# ON THE AUSLANDER-REITEN QUIVER OF AN INFINITESIMAL GROUP

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## 0. Introduction and Preliminaries

The investigation of the representations of finite group schemes over algebraically closed fields of positive characteristic typically proceeds in several steps. By general theory (cf. [29, (6.8)]) a finite algebraic group scheme $\mathcal{G}$ decomposes into a semidirect product $\mathcal{G} = \mathcal{G}^0 \times \mathcal{G}_{red}$ with a normal infinitesimal subgroup $\mathcal{G}^0$ and a reduced group $\mathcal{G}_{red}$. Accordingly, one begins by studying the representation theories of the constituents. These turn out to differ with regard to both, their methods and results. The final, and often most difficult, step amounts to fusing the results for infinitesimal and constant groups.

The classical modular representation theory of finite groups is fairly completely developed. One understands the Morita equivalence classes of the representation-finite and tame blocks (cf. [3, 7]), and the Auslander-Reiten quiver has also been determined (cf. [8]). Much of the success here rests on the availability of a comprehensive block theory that is based on Green's theory of vertices and sources.

By contrast, much less is known for the group algebras of infinitesimal algebraic groups. A general block theory remains elusive, and the most promising approach pursued so far is based on geometric methods related to the notion of a rank variety. In recent work Suslin-Friedlander-Bendel [26, 27] have extended this notion from infinitesimal groups of height $\leq 1$ to those of arbitrary height. The main purpose of the present note is to show how their results and those of [12, 17] can be exploited to study the stable Auslander- Reiten quiver of an infinitesimal group.

The paper is organized as follows. In the first section we show that the rank variety of an indecomposable module is an invariant of its stable Auslander-Reiten component. By combining work of Happel-Preiser-Ringel [20] with that of Erdmann-Skowroński [10] one then readily obtains the standard list of the possible tree classes of the AR-components: finite

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and infinite Dynkin diagrams, and Euclidean diagrams. Thus, Theorem 1.3 is the analogue of Webb's theorem [30] for group algebras of finite groups (see [9, 11] for infinitesimal groups of height $\leq 1$). In §2 we investigate the question to what extent the structure of an AR-component is determined by the dimension of its support variety. Components with one-dimensional variety are those containing periodic modules and are therefore either finite, or infinite tubes. In generalization of the main result of [13] we show that AR-components with support varieties of dimension $\geq 3$ are isomorphic to $\mathbb{Z}[A_\infty]$.

Following a brief discussion on the number of AR-components, we turn in the last two sections to special cases given by Frobenius kernels of reductive groups and supersolvable groups. For the former actions on varieties provide additional information, while a certain "linkage principle" is exploited in the investigation of the latter. For both classes of groups, one can reduce the list of possible tree classes by determining the finite and Euclidean components.

Throughout this paper we will be working over an algebraically closed field $k$ of characteristic $p > 0$. All $k$-vector spaces are assumed to be of finite dimension. Let $H$ be a Hopf algebra with counit $\varepsilon : H \longrightarrow k$ and antipode $\eta : H \longrightarrow H$. According to a result due to Sweedler (cf. [28, (5.1.6)]), there exists a one-dimensional subspace $\int_H^r \subset H$, the space of *right integrals* of $H$, such that

$$x \cdot h = x\,\varepsilon(h) \quad \forall\, h \in H,\ x \in \int_H^r .$$

The unique algebra homomorphism $\zeta : H \longrightarrow k$ satisfying $h \cdot x = \zeta(h)\,x$ for $h \in H$ and $x \in \int_H^r$ is called the *modular function* of $H$.

Given an algebra homomorphism $\lambda : H \longrightarrow k$ the convolution

$$\lambda * id_H : H \longrightarrow H \quad ; \quad h \mapsto \sum_{(h)} \lambda(h_{(1)})h_{(2)}$$

is an automorphism of $H$ of finite order. Every Hopf algebra is a (not necessarily symmetric) Frobenius algebra (cf. [24]). In case $H$ is cocommutative, $\nu := \zeta * id_H$ is a Nakayama automorphism of $H$ (cf. [18, (1.5)]).

For an $H$-module $M$ and an automorphism $\gamma : H \longrightarrow H$ of $k$-algebras we denote by $M^{(\gamma)}$ the $H$-module with underlying $k$-space $M$ and action given by

$$h \cdot m := \gamma^{-1}(h)m \quad \forall\, h \in H,\, m \in M.$$

In this fashion we obtain an action of the automorphism group $Aut_k(H)$ on $mod_H$, the category of $H$-modules. If $\lambda : H \longrightarrow k$ is an algebra homomorphism, then $k_\lambda$ denotes the corresponding one-dimensional $H$-module. Note that $\lambda \circ \eta$ is also a homomorphism, and that $M^{(\lambda * id_H)} \cong M \otimes_k k_{\lambda\circ\eta}$. In particular, we have $M^{(\nu)} \cong M \otimes_k k_{\zeta\circ\eta}$.

# 1. The Variety of an Auslander-Reiten Component

We refer the reader to [6, 22, 29] for general facts on algebraic $k$-groups. Given such a group $\mathcal{G}$, the $r^{th}$ Frobenius kernel of $\mathcal{G}$ will be denoted $\mathcal{G}_r$. If $\mathcal{G} = Spec_k(\mathcal{O}(\mathcal{G}))$ is a finite algebraic

$k$-group with function algebra $\mathcal{O}(\mathcal{G})$, then $H(\mathcal{G}) := \mathcal{O}(\mathcal{G})^*$ is its group algebra. By definition $H(\mathcal{G})$ is a finite dimensional cocommutative Hopf algebra.

We consider the $r^{th}$ Frobenius kernel ${}_{p^r}\alpha_k = Spec_k(k[T]/(T^{p^r}))$ of the additive group, and recall that, letting $\{X_0, \dots, X_{r-1}\} \subset H({}_{p^r}\alpha_k)$ be the set of linear functionals given by

$$X_i(T^j + (T^{p^r})) := \delta_{p^i,j},$$

we have $H({}_{p^r}\alpha_k) = k[X_0, \dots, X_{r-1}]/(X_0^p, \dots, X_{r-1}^p)$. In the sequel we shall denote the $p$-dimensional local $k$-algebra $k[X_{r-1}]/(X_{r-1}^p) \subset H({}_{p^r}\alpha_k)$ by $A_r$.

