Chromatic-Index Critical Multigraphs of Order 20

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Abstract

A multigraph M with maximum degree $\Delta(M)$ is called critical, if the chromatic index $\chi'(M) > \Delta(M)$ and $\chi'(M-e) = \chi'(M) - 1$ for each edge e of M. The weak critical graph conjecture [1, 7] claims that there exists a constant $c > 0$ such that every critical multigraph M with at most $c \cdot \Delta(M)$ vertices has odd order. We disprove this conjecture by constructing critical multigraphs of order 20 with maximum degree k for all $k \geq 5$.

1 Introduction

We consider connected multigraphs without loops. Throughout the paper we denote the edge set and the vertex set of a multigraph M by $E(M)$ and $V(M)$, the degree of a vertex v of M by $d_M(v)$, and the maximum degree of M by $\Delta(M)$. If $m \geq 1$ edges join the same pair v, w of vertices of M, then vw is called a *multiple edge* of *multiplicity* m or an *m-edge*. A *proper multiple edge* is a multiple edge of multiplicity at least 2 and a *graph* is a multigraph without proper multiple edges. An *r-colouring* of M is a function $\pi : E(M) \to \{1, ..., r\}$ so that $\pi(e) \neq \pi(f)$ for adjacent edges e and f. The *chromatic index* $\chi'(M)$ of a multigraph M is the smallest number r so that there is an r-colouring of M. A multigraph M is called critical, if $\chi'(M) > \Delta(M)$ and $\chi'(M-e) = \chi'(M) - 1$ for each edge e of M. We define M to be *k-critical*, if it is critical and $k = \Delta(M) = \chi'(M) - 1$. The Petersen graph will be denoted by P, and the graph obtained from P by deleting a vertex is called $P - v$.

Jakobsen [6] formulated the famous critical graph conjecture, which claims that every critical multigraph M has odd order. This conjecture was disproved by Goldberg [3] for $\Delta(M) = 3$, and Chetwynd and Fiol independently disproved it for $\Delta(M) = 4$ (see [1]). Recently, the author and Steffen [4] constructed k-critical graphs and multigraphs of even order for all $k \geq 4$. After the critical graph conjecture had been disproved, a new conjecture appeared in a paper by Chetwynd and Wilson [1].

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Conjecture 1.1 (Weak Critical Graph Conjecture) *There exists a constant $c > 0$ such that every critical multigraph M with at most $c \cdot \Delta(M)$ vertices has odd order.*

According to Chetwynd and Wilson [1], Yap [8], and Jensen and Toft [7], this conjecture was formulated by Goldberg [2], but in fact it does not appear in Goldberg's paper.

We disprove Conjecture 1.1 by proving the following theorem.

Theorem 1.2 *For all $k \geq 5$, there exists a k-critical multigraph of order 20.*

2 Basic Theorems

In this section we present several tools for constructing k-critical multigraphs.

Let M be a multigraph and S a set of edges of M. We say that N is obtained from M by *adding* S if the multiplicity of each edge of S increases by 1.

Lemma 2.1 ([4]) *Let $k \geq 0$ and M be a multigraph obtained from P by adding k 1-factors of P. Then $M - v$ is a $(k+3)$-critical multigraph for any $v \in V(M)$.*

A multigraph N is said to be obtained from M and M' by *Hajós-union* where v and v' are identified, if $v \in V(M)$, $v' \in V(M')$, and N is constructed from M and M' as follows:

1. One edge $uv \in E(M)$ and one edge $u'v' \in E(M')$ are removed.
2. The vertices v and v' are identified.
3. A new edge uu' is inserted.

Lemma 2.2 ([5]) *Let M and M' be two k-critical multigraphs and $v \in V(M)$, $v' \in V(M')$ two vertices such that $d_M(v) + d_{M'}(v') \leq k+2$. Then any multigraph obtained from M and M' by Hajós-union where v and v' are identified is k-critical.*

Let M be a multigraph on vertices $v_1, ..., v_n$ and with maximum vertex degree k. We call $s(M) = kn - \sum_{i=1}^{n} d_M(v_i)$ the *deficiency* of M.

Lemma 2.3 ([4]) *Let M be a k-critical multigraph, $k \geq 2$, and N an induced proper submultigraph of odd order. If $s(N) = k$ then the multigraph M^- obtained from M by contracting N to a single vertex is k-critical.*

Let M be a multigraph. We say that M^+ is obtained from M by *planting* an m-edge on the edge $uv \in E(M)$ if M^+ is obtained from M by removing uv and inserting vertices u^+, v^+ and the edges uu^+, vv^+ and an m-edge u^+v^+.

