A generalization of Hasminskii's theorem on existence of invariant measures for locally integrable drifts

V. Bogachev [*], **M. Röckner** [†]

Abstract

We prove existence of invariant measures for diffusions on $\mathbb{R}^n$ whose drift are merely locally in L^p with $p > n$. Our only global assumption is that a Lyapunov-type function exists. Subsequently, we generalize this result to the case where $\mathbb{R}^n$ is replaced by a Riemannian manifold.

AMS 1991 Subject classification:
Primary: 60H10, 47D07
Secondary: 47D06, 60J60
Keywords and phrases: invariant measures, elliptic equations, diffusion processes.
Running head: Existence of invariant measures

0 Introduction

A well-known theorem due to Hasminskii [1], [2] states that a diffusion process in $\mathbb{R}^n$ (or, more generally, in a smooth manifold) with locally Lipschitzian coefficients possesses an invariant probability provided there exists a certain Lyapunov-type function for the generator of this diffusion. In applications one of the best known cases is where the drift term is coercive. In recent years there has been a growing interest in diffusion processes with more general, possibly singular drifts. It has been discovered that this more general situation preserves many essential features of the locally Lipschitzian case. A generalization of Hasminskii's theorem has been obtained in [3, §4.3] for nondegenerate diffusion processes with continuous coefficients and certain classes of Lyapunov functions. Further extensions are found in [4], [5], [6], where the existence of invariant probabilities is established for diffusion

[*] Department of Mechanics and Mathematics, Moscow State University, 119899 Moscow, Russia
[†] Fakultät für Mathematik, Universität Bielefeld, D–33501 Bielefeld, Germany

processes with nondegenerate continuous diffusion terms satisfying certain uniform estimates and with locally bounded coercive drifts. On the other hand, it has been shown in [7] that any invariant measure of a nondegenerate diffusion in $\mathbb{R}^n$ (or in a smooth n-dimensional manifold) with drift locally integrable of order more than n admits a density locally in a Sobolev space from the same class as in the locally Lipschitzian case; under the same condition Varadhan's conjecture concerning the uniqueness of invariant probabilities for diffusions with non-smooth drifts has been settled positively in [8]. The purpose of this note is to apply the methods used in [7], [8] to replace the local Lipschitzness (or local boundedness) of the drift in the aforementioned theorem of Hasminskii by its local integrability. The formulations and proofs of our results are essentially analytical which seems to be natural from the point of view of the problems we consider; however, their probabilistic interpretation is straightforward. It is well-known that in the case of locally Lipschitzian coefficients any invariant measure μ for the diffusion ξ governed by the stochastic differential equation

$$d\xi_t = \sqrt{2A(\xi_t)}\, dw_t + B(\xi_t)dt$$

satisfies the elliptic equation

$$L^*_{A,B}\mu = 0 \tag{0.1}$$

with

$$L_{A,B}f = \mathrm{trace}(Af'') + (B, \nabla f), \quad f \in C_0^\infty(\mathbb{R}^n), \tag{0.2}$$

in the following weak sense:

$$\int\limits_{\mathbb{R}^n} L_{A,B}f\, d\mu = 0, \quad \forall\, f \in C_0^\infty(\mathbb{R}^n), \tag{0.3}$$

where implicitly we always assume that $|B|$, $A^{ij} \in L^1_{loc}(\mu)$ with $A = (A^{ij})$.

In many applications it becomes more convenient to work with the weak elliptic equation (0.3) instead of the invariance of measures in the sense of semigroups. However, as it will be mentioned below, under broad conditions (including those used in our paper) these two settings are equivalent.

For every $R > 0$ set $U_R := \{x \in \mathbb{R}^n : |x| < R\}$. If Ω is an open set in $\mathbb{R}^n$, we denote by $H^{p,1}(\Omega)$ the Sobolev class of functions which together with their generalized first order derivatives belong to $L^p(\Omega)$. The class $H^{p,1}(\Omega)$ is equipped with the Sobolev norm

$$\|f\|_{H^{p,1}(\Omega)} = \|f\|_{L^p(\Omega)} + \sum_{i=1}^n \|\partial_{x_i} f\|_{L^p(\Omega)}.$$

The symbol $C_0^\infty(\Omega)$ stands for the class of all infinitely differentiable functions on Ω with compact supports in Ω. Analogously, one defines the class $C_0^\infty(M)$ for a smooth finite dimensional manifold M. By $H^{p,1}_{loc}(\Omega)$ we denote the class of all functions f on Ω such that $\zeta f \in H^{p,1}(\Omega)$ for every $\zeta \in C_0^\infty(\Omega)$. Norms of vectors are denoted by $|\cdot|$ and norms of operators are denoted by $\|\cdot\|$.

1 Invariant measures on $\mathbb{R}^n$

In the proofs below we make use of the following result obtained in [7] (see Theorem 1 of the paper cited and the remarks following it).

We say that (0.1) is satisfied in an open set Ω if the equality in (0.3) holds for all $f \in C_0^\infty(\Omega)$.

Theorem 1.1 *Let μ be a finite (possibly, signed) measure on an open set $\Omega \subset \mathbb{R}^n$ satisfying the equation $L^*_{A,B}\mu = 0$ in Ω in the sense explained above, let $A = (A^{ij})$ be a mapping with values in the space of linear operators on $\mathbb{R}^n$ such that $A^{ij} \in C^\infty(\Omega)$ and $\det A \neq 0$, and let $B : \Omega \to \mathbb{R}^n$ satisfy one of the following two conditions: either $|B| \in L^\gamma_{loc}(\Omega, dx)$ or $|B| \in L^\gamma_{loc}(\Omega, \mu)$, where $\gamma > n$. Then μ has a density $F \in H^{\gamma,1}_{loc}(\Omega)$. In particular, F is Hölder continuous of order $1 - n/\gamma$.*

Note that if

$$\kappa := \|B\|_{L^\gamma(\Omega,\mathbb{R}^n)} + \sum_{i,j=1}^n \|A^{ij}\|_{H^{\gamma,1}(\Omega)} + \sup_{x\in\Omega} \|A(x)^{-1}\| < \infty, \tag{1.1}$$

then for any domain K with compact closure in Ω there exists $\lambda(K, \kappa)$ such that for every solution $\mu = F dx$ of the equation $L^*_{A,B}\mu = 0$ (in our sense), one has the estimate

$$\|F\|_{H^{\gamma,1}(K)} \leq \lambda(K, \kappa) \sup_{x\in K} |F(x)|, \tag{1.2}$$

provided the continuous version of F is chosen. Indeed, since $F \in H^{\gamma,1}_{loc}(\Omega) \subset H^{2,1}_{loc}(\Omega)$, by virtue of well-known results (see, e.g., [9, Ch. III]), this estimate holds for $\|F\|_{H^{2,1}(K)}$ replacing $\|F\|_{H^{\gamma,1}(K)}$. By the closed graph theorem and the inclusion $F \in H^{\gamma,1}(K)$, we get the stronger estimate (1.2). In fact, this estimate can be derived from the proof of [7, Theorem 1].

