THE MILGRAM NON-OPERAD

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Introduction

In [11] R.J. Milgram introduced geometric models J^n for the iterated loop-space operads $\Omega^n\Sigma^n$. Later J.M. Boardman, R.M. Vogt and P. May proved that n-fold loop spaces are closely related to E_n operads in general, and thus to the little cube operads (see [6], [7], [9]). Hence the question arises if the operad structure of $\Omega^n\Sigma^n$ translates to the geometric model J^n.

In [1] C. Balteanu, Z. Fiedorowicz, R. Schwänzl and R.M. Vogt construct an operad $\mathcal{M}_n$, which codifies n-fold monoidal categories, a categorial analog of n-fold loop spaces. They observe that an equivalent *preoperad* is embedded in $\mathcal{M}_n$, whose free space is of the same homotopy type as J^nX . For $n = 2$ the spaces are even homeomorphic (see [1, 3.12 – 14]).

Due to the underlying polytopes, the permutohedra, the Milgram-construction J^n is of some importance for the examination of coherent homotopy-commutativity. Similar to the associahedra introduced by Stasheff in [13], Williams uses the permutohedra and the Milgram-construction in [14] to define his notion of C_n-spaces, which is used in several subsequent papers, and is occuring in papers of McGibbon and Hemmi (see for example [10] and [8]).

In fact there exists an operad structure with the permutohedra as underlying spaces (this was pointed out to me by Clemens Berger and Zig Fiedorowicz), simply by using the convex extension of the permutation operad. But since the permutohedra are contractible this is an E_∞ operad in the sense of Boardman and Vogt (i.e. the symmetric group action does not need to be free). Therefore an algebra of this operad is homotopy equivalent to an infinite loop space and hence its associated monad can not be of the same homotopy type as the Milgram construction J^n which is just an n-fold loop space.

In [4] and [5] C. Berger conjectured an E_n operad structure of the permutohedra, whose associated monad is the Milgram construction J^n.

I will show that Berger's construction does not work. The first observation is that the would-be operad bears a structure far too strong, namely that of strictly abelian monoids. This collapse of structure is then used to show that the suggested structure does not define an operad at all. In fact the proof shows that the multiplication defined by Berger does not respect the degeneration conditions.

I would like to thank Zig Fiedorowicz and especially Clemens Berger for some helpful and enlightening discussions.

1. The Permutohedra and Berger's construction

Let Λ be the category of finite sets $\mathbf{n} = \{1, \ldots, n\}$ and injective maps. Each injective map $\varphi : \mathbf{n} \to \mathbf{m}$ has an unique decomposition of the form $\varphi = \varphi^{inc} \circ \varphi^\natural$, such that $\varphi^{inc} : \mathbf{n} \to \mathbf{m}$ is increasing and $\varphi^\natural : \mathbf{n} \to \mathbf{n}$ is a permutation.

Definition 1.1. The *Permutation preoperad* $\Sigma : \Lambda^{op} \to \mathbf{Top}$ is the functor with $\Sigma(\mathbf{n}) = \Sigma_n$ and $\varphi^* := \Sigma(\varphi) : \Sigma_m \to \Sigma_n$ given by $\sigma \mapsto (\sigma \circ \varphi)^\natural$ for $\varphi \in \Lambda(\mathbf{n}, \mathbf{m})$

The multiplications of the *Permutation Operad* Σ are given by

$$\begin{aligned}\gamma^{\Sigma}_{n;i_1,\dots,i_n} : \Sigma_n \times \Sigma_{i_1} \times \cdots \times \Sigma_{i_n} &\to \Sigma_{i_1+\cdots+i_n}\\ (\sigma; \tau_1, \dots, \tau_n) &\mapsto \sigma(i_1, \dots, i_n) \circ (\tau_1 \oplus \cdots \oplus \tau_n)\end{aligned}$$

where $\sigma(i_1, \dots, i_n)$ permutes the blocks (see [5, 1.15.(a)]) and $- \oplus - : \Sigma_n \times \Sigma_m \to \Sigma_{n+m}$ is the canonical product of permutations.