Given a $\mathcal{G}$-module $M$ and $r > 0$, we put

$$\hat{\mathcal{V}}_{\mathcal{G}_r}(M) := \{\varphi : {}_{p^r}\alpha_k \longrightarrow \mathcal{G} \ \ ; \ \ M|_{A_r} \text{ is not projective}\}.$$

Here $M|_{A_r}$ denotes the pullback of the module structure along the restriction $A_r \longrightarrow H(\mathcal{G})$ of the homomorphism $H({}_{p^r}\alpha_k) \longrightarrow H(\mathcal{G})$ that corresponds to $\varphi$. By abuse of notation this map will also be denoted $\varphi$. Note that $\hat{\mathcal{V}}_{\mathcal{G}_r}(M)$ is the variety of $k$-rational points of the support scheme $V_r(\mathcal{G})_M$ that was introduced in [27, §6]. Thanks to [27, (6.8)] we have

$$\dim \hat{\mathcal{V}}_{\mathcal{G}_r}(M) = \dim V_r(\mathcal{G})_M = \dim \mathcal{V}_{\mathcal{G}}(M),$$

where $\mathcal{V}_{\mathcal{G}}(M)$ denotes the support variety of the $H^{ev}(\mathcal{G}, k)$-module $\mathrm{Ext}^*_{H(\mathcal{G})}(M, M)$. In particular, $\dim \hat{\mathcal{V}}_{\mathcal{G}_r}(M)$ coincides with the complexity $c_{H(\mathcal{G})}(M)$ of $M$.

The *stable Auslander-Reiten quiver* of $H(\mathcal{G})$ will be denoted by $\Gamma_s(\mathcal{G})$. Recall that the vertices of $\Gamma_s(\mathcal{G})$ are the isoclasses of the non-projective indecomposable $\mathcal{G}$-modules. The arrows are induced by the so-called irreducible maps. We refer the reader to [1] concerning the general theory of these quivers. The set of irreducible maps between two $\mathcal{G}$-modules $M$ and $N$ will be denoted $Irr(M, N)$. If $M$ is a module for a self-injective $k$-algebra $\Lambda$, then we write $\Omega^n_\Lambda(M) = \Omega^n(M)$ for the $n^{th}$ syzygy of a minimal projective resolution of $M$. The resulting operator $\Omega_\Lambda$ is called the *Heller operator*. Important for our purposes is its connection with the *Auslander-Reiten translation* $\tau$ of $\Gamma_s(\mathcal{G})$. It follows from [3, p.138] that

$$\tau(M) \cong \Omega^2(M^{(\nu)})$$

for every non-projective indecomposable $H(\mathcal{G})$-module $M$.

**Proposition 1.1** *Let $\Theta \subset \Gamma_s(\mathcal{G})$ be a component. Given $\mathcal{G}$-modules $M$ and $N$ such that $[M], [N] \in \Theta$, we have $\hat{\mathcal{V}}_{\mathcal{G}_r}(M) = \hat{\mathcal{V}}_{\mathcal{G}_r}(N)$.*

*Proof.* We first show that $\hat{\mathcal{V}}_{\mathcal{G}_r}(M) = \hat{\mathcal{V}}_{\mathcal{G}_r}(\tau(M))$. According to [27, (7.3)] $H(\mathcal{G})|_{A_r}$ is projective for every non-trivial element $\varphi \in \hat{\mathcal{V}}_{\mathcal{G}_r}(M)$. From the isomorphism

$$\Omega^2_{H(\mathcal{G})}(M)|_{A_r} \cong \Omega^2_{A_r}(M|_{A_r}) \oplus (\text{proj})$$

we obtain the identity $\hat{\mathcal{V}}_{\mathcal{G}_r}(M) = \hat{\mathcal{V}}_{\mathcal{G}_r}(\Omega^2_{H(\mathcal{G})}(M))$. Our earlier observations now yield

$$\tau(M) \cong \Omega^2_{H(\mathcal{G})}(M^{(\nu)}) \cong \Omega^2_{H(\mathcal{G})}(M \otimes_k k_{\zeta\circ\eta}) \cong \Omega^2_{H(\mathcal{G})}(M) \otimes_k k_{\zeta\circ\eta},$$

with the last isomorphism following from the fact that $M \mapsto M \otimes_k k_{\zeta\circ\eta}$ is an auto-equivalence of $mod_{H(\mathcal{G})}$. Since $H({}_{p^r}\alpha_k)$ is a local Hopf algebra, we have $(\Omega^2_{H(\mathcal{G})}(M) \otimes_k k_\zeta)|_{H({}_{p^r}\alpha_k)} \cong \Omega^2_{H(\mathcal{G})}(M)|_{H({}_{p^r}\alpha_k)}$, whence $(\Omega^2_{H(\mathcal{G})}(M) \otimes_k k_\zeta)|_{A_r} \cong \Omega^2_{H(\mathcal{G})}(M)|_{A_r}$ and

$$\hat{\mathcal{V}}_{\mathcal{G}_r}(\tau(M)) = \hat{\mathcal{V}}_{\mathcal{G}_r}(\Omega^2_{H(\mathcal{G})}(M) \otimes_k k_{\zeta\circ\eta}) = \hat{\mathcal{V}}_{\mathcal{G}_r}(\Omega^2_{H(\mathcal{G})}(M)) = \hat{\mathcal{V}}_{\mathcal{G}_r}(M).$$

Now let

$$(0) \longrightarrow \tau(M) \longrightarrow X_M \longrightarrow M \longrightarrow (0)$$

be the almost split sequence terminating in $M$. Directly from the definition we obtain $\hat{\mathcal{V}}_{\mathcal{G}_r}(X_M) \subset \hat{\mathcal{V}}_{\mathcal{G}_r}(\tau(M)) \cup \hat{\mathcal{V}}_{\mathcal{G}_r}(M) = \hat{\mathcal{V}}_{\mathcal{G}_r}(M)$. If $Irr(N, M) \neq \emptyset$, then $N$ is a direct summand of $X_M$, whence $\hat{\mathcal{V}}_{\mathcal{G}_r}(N) \subset \hat{\mathcal{V}}_{\mathcal{G}_r}(M)$. By considering the almost split sequence

$$(0) \longrightarrow N \longrightarrow X_{\tau^{-1}(N)} \longrightarrow \tau^{-1}(N) \longrightarrow (0)$$

we conclude that $\hat{\mathcal{V}}_{\mathcal{G}_r}(M) \subset \hat{\mathcal{V}}_{\mathcal{G}_r}(N)$ whenever $Irr(N, M) \neq \emptyset$. Since $\Theta$ is connected, this yields the desired result. $\square$

**Definition.** Let $\Theta \subset \Gamma_s(\mathcal{G})$ be a component. Given $r \in \mathbb{N}$, we define

$$\hat{\mathcal{V}}_{\mathcal{G}_r}(\Theta) := \hat{\mathcal{V}}_{\mathcal{G}_r}(M) \qquad [M] \in \Theta.$$

In the sequel we will be working with certain induced modules. Given a homomorphism $\varphi : {}_{p^r}\alpha_k \longrightarrow \mathcal{G}$, we put $M_\varphi := H(\mathcal{G}) \otimes_{A_r} k$. For future reference we collect a few elementary properties:

**Lemma 1.2** *Let $\varphi : {}_{p^r}\alpha_k \longrightarrow \mathcal{G}$ be a non-trivial homomorphism. Then the module $M_\varphi$ has the following properties:*

(1) $\Omega^2_{H(\mathcal{G})}(M_\varphi) \oplus (\mathrm{proj}) \cong M_\varphi$.

