Characteristic tilting modules over a class of quasi–hereditary algebras

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Abstract

With each finite directed quiver Q a quasi–hereditary algebra, the so–called twisted double of the path algebra kQ, is associated (This class of algebras contains certain interesting ones). Charateristic tilting modules over this class of quasi–hereditary algebras are constructed. Their endomorphism algebras are explicitly described. It turns out that this class of quasi–hereditary algebras is closed under taking Ringel dual.

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Quasi–hereditary algebras are introduced by Scott in [15] in order to study highest weight categories in representation theory of semisimple Lie algebras and algebraic groups (see also [3]). Many important algebras such as hereditary algebras, Schur algebras, and algebras to blocks of the category $\mathcal{O}$ in [2] are typical examples of quasi–hereditary algebras.

In the study of a quasi–hereditary algebra A (with respect to a partial order on the set Λ of simple A–modules), the central notions are the standard modules $\{\Delta(\lambda) : \lambda \in \Lambda\}$ and the costandard modules $\{\nabla(\lambda) : \lambda \in \Lambda\}$. Of main interest are the categories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ of A–modules which have a Δ-filtration and ∇-filtration respectively. It is proved by Ringel in [14] that $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ contains only finitely many isomorphism classes of indecomposable modules, which may be labelled by $T(\lambda)$, $\lambda \in \Lambda$. These modules $T(\lambda)$, which now are called tilting modules, soon become an important object in representation theory of Lie algebras, algebraic groups and quantum groups (see, for example, [7], [8], [1] and [13]). Moreover, the module $T := \oplus T(\lambda)$ is called the characteristic tilting module of A whose endomorphism algebra, called the Ringel dual of A, is again quasi–hereditary.

In general, it seems to be difficult to describe characteristic tilting modules and their endomorphism algebras. However, these questions could be dealt with for some special classes of quasi–hereditary algebras (see, for example, [4], [5], [7], [8], [12] and

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[17]). In the present paper we deal with a class of quasi–hereditary algebras which are twisted doubles of hereditary algebras (This class of algebras covers quadratic duals of all Temperley–Lieb algebras). The characteristic tilting modules and the Ringel duals of these algebras are explicitly described. It turns out that this class of quasi–hereditary algebras is closed under taking Ringel dual. This generalizes the main results in [4, Section 4]. Moreover, the method presented in this paper simplifies the proofs there.

1. Preliminaries

In this section we recall some definitons and include some basic results needed in the sequel.

1.1 Let A be a finite dimensional algebra over a field k. The category of all finitely generated left A–modules will be denoted by A–mod, and the composition of maps $f : M_1 \longrightarrow M_2$ and $g : M_2 \longrightarrow M_3$ is denoted by gf. Given a class Θ of A–modules, we denote by $\mathcal{F}(\Theta)$ the full subcategory of all A–modules which have a Θ–filtration, that is, a filtration

$$0 = M_t \subset M_{t-1} \subset \cdots \subset M_1 \subset M_0 = M$$

such that each factor M_{i-1}/M_i is isomorphic to one object in Θ for $1 \leq i \leq t$. The modules in $\mathcal{F}(\Theta)$ are called Θ-good modules, and the category $\mathcal{F}(\Theta)$ is called the Θ-good module category.

Let Λ be a finite poset in bijective correspondence with the isomorphism classes of simple A–modules. For each $\lambda \in \Lambda$, let $E(\lambda)$ be a simple module in the isomorphism class corresponding to λ and $P(\lambda)$ (or $P_A(\lambda)$) a projective cover of $E(\lambda)$ and denote by $\Delta(\lambda)$ the maximal factor module of $P(\lambda)$ with composition factors of the form $E(\mu)$, $\mu \leq \lambda$. Dually, let $Q(\lambda)$ (or $Q_A(\lambda)$) be an injective hull of $E(\lambda)$ and denote by $\nabla(\lambda)$ the maximal submodule of $Q(\lambda)$ with the composition factors of the form $E(\mu)$, $\mu \leq \lambda$. Let Δ (respectively, ∇) be the full subcategory consisting of all $\Delta(\lambda)$, $\lambda \in \Lambda$ (respectively, all $\nabla(\lambda)$, $\lambda \in \Lambda$). We call modules in Δ standard modules and ones in ∇ costandard modules.