The product $\Sigma_{i_1} \oplus \cdots \oplus \Sigma_{i_n} \subset \Sigma_{i_1+\cdots+i_n}$ will be denoted with $\Sigma(i_1, \dots, i_n)$.

Like Berger I will use the right action of the symmetric groups on spaces. One can switch to the left action by using the inverse permutations instead. For more details about (pre)operads in general the reader is referred to [4], [5] or [9].

Definition 1.2. For $n \geq 1$ the symmetric group Σ_n acts on $\mathbb{R}^n$ from the right by permuting the coordinates

$$(x_1, \dots, x_n)\sigma = \big(x_{\sigma(1)}, \dots x_{\sigma(n)}\big).$$

The *n-th Permutohedron* $P_n \subset \mathbb{R}^n$ is the convex hull of the orbit of $(1, 2, \dots, n) \in \mathbb{R}^n$ under this operation.

The convex hull of the orbit of $\Sigma(i_1, \dots, i_r) \subset \Sigma_n$ will be denoted with $P(i_1, \dots, i_r) \subset P_n$.

The geometric and simplicial properties of these polygons were examined in [11], [14] and [2]. Here I will give only a rough sketch of the few details I will use.

P_1 consists of only one point, P_2 is homeomorphic to the unit interval in $\mathbb{R}$ and P_2 to the hexagon in $\mathbb{R}^2$. In general P_n is a $(n-1)$ dimensional polytope.

Remark 1.3. In [11] and [12] Milgram called the n-th permutohedron $C(n-1)$ instead of P_n.

Obviously there exists a right Σ_n-action on P_n given by permuting the coordinates. The vertices are mapped to vertices, and for each $\sigma \in \Sigma_n$ the map $P_n \to P_n$ with $x \mapsto x\sigma$ is a homeomorphism. But unfortunaltely this action is not free, since the barycenter of each permutohedron is a fixed point.

An arbitrary point $x \in P_n$ will be denoted by a linear combination of permutations

$$x = \sum_{\sigma \in \Sigma_n} t_\sigma \sigma \qquad \text{with} \qquad \sum_{\sigma \in \Sigma_n} t_\sigma = 1.$$

If $x = \sum_{\sigma \in \Sigma_n} s_\sigma \sigma$ and $y = \sum_{\tau \in \Sigma_m} t_\tau \tau$ are points of P_n resp. P_m, the point $x \oplus y \in P_{n+m}$ is given by

$$x \oplus y = \sum_{\sigma \in \Sigma_n} \sum_{\tau \in \Sigma_m} s_\sigma t_\tau \sigma \oplus \tau.$$

In [11] Milgram defined maps $I_k : P_k \times P_{n-k} \to P_n$ given by $(x, y) \mapsto x \oplus y$ mapping the product of two permutohedra into certain faces of a higher dimensional permutohedron. More general the codimension $(r-1)$ faces of P_n are in one-to-one correspondence with the ordered partitions of $\{1, \dots, n\}$ of type $(i_1, \dots, i_r)$ with $i_1 + \cdots + i_r = n, i_k \geq 1$.

Remark 1.4. Each partition of type $(i_1, \dots, i_r)$ can be interpreted as a permutation of $i_1 + \cdots + i_r$ elements. If the classes are given by

$$\{j_1, \dots, j_{i_1}\}, \{j_{i_1+1}, \dots, j_{i_1+i_2}\}, \dots, \{j_{i_1+\cdots+i_{r-1}+1}, \dots, i_{j_1+\cdots+j_r}\}$$

with $j_k < j_l$ for $i_m \le k < l < i_{m+1}$, then the corresponding permutation is given by $k \mapsto j_k$ for $1 \le k \le i_1 + \cdots + i_r$.

The converse does not hold, since the same permutation can be associated to different partitions. For example the identity in Σ_3 corresponds to the partion $\{1\}, \{2, 3\}$ and to $\{1, 2\}, \{3\}$.