(2) $M_\varphi^{(\nu)} \cong M_\varphi$.

*Proof.* (1). Since $A_r \cong k[X]/(X^p)$ we have $\Omega^2_{A_r}(k) \cong k$. Thanks to [27, (7.3)] $H(\mathcal{G})$ is a projective $A_r$-module, so that general properties of the Heller operator yield

$$M_\varphi \cong H(\mathcal{G}) \otimes_{A_r} \Omega^2_{A_r}(k) \cong \Omega^2_{H(\mathcal{G})}(M_\varphi) \oplus (\mathrm{proj}).$$

(2). Recall that $\nu = \zeta * id_{H(\mathcal{G})}$, where $\zeta$ is the modular function of $H(\mathcal{G})$. Since $H({}_{p^r}\alpha_k)$ is local, we have $\zeta \circ \varphi = \varepsilon$, so that $\nu \circ \varphi = \varphi$. It follows that there is a map $M_\varphi \longrightarrow M_\varphi$ sending $h \otimes \alpha$ to $\nu(h) \otimes \alpha$. This map is the desired isomorphism $M_\varphi^{(\nu)} \cong M_\varphi$. $\square$

Our next result, which generalizes earlier work on groups of height $\leq 1$ (cf. [9, 11]), is the analogue of Webb's theorem [30, Thm.A] for infinitesimal groups.

**Theorem 1.3** *Let $\Theta \subset \Gamma_s(\mathcal{G})$ be a component of the stable Auslander-Reiten quiver of an infinitesimal group $\mathcal{G}$. Then the tree class $T_\Theta$ of $\Theta$ is either a finite Dynkin diagram, or an infinite Dynkin diagram, or a Euclidean diagram.*

*Proof.* Let $r$ be the height of $\mathcal{G}$. Then we have $\hat{\mathcal{V}}_{\mathcal{G}_r}(\Theta) \neq \{0\}$, and for a non-trivial element $\varphi : {}_{p^r}\alpha_k \longrightarrow \mathcal{G}$ of $\hat{\mathcal{V}}_{\mathcal{G}_r}(\Theta)$, we consider the module $M_\varphi$. According to (1.2) there exists a projective $H(\mathcal{G})$-module $P$ such that $\tau(M_\varphi) \oplus P \cong M_\varphi$.

Let $[M]$ be an element of $\Theta$. Then $M$ is a non-injective module for the local, self-injective algebra $A_r$. Consequently,

$$(0) \neq Ext^1_{A_r}(k, M) \cong Ext^1_{H(\mathcal{G})}(M_\varphi, M)$$

by Frobenius reciprocity. We may now apply [9, (1.5)] to obtain our result. $\square$

*Remark.* Since $k$ is algebraically closed, the labels of the edges of the possible tree class have the form $(n, n)$. Hence the following tree classes can occur:

(a) finite Dynkin diagrams $A_n$, $D_n$, $E_6$, $E_7$, $E_8$.

(b) infinite Dynkin diagrams $A_\infty$, $A_\infty^\infty$, $D_\infty$.

(c) Euclidean diagrams $\tilde{A}_{12}$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$.

## 2. Dimensions of Auslander-Reiten Components

In the previous section we have associated to each component $\Theta \subset \Gamma_s(\mathcal{G})$ conical, affine varieties $\hat{\mathcal{V}}_{\mathcal{G}_r}(\Theta)$. If $r$ is the height of the infinitesimal $k$-group $\mathcal{G}$, then, as we shall demonstrate below, the dimension of $\hat{\mathcal{V}}_{\mathcal{G}_r}(\Theta)$ is related to the the structure of the component. Recall that an indecomposable $H(\mathcal{G})$-module $M$ is said to be *periodic* if there exists $n \in \mathbb{N}$ such that $\Omega^n_{H(\mathcal{G})}(M) \cong M$. Since the Nakayama automorphism $\nu$ of $H(\mathcal{G})$ has finite order, $M$ is periodic if and only if it is $\tau$*-periodic* in the sense that $\tau^m(M) \cong M$ for some $m \in \mathbb{N}$. The tree class of a component $\Theta \subset \Gamma_s(\mathcal{G})$ will be denoted $T_\Theta$. We begin by studying components with one-dimensional support.

**Proposition 2.1** *Let $\mathcal{G}$ be an infinitesimal $k$-group of height $r$.*

(1) *If $M$ be a $\mathcal{G}$-module of complexity $1$, then $M$ is periodic.*

(2) *Let $\Theta \subset \Gamma_s(\mathcal{G})$ be a component. Then $\dim \hat{\mathcal{V}}_{\mathcal{G}_r}(\Theta) = 1$ if and only if either $T_\Theta$ is a finite Dynkin diagram, or $\Theta$ is an infinite tube.*

*Proof.* (1). It follows from [19, (1.1)] and [4, (5.7.2)] that the complexity $c_{H(\mathcal{G})}(M)$ coincides with the dimension of the cohomological support variety $\mathcal{V}_\mathcal{G}(M)$. By considering a non-nilpotent element of the corresponding coordinate ring, we infer from our present assumption the existence of an element $[\zeta] \in H^{2n}(\mathcal{G}, k)$ such that its zero locus $Z(\zeta)$ meets $\mathcal{V}_\mathcal{G}(M)$ in $\{0\}$. Let $L_\zeta$ be the kernel of the corresponding map $\Omega^{2n}_{H(\mathcal{G})}(k) \longrightarrow k$. It was shown in the proof of [27, (7.5)] that $\mathcal{V}_\mathcal{G}(L_\zeta) = Z(\zeta)$. In view of [27, (7.2)] the arguments of [4, (5.10.4)] may now be adopted verbatim to conclude the proof.

(2). Suppose that $\dim \hat{\mathcal{V}}_{\mathcal{G}_r}(\Theta) = 1$ and let $[M] \in \Theta$ be a vertex. According to [27, (6.8)], we have $c_{H(\mathcal{G})}(M) = \dim \mathcal{V}_\mathcal{G}(M) = 1$ and (1) implies that $M$ is periodic. Thus, $\Theta$ consists of $\tau$-periodic modules, and the main result of [20] shows that $\Theta$ has the asserted form.