The algebra A is said to be quasi–hereditary with respect to $(\Lambda, \leq)$ if for each $\lambda \in \Lambda$ we have

(i) $\mathrm{End}_A(\Delta(\lambda))$ is a division ring;

(ii) $P(\lambda) \in \mathcal{F}(\Delta)$, and moreover, $P(\lambda)$ has a Δ–filtration with factors $\Delta(\mu)$ for $\mu \geq \lambda$ in which $\Delta(\lambda)$ occurs exactly once.

For a quasi–hereditary algebra A with respect to a poset Λ we call the elements in Λ weights and Λ the weight poset of A. By (A, Λ) we denote a quasi–hereditary algebra A with the weight poset Λ.

If a quasi–hereditary algebra has a duality δ on the category A–mod which fixes simple modules, we call it a BGG–algebra (see [9], [3]).

1.2 Let A be a quasi–hereditary algrbra with the weight poset Λ. By [14], there are indecomposable modules $T(\lambda)$, $\lambda \in \Lambda$, and exact sequences

$$0 \longrightarrow \Delta(\lambda) \longrightarrow T(\lambda) \longrightarrow X(\lambda) \longrightarrow 0$$

and

$$0 \longrightarrow Y(\lambda) \longrightarrow T(\lambda) \longrightarrow \nabla(\lambda) \longrightarrow 0$$

where $X(\lambda)$ is filtered with factors $\Delta(\mu)$, $\mu < \lambda$, and $Y(\lambda)$ is filtered with factors $\nabla(\mu)$, $\mu < \lambda$, such that $T = \oplus_{\lambda\in\Lambda}T(\lambda)$ satisfies

$$\text{add}T = \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla).$$

The module T is a generalized tilting and cotilting modules and called the characteristic tilting module of A. The algebra $\mathcal{R}(\mathcal{A}) := (\text{End}_A(T))^{op}$ is called the Ringel dual of A (Note that the Ringel dual is defined to be the opposite algebra of $\text{End}_A(T)$ since we compose morphisms from right to left). This algebra is shown to be again quasi–hereditary with weight poset Λ^{op} whose standard modules are $\text{Hom}_A(T, \nabla(\lambda))$, $\lambda \in \Lambda$ (see [14, Theorem 6]).

Lemma. *Let A be quasi–hereditary. Then the following conditions are equivalent:*

(i) The projective dimension of any standard module is at most 1.

(ii) The projective dimension of the characteristic module T is at most 1.

(iii) The subcategory $\mathcal{F}(\nabla)$ is closed under factor modules.

(iv) All $X(\lambda)$, $\lambda \in \Lambda$, lie in add T.

Proof. The eqivalence of (i), (ii) and (iii) has been proved in [6, Lemma 4.1]. It is obvious that (iii) implies (iv). We now prove that (iv) implies (i).

Suppose that all $X(\lambda)$ lie in add$T$. Then there holds that $\text{Ext}^i_A(X, Y) = 0$ for all $X, Y \in \mathcal{F}(\Delta)$ and $i \geq 2$ since $T(\lambda)$'s are Ext-injective objects in $\mathcal{F}(\Delta)$. For each $\lambda \in \Lambda$, there is an exact sequence

$$0 \longrightarrow Q(\lambda) \longrightarrow P(\lambda) \longrightarrow \Delta(\lambda) \longrightarrow 0,$$

where $Q(\lambda)$ denotes the kernel of the canonical projection $P(\lambda) \to \Delta(\lambda)$.

Let $X \in \mathcal{F}(\Delta)$ and apply $\text{Hom}_A(-, X)$ to the sequence above, we obtain the following exact sequences ($i \geq 1$)

$$\longrightarrow \text{Ext}^i_A(P(\lambda), X) \longrightarrow \text{Ext}^i_A(Q(\lambda), X) \longrightarrow \text{Ext}^{i+1}_A(\Delta(\lambda), X) \longrightarrow$$

This implies that $\text{Ext}^i_A(Q(\lambda), X) = 0$ for all $i \geq 1$, that is, $Q(\lambda)$ is Ext–projective in $\mathcal{F}(\Delta)$, thus a projective A–module. This finishes the proof.

Dually, we have the following lemma (see [6, Lemma 4.1*]).

Lemma*. *Let A be quasi–hereditary. Then the following conditions are equivalent:*

(i) The injective dimension of any costandard module is at most 1.

(ii) The injective dimension of the characteristic module T is at most 1.