The main idea of our method of proof is contained in the next lemma which is of interest in its own right.

Lemma 1.2 *Let $\{\mu_k\}$ be a sequence of probability measures on $\mathbb{R}^n$ such that*

$$L^*_{A_k,B_k}\mu_k = 0,$$

in the sense of (0.3), where $A_k = (A^{ij}_k)$ are C^∞-mappings on $\mathbb{R}^n$ with values in the nonnegative symmetric linear operators on $\mathbb{R}^n$ and $B_k = (B^i_k)$ are Borel mappings on $\mathbb{R}^n$ such that $|B_k| \in L^1_{loc}(\mu_k)$. Suppose that, for every $R > 0$, there exist $\alpha_R > n$, $\beta_R > 0$ and $\gamma_R > 0$ such that

$$\kappa_R := \sup_k \int_{U_R} \Big[|B_k(x)|^{\alpha_R} + \sum_{i,j=1}^n \|\nabla A^{ij}_k(x)\|^{\alpha_R}\Big]\, dx < \infty \tag{1.3}$$

and

$$\beta_R I \le A_k(x) \le \gamma_R I, \quad \forall\, k,\ \forall\, x \in U_R. \tag{1.4}$$

Assume, in addition, that there exists a function $V \in C^2(\mathbb{R}^n)$ such that

$$\lim_{|x|\to\infty} V(x) = +\infty \tag{1.5}$$

and

$$\lim_{|x|\to\infty} \sup_k L_{A_k,B_k} V(x) = -\infty. \tag{1.6}$$

Then:

(i) *The measures μ_k admit continuous densities p_k such that, for every $R > 1$, the functions $p_k|_{U_{R-1}}$ are uniformly bounded, uniformly Hölder continuous and uniformly bounded in $H^{\alpha_R,1}(U_{R-1})$.*

(ii) *The sequence $\{\mu_k\}$ is relatively weakly compact.*

(iii) $\inf_k \inf_{x\in U_R} p_k(x) > 0$ *for every $R > 0$.*

Furthermore, assertions (i) – (iii) *remain valid for all $k > R$ if, for each k, the measure μ_k vanishes outside U_k and satisfies the equation $L^*_{A_k,B_k}\mu_k = 0$ only in U_k.*

Proof. As already noted, the existence of continuous densities p_k such that $p_k \in H^{\alpha_R,1}_{loc}(U_R)$ is proved in [7]. In particular, the functions p_k are locally Hölder continuous. Now assertion (i) follows from [10] (or from the analogous results in [11]; see also [12]). According to [10, Corollary 5.3] (see also [13, Theorem 8.20 and a remark on p. 197]), given $0 < R < R'$, there is $C(R,R') > 0$ (depending only on R, R', $\beta_{R'}$, $\gamma_{R'}$, and $\kappa_{R'}$) such that for every nonnegative function g_k from $H^{\alpha_{R'},1}_{loc}(\mathbb{R}^n)$ satisfying the equation

$$\partial_{x_i}\Big(A^{ij}_k(x)\partial_{x_j} g_k - B^i_k g_k + \partial_{x_j} A^{ij}_k g_k\Big) = 0$$

in $U_{R'}$ in the weak sense, one has

$$\sup_{U_R} g_k \le C(R,R') \inf_{U_R} g_k. \tag{1.7}$$

Applying this assertion to p_k, we conclude that

$$\sup_{U_R} p_k \le C(R,R+1) \inf_{U_R} p_k \le \frac{C(R,R+1)}{\mathrm{Vol}\,(U_R)} \int_{U_R} p_k\,dx \le \frac{C(R,R+1)}{\mathrm{Vol}\,(U_R)}.$$

Therefore, the sequence $\{p_k\}$ is uniformly bounded on U_R. In addition, applying [9, Theorem 14.1, Ch. 3, p. 201] or [10, Corollary 5.5], we get that the functions p_k on U_R are uniformly Hölder continuous, i.e., there exist $\delta > 0$ and $K > 0$ such that for all k one has

$$|p_k(x) - p_k(y)| \le K|x-y|^\delta, \qquad \forall x,\ y \in U_R.$$

According to what has been said above (see (1.2)), the sequence $\{p_k|_{U_R}\}$ is uniformly bounded in $H^{\alpha_R,1}(U_{R-1})$.

In order to prove claim (ii) let us take a ball $U = U_r$ such that

$$L_{A_k,B_k}V(x) \le -1, \quad \forall k,\ \forall x \notin U.$$

Let us take a function $\psi \in C^2(\mathbb{R}^1)$ such that $\psi(t) = t$ if $t \le 1$, $\psi(t) = 2$ if $t \ge 3$, $\psi'(t) \ge 0$ and $\psi''(t) \le 0$. Put $\psi_j(t) = j\psi(t/j)$, $j \in \mathbb{N}$. Note that

$$\int_{\mathbb{R}^n} L_{A_k,B_k}(\psi_j \circ V)\, d\mu_k = 0, \quad \forall\, k,\ j. \tag{1.8}$$

Indeed, it follows from the condition and Lebesgue's dominated convergence theorem that $\int L_{A_k,B_k}\varphi\, d\mu_k = 0$ for every $\varphi \in C_0^2(\mathbb{R}^n)$. Hence, the same is true for every $\varphi \in C^2(\mathbb{R}^n)$ such that $\varphi = const$ outside some ball (since $\varphi - const$ has compact support and $L_{A_k,B_k}1 = 0$). Noting that $\psi_j \circ V = 2j$ outside the ball U_{m_j}, where $m_j > r$ is chosen in a such a way that $V(x) \ge 3j$ if $x \notin U_{m_j}$, we arrive at (1.8). By assertion (i) and conditions (1.3), (1.4), we have

$$S := \sup_k \int_U |L_{A_k,B_k}V|\, d\mu_k < \infty. \tag{1.9}$$

Let us fix j_0 such that $j_0 > \sup_{x\in U} V(x)$. Below we deal only with $j \ge j_0$. Note that

$$L_{A_k,B_k}(\psi_j \circ V) = \psi_j'(V)L_{A_k,B_k}V + \psi_j''(V)(A_k\nabla V, \nabla V). \tag{1.10}$$

Therefore, for all $j \ge j_0$, one has

$$L_{A_k,B_k}(\psi_j \circ V) \le \psi_j'(V)L_{A_k,B_k}V(x) \le 0 \quad \text{if } V(x) \ge j.$$