If $\Theta$ is an infinite tube, then all modules belonging to $\Theta$ are periodic, so that $\dim \hat{\mathcal{V}}_{\mathcal{G}_r}(\Theta) = 1$. Supposing that $T_\Theta$ is a finite Dynkin diagram, we let $\varphi \in \hat{\mathcal{V}}_{\mathcal{G}_r}(\Theta) \setminus \{0\}$. As before we consider the $\mathcal{G}$-module $M_\varphi := H(\mathcal{G}) \otimes_{A_r} k$. Thanks to [10, (3.2)] the function

$$d_\varphi : \Theta \longrightarrow \mathbb{N} \ \ ; \ \ [X] \mapsto \dim_k Ext^1_{H(\mathcal{G})}(M_\varphi, X)$$

is subadditive. In view of (1.2) $d_\varphi \circ \tau = \tau$ holds, so that $d_\varphi$ defines a subadditive function on the tree class $T_\Theta$. We may now apply [3, (4.5.8)] to see that $d_\varphi$ is not additive. Owing to [7, (I.8.8)] this implies the existence of a vertex $[X] \in \Theta$ such that $X$ or $\Omega^{-1}_{H(\mathcal{G})}(X)$ is a direct summand of $M_\varphi$. Consequently, $X$ is periodic and $\dim \hat{\mathcal{V}}_{\mathcal{G}_r}(\Theta) = \dim \hat{\mathcal{V}}_{\mathcal{G}_r}(X) = 1$. $\square$

We say that a component $\Theta \subset \Gamma_s(\mathcal{G})$ is *periodic* if it contains a periodic module. In view of (2.1) these are precisely the components with $\dim \hat{\mathcal{V}}_{\mathcal{G}_r}(\Theta) = 1$ ($r =$ height of $\mathcal{G}$). Our next result generalizes [13, (2.1)] to our present context.

**Theorem 2.2** *Let $\mathcal{G}$ be an infinitesimal $k$-group of height $r$. If $\Theta \not\cong \mathbb{Z}[A_\infty]$ is a nonperiodic component, then* $\dim \hat{\mathcal{V}}_{\mathcal{G}_r}(\Theta) = 2$.

*Proof.* By combining (1.3) with (2.1) we see that the tree class $T_\Theta$ is either a Euclidean diagram, or an infinite Dynkin diagram.

If $T_\Theta$ is a Euclidean diagram, then we may argue as in the proof of [13, (2.1)] to see that $\dim \hat{\mathcal{V}}_{\mathcal{G}_r}(\Theta) \leq 2$.

In view of (1.3) it thus remains to consider the cases where $T_\Theta \cong A_\infty^\infty$ or $T_\Theta \cong D_\infty$. Assuming this to be the case, we pick a nontrivial map $\varphi : {}_{p^r}\alpha_k \longrightarrow \mathcal{G}$ and put $M_\varphi := H(\mathcal{G}) \otimes_{A_r} k$. Thanks to (1.2) there are projective $H(\mathcal{G})$-modules $P$, $Q$ such that

$$\tau(M_\varphi) \oplus P \cong M_\varphi \cong \tau^{-1}(M_\varphi) \oplus Q.$$

Now let $[M_0] \in \Theta$ be a vertex with two predecessors. Suppose there is an injective irreducible map $g : M_0 \longrightarrow N_0$. By choice of $M_0$, the morphism $g$ is properly irreducible. Since $A := \operatorname{coker} g$ is not projective, there exists a non-zero element $\varphi \in \hat{\mathcal{V}}_{\mathcal{G}_r}(A)$. Observing $Ext^1_{H(\mathcal{G})}(M_\varphi, A) \cong Ext^1_{A_r}(k, A) \neq (0)$ we apply [8, (1.5)] to see that $A$ is an image of $M_\varphi$ or $\Omega_{H(\mathcal{G})}(M_\varphi)$. Thus, setting $q := max\{\dim_k M_\varphi, \dim_k \Omega_{H(\mathcal{G})}(M_\varphi)\}$ we obtain

$$\dim_k N_0 \leq \dim_k M_0 + q.$$

If $g$ is surjective, then the dual map $g^* : N_0^* \longrightarrow M_0^*$ is injective with cokernel $A \cong (\ker g)^*$. Since $M \mapsto M^*$ is an anti-equivalence on $mod_{H(\mathcal{G})}$, the above arguments show

$$\dim_k M_0 \leq \dim_k N_0 + q.$$

By considering a walk $\tau(M_0) \to N_0 \to M_0$ consisting of properly irreducible maps, we thus obtain $\dim_k \tau(M_0) \leq \dim_k M_0 + 2q$. Repeated application then shows

$$\dim_k \Omega^{2n}_{H(\mathcal{G})}(M_0) = \dim_k \tau^n(M_0) \leq \dim_k M_0 + 2nq \quad \forall\, n \geq 1.$$

By considering $[\Omega_{H(\mathcal{G})}(M_0)] \in \Omega_{H(\mathcal{G})}(\Theta) \cong \Theta$ we obtain

$$\dim_k \Omega^n_{H(\mathcal{G})}(M_0) \leq \dim_k M_0 + nq'$$

for some $q' > 0$. Consequently, $\dim \hat{\mathcal{V}}_{\mathcal{G}_r}(\Theta) = c_{H(\mathcal{G})}(M_0) \leq 2$, as desired. $\square$

# 3. The Number of AR-Components

A well-known conjecture asserts that an algebra of infinite representation type admits infinitely many Auslander-Reiten components. In this section we employ the methods of [14] to verify this conjecture for blocks of infinitesimal groups. Throughout this section, we fix an infinitesimal algebraic $k$-group $\mathcal{G}$. Given an Auslander-Reiten component $\Theta \subset \Gamma_s(\mathcal{G})$ and an integer $d > 0$, we consider

$$\Theta(d) := \{[M] \in \Theta \ ; \ dim_k M \leq d\}.$$

**Lemma 3.1** *Let $\mathcal{T} \subset \Theta$ be a $\tau$-orbit. Then $\Theta(d) \cap \mathcal{T}$ is finite.*

*Proof.* According to [19, (1.1)] the cohomology ring $H^{ev}(\mathcal{G}, k) := \bigoplus_{i=0}^{\infty} H^{2i}(\mathcal{G}, k)$ is a finitely generated $k$-algebra. The same result shows that $Ext^*_{H(\mathcal{G})}(M, M)$ is a finitely generated $H^{ev}(\mathcal{G}, k)$-module. Consequently, the proof may be completed by adopting the arguments of [14, (1.1)]. $\square$

**Theorem 3.2** *Let $\Theta \subset \Gamma_s(\mathcal{G})$ be an Auslander-Reiten component. Then $\Theta(d)$ is finite.*

*Proof.* In view of (3.1) it suffices to verify the statement for the case where $\Theta$ has infinitely many $\tau$-orbits. Consequently, we have $\Theta \cong \mathbb{Z}[A_\infty]$, $\mathbb{Z}[A_\infty^\infty]$, $\mathbb{Z}[D_\infty]$. For the latter two types, the proof of [14, (1.2)] yields the desired result. If $\Theta \cong \mathbb{Z}[A_\infty]$, then the set of vertices is $\mathbb{Z} \times \mathbb{N}$. We have arrows $(j, n) \to (j, n+1)$ pointing towards infinity, and arrows $(j, n+1) \to (j-1, n)$ poining towards the end. The Auslander-Reiten translation is given by $\tau(j, n) = (j+1, n)$. Since the component $\Theta$ contains only finitely many meshes that are associated to projective indecomposable modules, there exists a vertex $(a, b)$ such that all projective meshes occur inside the region $\Upsilon(a, b) := \{(\ell, m) \ ; \ m \leq b, \ a + m - b \leq \ell \leq a + b - m\}$. It follows from the mesh relations that all arrows pointing towards infinity that originate in vertices $(\ell, m)$ of $\Theta$ satisfying $\ell \leq a + m - b$ are injective, while those pointing towards the end terminating in vertices $(\ell, m)$ satisfying $\ell \geq a + b - m$ are surjective. Consequently, $\Theta(d) \subset \Upsilon(a, b + d) \cup \{(\ell, m) \ ; \ m \leq d\}$ is contained in a finite union of $\tau$-orbits. The assertion now follows from (3.1). $\square$