Lemma 2.4 *Let M be a k-critical multigraph, $k \geq 2$, and let M^+ be a multigraph obtained from M by planting a $(k-1)$-edge on an edge $uv \in E(M)$. Then M^+ is a k-critical multigraph.*

Proof. Let T be the multigraph with three vertices t, u^+, v^+, edges tu^+, tv^+, and a $(k-1)$-edge u^+v^+. Clearly, T is a k-critical multigraph. The multigraph M^+ is obtained from M and T by Hajós-union where t and u are identified, the edges uv and tv^+ are removed, and a new edge vv^+ is inserted. By Lemma 2.2, M^+ is a k-critical multigraph.

□

3 Construction

Let the vertices and edges of P be labeled as in Figure 1.

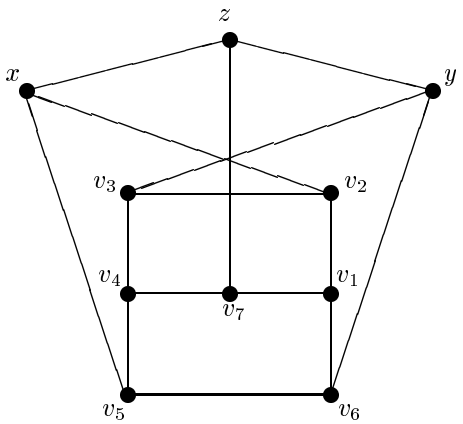

Figure 1: The Petersen Graph

Then the edge sets $F_1 = \{v_2v_3, v_1v_6, v_4v_7, v_5x, zy\}$ and $F_2 = \{v_2v_3, v_4v_5, v_1v_7, v_6y, zx\}$ are 1-factors of P. For $k \geq 5$ and $1 \leq j \leq k-2$, let P_k^j be the multigraph obtained from P by adding $j-1$ copies of F_1 and $k-j-2$ copies of F_2. By Lemma 2.1, the multigraph $H_k^j = P_k^j - z$ is k-critical. Furthermore, $d_{H_k^j}(v_7) = k-1$, $d_{H_k^j}(x) = j+1$, and $d_{H_k^j}(y) = k-j$.

Let $1 \leq l \leq k-4$. The multigraphs H_k^l, H_k^{l+1} and H_k^{l+2} are joined by Hajós-union to obtain a new k-critical multigraph. To avoid confusion we indicate the vertices of H_k^j with an upper index j. Firstly, H_k^l and H_k^{l+1} are joined by Hajós-union where the vertices y^l and x^{l+1} are identified, the edges $v_3^l y^l$ and $x^{l+1}v_2^{l+1}$ are removed, and a new edge $v_3^l v_2^{l+1}$ is inserted. Secondly, this multigraph and H_k^{l+2} are joined by Hajós-union where the vertices y^{l+1} and x^{l+2} are identified, the edges $v_3^{l+1}y^{l+1}$ and $x^{l+2}v_2^{l+2}$ are removed, and a new edge $v_3^{l+1}v_2^{l+2}$ is inserted. The obtained multigraph is called M_k^l. By Lemma 2.2, M_k^l is k-critical. The submultigraphs T^l induced by y^l,v_6^l,v_5^{l+1} and T^{l+1} induced by y^{l+1},v_6^{l+1},v_5^{l+2} have deficiency k. Contracting T^l to a single vertex t^l and T^{l+1} to a single vertex t^{l+1} yields a new multigraph N_k^l of order 21, where $d_{N_k^l}(x^l) = l+1$ and $d_{N_k^l}(y^{l+2}) = k-l-2$. Lemma 2.3 implies that N_k^l is k-critical. Let Z_k^l be the multigraph obtained from N_k^l

by identifying the vertices x^l and y^{l+2} to a single vertex v^l. Clearly, Z_k^l has order 20, maximum degree k, and it has no k-colouring.