If $V(x) \le j$, one has $L_{A_k,B_k}(\psi_j \circ V)(x) = L_{A_k,B_k}V(x)$. Finally,

$$L_{A_k,B_k}(\psi_j \circ V)(x) = 0 \quad \text{if } x \notin U_{m_j}.$$

Let

$$R_j = \max\{R \in \mathbb{N} : \sup_{x\in U_R} V(x) \le j\},$$

i.e. U_{R_j} is the maximal ball U_R in $\{V \le j\}$ with an integer radius. Clearly, $R_j \to \infty$ as $j \to \infty$ by condition (1.5). Note that the zero integral of $L_{A_k,B_k}(\psi_j \circ V)$ over $\mathbb{R}^n$ in (1.8) is the sum of the integrals over the sets $U \cap \{V \le j\}$, $U_{m_j} \cap \{V \le j\}\backslash U$, and $U_{m_j} \cap \{V > j\}$. Since $L_{A_k,B_k}(\psi_j \circ V) = L_{A_k,B_k}V$ on $\{V \le j\}$, we obtain from (1.9) that

$$\Big|\int_{U\cap\{V\le j\}} L_{A_k,B_k}(\psi_j \circ V)\, d\mu_k\Big| \le S.$$

The two other integrals are nonpositive, since $L_{A_k,B_k}(\psi_j \circ V) \le 0$ on the set $\{V > j\}$ and $L_{A_k,B_k}(\psi_j \circ V) = L_{A_k,B_k}V \le -1$ on the set $\{V \le j\}\backslash U$. Therefore,

$$\Bigl|\int\limits_{U_{m_j}\cap\{V\le j\}\backslash U} L_{A_k,B_k}(\psi_j \circ V)\, d\mu_k\Bigr| \le S, \qquad \Bigl|\int\limits_{U_{m_j}\cap\{V> j\}} L_{A_k,B_k}(\psi_j \circ V)\, d\mu_k\Bigr| \le S.$$

Using that $U_{R_j} \subset U_{m_j} \cap \{V \le j\}$ and that $L_{A_k,B_k}(\psi_j \circ V) = L_{A_k,B_k}V \le 0$ on the set $U_{m_j} \cap \{V \le j\}\backslash U$, this yields

$$0 \le \int\limits_{U_{R_j}\backslash U} (-L_{A_k,B_k}V)\, d\mu_k \le S, \quad \forall\, k, \forall\, j \ge j_0. \tag{1.11}$$

Hence

$$\sup_k \int\limits_{\mathbb{R}^n} |L_{A_k,B_k}V|\, d\mu_k \le 2S. \tag{1.12}$$

Taking into account (1.6), we conclude that the sequence $\{\mu_k\}$ is uniformly tight (i.e., for every $\varepsilon > 0$ there is a number $R > 0$ such that $\mu_k(\mathbb{R}^n\backslash U_R) < \varepsilon$ for all k), hence, is relatively weakly compact.

Let us now show (iii), i.e., that $\{p_k\}$ is uniformly separated from zero on U_R. Since p_k is not identically zero on $\mathbb{R}^n$, it follows by (1.7) that p_k is strictly positive. Suppose that $\lim\limits_{j\to\infty} \inf\limits_{U_R} p_{k_j} = 0$. As we have proved above, $\{p_{k_j}\}$ contains a subsequence that converges uniformly on every ball. Applying (1.7), we conclude that it converges locally uniformly to zero. This contradicts the fact that the sequence of probability measures $\{\mu_k\}$ is relatively weakly compact.

Let us prove the last claim. As above, the functions p_k, $k \ge m$, are uniformly bounded in $H^{\alpha_m,1}(U_{m-1})$ and uniformly Hölder continuous. The reasoning used to show (1.8) gives the equality

$$\int\limits_{\mathbb{R}^n} L_{A,B}(\psi_j \circ V)\, d\mu_k = 0, \quad \forall\, j,\ \forall k > m_j. \tag{1.13}$$

In the same manner as above, (1.13) leads to the estimate

$$\Bigl|\int\limits_{U_{R_j}\backslash U} L_{A,B}V\, d\mu_k\Bigr| \le S, \quad \forall\, j \ge j_0,\ \forall\, k > m_j, \tag{1.14}$$

where $U = U_r$ is such that

$$LV(x) \le -1, \quad \forall\, x \notin U.$$

Since $LV(x) \to -\infty$ as $|x| \to \infty$, we conclude by (1.14) that the sequence $\{\mu_k\}$ is uniformly tight. Assertion (iii) follows by the same reasoning as in the first part of this lemma. □

Corollary 1.3 *In the situation of Lemma* 1.2, *the sequence* $\{\mu_k\}$ *is relatively compact in the variation norm.*

Proof. It follows from Lemma 1.2 that every subsequence of this sequence has a subsequence $\{\nu_k\}$ such that $\{\nu_k\}$ converges weakly to some probability measure ν and the continuous densities p_k of the measures ν_k converge locally uniformly to a continuous limit p. Clearly, p serves as a density for ν. By Scheffé's theorem (see [14, p. 224, Appendix II]),

$$\int_{\mathbb{R}^n} |p_k(x) - p(x)|\, dx \to 0,$$

which is our claim. □

Corollary 1.4 *Suppose the conditions specified above in Lemma* 1.2 *are satisfied except for* (1.6) *which is replaced by*

$$\lim_{|x|\to\infty} \sup_k (B_k(x), x) = -\infty. \tag{1.15}$$

Assume, in addition, that $\sup\limits_R \gamma_R < \infty$. *Then, for every* $p \geq 1$, *one has*

$$M_p := \sup_k \int_{\mathbb{R}^n} (x,x)^p\, \mu_k(dx) < \infty. \tag{1.16}$$

Proof. We may assume that $p \in \mathbb{N}$. By (1.15) the function $V(x) = (x,x)^p$ satisfies the conditions of the previous lemma, since

$$\begin{aligned} L_{A_k,B_k} V(x) &= 4p(p-1)(x,x)^{p-2} \sum_{i,j=1}^n A_k^{ij}(x) x_i x_j \\ &\quad + 2p(x,x)^{p-1} \operatorname{trace} A_k(x) + 2p(x,x)^{p-1}(x, B_k(x)) \\ &\quad \leq (x,x)^{p-1} \Big[4p(p-1)\|A_k(x)\| + 2pn\|A_k(x)\| + 2p(x, B_k(x)) \Big]. \end{aligned} \tag{1.17}$$

Hence by (1.12)

$$\sup_k \int_{\mathbb{R}^n} |L_{A_k,B_k} V(x)|\, \mu_k(dx) < \infty.$$

Applying (1.15), (1.17), and the uniform boundedness of $\|A_k(x)\|$, we conclude that $M_{p-1} < \infty$. □

Remark 1.5 (i) We observe that the proof for the tightness of $\{\mu_k\}$ only used (1.5) and (1.6). Hence assertion (ii) of Lemma 1.2 is valid under (1.5) and (1.6) for measures μ_k satisfying equations $L^*_{A_k,B_k}\mu_k = 0$ with arbitrary coefficients A_k and B_k that are locally μ_k-integrable.