**Corollary 3.3** *Let $\mathcal{B} \subset H(\mathcal{G})$ be a block of infinite representation type. Then $\Gamma_s(\mathcal{B})$ possesses infinitely many components.*

*Proof.* This is a direct consequence of (3.2) and the second Brauer-Thrall conjecture. $\square$

# 4. Frobenius Kernels of Reductive Groups

Throughout this section we consider a reductive group scheme $\mathcal{G}$, defined over the algebaically closed field $k$ whose characteristic $p$ is assumed to be $\geq 5$. We refer the reader to [22] concerning the representation theory of reductive groups.

**Theorem 4.1** *Let $\Theta \subset \Gamma_s(\mathcal{G}_r)$ be a component. Then $\Theta$ is isomorphic to one of the following types: $\mathbb{Z}[A_\infty]$, $\mathbb{Z}[A_\infty]/(\tau^\ell)$, $\mathbb{Z}[A_\infty^\infty]$, $\mathbb{Z}[D_\infty]$, $\mathbb{Z}[\tilde{A}_{12}]$.*

*Proof.* We proceed according to the dimension of the support variety of $\Theta$. If $\dim \hat{\mathcal{V}}_{\mathcal{G}_r}(\Theta) = 1$, then (2.1) shows that either $\Theta \cong \mathbb{Z}[A_\infty]/(\tau^\ell)$ is an infinite tube, or the tree class $T_\Theta$ is a finite Dynkin diagram. In the latter case $\Theta$ is finite, and in virtue of [1, (VII.2.1)] $\Theta$ is the set of nonprojective indecomposables of a block $\mathcal{B} \subset H(\mathcal{G}_r)$ of finite representation type. This, however, contradicts [15, (7.1)].

In view of (1.3) we may now conclude the proof of our result by showing that every component $\Theta$, which is either isomorphic to $\mathbb{Z}[\tilde{A}_n]$ or has Euclidean tree class is isomorphic to $\mathbb{Z}[\tilde{A}_{12}]$. In view of [30, Thm.A] and [5, p.155] such a component $\Theta$ is attached to a principal indecomposable module. By the same token $\Omega(\Theta) \cong \Theta$ also has this property, so that $\Theta$ contains a simple vertex $[S]$. Since $\Theta$ has a tree class which is either $A_\infty^\infty$ or of Euclidean type, (2.2) implies that $\dim \hat{\mathcal{V}}_{\mathcal{G}_r}(S) = 2$. The arguments of the proof of [15, (7.1)] now show that there exists a subgroup $\mathcal{H} \subset \mathcal{G}$ and a projective $\mathcal{H}_r$-module $P$ such that

(a) $\mathcal{G}_r \cong SL(2)_r \times \mathcal{H}_r$, and

(b) there is a decomposition $S \cong L(\lambda) \otimes_k P$, with a simple $SL(2)_r$-module $L(\lambda)$.

The module $L(\lambda)$ is defined by the highest weight $\lambda \in \{0, \ldots, p^r - 1\}$. We write

$$\lambda = \sum_{i=0}^{r-1} \lambda_i p^i,$$

with $\lambda_i \in \{0, \ldots, p-1\}$. Since $\hat{\mathcal{V}}_{SL(2)_r}(L(\lambda)) = \hat{\mathcal{V}}_{\mathcal{G}_r}(S)$ has dimension 2, we readily obtain from [27, (7.8)] that there exist at most two indices $i, j \in \{0, \ldots, r-1\}$ such that $\lambda_i \neq p - 1 \neq \lambda_j$. In case two indices differ from $p-1$, the same result shows that

$$\hat{\mathcal{V}}_{SL(2)_r}(L(\lambda)) \cong \{(x, y) \in \hat{\mathcal{V}}_{SL(2)_1}(k) \times \hat{\mathcal{V}}_{SL(2)_1}(k) \ ; \ [x, y] = 0\}$$

is the commuting variety of $\hat{\mathcal{V}}_{SL(2)_1}(k)$. We observe that the algebraic group $G := SL(2)(k) \times (k \setminus \{0\})$ operates on this set via conjugation and scalar multiplication. Setting $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ we see that there are four $G$-orbits, given by $(e, e)$, $(e, 0)$, $(0, e)$, and $(0, 0)$. Since the orbit passing through $(e, e)$ has dimension 3, we have reached a contradiction.

Hence there is exactly one index $i \in \{0, \ldots, r-1\}$ such that $\lambda_i \neq p-1$. We consider the projective cover $P(\lambda)$ of $L(\lambda)$. Owing to [2, Lemma 6] (cf. also [25, Satz4]) we have

$$P(\lambda) \cong P(\lambda_i)^{[i]} \otimes_k \bigotimes_{j \neq i} L(p-1)^{[j]},$$

with the second factor being projective and simple. Here $M^{[i]}$ denotes the $i^{th}$ Frobenius twist of the $\mathcal{G}$-module $M$. From the $s\ell(2)$-theory we now obtain a filtration of $P(\lambda_i)^{[i]}$ with factors (from top to bottom) given by $L(\lambda_i)^{[i]}$, $L(p-2-\lambda_i)^{[i]} \oplus L(p-2-\lambda_i)^{[i]}$, and $L(\lambda_i)^{[i]}$. Upon tensoring with $\bigotimes_{j \neq i} L(p-1)^{[j]}$, Steinberg's twisted tensor product theorem [22, (II.3.17)] readily shows that

$$\mathrm{Rad}(P(\lambda))/\mathrm{Soc}(P(\lambda)) \cong 2(L(p-2-\lambda_i)^{[i]} \otimes_k \bigotimes_{j \neq i} L(p-1)^{[j]}).$$

Accordingly, the vertex $[\mathrm{Rad}(P(\lambda))]$ has a successor in $\Theta$ of multiplicity 2. This readily implies that $\Theta \cong \mathbb{Z}[\tilde{A}_{12}]$. $\square$

*Remarks.* (1) The proof of (4.1) in conjunction with [25, Satz6] also shows that Euclidean components can occur in wild blocks whenever $r \geq 2$.

(2) The arguments of (4.1) also imply that the components of $\Gamma_s(\mathcal{G}_r)$, that are attached to a principal indecomposable module, are isomorphic to $\mathbb{Z}[\tilde{A}_{12}]$ or $\mathbb{Z}[A_\infty]$.