The vertices of the codimension $(r-1)$ face, corresponding to a partition of type $(i_1, \dots, i_r)$, with associated permutation $\sigma \in \Sigma_n$, are given by the left coset $\Sigma(i_1, \dots, i_r)\sigma$. In addition there is a homeomorphism $I_\sigma : P_{i_1} \times \cdots \times P_{i_r} \to P(i_1, \dots, i_r)\sigma \subset P_n$ with

$$(x_1, \dots, x_r) \mapsto (x_1 \oplus \cdots \oplus x_r)\sigma.$$

Using the *weak Bruhat order* on the symmetric groups, the 1-skeleton of these faces can be oriented.

Definition 1.5. The *inversion index* $inv(\sigma)$ of a permutation $\sigma \in \Sigma_n$ is the number of ordered pairs $(i, j), 1 \le i < j \le n$, whose orders are inverted by σ, i.e. $\sigma(i) > \sigma(j)$.

The *Weak Bruhat Order* of Σ_n is the partial order generated by $\sigma < \tau$, if τ is the composition of σ and a transposition of two subsequent numbers, that is $\tau = \sigma \circ (i, i+1)$, and $inv(\sigma) < inv(\tau)$.

Example 1.6. Σ_3 is given by the Poset

$$\begin{tikzcd}
 & (123) \arrow[dl] \arrow[dr] & \\
(213) \arrow[d] & & (132) \arrow[d] \\
(231) \arrow[dr] & & (312) \arrow[dl] \\
 & (321) &
\end{tikzcd}$$

Here and in the following the permutation

$$\begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}$$

will be denoted with $(\sigma(1), \sigma(2), \dots, \sigma(n))$ (the commas will be left, if unnecessary).

Applying this partial order to the permutohedra the edges will be oriented. The face corresponding to a certain partition of type $(i_1, \dots, i_r)$ and associated permutation σ has exactly one initial vertex, given by σ. It also has exactly one terminal vertex, given by the permutation that turns the classes of the partition "upside-down", i.e.

$$l \mapsto k_{l'} \quad \text{with} \quad l' = 2(i_1 + \dots i_l) + i_{l+1} - l + 1 \quad \text{if} \quad i_l < l \le i_{l+1}.$$

Definition 1.7. An interval $[\sigma_1, \sigma_2]$ in Σ_n with the weak Bruhat order is called *admissible*, if it is the vertex set of some coset $\Sigma(i_1, \dots, i_r)\sigma$. Hence σ_1 is the initial and σ_2 the terminal vertex of a face of P_n.

The *geodesic* of an arbitrary interval $[\sigma_1, \sigma_2]$ of the weak Bruhat order in Σ_n, is the average of all oriented edge-paths between σ_1 and σ_2 in P_n. The *barycenter* of $[\sigma_1, \sigma_2]$ is the barycenter of its geodesic.

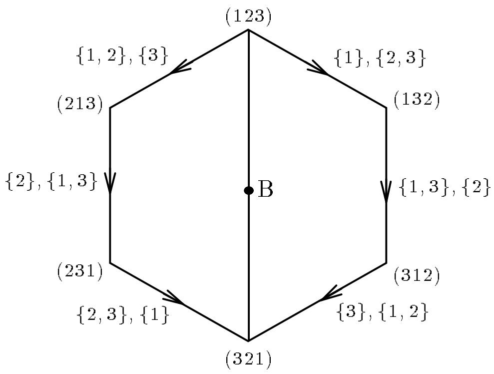

FIGURE 1. The geodesic and barycenter of P_3

The barycenter of an admissible interval coincides with the barycenter of the corresponding cell. For example the geodesic of $[(123),(213)] \subset \Sigma_3$ is the corresponding edge of P_3, and its barycenter the barycenter of the interval. The geodesic of $[(123),(321)] \subset \Sigma_3$ is the line between (123) and (321) and its barycenter is the point B. The geodesic of the non-admissible intervall $[(123),(231)] \subset \Sigma_3$ consists of the two edges $[(123),(213)]$ and $[(213),(231)]$ of P_3. Hence its barycenter is the point $(213) \in P_3$ (see Fig. 1).

To define an operad-multiplication on the permutohedra we have to define maps $P_r \times P_{i_1} \times \cdots \times P_{i_r} \to P_{i_1+\cdots+i_r}$ which satisfy the associativity conditions. But the intention to formulate an operad, whose associated monad is the Milgram construction gives certain additional restrictions to the choice of the multiplication.