Theorem 3.1 *For each $k \geq 5$ and $1 \leq l \leq k-4$, the multigraph Z_k^l is k-critical.*

Proof. The edge sets
$F_3 = \{v^l v_5^l, v_4^l v_7^l, v_1^l t^l, v_4^{l+1} v_7^{l+1}, v_1^{l+1} t^{l+1}, v_4^{l+2} v_7^{l+2}, v_1^{l+2} v_6^{l+2}, v_2^l v_3^l, v_2^{l+1} v_3^{l+1}, v_2^{l+2} v_3^{l+2}\}$ and
$F_4 = \{v_5^l v_4^l, v_7^l v_1^l, t^l v_4^{l+1}, v_7^{l+1} v_1^{l+1}, t^{l+1} v_4^{l+2}, v_7^{l+2} v_1^{l+2}, v_6^{l+2} v^l, v_2^l v_3^l, v_2^{l+1} v_3^{l+1}, v_2^{l+2} v_3^{l+2}\}$ are
1-factors of Z_5^1. It is easy to see that Z_k^l is isomorphic to the multigraph obtained from Z_5^1 by adding $l-1$ copies of F_3 and $k-4-l$ copies of F_4. Since every edge of Z_k^l can be considered as an edge of Z_5^1, it is sufficient to prove that Z_5^1 is a 5-critical multigraph.

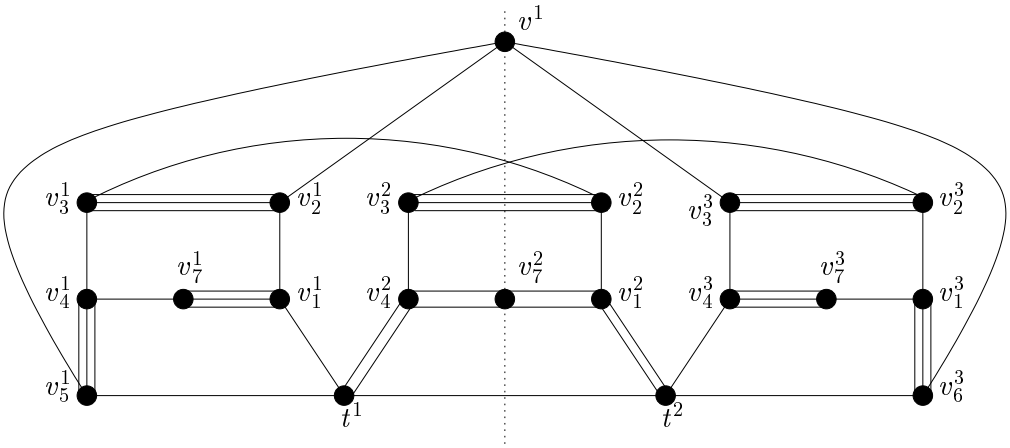

Figure 2: The multigraph Z_5^1 with a dotted drawn axis of symmetry

Let $V_1 = \{v_1^1, v_2^1, v_7^1, v^1, v_3^3, v_4^3, v_7^3\}$, $V_2 = \{v_1^2, v_7^2, t^2, v_1^3, v_2^3, v_3^3, v_4^3, v_6^3, v_7^3\}$, and
$V_3 = \{v_4^2, v_7^2, t^1, v_1^1, v_2^1, v_3^1, v_4^1, v_5^1, v_7^1\}$. Then for any edge e of Z_5^1 there exists $i \in \{1,2,3\}$ such that $e \in E(Z_5^1 - V_i)$.

Case 1: $e \in E(Z_5^1 - V_1)$.
For $l = 1$ the above defined edge sets F_3 and F_4 are 1-factors of Z_5^1 where each edge which is contained in both, F_3 and F_4, is a proper multiple edge in Z_5^1. Hence, the multigraph $Z_1 = (Z_5^1 - F_3) - F_4$ (see Figure 3) is well defined and has no 3-colouring. Two vertices of $Z_1 - V_1$ are adjacent, if and only if they are adjacent in $Z_5^1 - V_1$. Therefore, e can be considered as an edge of $Z_1 - V_1$. The multigraph $Z_1 - V_1$ is 3-critical, since it is obtained from $P - v$ by planting a 2-edge on two edges of $P - v$, respectively. Obviously, any 3-colouring of $(Z_1 - V_1) - e$ can be extended to a 3-colouring of $Z_1 - e$. By colouring F_3 and F_4 with a new colour, respectively, we obtain a 5-colouring of $Z_5^1 - e$.

