix} s & & 0 & \\ & t & & \\ & & x & y \\ 0 & & z & v \end{pmatrix} \qquad \text{with } \ s,\, t,\, (xv - yz) \not\equiv 0 \mod p.$$

An easy calculation shows that every module $\varphi^{-1}M_9$ is isomorphic to one of the following form:

$$M_9(a,b) := \langle\, p^2e_1,\, p(ae_1+e_2),\, pe_1+e_3,\, abe_1+be_2+e_3' \,\rangle \quad \text{with } a,b \not\equiv 0 \mod p.$$

These modules depend on $a$ only modulo $p^2$ and on $b$ only modulo $p$. Using Theorem 3.11 we see that the modules $M_9(a,b)$ and $M_9(\bar a,\bar b)$ are isomorphic if and only if $a \equiv \pm\bar a \mod p^2$ and $b \equiv \pm\bar b \mod p$.

For $p \neq 2$ the action of the Picard group on this genus is not transitive:

**Proposition 3.12.** *For $p \neq 2$ the action of the Picard group decomposes the genus of $M_9$ into two orbits: the one of $M_9 = M_9(1,1)$ and that of $M_9(1,-1)$.*

*Proof.* Using similar methods as in the proof of Proposition 3.9 we can show that $M(a,b) \otimes_{A(C_{p^2})} M_9(1,\pm1)$ is isomorphic to the submodule of $C(M_9)$

$$\langle\, p^2e_1,\, p(ae_1+e_2),\, abpe_1+e_3,\, \pm abe_1 \pm be_2 + e_3' \,\rangle.$$

Choose an automorphism of $C(M_9)$ of the form

$$\begin{pmatrix} 1 & & 0 & \\ & 1 & & \\ & & ab & p \\ 0 & & rp & x \end{pmatrix} \qquad \text{such that } abx = 1 + rp^2.$$

It maps the module above isomorphically onto $M_9(a, \pm ab^2)$. For $p \neq 2$ we need both signs to get all modules in the genus of $M_9$. ◻

## 4. Groups with Square Free Order

Groups with square free order are precisely those whose Burnside rings have at most double points in their prime ideal spectrum. Some of the nice properties we have encountered for $A(C_p)$ hold for the representation theory of all these rings.

**Theorem 4.1.** *Let $G$ be a finite group whose order is not divisible by a square. Then:*

(i) *Every indecomposable representation of $A(G)$ is isomorphic to an ideal in $A(G)$.*
(ii) *Every indecomposable $A(G)$-representation is a projective module over a ring extension $A(G) \subset \Lambda \subset C(G)$.*
(iii) *The Picard group $\operatorname{Pic} A(G)$ acts transitively on each indecomposable genus.*

*Proof.* We want to use various results of Bass [1]. For this, we have to show that every ring extension $A(G) \subset \Lambda \subset C(G)$ is a Gorenstein ring. This is a local property. For a regular point $\mathfrak{p} \in \operatorname{Spec}\Lambda$ the localized ring $\Lambda_{\mathfrak{p}}$ is regular and in particular Gorenstein. For a double point $\mathfrak{p} \in \operatorname{Spec}\Lambda$ the ideal $\mathfrak{p}$ is of the form

$$\mathfrak{p} = pe_iC(G) \cap \Lambda = pe_jC(G) \cap \Lambda \quad \text{for primitive idempotents } e_i \neq e_j \text{ in } C(G).$$

For every $k \neq i,j$ the intersection $pe_kC(G) \cap \Lambda$ is not equal to $\mathfrak{p}$, because otherwise $\mathfrak{p}$ would be a crossing of an order higher than 2. Hence, there is an element in $\Lambda$

$$le_k + ne_i + me_j + r \quad \text{such that} \quad l \equiv 0(p),\ n,m \not\equiv 0(p) \text{ and } r \in \sum_{t \neq i,j,k} e_tC(G).$$

We can suppose that the coefficient $l$ of $e_k$ is divisible by the order of $G$. If this is not the case we only need to multiply by a suitable number prime to $p$. Furthermore, since $|G|C(G) \subset A(G)$, we can suppose that $l = 0$. The product of these elements for all $k \neq i,j$ is an element $x \in \Lambda \cap (e_i + e_j)C(G)$ which is not contained in $\mathfrak{p}$. For every $\lambda = \sum_t \lambda_t e_t \in \Lambda$ we get after localization with respect to $\Lambda \backslash \mathfrak{p}$ the equality

$$\frac{\lambda}{1} = \frac{\lambda x}{x} = \frac{(\lambda_i e_i + \lambda_j e_j)x}{x} = \frac{\lambda_i e_i + \lambda_j e_j}{1}.$$

This implies that $\Lambda_\mathfrak{p}$ is generated by at most two elements. Moreover, the image of $p \in A(G)$ in $\mathfrak{p}\Lambda_\mathfrak{p}$ is not a zero-divisor. By a criterion of Bass [1, 6.4] this implies that $\Lambda_\mathfrak{p}$ is a Gorenstein ring.

Now the first statement follows from [1, 7.3]. The second statement is a consequence of the above and [1, 7.2].

Let $M$ be an indecomposable $A(G)$-representation which is projective for a ring extension $\Lambda$ of $A(G)$ in $C(G)$. By the second statement $\operatorname{Pic}\Lambda$ acts transitively on the genus of $M$. The injection $A(G) \subset \Lambda$ induces a surjective homomorphism

$$\begin{array}{ccc} \operatorname{Pic} A(G) & \longrightarrow & \operatorname{Pic}\Lambda \\ N & \longmapsto & \Lambda \otimes_{A(G)} N \end{array}$$

[5, 49.25 and 55.26]. This proves the third part. □

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