The operad structure has to extend the permutation operad Σ. This can be done very easily by mapping the vertices of the product $P_r \times P_{i_1} \times \cdots \times P_{i_r}$ to the corresponding vertices of $P_{i_1+\cdots+i_r}$, given by Σ. But the extension of this map can be done in two different ways.

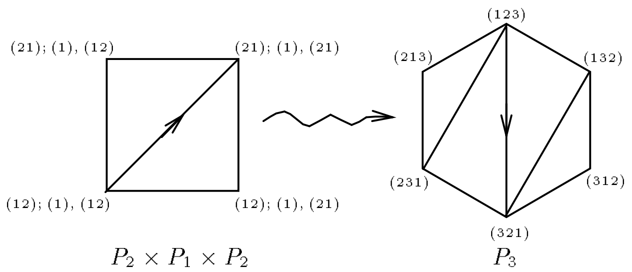

FIGURE 2. The convex multiplication

The first possibility is the *convex* extension of the permutation operad, by mapping $P_r \times P_{i_1} \times \cdots \times P_{i_r}$ to the convex hull of the image vertices (see Fig. 1). This construction does in fact define an operad whose n-th space is P_n. It is E_∞ in the sense of Boardman and Vogt, i.e. its underlying spaces are contractible but the actions of the symmetric groups does not need to be free. Thus its algebras are homotopy equivalent to infinte loop spaces (cmp. [7] section VI.3). But since the Milgram construction is only a model for n-fold loop spaces the associated monad of the convex extension is of the wrong homotopy type.

Berger tried to get the correct operad by application of two changes to the convex extension of the permutation operad. First he deformed the multiplication such that it respects the relations on the borders of the permutohedra, given in Milgrams construction. But this does not affect the homotopy type of the spaces. In a second step he made the action of the symmetric group free, which would give the correct homotopy type.

In the Milgram construction the border of $P_r \times P_{i_1} \times \cdots \times P_{i_r}$ has to be mapped to the border of $P_{i_1+\cdots+i_r}$. Berger did this by using a *cubical* extension of the symmetric operad instead of the convex one. He defined the operad-multiplication $\gamma^P_{n;i_1,\ldots,i_n} : P_n \times P_{i_1} \times \cdots \times P_{i_n} \to P_{i_1+\cdots+P_{i_n}}$ as the affine extension of the permutation operad, such that the barycenter of any interval in $\Sigma_{i_1+\cdots+i_n}$ is mapped to the barycenter of its geodesics in $P_{i_1+\cdots+i_n}$ (see Fig. 1). Hence for $n = 2$ and $i_1 + i_2 = 3$ the squares A,B,C and D of $P_2 \times P_1 \times P_2$, resp. $P_2 \times P_2 \times P_1$ are mapped to the corresponding segments of P_3.

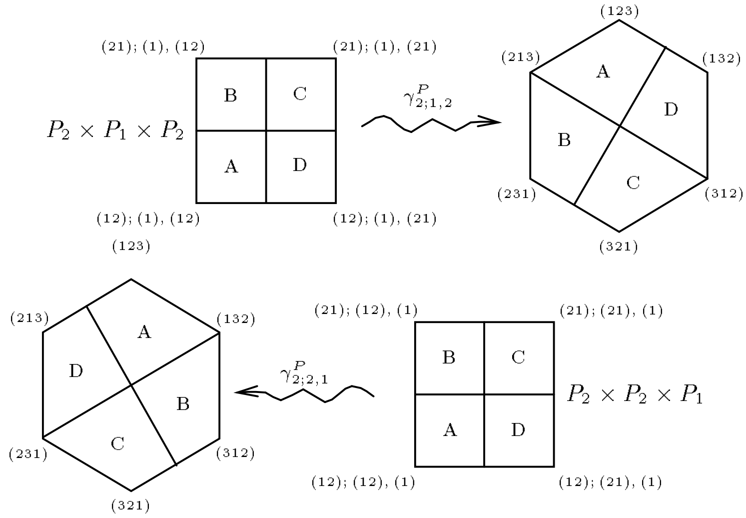

FIGURE 3. The cubical multiplication

The second step, in which the symmetric group action is made free, is done in the definition of the would-be operad. For each permutation in Σ_n a copy of P_n is added and an appropriate quotient of the space $P_n \times \Sigma_n$ is taken to be the n-th spce of the new operad.