(ii) Clearly, conditions (1.3) and (1.4) can be imposed locally, i.e., for every point x there exist a neighborhood W of x and numbers $\alpha_W > n$, $\beta_W > 0$, and $\gamma_W > 0$ such that (1.3) and (1.4) hold with W, α_W, β_W, and γ_W replacing U_R, α_R, β_R, and γ_R, respectively.

Theorem 1.6 *Suppose that $A = (A^{ij})$ is a mapping on $\mathbb{R}^n$ with values in the nonnegative symmetric linear operators on $\mathbb{R}^n$ and let B be a Borel mapping on $\mathbb{R}^n$ such that, for every ball U_R, there exist numbers $\alpha_R > n$ and $\beta_R > 0$ such that $\beta_R I \le A(x)$ for all $x \in U_R$ and $A^{ij}|_{U_R} \in H^{\alpha_R,1}(U_R)$, $\big|B|_{U_R}\big| \in L^{\alpha_R}(U_R)$. Assume, in addition, that there exists a function $V \in C^2(\mathbb{R}^n)$ such that*

$$V(x) \to +\infty \quad \textit{and} \quad L_{A,B}V(x) \to -\infty \quad \textit{as } |x| \to \infty. \tag{1.18}$$

Then there exists a probability measure μ such that

$$|B| \in L^1_{loc}(\mu) \quad \textit{and} \quad L^*_{A,B}\mu = 0$$

in the sense of (0.3). *Moreover, μ admits a continuous strictly positive density p such that $p|_{U_{R-1}} \in H^{\alpha_R,1}(U_{R-1})$ for every $R > 1$.*

Proof. Let us first find a sequence of nonnegative functions p_k that are not identically zero and satisfy (in the weak sense) our equation in the balls U_k, $k > 1$. According to [10, Theorem 3.2 and the remark at the end of §3], for every k there exists a nonnegative function $p_k \in H^{\alpha_{k+1},1}(U_k)$ such that $p_k = 1$ on the boundary of U_k (in the sense explained in [10]) and in the interior of U_k one has

$$\partial_{x_i}\Big(A^{ij}\partial_{x_j}p_k - B^i p_k + (\partial_{x_j}A^{ij})p_k\Big) = 0 \tag{1.19}$$

in the weak sense. The fact that these solutions are nonnegative follows by the weak maximum principle [15, Theorem 7]. Our case corresponds to $\gamma = b^i = b = 0$ in the theorem cited (see also [10, Exercise 8.1]). It is worth mentioning that the existence of the functions p_k with the aforementioned properties can be derived from the solvability of the boundary problem for smooth coefficients. Indeed, there exists a sequence of smooth mappings $A_m = (A^{ij}_m)^n_{i,j=1}$ on U_{k+1} with values in the space of the symmetric linear operators on $\mathbb{R}^n$ such that the A_m's are uniformly nondegenerate and converge to A in the norm of the Sobolev class $H^{\alpha_{k+1},1}(U_{k+1})$. In addition, there exists a sequence of smooth mappings $B_m = (B^i_m)^n_{i=1}$ convergent to B in $L^{\alpha_{k+1}}(U_{k+1}, \mathbb{R}^n)$. Let us denote by f_m the solutions of the boundary value problems

$$\partial_{x_i}\Big(A^{ij}_m(x)\partial_{x_j}f_m - B^i f_m + \partial_{x_j}A^{ij}_m f_m\Big) = 0$$

with the boundary conditions $f_m = 1$ on the boundary of U_{k+1}. Such solutions exist according to [13, Theorem 8.3 and the remark at the end of §8.2]. By the Harnack inequality (see (1.7)), the sequence of solutions f_m obtained in this way is uniformly bounded. In addition, the quantities on the right-hand side in (1.1), corresponding to the coefficients A_m and B_m, are uniformly bounded as well. Hence, by (1.2), the sequence $\{f_m\}$ is bounded in the space $H^{\alpha_{k+1},1}(U_k)$. Choosing a subsequence from $\{f_m\}$ that is uniformly convergent on U_k, we conclude that its limit can be taken for p_k, since it satisfies equation (1.19) in U_k in the weak sense. It follows from our reasoning that $p_k \in H^{\alpha_{k+1},1}(U_k)$. Finally, note that $\inf\limits_{U_k} p_k > 0$, since $\inf\limits_m \inf\limits_{U_k} f_m > 0$ by (1.7).

Multiplying the solutions p_k by positive constants and letting $p_k = 0$ outside U_k, we get probability measures μ_k satisfying the elliptic equation $L^*_{A,B}\mu_k = 0$ in U_k. By Lemma 1.2 the functions p_k, $k \geq m$, are uniformly bounded in $H^{\alpha_m,1}(U_{m-1})$ and uniformly Hölder continuous. In addition, the sequence $\{\mu_k\}$ contains a subsequence $\{\mu_{k_i}\}$ convergent in the variation norm to a probability measure μ with a strictly positive continuous density p such that $p \in H^{\alpha_R,1}(U_{R-1})$ for every $R > 1$. Therefore, for any $\varphi \in C_0^\infty(\mathbb{R}^n)$, we get by Lebesgue's dominated convergence theorem

$$0 = \lim_{k\to\infty} \int\limits_{\mathbb{R}^n} L_{A,B}\varphi(x)p_k(x)\,dx = \int\limits_{\mathbb{R}^n} L_{A,B}\varphi(x)p(x)\,dx.$$

□

Corollary 1.7 *The assertion of the previous theorem is true if A is a locally Lipschitzian uniformly bounded mapping with values in nondegenerate nonnegative symmetric operators on $\mathbb{R}^n$ and B is a Borel mapping such that $|B| \in L^\alpha_{loc}(\mathbb{R}^n)$, where $\alpha > n$, and $\lim\limits_{|x|\to\infty} (B(x), x) = -\infty$.*

Proof. It suffices to take $V(x) = (x, x)$. □

For example, the previous corollary applies to the situation where $A = I$ and $B(x) = -k(x)x$, where $k \in L^\alpha_{loc}(\mathbb{R}^n)$ is such that

$$k(x)|x|^2 \to +\infty \quad \text{as} \quad |x| \to \infty.$$

More generally, if $A = I$, then it suffices to have the weaker estimate

$$\limsup_{|x|\to\infty} (x, x)^{\gamma-1}\Big[2(\gamma-1) + n + (B(x), x)\Big] = -\infty$$

for some $\gamma \geq 1$ (then the function $V(x) = (x, x)^\gamma$ can be used).