(iii) The subcategory $\mathcal{F}(\Delta)$ is closed under submodules.

(iv) All $Y(\lambda)$, $\lambda \in \Lambda$, lie in add T.

1.3 Let $Q = (Q_0, Q_1)$ be a finite quiver. For each path w in Q, we denote by $s(w)$ and $t(w)$ the starting and terminal vertex of w respectively, and by $l(w)$ the length of w.

From now on, we suppose that Q is a directed quiver and that $Q_0 = \{1, 2, \cdots, n\}$ is such that an arrow $\alpha : i \to j$ implies $i > j$. With Q we now associate an algebra $\mathcal{A}(Q)$ as follows. By $\overline{Q}$ we denote the quiver obtained from Q by formally adding opposite arrows $\alpha' : j \to i$ whenever $\alpha : i \to j$ is an arrow in Q. We define $\mathcal{A}(Q)$ to be the algebra $k\overline{Q}/I$, where I denotes the ideal of the path algebra $k\overline{Q}$ generated by elements of the following two types

(1) $\alpha'\alpha - \sum_{\gamma \in Q_1, t(\gamma)=s(\alpha)} \gamma\gamma'$, $\quad \alpha \in Q_1$,
(2) $\beta'\alpha$, $\quad \alpha, \beta \in Q_1$ with $t(\alpha) = t(\beta)$ and $\alpha \neq \beta$.

For example, if $Q = Q_n$ denotes the following quiver

$$1 \xleftarrow{\alpha_1} 2 \xleftarrow{\alpha_2} 3 \longleftarrow \quad \cdots \; n-1 \xleftarrow{\alpha_{n-1}} n$$

then the algebra $\mathcal{A}_n := \mathcal{A}(Q_n)$ is defined by the quvier $\overline{Q_n}$

$$\begin{tikzcd} 1 \arrow[r, shift right, "\alpha_1'"'] & 2 \arrow[l, shift right, "\alpha_1"'] \arrow[r, shift right, "\alpha_2'"'] & 3 \arrow[l, shift right, "\alpha_2"'] \arrow[r, shift right] & \cdots \arrow[l, shift right] & n-1 \arrow[r, shift right, "\alpha_{n-1}'"'] & n \arrow[l, shift right, "\alpha_{n-1}"'] \end{tikzcd}$$

modulo relations: $\alpha'_{n-1}\alpha_{n-1} = 0$, $\alpha'_i\alpha_i = \alpha_{i+1}\alpha'_{i+1}$ for $1 \le i \le n-2$. It is known that the algebra $\mathcal{A}_n$ is isomorphic to the Auslander algebra of $k[X]/(X^n)$, where $k[X]$ is the polynomial ring with one variable X. Further, each block of Temperley–Lieb algebras is Morita equivalent to the quadratic dual of certain $\mathcal{A}_n$ (see [16]).

Note that if Q is such that its underlying diagram is a tree, the algebra $\mathcal{A}(Q)$ is then the twisted double incidence algebra of a poset defined in [4]. In general, the algebra $\mathcal{A}(Q)$ is the twisted double of the path algebra $kQ =: B$ with certain labelling (see a general construction in [12]). According to [12, Proposition 9.1], $\mathcal{A}(Q)$ is quasi–hereditary with the natural ordering $\{1 < 2 < \cdots < n\}$. In fact, $\mathcal{A}(Q)$ admits a triangular decomposition $B \otimes_S B^{op}$ (for definition see [10]), where S denotes the maximal semisimple subalgebra $\oplus_{i=1}^n ke_i$ of $\mathcal{A}(Q)$. Moreover, $\mathcal{A}(Q)$ admits an anti-isomorphism ε which exchanges α with α', $\alpha \in Q$. Thus we have the following proposition.