Here I will only describe the suggested construction for the operad $J^{(2)}$ corresponding to J_2, i.e. 2-fold loop spaces.

Definition 1.8. (cmp. [5, 2.12]) Let $J^{(2)}_n$ be the quotient space of $P_n \times \Sigma_n$ under the relation

$$(x\tau; \sigma) \sim (x; \tau\sigma)$$

for any partition $\tau \in \Sigma_n$ of type $(i_1, \ldots, i_r), i_1 + \cdots + i_r = n$, $x \in P(i_1, \ldots, i_r)$ and $\sigma \in \Sigma_n$.

The action of $\varphi \in \Lambda(\mathbf{n}, \mathbf{m})$ is induced by

$$\begin{aligned} \varphi^* : P_m \times \Sigma_m &\to P_n \times \Sigma_n \\ (x; \sigma) &\mapsto \big(x(\sigma\varphi)^{inc}; (\sigma\varphi)^{\natural}\big). \end{aligned}$$

Following [5, 2.14] the Λ structure and the maps $\gamma_{n;i_1,\ldots,i_r} := \gamma^P_{i_1,\ldots,i_r} \times \gamma^\Sigma_{i_1,\ldots,i_r}$ induce an E_2-operad structure on the $J^{(2)}_n$.

2. The commutativity of Berger's construction

In the following we assume that Berger's construction defines an operad. If $(X,*)$ is a $J^{(2)}$-algebra, there exist maps $F_n : P_n \times \Sigma_n \times X^n \to X$. These fulfil certain conditions, induced by the operad structure and the relations on $J_n^{(2)}$. Used here are

1. the associativity condition
$$F_n\big(s; F_{i_1} = (t_1; x_1^1, \dots, x_{i_1}^1), \dots, F_{i_n}(t_n; x_1^n, \dots, x_{i_n}^n)\big)$$
$$= F_{i_1+\dots+i_n}\big(\gamma_{n;i_1,\dots,i_n}(s; t_1, \dots, t_n); x_1^1, \dots x_{i_n}^n\big),$$
2. the degeneration relations on the associated monad, induced by the two maps $\mathbf{1} \to \mathbf{2}$ $F_2(s; x, *) = F_1(*; x) = x = F_2(s; *, x)$
3. and the Σ_n-equivariance: $F_n(s\sigma; x_1, \dots, x_n) = F_n\big(s; x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}\big)$.

Remark 2.1. Since only the identities of Σ_n are used, in the future this coordinate will be dropped.

There is a multiplication on X, given by $xy = F_2\big((12); x, y\big)$. The associativity of the permutation operad shows the associativity of this multiplication. Since the degeneration relations hold $*$ is a unit. Therefore X is a associative monoid with strict unit.

Obviously the map $\phi : I \to P_2$ with $t \mapsto (12)(1-t) + (21)t$ is a homeomorphism. Thus there exists a homotopy $h_t : X \times X \to X$ with
$$h_t(x,y) = F_2\big((12)(t-1) + (21)t; x, y\big),$$
running from $h_0(x,y) = xy$ to $h_1(x,y) = F_2\big((21); x, y)\big) = F_2\big((12); y, x\big) = yx$.

Via F_3 and the degenerations one gets even stricter conditions for the commuting homotopy h_t. The maps $\gamma_{2;1,2}^P$ and $\gamma_{2;2,1}^P$ are homeomorphisms. Therefore there exist $s_i, t_i \in P_2, i = 1,2$ for each $r \in P_3$ such that
$$\gamma_{2:2,1}^P\big(s_2; t_2, (1)\big) = r = \gamma_{2;1,2}^P\big(s_1; (1), t_1\big).$$
Hence for $x, y \in X$ one gets
$$F_2(s_2; x, y) = F_3(r; *, x, y) = F_2(t_1; x, y)$$
In the first case the homotopy h_t is mapped to the edges $[(123),(132)]$ and $[(231),(321)]$ (cmp. Fig. 2). In the second case h_t is mapped to the geodesics of the intervals $[(123),(312)]$ and $[(213),(321)]$, such that the center point of the homotopy $h_{\frac{1}{2}}(x,y)$ is mapped to (132) and (231). Thus the homotopy needs to be equal to xy on its first half and equal to yx on the second half (In the first case the edge $[(213),(231)]$ is mapped to xy and the edge $[(132),(312)]$ to yx). Thus $(X,*)$ must be an abelian monoid.