Another example covered by Theorem 1.6 is this: $A = I$ and for some $\gamma \geq 1$ one has $(B(x), x) \leq -r < -n+2-2\gamma$ outside some ball (of course, we assume that $|B|$ is locally integrable in power bigger than n). Then we take the function $V(x) = (x, x)^\gamma$. The same coercivity condition was assumed in [5] for locally bounded B (in order to get the equation from [5] one should take $A = I/2$ in our case). Note that, as shown in [5], to this case one can reduce the more general case where the diffusion term is continuous, nondegenerate, and satisfies certain uniform bounds, provided that the drift is locally bounded and coercive and that the corresponding stochastic differential equation has a unique weak solution.

Corollary 1.8 *Suppose that* $A = (A^{ij})$ *is a continuous mapping on* $\mathbb{R}^n$ *with values in the nondegenerate nonnegative symmetric linear operators on* $\mathbb{R}^n$ *and let* B *be a continuous mapping on* $\mathbb{R}^n$. *Assume also that there exists a function* $V \in C^2(\mathbb{R}^n)$ *such that* (1.18) *is fulfilled. Then there exists a probability measure* μ *such that*

$$L^*_{A,B}\mu = 0$$

in the sense of (0.3).

Proof. It is readily seen that there exists a sequence of smooth mappings A_k with values in the nonnegative symmetric operators on $\mathbb{R}^n$ such that condition (1.6) is satisfied with our function V and the mappings A_k converge to A uniformly on compact sets. Let μ_k be the probability measure solving the equation $L^*_{A_k,B}\mu_k = 0$. According to Remark 1.5 the sequence $\{\mu_k\}$ is uniformly tight. Hence it contains a subsequence that converges weakly to some probability measure μ on $\mathbb{R}^n$. For any $\varphi \in C_0^\infty(\mathbb{R}^n)$, by the continuity of B and the uniform convergence of A_k to A on the support of φ, we get that $L_{A_k,B}\varphi \to L_{A,B}\varphi$ uniformly on the support of φ, whence we conclude that $L^*_{A,B}\mu = 0$. Note that the continuity of B was only used to show that any weak cluster point of $\{\mu_k\}$ is a solution of the elliptic equation in the weak sense. $\square$

Remark 1.9 It is obvious that the Borel measurability of B can be replaced by its Lebesgue measurability and that our solution does not depend on a Lebesgue equivalent version of B.

Remark 1.10 (i) Let μ be a probability measure as in Theorem 1.6. By [16, Theorem 2.1 and Proposition 2.4] (which apply with $\chi_n := \frac{1}{n}V$, $n \in \mathbb{N}$, in the condition stated there, since the coefficients A^{ij} are locally Hölder continuous by the condition $\alpha_R > n$ and the Sobolev embedding theorem), the closure of $L_{A,B}$ on $L^1(\mu)$ generates a strongly continuous semigroup $(T_t^\mu)_{t\geq 0}$ on $L^1(\mu)$ such that μ is invariant for $(T_t^\mu)_{t\geq 0}$ in the usual sense, i.e.,

$$\int_{\mathbb{R}^n} T_t f \, d\mu = \int_{\mathbb{R}^n} f \, d\mu, \quad \forall\, f \in L^1(\mu).$$

In addition, there exists a Markov process associated (in a certain sense) with this semigroup (see [16]). However, in the case of a locally unbounded drift, such a process may be defined not for every starting point (see [17] about connections between Markovian semigroups and diffusions). Thus, in our case the existence of an invariant probability yields the existence of a corresponding diffusion. This is in contrast to [3], [4] – [6], where, in the case of a nondegenerate continuous diffusion term and a locally bounded drift, the existence and uniqueness of the process in the classical sense was assumed in advance. Note that in [2], [4] – [6] the Harnack inequality was employed for parabolic equations. It is worth noting that according to [16], even without the condition of the existence of a Lyapunov function, for any probability measure μ, satisfying the equation $L^*_{A,B}\mu = 0$ in the weak sense, $L_{A,B}$ has an extension that generates a strongly continuous semigroup $(T_t)_{t\geq 0}$ on $L^1(\mu)$ with generator G equal to $L_{A,B}$ on $C_0^\infty(\mathbb{R}^n)$ such that μ is a *subinvariant measure* for this semigroup, i.e.,

$$\int\limits_{\mathbb{R}^n} T_t f(x)\,\mu(dx) \leq \int\limits_{\mathbb{R}^n} f(x)\,\mu(dx)$$

for all nonnegative bounded measurable functions f (cf. [16, Theorem 1.5a]). However, in general, as shown in [16, Example 1.12], the measure μ may fail to be invariant for $(T_t)_{t\geq 0}$. Thus, our weak solutions of (0.1) turn out to be a more general object than the invariant measures in the classical sense, but this distinction does not appear under our condition of the existence of a Lyapunov function.

(ii) If A is infinitely differentiable, then in the situation of Theorem 1.6 there is only one probability μ such that $L^*_{A,B}\mu = 0$. Indeed, let μ be such a probability. As noted in (i), the closure of $L_{A,B}$ on $L^1(\mu)$ generates a strongly continuous semigroup $(T_t^\mu)_{t\geq 0}$ on $L^1(\mu)$. Hence there is only one such μ, which is proved by a reasoning similar to the one given in the remarks following [8, Theorem 1.2] (see also [16, Proposition 2.8]). In this case by the same arguments that prove [8, Theorem 1.3(iv)] one can show that μ is the unique invariant probability of the aforementioned semigroup $(T_t^\mu)_{t\geq 0}$. It is very likely that the uniqueness statement is still true under the more general conditions mentioned in Theorem 1.6.