Proposition. *The algebra $\mathcal{A}(Q)$ is a BGG-algebra.*

Corollary . *(1) For each $1 \le i \le n$, the standard module $\Delta(i)$ over $\mathcal{A}(Q)$ is the projective B-module $P_B(i)$. Moreover, there holds that*

$$0 \longrightarrow \bigoplus_{t(\alpha)=i} P_{\mathcal{A}(Q)}(s(\alpha)) \longrightarrow P_{\mathcal{A}(Q)}(i) \longrightarrow \Delta(i) \longrightarrow 0,$$

where the sum is taken over all arrows α in Q with terminal vertex i. Dually, the costandard module $\nabla(i)$ over $\mathcal{A}(Q)$ is the injective B^{op}-module $Q_{B^{op}}(i)$, and there holds that

$$0 \longrightarrow \nabla(i) \longrightarrow I_{\mathcal{A}(Q)}(i) \longrightarrow \bigoplus_{t(\alpha)=i} I_{\mathcal{A}(Q)}(s(\alpha)) \longrightarrow 0.$$

(2)For each $1 \leq i \leq n$, there are exact sequences

$$0 \longrightarrow \Delta(i) \longrightarrow T(i) \longrightarrow X(i) \longrightarrow 0$$

$$0 \longrightarrow Y(i) \longrightarrow T(i) \longrightarrow \nabla(i) \longrightarrow 0$$

with $X(i), Y(i) \in \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$.

(3) There holds that

$$gl.dim.\mathcal{A}(Q) = \begin{cases} 0 & \textit{if } \mathcal{A}(Q) \textit{ is semisimple,} \\ 2 & \textit{otherwise} \end{cases}$$

and

$$dim_k \mathcal{A}(Q) = \sum_{i=1}^{n} (dim_k \Delta(i))^2 = \sum_{i=1}^{n} (dim_k P_B(i))^2.$$

Proof. (1) This statement follows directly from the construction of $\mathcal{A}(Q)$.

(2) This statement is a direct implication of (1) and the two lemmas in 1.2.

(3) For the first equality we refer to Section 3 in [4] (see also [11]), and for the second one to [9].

In the following two sections we are going to describe the characteristic tilting module and the Ringel dual of $\mathcal{A}(Q)$.

2. Construction of characteristic tilting modules

This section is devoted to the construction of the characteristic tilting module over the algebra $\mathcal{A}(Q)$ for arbitrary directed quiver Q.

2.1 Let Q be a directed quiver and $\mathcal{A}(Q)$ the algebra defined in 1.3. For each $1 \leq i \leq n$, we now construct an $\mathcal{A}(Q)$-module $K(i)$ as follows. As a k-vector space, $K(i)$ has a basis

$$\mathcal{S}(i) := \{(w, v) | v, w \text{ are paths in } Q \text{ with } s(v) = i \text{ and } t(v) = s(w)\}$$

For each $j \leq i$, we set $K(i)_j$ the subspace of $K(i)$ whose basis elements are $(w, v) \in \mathcal{S}$ with $t(w) = j$, and for $j > i$, we set $K(i)_j = 0$. For each arrow $\alpha : j \to l$ with $j, l \leq i$, we define a k-linear map $K(i)_\alpha : K(i)_j \to K(i)_l$ by sending (w, v) to $(\alpha w, v)$ and a k-linear map $K(i)_{\alpha'} : K(i)_l \to K(i)_j$ given by

$$(w, v) \longmapsto \begin{cases} (w_1, v_1) & \text{if } v = \beta v_1 \text{ and } w\beta = \alpha w_1 \text{ for some arrow } \beta \\ 0 & \text{otherwise,} \end{cases}$$

and for $\alpha : j \to l$ with $j > i$ or $l > i$, we define both $K(i)_\alpha$ and $K(i)_{\alpha'}$ to be zero. It is easy to check that $K(i) = \oplus_{j=1}^{n} K(i)_j$ becomes an $\mathcal{A}(Q)$-module.

Proposition. *(1) For each $1 \leq i \leq n$, there is an exact sequence*

$$0 \longrightarrow \Delta(i) \longrightarrow K(i) \longrightarrow \bigoplus_{s(\alpha)=i} K(t(\alpha)) \longrightarrow 0,$$

where the sum is taken over all arrows α in Q with starting vertex i. Dually, there is an exact sequence

$$0 \longrightarrow \bigoplus_{s(\alpha)=i} K(t(\alpha)) \longrightarrow K(i) \longrightarrow \nabla(i) \longrightarrow 0.$$

In particular, all $K(i)$ lie in $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$.

(2) Each $K(i)$ is indecomposable, thus isomorphic to $T_{\mathcal{A}(Q)}(i) =: T(i)$.