**Corollary 4.2** *Let $\mathcal{G}$ be almost simple, $r \in \mathbb{N}$. If $H(\mathcal{G}_r)$ admits a tame block, or if $\Gamma_s(\mathcal{G}_r)$ possesses a Euclidean component, then $\mathcal{G} \cong SL(2)$ or $\mathcal{G} \cong PSL(2)$.*

*Proof.* In either case there exists a simple $H(\mathcal{G}_r)$-module $S$ whose support variety has dimension 2. The proof of [15, (7.1)] then shows that $\mathcal{G}$ has rank 1, and is therefore of the asserted form. $\square$

# 5. Supersolvable Infinitesimal Groups

Throughout this section we assume that $p \geq 3$. We shall be studying the stable Auslander-Reiten quiver $\Gamma_s(\mathcal{G})$ of a supersolvable infinitesimal $k$-group $\mathcal{G}$. In contrast to the foregoing sections, module varieties will not play a rôle here. Instead we shall exploit detailed information the block structure of $H(\mathcal{G})$.

Recall that the set of algebra homomorphisms from $H(\mathcal{G})$ to $k$ coincides with $G(\mathcal{O}(\mathcal{G}))$, the set of group-like elements of the function algebra $\mathcal{O}(\mathcal{G})$. Since $\mathcal{O}(\mathcal{G})$ is a free $k[G(\mathcal{O}(\mathcal{G}))]$-module, it follows that $G(\mathcal{O}(\mathcal{G}))$ is an abelian $p$-group. Given an $H(\mathcal{G})$-module $M$ and $\lambda \in G(\mathcal{O}(\mathcal{G}))$ we will write $M^{(\lambda)}$ instead of $M^{(\lambda * id_{H(\mathcal{G})})}$.

We will say that a component $\Theta \subset \Gamma_s(\mathcal{G})$ is *Euclidean* if $\Theta$ either has a Euclidean tree class, or $\Theta \cong \mathbb{Z}[\tilde{A}_n]$. In the latter case $\Theta$ has tree class $A_\infty^\infty$.

**Lemma 5.1** *Let $S$ and $T$ be simple $H(\mathcal{G})$-modules. If there exists $n \in \mathbb{Z} \setminus \{0\}$ such that $\tau^n(S) = T$, then $S$ and $T$ are periodic.*

*Proof.* Without loss of generality we may assume that $n > 0$. By assumption $S$ and $T$ belong to the same block of $H(\mathcal{G})$, and [17, (2.4)] thus provides an element $\gamma \in G(\mathcal{O}(\mathcal{G}))$ such that $\Omega^{2n}(S) \cong S^{(\gamma)}$. Since $\Omega$ commutes with the operator $[M] \mapsto [M^{(\gamma)}]$ and the latter has finite order, we conclude that there is $m \in \mathbb{N}$ such that $\Omega^m(S) \cong S$. Consequently, $S$ and $T$ are periodic. $\square$

Let $\mathcal{B} \subset H(\mathcal{G})$ be a block. As $\mathcal{G}$ is supersolvable, a consecutive application of [17, (2.4)] and [12, (4.2)] shows that a suitable subgroup $G(\mathcal{B}) \subset G(\mathcal{O}(\mathcal{G}))$ operates transitively on the set $\mathcal{S}(\mathcal{B})$ of isoclasses of simple $\mathcal{B}$-modules. By the same token all principal indecomposable $\mathcal{B}$-modules have the same length $\ell_{\mathcal{B}}$. In view of [17, (2.1),(2.3)] $\ell_{\mathcal{B}}$ is a $p$-power. Letting $\ell(M)$ be the length of a $\mathcal{G}$-module $M$, it follows that

$$\ell(\tau(M)) = \ell(\Omega^2(M)) \equiv \ell(M) \qquad \mathrm{mod}(\ell_{\mathcal{B}})$$

for every indecomposable $\mathcal{B}$-module $M$.

If $\Theta \subset \Gamma_s(\mathcal{G})$ is a component with orbit graph $\bar{T}_\Theta$, then the foregoing observations show that the length-function induces a map

$$\bar{T}_\Theta \longrightarrow \{0, \ldots, \ell_{\mathcal{B}} - 1\} \quad ; \quad x \mapsto \ell_x,$$

where $\ell_x \equiv \ell(M) \mod(\ell_{\mathcal{B}})$ for every $[M] \in x$. Moreover, by composing the length function with the covering map $\mathbb{Z}[T_\Theta] \longrightarrow \Theta$ we obtain a similar map on the orbit graph $T_\Theta$ of $\mathbb{Z}[T_\Theta]$.

**Theorem 5.2** *The stable Auslander-Reiten quiver $\Gamma_s(\mathcal{G})$ does not contain any Euclidean components.*

*Proof.* We begin with a few general observations. If $\Theta$ is a Euclidean component, then we either have $\Theta \cong \mathbb{Z}[\tilde{A}_n]$ or $T_\Theta$ is a Euclidean diagram. Thanks to [30, (2.4)] and [5, p.155] $\Theta$ is attached to a principal indecomposable module $P$. Since $\Omega(\Theta) \cong \Theta$ also enjoys this property, [17, (2.4)] provides $\lambda \in G(\mathcal{O}(\mathcal{G}))$ such that the restriction of the map

$$\Gamma_s(\mathcal{G}) \longrightarrow \Gamma_s(\mathcal{G}) \quad ; \quad [M] \mapsto [\Omega(M^{(\lambda)})]$$

defines an automorphism $\varphi$ of the component $\Theta$.

The arguments of [12, (4.9)] may be adopted verbatim to see that $\Theta \not\cong \mathbb{Z}[\tilde{A}_n]$. According to [3, (4.15.5)] $\varphi$ induces an automorphism $\psi : \mathbb{Z}[T_\Theta] \longrightarrow \mathbb{Z}[T_\Theta]$. Let $\eta : T_\Theta \longrightarrow T_\Theta$ be the automorphism that sends the $\tau$-orbit of $(n, x)$ to that of $\psi(n, x)$. As $\Theta$ does not contain any periodic modules, the map $\eta$ is fixed-point free. Consequently, components of tree class $(\tilde{E}_n)_{6 \leq n \leq 8}$ do not occur. If $\Theta$ has tree class $\tilde{A}_{12}$ then [7, (IV.3.8.2)] shows that the principal indecomposable module $P$ attached to $\Theta$ has length 4. Since $p \geq 3$ this contradicts [17, (2.3)].

We finally assume that $T_\Theta = \tilde{D}_n$. Since $\eta$ is a fixed-point free automorphism, it readily follows that $n$ is odd. We may now apply [12, (2.1)] to see that $\Theta \cong \mathbb{Z}[\tilde{D}_n]$. By the same token, $\mathbb{Z}[\tilde{D}_n]$ does not possess any automorphisms of order a $p$-power. Since $ord(\varphi^2 \circ \tau^{-1}|_\Theta)$ is a $p$-power, we conclude that $\varphi^2 = \tau|_\Theta$. Another application of [12, (2.1)] now yields $n = 5$.