If we write the stochastic equation in the form

$$d\xi_t = A(\xi_t)\,dw_t + B(\xi_t)\,dt$$

with a mapping A taking values in the positive definite operators, then the corresponding elliptic operator has the form

$$Lf(x) = \frac{1}{2}\mathrm{trace}\,\Big(A^2(x)f''(x)\Big) + (\nabla f, B).$$

Due to our differentiability conditions on A, this operator can be written in Hörmander form
$$L = \frac{1}{2}\sum_{j=1}^{n} X_j^2 + X_0, \tag{1.20}$$
where $X_0,\ X_1,\dots,X_n$ are vector fields, $|X_0|$ is locally integrable in power bigger than n, and X_j, $j \geq 1$, belong to the Sobolev class $H^{\alpha,1}_{loc}$ with $\alpha > n$. However, our operator (0.2) can be also written in Hörmander form (1.20) with the same type of conditions on X_j. To this end, it suffices to show that if $A = (A^{ij})$ is a mapping with values in the symmetric positive definite operators such that it is locally uniformly nondegenerate and $A^{ij} \in H^{\alpha,1}_{loc}(\mathbb{R}^n)$, where $\alpha > n$, then $\sqrt{A} = (C^{ij})$ is a mapping of the same type. Indeed, $\sqrt{A}$ is locally uniformly nondegenerate. Let us verify that $C^{ij} \in H^{\alpha,1}_{loc}(\mathbb{R}^n)$. By the Sobolev embedding theorem, the mapping A has a continuous version with which we work further on. Since the property to be shown is local, we may assume (dividing by a positive constant) that the norm of A belongs to an interval $[c,1]$, where $c > 0$. Recall that $\sqrt{A}$ can be obtained as the sum of the series $\sum\limits_{n=0}^{\infty} c_n(I-A)^n$ convergent absolutely in norm, where the coefficients c_n are given by the absolutely convergent expansion $\sqrt{1-z} = \sum\limits_{n=0}^{\infty} c_n z^n$, $z \in [0,1]$, i.e., $c_n = (n!)^{-1}\sqrt{1-z}^{(n)}|_{z=0}$. Note that if T is a matrix-valued mapping with the entries in $H^{\alpha,1}(\mathbb{R}^n)$, then $\partial_{x_i}T^n$ is the sum of the products $T^k(\partial_{x_i}T)T^{n-k-1}$, $k = 0,1,\dots,n-1$. Hence, $\|\partial_{x_i}T^n\| \leq n\|T\|^{n-1}\| \,\|\partial_{x_i}T\|$. Taking $T = I - A$ and using the condition $\|I-A\| \leq 1-c$, we conclude that the series $\sum\limits_{n=0}^{\infty} \|c_n\partial_{x_i}(I-A)^n\|$ converges in $L^\alpha(U_R)$, whence the claim.

2 Invariant measures on manifolds

The reader might have already noticed that our previous results extend with the same proofs to the case of a finite dimensional smooth manifold M. In this case, the equation $L^*\mu = 0$ is understood again in the weak sense
$$\int L\varphi\, d\mu = 0, \quad \forall\, \varphi \in C_0^\infty(M), \tag{2.1}$$
provided μ is a locally finite Borel measure on M and L is an elliptic differential operator such that $L\varphi \in L^1(\mu)$ for every $\varphi \in C_0^\infty(M)$. In a similar way one defines the equation $L^*\mu = 0$ for a measure μ on an open set $\Omega \subset M$. Concerning diffusions on smooth manifolds generated by second order elliptic operators, we refer to [18], [19], [20], [21], [22].

Let us give extensions of the results obtained in the previous section to the case of a manifold. We shall start with the corresponding analogue of Lemma 1.2 combined with its corollary. Throughout this section M is a connected n-dimensional smooth manifold without boundary.

Lemma 2.1 *Let $\{\mu_k\}$ be a sequence of probability measures on M such that*

$$L_k^*\mu_k = 0,$$

in the sense of (2.1), *where the operators L_k have in local coordinates the same form as L_{A_k,B_k} in Lemma* 1.2 *and satisfy the uniform local estimates* (1.3) *and* (1.4) *for every set U_R that has a compact closure in a local chart. Assume that there exist a sequence of open sets $U_j \subset M$ with compact closures $\overline{U_j} \subset U_{j+1}$ and $M = \bigcup_{j=1}^{\infty} U_j$ and a nonnegative function $V \in C^2(M)$ such that $\lim\limits_{j\to\infty} \inf\limits_{x\notin U_j} V(x) = +\infty$ and*

$$\lim_{j\to\infty} \sup_k \sup_{x\notin U_j} L_k V(x) = -\infty. \tag{2.2}$$

Then:

(i) *The measures μ_k admit continuous densities p_k with respect to a fixed Riemannian volume on M such that, for every sufficiently small coordinate neighborhood W, the functions $p_k|_W$ are uniformly bounded, uniformly Hölder continuous and uniformly bounded in the Sobolev space $H^{\alpha_R,1}(W)$.*

(ii) *The sequence $\{\mu_k\}$ is relatively compact in the variation norm.*

(iii) $\inf\limits_k \inf\limits_{x\in V} p_k(x) > 0$ *for every sufficiently small coordinate neighborhood W.*

Furthermore, assertions (i) – (iii) *remain valid if, for every k, the measure μ_k vanishes outside U_k and satisfies the equation $L_k^*\mu_k = 0$ only in U_k.*

Proof. All these assertions are proved by the same reasoning as above in the case of $\mathbb{R}^n$; the only difference is that instead of the balls U_k we consider open sets U_k with compact closures. Relationship (1.10) makes sense and is valid in local coordinates, which is enough for the subsequent estimates that involve only L_kV, $L_k(\psi_j \circ V)$, and V, hence are written without reference to local coordinates. It should be emphasized that if we write the operators L_k in Hörmander form (1.20), then relationship (1.10) takes the following coordinate-free form:

$$L_k(\psi_j \circ V) = \psi_j'(V)L_kV + \psi_j''(V)\sum_{i=1}^{n}(\partial_{X_i}V)^2.$$

Clearly, the right-hand side is majorized by $\psi_j'(V)L_kV$. Now the reasoning that follows (1.10) can be applied to the foregoing relationship. □

Remark 2.2 Note that if M is equipped with a Riemannian metric and λ is the corresponding Riemannian volume, then $\mu_k = \varrho_k \cdot \lambda$, where the functions ϱ_k are uniformly Hölder continuous on every region U with compact closure in M and, in addition, the sequence $\{\varrho_k\}$ is bounded in the Sobolev space $H^{\alpha,1}(U)$ for some $\alpha = \alpha(U) > n$. In other words, one has the same picture as in the flat case.

Theorem 2.3 *Let M be an n-dimensional connected smooth manifold without boundary and let L be a second order elliptic operator on M which in local coordinates has the form specified in Theorem 1.6 including the analogous local integrability conditions, i.e., every x has a neighborhood U such that in local coordinates one has (1.3) and (1.4), where U_R, α_R, β_R, and γ_R are replaced by U, $\alpha_U > n$, $\beta_U > 0$, and $\gamma_U > 0$, respectively. Assume that there exist a sequence of open sets $U_j \subset M$ with compact closures $\overline{U_j} \subset U_{j+1}$ and $M = \bigcup_{j=1}^{\infty} U_j$ and a nonnegative function $V \in C^2(M)$ such that*

$$\lim_{j\to\infty} \inf_{x\notin U_j} V(x) = +\infty \quad \textit{and} \quad \lim_{j\to\infty} \sup_{x\notin U_j} LV(x) = -\infty. \tag{2.3}$$

Then there exists a Borel probability measure μ on M such that

$$L^*\mu = 0.$$

In addition, in every sufficiently small neighborhood W in local coordinates, μ has a continuous strictly positive density of the class $H^{\alpha_W,1}(W)$ with $\alpha_W > n$.