Proof. (1) One easily observes that the subspace N of $K(i)$ spanned by $(w, e_i) \in \mathcal{S}(i)$ is a submodule of $K(i)$ which is isomorphic to $\Delta(i)$. Then each arrow $\alpha : i \to j = t(\alpha)$ provides a direct summand U_α of the quotient module $K(i)/N$ which is spanned by the elements of the form $(w, v\alpha) + N$ and isomorphic to $K(j)$ by the construction. Further, one sees that $K(i)/N = \oplus_{s(\alpha)=i} U_\alpha \cong \oplus_{s(\alpha)=i} K(t(\alpha))$. This shows the existence of the demanded exact sequence.

Dually, each arrow $\alpha : i \to j = t(\alpha)$ gives rise to a submodule V_α of $K(i)$ which is spanned by all elements of the forms $(w\alpha, e_i)$ and $(u, v\alpha)$ with $l(u) \geq 1$ and isomorphic to $K(j)$. Since $V_\alpha \cap (\sum_{\alpha' \neq \alpha} V_{\alpha'}) = 0$, one has that $V := \sum_{s(\alpha)=i} V_\alpha = \oplus_{s(\alpha)=i} V_\alpha \cong \oplus_{s(\alpha)=i} K(t(\alpha))$. Further, the quotient $K(i)/V$ has a basis consisting of elements $(e_j, v) + V$, and is isomorphic to $\nabla(i)$. This implies the existence of the dual exact sequence.

(2) For each $1 \leq i \leq n$, it follows from the construction that $K(i)_i = k(e_i, e_i)$ is one-dimensional. Suppose that M is an indecomposable summand of $K(i)$ such that $M_i = K(i)_i = k(e_i, e_i)$. By (1) and 1.2, M also lies in $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ and is then isomorphic to $T(i)$. Hence M has also a composition factor isomorphic to $\nabla(i)$. This implies that all the elements $(e_j, v) \in \mathcal{S}(i)$ are contained in M. Thus, for each element $(w, v) \in \mathcal{S}(i)$ with $w = \alpha_m \cdots \alpha_2 \alpha_1$, one has that $(w, v) = K(i)_{\alpha_m} \cdots K(i)_{\alpha_1}(e_{t(v)}, v)$ lies in M, that is, $\mathcal{S}(i)$ is contained in M. Hence $K(i) = M$ is indecomposable. This finishes the proof.

3. Ringel duals

Our purpose in this section is to describe the Ringel dual of $\mathcal{A}(Q)$. We shall show that the Ringel dual of $\mathcal{A}(Q)$ is isomorphic to the algebra $\mathcal{A}(Q^{op})$ associated with Q^{op}.

3.1 Let Q be a directed quiver and $K(i)$, $1 \leq i \leq n$, the modules over $\mathcal{A}(Q)$ constructeded in Section 2. In the following we simply identify $T(i)$ with $K(i)$ for $1 \leq i \leq n$. For each arrow $\alpha : i \to j$ in Q, we define A-module homomorphisms $l_\alpha : T(j) \to T(i)$ by sending (w, v) to $(w\beta, v_1)$, where $(w, v) \in \mathcal{S}(j)$ and $\beta \in Q$ is such that $v\alpha = \beta v_1$, and $p_\alpha : T(i) \to T(j)$ given by

$$\mathcal{S}(i) \ni (t, u) \longmapsto \begin{cases} (t, u_1) & \text{if } u = u_1\alpha \\ 0 & \text{otherwise.} \end{cases}$$

Lemma. *(1) For each pair of arrows α, β in Q with $s(\alpha) = s(\beta)$, there holds that*

$$p_\beta l_\alpha = \begin{cases} \sum_{\gamma \in Q, s(\gamma)=t(\alpha)} l_\gamma p_\gamma & \text{if } \beta = \alpha, \\ 0 & \text{if } \beta \neq \alpha. \end{cases}$$

(2) For each $1 \leq i \leq n$, there are exact sequences

$$0 \longrightarrow \Delta(i) \longrightarrow T(i) \stackrel{p_i:=(p_\alpha)_\alpha}{\longrightarrow} \bigoplus_{s(\alpha)=i} T(t(\alpha)) \longrightarrow 0$$

and

$$0 \longrightarrow \bigoplus_{s(\alpha)=i} T(t(\alpha)) \stackrel{l_i:=(l_\alpha)_\alpha}{\longrightarrow} T(i) \longrightarrow \nabla(i) \longrightarrow 0.$$

This lemma follows directly from the definition of l_α's and p_α's and Proposition 2.1.