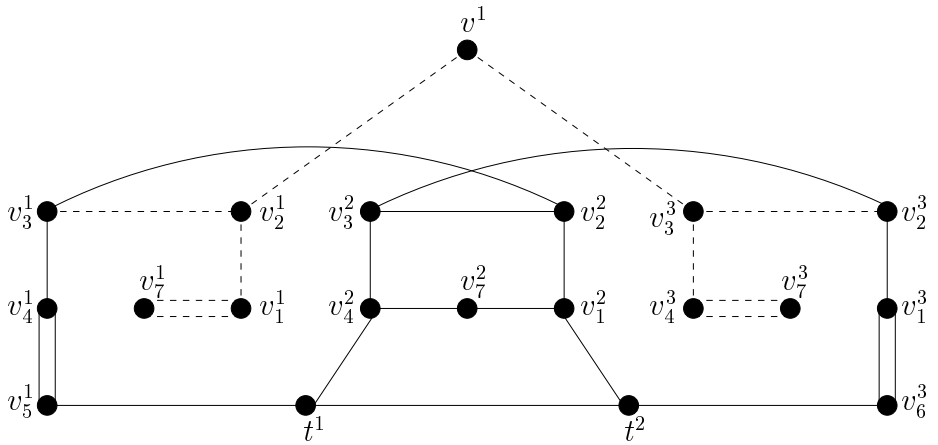

Figure 3: The multigraph Z_1 where the edges incident with a vertex of V_1 are drawn dashed

Case 2: $e \in E(Z_5^1 - V_2)$.
The edge sets $F_5 = \{v^1v_3^3, v_2^3v_3^2, v_2^2v_1^2, v_7^2v_4^2, t^1t^2, v_4^3v_7^3, v_1^3v_6^3, v_2^1v_3^1, v_4^1v_5^1, v_1^1v_7^1\}$ and
$F_6 = \{v_3^3v_2^3, v_3^2v_2^2, v_1^2v_7^2, v_4^2t^1, t^2v_4^3, v_7^3v_1^3, v_6^3v^1, v_2^1v_3^1, v_4^1v_5^1, v_1^1v_7^1\}$ are 1-factors of Z_5^1 where each edge which is contained in both, F_5 and F_6, is a proper multiple edge in Z_5^1. Hence, the multigraph $Z_2 = (Z_5^1 - F_5) - F_6$ is well defined and has no 3-colouring.

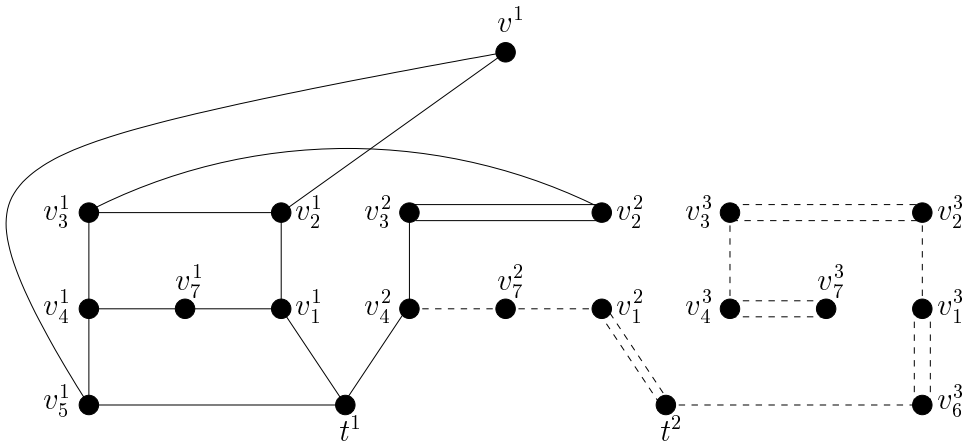

Figure 4: The multigraph Z_2 where the edges incident with a vertex of V_2 are drawn dashed

Two vertices of $Z_2 - V_2$ are adjacent, if and only if they are adjacent in $Z_5^1 - V_2$. Therefore, e can be considered as an edge of $Z_2 - V_2$. The multigraph $Z_2 - V_2$ is 3-critical, since it is obtained from $P - v$ by planting a 2-edge on one edge of $P - v$. Obviously, any 3-colouring of $(Z_2 - V_2) - e$ can be extended to a 3-colouring of $Z_2 - e$. By colouring F_5

and F_6 with a new colour, respectively, we obtain a 5-colouring of $Z_5^1 - e$.

Case 3: $e \in E(Z_5^1 - V_3)$.
By the symmetry of Z_5^1 (see Figure 2), there is an isomorphism of Z_5^1 where the image of each edge of $Z_5^1 - V_3$ is an edge of $Z_5^1 - V_2$. Hence, Case 3 can be reduced to Case 2.

Since there is a 5-colouring of $Z_5^1 - e$ for any edge $e \in E(Z_5^1)$, the multigraph Z_5^1 is 5-critical.

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