Proof. The proof is the same as in Theorem 1.6 due to the fact that the local regularity results from [7], [10], and [15], on which our considerations rely, extend automatically to the manifold case. In order to apply the existence result for the boundary value problems in relatively compact domains in M, we choose open sets $D_j \subset M$ with smooth boundaries such that $\overline{U_j} \subset D_j \subset \overline{D_j} \subset D_{j+1} \subset U_{j+1}$ (this is necessary in order to find solutions with the boundary condition 1 on D_j). Clearly, this is possible, since U_j can be covered by finitely many closed regions $W_i \subset U_{j+1}$, $i = 1, \ldots, d$, that are diffeomorphic to closed balls in $\mathbb{R}^n$. By using a smooth partition of unity and standard smoothing (see, e.g., [23] and [21]), one can enclose $\bigcup_{i=1}^{d} W_i$ in a region that has a smooth boundary and a compact closure in U_{j+1}.

The existence of probability measures μ_k satisfying the equation $L^*\mu_k = 0$ in D_k is proved similarly to the flat case. Namely, for fixed k, we approximate L by elliptic operators L_m with smooth coefficients in such a way that in finitely many charts covering $\overline{D_{k+1}}$, the second order coefficients of the operators L_m are nondegenerate and converge to the second order coefficients of L in the corresponding $H^{\alpha,1}$-norms and the first order coefficients of the L_m's converge to those of L in the L^α-norm, where $\alpha > n$. We may assume that M is equipped with a Riemannian metric. Let λ be the corresponding Riemannian volume. By $\widehat{L_m}$ we denote the formal adjoint of L_m in $L^2(\lambda)$. Then $\widehat{L_m}$ is a second order elliptic operator such that

$$\int_M \varphi L_m \psi \, d\lambda = \int_M \psi \widehat{L_m} \varphi \, d\lambda$$

for all smooth functions φ and ψ with compact supports. In addition, the second order parts of $\widehat{L_m}$ are nondegenerate in local coordinates. The boundary problem on D_{k+1}

$$\widehat{L_m} f = 0, \qquad f|_{\partial D_{k+1}} = 1, \tag{2.4}$$

has a smooth solution f_m (note that ∂D_{k+1} is nonempty, since M is connected). Indeed, as shown in [24, Section 3.1.1] (in particular, see [24, 3.1.1, Remark 8]), the operator L_m between suitably chosen Sobolev spaces is Fredholm with index zero. The boundary problem $\widehat{L_m} f = 0$, $f|_{\partial D_{k+1}} = 0$, has only trivial solution. This follows from the weak maximum principle, which is valid by the same reasoning as in the flat case in [15] (note that in local coordinates the operator $\widehat{L_m}$ has form (1.19) due to the smoothness and positivity of the densities of the measure λ with respect to the corresponding Lebesgue measures). Therefore, for every smooth function g on M, the boundary problem $\widehat{L_m} f = g$, $f|_{\partial D_{k+1}} = 0$, is solvable, whence the solvability of problem (2.4) follows. Applying estimate (1.7), which extend to the manifold case, we conclude as in the flat case that $\inf\limits_m \inf\limits_{U_k} f_m > 0$. By analogy with the case of $\mathbb{R}^n$, we get p_k as the uniform limit on D_k of a subsequence of $\{f_m\}$. Since f_m is smooth, we get for every $\varphi \in C_0^\infty(D_k)$ the equality

$$\int\limits_M L_m \varphi \, f_m \, d\lambda = \int\limits_M \varphi \, \widehat{L_m} f_m \, d\lambda,$$

i.e., the measure $\nu_m := f_m \cdot \lambda$ on D_k satisfies the elliptic equation $L_m^* \nu_m = 0$ in the weak sense. Then we put $p_k = 0$ outside D_k and take the probability measures $\mu_k = c_k p_k \cdot \lambda$ with suitably chosen constants c_k. $\square$

Remark 2.4 If the coefficients of L are continuous, then the manifold case analogue of Corollary 1.8 is valid.

Note that if M is not connected, then the theorem above can be applied to its connected components.

Clearly, the condition in Theorem 2.3 implies that M is not compact (otherwise V would be bounded). For compact manifolds, the situation is simpler.

Theorem 2.5 *Let M be a compact n-dimensional smooth manifold without boundary and let L be a second order elliptic operator on M which in local coordinates has a second order part $A = (A^{ij})$ and a first order part $B = (B^i)$ such that A is locally uniformly nondegenerate, $A^{ij} \in H^{\alpha,1}_{loc}$ and $B^i \in L^\alpha_{loc}$, where $\alpha > n$. Then there exists a Borel probability measure μ on M such that $L^*\mu = 0$. In addition, in local coordinates, μ has a density of the class $H^{\alpha,1}$ which is continuous and strictly positive.*

Proof. It suffices to approximate L by elliptic operators L_k in such a way that the corresponding operators $A_k = (A_k^{ij})$ are uniformly nondegenerate, $A_k^{ij} \to A^{ij}$ in the

Sobolev norm and $B^i_k \to B^i$ in L^α (note that by the compactness of M, we deal actually with finitely many charts). For every k, there exists a probability measure μ_k on M with a smooth density p_k in local coordinates such that $L_k^*\mu_k = 0$ (see [20, Proposition V.4.5]). Then, by the same reasoning as in the flat case, the sequence $\{p_k\}$ is bounded $H^{\alpha,1}(M)$. Hence, it contains a subsequence convergent weakly in $H^{\alpha,1}(M)$ to some probability density $p \in H^{\alpha,1}(M)$. Clearly, $L^*\mu = 0$. □

Note that if M is a Riemannian manifold of dimension n and L has representation (1.20) with some vector fields $X_0, \ldots, X_n$, then the local conditions in Theorem 2.3 can be written without reference to local coordinates. Namely, we need that X_0 be in $L^\alpha_{loc}(\lambda, TM)$ and X_j, $j = 1, \ldots, n$, be in the Sobolev class $H^{\alpha,1}_{loc}(M, TM)$, where $\alpha > n$, λ is the Riemannian volume, and TM is the tangent bundle.

Remark 2.6 As noted by W. Stannat, similarly to Remark 1.10, if μ is a probability measure constructed in Theorem 2.3, then, extending the results proved in [16] to the manifold case, one gets that the closure of L on $L^1(\mu)$ generates a strongly continuous semigroup $(T^\mu_t)_{t\geq 0}$ on $L^1(\mu)$ such that μ is invariant for $(T^\mu_t)_{t\geq 0}$ in the usual sense and, in addition, there exists a Markov process associated with this semigroup.