3.2 In order to describe the Ringel dual $\mathcal{R}(\mathcal{A}(Q))$ of $\mathcal{A}(Q)$, we now determine the irreducible morphisms in $\mathcal{T} := \mathrm{add}(\oplus_{i=1}^n T(i))$. The following lemma due to C.C. Xi is helpful.

Lemma ([5, Lemma 2.2]). *Let A be a quasi-hereditary algebra with the weight poset Λ. Suppose λ is a maximal element in Λ. Then*

1) The map p_λ in the exact sequence

$$0 \longrightarrow \Delta_A(\lambda) \longrightarrow T_A(\lambda) \stackrel{p_\lambda}{\longrightarrow} X_A(\lambda) \longrightarrow 0$$

is a relative source map for $T_A(\lambda)$ in $\mathcal{F}_A(\Delta)$ if $X_A(\lambda) \neq 0$.

2) Dually, the map l_λ in the exact sequence

$$0 \longrightarrow Y_A(\lambda) \stackrel{l_\lambda}{\longrightarrow} T_A(i) \longrightarrow \nabla_A(i) \longrightarrow 0$$

is a relative sink map for $T_A(\lambda)$ in $\mathcal{F}_A(\nabla)$ if $Y_A(\lambda) \neq 0$.

3.3 For any two vertices $j \leq i$ in Q, we denote by a_{ji} the number of all arrows in Q from i to j and by b_{ji} the number of all paths in Q from i to j. Furhter, for $1 \leq i, j \leq n$, we let $\mathrm{Irr}_{\mathcal{T}}(T(i), T(j)) := \mathrm{rad}_{\mathcal{T}}(T(i), T(j))/\mathrm{rad}^2_{\mathcal{T}}(T(i), T(j))$ be the bimodule of irreducible maps from $T(i)$ to $T(j)$ in $\mathcal{T}$.

Lemma. *For $j < i$ in Q, the residue classes of all l_α (resp. p_α), $\alpha \in Q$ with $s(\alpha) = i$ and $t(\alpha) = j$, form a k–basis of $Irr_{\mathcal{T}}(T(j), T(i))$ (resp. $Irr_{\mathcal{T}}(T(i), T(j))$). In particular, there holds that*

$$dim_k Irr_{\mathcal{T}}(T(j), T(i)) = dim_k Irr_{\mathcal{T}}(T(i), T(j)) = a_{ji}.$$

Proof. For each $1 \leq i \leq n$, we set $\mathcal{T}_i := \mathrm{add}(\oplus_{j=1}^i T(j))$. We claim that for $j, l \leq i$, there holds that

$$\mathrm{Irr}_{\mathcal{T}_i}(T(l), T(j)) = \mathrm{Irr}_{\mathcal{T}_{i+1}}(T(l), T(j)).$$

Indeed, it suffices to prove that every morphism from $T(l)$ to $T(j)$ which factors through $T(i+1)$ lies in $\mathrm{rad}^2_{\mathcal{T}_i}(T(l), T(j))$. Let $f = hg : T(l) \to T(j)$, where $g : T(l) \to T(i+1)$ and $h : T(i+1) \to T(j)$. Since $\mathrm{Hom}_{\mathcal{A}(Q)}(T(l), \nabla(i+1)) = 0$, it follows that g factor through l_{i+1}, that is, $g = \sum_{s(\alpha)=i+1} l_\alpha g_\alpha$, where $g_\alpha : T(l) \to T(t(\alpha))$. Similarly, one has that $h = \sum_{s(\alpha)=i+1} h_\alpha p_\alpha$ for some $h_\alpha : T(t(\alpha)) \to T(j)$. By Lemma 3.1, one finally obtains that

$$f = hg = \sum_{s(\alpha)=i+1} h_\alpha p_\alpha l_\alpha g_\alpha = \sum_{s(\alpha)=i+1} \sum_{s(\gamma)=t(\alpha)} h_\alpha l_\gamma p_\gamma g_\alpha \in \mathrm{rad}^2_{\mathcal{T}_i}(T(l), T(j)).$$

The lemma then follows from Lemma 3.2 and the fact that $\bigoplus_{j=1}^{i} T(j)$ is the characteristic tilting module of the quasi-hereditary algebra $\mathcal{A}(Q)/\mathcal{A}(Q)\varepsilon_i\mathcal{A}(Q)$, where $\varepsilon_i = (e_{i+1} + \cdots + e_n)$.