We let $\mathcal{B}$ be the block associated to $\Theta$ and consider the orbit graph $\bar{T}_\Theta \cong \tilde{D}_5$ of $\Theta$. By our introductory remarks $\Theta$ contains a simple vertex $[S]$, and the vertex $[Rad(P)]$ defined by the radical of of a principal indecomposable $\mathcal{B}$-module $P$. These two modules belong to orbits $x_S$, $x_{Rad(P)} \in \tilde{D}_5$ with $\ell_{x_S} = 1$, and $\ell_{x_{Rad(P)}} = \ell_{\mathcal{B}} - 1$.

If $[S]$ has 3 predecessors in $\Theta$, then there exists a vertex $[X] \in \Theta$ and a principal indecomposable $\mathcal{B}$-module $Q$ such that

$$(0) \longrightarrow \tau(X) \longrightarrow S \oplus Q \longrightarrow X \longrightarrow (0)$$

is the almost split sequence terminating in $X$. Hence the above sequence is the standard almost split sequence for $Q$, and $S$ is isomorphic to the heart $H(Q) := Rad(Q)/Soc(Q)$ of $Q$. Consequently, $Q$ is a uniserial module of length 3. By the linkage principle for $\mathcal{B}$ [17, (2.3)] every principal indecomposable $\mathcal{B}$-module is uniserial. Thus, $\mathcal{B}$ is a Nakayama algebra, a contradiction.

Accordingly, $[S]$ is located at an end of $\Theta$, so that the heart $H(P)$ is indecomposable. Hence $[H(P)] \in \Theta$ belongs to an orbit $x_{H(P)}$ with $\ell_{x_{H(P)}} = \ell_{\mathcal{B}} - 2$.

By definition we have $\ell_{\eta(x)} = \ell_{\mathcal{B}} - \ell_x$ for every $x \in \tilde{D}_5$. So far, we have identified orbits $x_S$ and $x_{Rad(P)}$ of lengths 1 and $\ell_{\mathcal{B}}-1$, respectively, that are located at ends, and $x_{H(P)}$, $\eta(x_{H(P)})$ of lengths $\ell_{\mathcal{B}}-2$ and 2, respectively, that are not located at ends. From the additivity of the length function we obtain that the lengths of the end vertices adjacent to $x_{H(P)}$ are $\ell_{\mathcal{B}} - 1$, while those of the ends adjacent to $\eta(x_{H(P)})$ are 1.

Since $\ell_{\mathcal{B}}$ is odd, and $\mathcal{B}$ is not a Nakayama algebra, we have $\ell_{\mathcal{B}} \geq 5$. By standard properties of additive functions every non-periodic vertex of $\Gamma_s(\mathcal{B})$ is of the form $[M^{(\gamma)}]$ for $[M] \in \Theta$ and a suitable $\gamma \in G(\mathcal{O}(\mathcal{G}))$ (cf. for instance [12, (4.6)]).

Suppose there is a non-periodic, indecomposable $\mathcal{B}$-module of length 2. Then $\Theta$ contains a vertex $[M_2]$ of length 2, and the above observations show that $[M_2]$ has 3 predecessors in $\Theta$. There results an almost split sequence

$$(0) \longrightarrow \tau(X) \longrightarrow M_2 \oplus Q \longrightarrow X \longrightarrow (0),$$

where $Q = (0)$ or $Q$ principal indecomposable. In the former case $X$ and $\tau(X)$ are simple, and (5.1) provides a contradiction. Alternatively, $H(Q) \cong M_2$, so that $\ell_{\mathcal{B}} = 4$. As a result, all $\mathcal{B}$-modules of length 2 are periodic.

Since $\mathrm{Soc}(P)$ is simple, every submodule $M \subset P$ is indecomposable. Let $M_2 \subset M_3$ be submodules of $P$ of lengths 2 and 3, respectively. By the above $M_2$ is periodic, while the simple module $M_3/M_2$ is not periodic. It follows that $M_3$ is not periodic either. Consequently, $\Theta$ contains a vertex of length 3. Since the possible lengths are 1, 2, $\ell_{\mathcal{B}} - 1$ and $\ell_{\mathcal{B}} - 2$, we see that $\ell_{\mathcal{B}} = 5$. But then $\Omega^{-1}(M_3)$ is a non-periodic indecomposable $\mathcal{B}$-module of length 2, a contradiction. □

We turn to the determination of the finite components of $\Gamma_s(\mathcal{G})$ and the structure of the representation-finite blocks. According to [16, (2.1),(2.7)] representation-finite infinitesimal groups are supersolvable with all blocks being Nakayama algebras. Our next result provides a generalization for arbitrary blocks of supersolvable infinitesimal groups.

**Theorem 5.3** *Let $\mathcal{B} \subset H(\mathcal{G})$ be a block, $\Theta \subset \Gamma_s(\mathcal{G})$ a component.*

(1) *If $\mathcal{B}$ is representation-finite, then $\mathcal{B}$ is a Nakayama algebra.*

(2) *If $\Theta$ is finite, then there exist $r > 0$, $s \geq 0$ such that $\Theta \cong \mathbb{Z}[A_{p^r-1}]/(\tau^{p^s})$.*

*Proof.* (1). Recall that $\mathcal{B}$ is transitive, in the sense that the subgroup $G(\mathcal{B}) \subset G(\mathcal{O}(\mathcal{G}))$ operates transitively on the set $\mathcal{S}(\mathcal{B})$ of isoclasses of simple $\mathcal{B}$-modules via $\lambda \cdot [S] := [S^{(\lambda)}]$. Since $G(\mathcal{B})$ operates on $\mathrm{mod}_{\mathcal{B}}$ via auto-equivalences, the aforementioned operation induces an action of $G(\mathcal{B})$ on the Gabriel quiver $Q(\mathcal{B})$ of $\mathcal{B}$. Let $Q^s(\mathcal{B}) = Q(\mathcal{B}) \times \{0, 1\}$ be the separated quiver of $\mathcal{B}$. If we define

$$\lambda \cdot (x, i) := (\lambda \cdot x, i) \quad \forall\, x \in Q(\mathcal{B}),\, i \in \{0, 1\},$$

then $G(\mathcal{B})$ operates on $Q^s(\mathcal{B})$ via quiver automorphisms, each of which has order $p$-power. The same holds for the underlying undirected graph $\bar{Q}^s(\mathcal{B})$. Since $G(\mathcal{B})$ is abelian, an element operates either trivially or without fixed-points.