In certain cases, when dealing with a Riemannian manifold, one can take for a Lyapunov function V the function $r(x) = d(x,a)^k$, where a is a fixed point and d is the distance in M (see, e.g., [19, Ch. IX, §6]). However, in general the sets $\{r \leq c\}$ may not be compact for such a function. In addition, such r need not be differentiable everywhere. If M has Ricci curvature bounded below, the latter difficulty is overcome by considering r ouside the set of its singularities (see [19, Ch. IX, §6], [22], [25], where similar ideas were used in the investigation of non-explosion and reccurence of diffusions on manifolds). In particular, a sufficient condition for the existence of a solution to our equation is the condition $\limsup_{r\to\infty} r^p Lr < 0$ for some $p > 0$ (then the function $V = r^{2p+2}$ can be used). If $L = \Delta + X_0$, where Δ is the Laplace–Beltrami operator on M, then as a certain analogue of Corollary 1.7 the following condition can be used: $\lim_{r\to\infty} \partial_{X_0} r = -\infty$. Indeed, since M has Ricci curvature bounded below, Δr is bounded above (in the sense of distributions) by a result of Yau. These questions will be investigated in a forthcoming joint work with F.Y. Wang. Infinite dimensional generalizations will be treated in a separate paper.

Remark 2.7 For other existence results for invariant measures (also in infinite dimensions), see, e.g., [26], [27], [28], [29]. We also refer to [30] and the references therein.

We thank W. Stannat and A. Yu. Veretennikov for useful discussions. This work was partially supported by the Russian Fundamental Research Foundation grants

96–15–96865 and 97–01–00932, grant DFG 436 RUS 113/343/0(R), grant INTAS 94–0378, grant INTAS–RFBR 95-0099, the SFB–343, and EU–TMR–Project ERB–FMRX–CT96–0075.

References

[1] R. Z. Hasminskii, Ergodic properties of reccurent diffusion processes and stabilization of the solution of the Cauchy problem for parabolic equations, Theor. Probab. Appl. 5 (1960), 179–196.

[2] R. Z. Hasminskii, Stochastic stability of differential equations, Sijthoff and Noordhoff, 1980.

[3] A. V. Skorohod, Asymptotic methods in the theory of stochastic differential equations, Naukova Dumka, Kiev, 1987 (in Russian); English transl.: 1989.

[4] A. Yu. Veretennikov, Bounds for the mixing rate in the theory of stochastic equations, Theory Probab. Appl. 32 (1987), n 2, 273–281.

[5] A. Yu. Veretennikov, On polynomial mixing bounds for stochastic differential equations, Stochastic Processes Appl. 70 (1997), 115–127.

[6] A. Yu. Veretennikov, On polynomial mixing and convergence rate for stochastic difference and differential equations, Preprint N 393 Weierstrass Institute Angew. Analysis und Stoch. (1998).

[7] V. I. Bogachev, N. V. Krylov, M. Röckner, Elliptic regularity and essential self-adjointness of Dirichlet operators, Preprint 96–038 SFB 343 (1996); to appear in Annali Scuola Norm. Super. di Pisa.

[8] S. Albeverio, V. Bogachev, M. Röckner, On uniqueness of invariant measures for finite and infinite dimensional diffusions, Preprint 97–057 SFB 343, Bielefeld Univ. (1997); to appear in Comm. Pure Appl. Math.

[9] O. A. Ladyz'enskaya, N. M. Ural'tseva, "Linear and quasilinear elliptic equations", New York, Academic Press, 1968.

[10] N. S. Trudinger, Linear elliptic operators with measurable coefficients, Ann. Scuola Normale Super. Pisa (3) 27 (1973), 265–308.

[11] G. Stampacchia, Équations elliptiques du second ordre à coefficients discontinus, Les Presses de l'Université de Montréal, 1966.

[12] R. M. Hervé, M. Hervé, Les fonctions surharmoniques associées à un opérateur elliptique du second ordre à coefficients discontinus, Ann. Inst. Fourier (Grenoble) 19 (1968), 305–359.

[13] D. Gilbarg, N. S. Trudinger, Elliptic partial differential equations of second order, Springer, 1977.

[14] P. Billingsley, Convergence of probability measures, New York, Wiley, 1968.

[15] N. S. Trudinger, Maximum principles for linear, non-uniformly elliptic operators with measurable coefficients, Math. Z. 156 (1977), 291–301.

[16] W. Stannat, Nonsymmetric Dirichlet operators on L^1: existence, uniqueness and associated Markov processes, Preprint N 97–090 SFB 343, Bielefeld Univ. (1997).

[17] Z.-M. Ma, M. Röckner, Introduction to the theory of (non-symmetric) Dirichlet forms, Springer, 1992.

[18] R. Azencott, Behavior of diffusion semi-groups at infinity, Bull. Soc. math. France 102 (1974), 193–240.

[19] D. Elworthy, Stochastic differential equations on manifolds, Cambridge Univ. Press, 1982.

[20] N. Ikeda, S. Watanabe, Stochastic differential equations and diffusion processes, North–Holland, 1981.

[21] S. A. Molchanov, Strong Feller property of diffusion processes on smooth manifolds, Theory Probab. Appl. 13 (1968), 471–475.

[22] N. T. Varopoulos, Potential theory and diffusion on Riemannian manifolds, Conference on harmonic analysis in honor of Antoni Zygmund (Chicago, 1981), V. II, pp. 821–837, Wadsworth, 1983.

[23] M. W. Hirsch, Differential topology, Springer–Verlag, New York – Berlin, 1976.

[24] S. Rempel, B.-W. Schulze, Index theory of elliptic boundary problems, Akademie–Verlag, Berlin, 1982.

[25] S.-T. Yau, On the heat kernel of a complete Riemannian manifold, J. Math. Pures Appl. 57 (1978), 191–201.

[26] V. I. Bogachev, M. Röckner, Regularity of invariant measures on finite and infinite dimensional spaces and applications, J. Funct. Anal. 133 (1995), 168–223.

[27] V. I. Bogachev, M. Röckner, T. S. Zhang, Existence of invariant measures for diffusions with singular drifts, Preprint 97–072 SFB 343, Bielefeld Univ. (1997). To appear in: Appl. Math. Optim.

[28] G. Leha, G. Ritter, A. Wakolbinger, An improved Lyapunov–function approach to the behavior of diffusion processes in Hilbert spaces, Stoch. Anal. Appl. 15 (1997), n 1, 59–89.

[29] I. Shigekawa, Existence of invariant measures of diffusions on an abstract Wiener space, Osaka J. Math. 24 (1987), n 1, 37–59.

[30] G. Da Prato, J. Zabczyk, Ergodicity for infinite dimensional systems, Cambridge, Cambridge University Press, 1996.