3.4 Lemma. *For $j \leq i$ in Q, there holds that*

$$dim_k Hom_{\mathcal{A}(Q)}(\Delta(j), T(i)) = dim_k Hom_{\mathcal{A}(Q)}(T(i), \nabla(j)) = b_{ji}.$$

Proof. By the duality, it is enough to show that $\dim_k \mathrm{Hom}_{\mathcal{A}(Q)}(\Delta(j), T(i)) = b_{ji}$. If $j = i$, it holds obviously that $\dim_k \mathrm{Hom}_{\mathcal{A}(Q)}(\Delta(i), T(i)) = 1 = b_{ii}$. Let $j < i$ and apply $\mathrm{Hom}_{\mathcal{A}(Q)}(\Delta(j), -)$ to the exact sequence

$$0 \longrightarrow \bigoplus_{s(\alpha)=i} T(t(\alpha)) \xrightarrow{l_i} T(i) \longrightarrow \nabla(i) \longrightarrow 0,$$

we have that

$$\begin{aligned} \dim_k \mathrm{Hom}_{\mathcal{A}(Q)}(\Delta(j), T(i)) &= \dim_k \mathrm{Hom}_{\mathcal{A}(Q)}(\Delta(j), \oplus_{s(\alpha)=i} T(t(\alpha))) \\ &= \textstyle\sum_{j \leq s < i} a_{si} \dim_k \mathrm{Hom}_{\mathcal{A}(Q)}(\Delta(j), T(s)) \end{aligned}$$

since $\mathrm{Hom}_{\mathcal{A}(Q)}(\Delta(j), \nabla(i)) = 0$. The required equality then follows from an inductive argument.

3.5 By Q^{op} we denote the opposite quiver of Q, and by $\mathcal{A}(Q^{op})$ the algebra associated with Q^{op}. For each arrow $\alpha : i \to j$ in Q, we denote by α^{op} the corresponding arrow in Q^{op} from j to i. With the help of Lemmas 3.1–3.4, we have the following theorem.

Theorem. *The Ringel dual $\mathcal{R}(\mathcal{A}(Q)) := (End_{\mathcal{A}(Q)}(T))^{op}$ of $\mathcal{A}(Q)$ is isomorphic to $\mathcal{A}(Q^{op})$.*

Proof. By Lemma 3.1, the correspondence $\alpha^{op} \longmapsto p_\alpha$, $(\alpha^{op})' \longmapsto l_\alpha$, $\alpha \in Q_1$, induces a surjective algebra homomorphism $\Phi : \mathcal{A}(Q^{op}) \longrightarrow \mathcal{R}(\mathcal{A}(Q))$. Since $\mathcal{A}(Q)$ is a BGG-algebra, so is its Ringel dual $\mathcal{R}(\mathcal{A}(Q))$. Again, $\mathcal{A}(Q^{op})$ is a BGG-algebra. From Corollary 1.3 (3) it follows that

$$\dim_k \mathcal{A}(Q^{op}) = \sum_{i=1}^{n} (\dim_k \Delta_{\mathcal{A}(Q^{op})}(i))^2$$

and

$$\dim_k \mathcal{R}(\mathcal{A}(Q)) = \sum_{i=1}^{n} (\dim_k \Delta_{\mathcal{R}(\mathcal{A}(Q))}(i))^2.$$

On one hand, the standard module $\Delta_{\mathcal{A}(Q^{op})}(i)$ is just projective B^{op}-module $P_{B^{op}}(i)$, and thus its dimension is $\sum_{l=i}^{n} b_{il}$.

On the other hand, by 1.2, there holds that $\Delta_{\mathcal{R}((\mathcal{A}(Q))}(i) \cong \mathrm{Hom}_{\mathcal{A}(Q)}(T, \nabla(i))$. From Lemma 3.4 it follows that

$$\dim_k \Delta_{\mathcal{R}(\mathcal{A}(Q))}(i) = \sum_{i=1}^{n} \dim_k \mathrm{Hom}_{\mathcal{A}(Q)}(T, \nabla(i)) = \sum_{l=i}^{n} b_{il}.$$

Therefore, $\dim_k \mathcal{A}(Q^{op}) = \dim_k \mathcal{R}(\mathcal{A}(Q))$. This implies that Φ is an isomorphism. This finishes the proof.

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