Owing to [1, (X.2.6)] the connected components of the graph $\bar{Q}^s(\mathcal{B})$ are finite Dynkin diagrams of types $A_n$, $D_n$, $E_6$, $E_7$, or $E_8$. By the above $G(\mathcal{B})$ operates on the set $\mathcal{C}$ of components of $\bar{Q}^s(\mathcal{B})$ via permutations. Suppose that $\lambda \in G(\mathcal{B})$ fixes a component $C \in \mathcal{C}$. If $C$ has type $D_n$, $E_6$, $E_7$, or $E_8$, then the corresponding automorphism $\hat{\lambda} : \bar{Q}^s(\mathcal{B}) \longrightarrow \bar{Q}^s(\mathcal{B})$ has a fixed-point, and $\lambda$ operates trivially on $\bar{Q}^s(\mathcal{B})$. Alternatively, $\hat{\lambda}|_C$ has order $\leq 2$, which, in view of $p \neq 2$, entails $\hat{\lambda}|_C = id_C$. Thus, $\lambda$ operates trivially in this case as well. Our discussion shows that the number of components of $\bar{Q}^s(\mathcal{B})$ equals the number of simple $\mathcal{B}$-modules, so that each component of $\bar{Q}^s(\mathcal{B})$ has exactly two elements. Since $\dim_k Ext^1_{\mathcal{B}}(S,T) \leq 1$ for any two simple $\mathcal{B}$-module $S$ and $T$, we see that

$$\sum_{[T]\in\mathcal{S}(\mathcal{B})} \dim_k Ext^1_{\mathcal{B}}(S,T) \leq 1 \quad \text{and} \quad \sum_{[T]\in\mathcal{S}(\mathcal{B})} \dim_k Ext^1_{\mathcal{B}}(T,S) \leq 1$$

for every simple $\mathcal{B}$-module $S$. Thanks to [21, Thm.9] this shows that $\mathcal{B}$ is a Nakayama algebra.

(2). If $\Theta \subset \Gamma_s(\mathcal{G})$ is finite, then [1, (VII.2.1)] provides a block $\mathcal{B} \subset H(\mathcal{G})$ such that $\Theta$ is the set of isoclasses of the nonprojective indecomposable $\mathcal{B}$-modules. By (1) the block $\mathcal{B}$ is a Nakayama algebra. According to [1, p.253] this implies $\Theta \cong \mathbb{Z}[A_n]/(\tau^q)$, where $q := card(\mathcal{S}(\mathcal{B}))$ and $n+1$ is the Loewy length of $\mathcal{B}$. Thanks to [17, (1.1)] and the transitivity of $\mathcal{B}$, both of these parameters are $p$-powers. $\square$

**Corollary 5.4** *Let $\mathcal{B} \subset H(\mathcal{G})$ be a representation-finite block. Then $\mathcal{B}$ is Morita equivalent to $k[\tilde{A}_{p^s-1}]/I_{p^r}$, where $I_{p^r}$ is generated by all paths of length $p^r$.*

*Proof.* According to (5.3) there exist $r, s \geq 0$ such that $\mathcal{B}$ is a Nakayama algebra of Loewy length $p^r$, and with $p^s$ simple modules. By combining [23, Satz 8] with [1, (IV.2.13)] we see that the basic algebra of $\mathcal{B}$ is isomorphic to $k[\tilde{A}_{p^s-1}]/I_{p^r}$. $\square$

We have seen in §2 that "most" AR-components are of type $\mathbb{Z}[A_\infty]$. The following result shows that simple vertices of such components are quasi-simple.

**Proposition 5.5** *Let $\Theta \cong \mathbb{Z}[A_\infty]$ be a component of $\Gamma_s(\mathcal{G})$. Then $\Theta$ contains at most one simple vertex $[S]$. Such a vertex is quasi-simple, and the heart $H(P)$ of its projective cover $P$ is indecomposable.*

*Proof.* Let $\Upsilon$ be the set of all those components of $\Gamma_s(\mathcal{G})$ that are isomorphic to $\mathbb{Z}[A_\infty]$ and contain a simple vertex $[S]$ of quasi-length $ql([S]) \geq 2$. Given $\Theta \in \Upsilon$ we define

$$q_\Theta := \min\{ql([S]) \ ; \ [S] \in \Theta \text{ simple with } ql([S]) \geq 2\}.$$

Assuming that $\Upsilon \neq \emptyset$ we put $r := \min\{q_\Theta \ ; \ \Theta \in \Upsilon\}$. Let $\Theta_0 \in \Upsilon$ be a component with $q_{\Theta_0} = r$. There exists a simple vertex $[S_0] \in \Theta_0$ such that $ql([S_0]) = r$. Since every irreducible map terminating (originating) in $S_0$ is surjective (injective), we conclude from the mesh relations that there exist principal indecomposable modules $P_1$, $P_2$ such that each $[Rad(P_i)] \in \Theta_0$ has quasi-length $< r$. Consequently, the component $\Omega^{-1}(\Theta_0) \cong \Theta_0$ contains simple vertices of quasi-length $< r$. By choice of $r$ we obtain $ql([Rad(P_1)]) = 1 = ql([Rad(P_2)])$. If $r > 2$, then

$\Omega^{-1}(\Theta_0)$ contains two simple vertices of quasilength 1. By (5.1) the corresponding simple modules are periodic, a contradiction. Consequently, $r = 2$, and $[S_0]$ is the only successor of $[Rad(P_1)]$. From the standard AR-sequence we obtain $H(P_1) \cong S_0$, so that $P_1$ is uniserial. By transitivity, this holds for every principal indecomposable module of the block associated to $S_0$. Hence this block is a Nakayama algebra, a contradiction.

As a result of our discussion, $\Upsilon = \emptyset$. Hence every simple vertex of a $\mathbb{Z}[A_\infty]$-component is quasi-simple, and another application of (5.1) shows that such components contain at most one simple vertex. $\square$

Our final result is concerned with the special case, where the underlying infinitesimal group $\mathcal{G}$ is unipotent. So far, these groups form the only class whose Auslander-Reiten quivers are completely understood.

**Corollary 5.6** *Let $\Theta \subset \Gamma_s(\mathcal{G})$ be a component of the AR-quiver of the unipotent infinitesimal group $\mathcal{G}$. Then $\Theta \cong \mathbb{Z}[A_{p^n}]/(\tau)$, $\mathbb{Z}[A_\infty]/(\tau^r)$, or $\mathbb{Z}[A_\infty]$.*

*Proof.* If $\Theta$ is periodic, then it is either finite, or isomorphic to $\mathbb{Z}[A_\infty]/(\tau^r)$. In the former case (5.3) implies that $\Theta \cong \mathbb{Z}[A_n]/(\tau^r)$. Since $H(\mathcal{G})$ is local, we have $r = 1$.

Alternatively, a consecutive application of (1.3) and (5.2) shows that $\Theta \cong \mathbb{Z}[A_\infty]$, $\mathbb{Z}[A_\infty^\infty]$, or $\mathbb{Z}[D_\infty]$. It therefore remains to rule out the last two types. In view of [27, §6,§7] we may adopt the arguments of [9, (3.3)] to see that there are no such components. $\square$


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