Remark 2.2. Since the permutation coordinate wasn't used, the "liberation" of the symmetric group action is not involved in the failure of the suggested construction. In fact the cubical extension of the multiplication does not fulfil the needed degeneration properties.

Now let X be a 2-connected CW-complex with non-degenrate base point $*$ (i.e. the inclusion $* \hookrightarrow X$ is a closed cofibration). Then the two-fold Moore loop space $Y := \Omega_M^2 X$ of X is a connected CW-complex. The canonical evaluation map $e : \Sigma^2\Omega_M^2 X \to X$ induces a homomorphism of monoids $\Omega_M^2 e : \Omega_M^2\Sigma^2 Y \to Y$.

If $J^{(2)}$ is an E_2 operad, whose associated monad is the Milgram construction, there exists a homomorphism of monoids $\psi : J^{(2)}Y \to \Omega_M^2\Sigma^2 Y$ (the map is given in [11, 5.2.]). Therefore

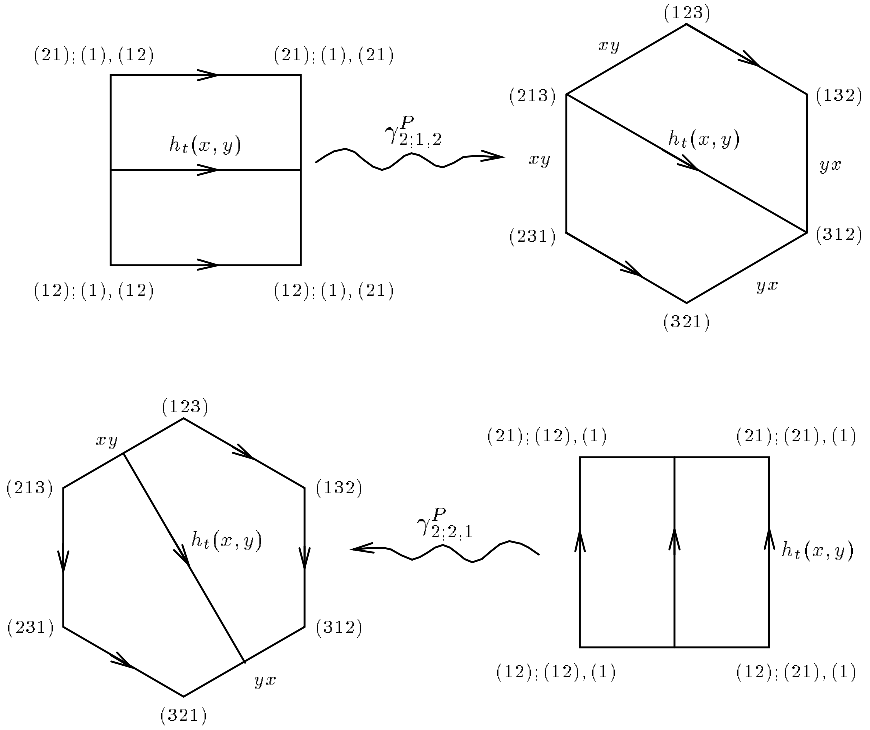

FIGURE 4. The commuting homotopy

we get a homomorphism $\varphi : J^{(2)}Y \to Y$. From the construction of ψ in [11] one can see that the diagram

$$\begin{tikzcd} J^{(2)}Y \arrow[r, "\psi"] & \Omega^2_M\Sigma^2 Y \arrow[r, "\Omega^2_M e"] & Y \\ & Y \arrow[ul, hook] \arrow[u, hook] \arrow[ur, equal] & \end{tikzcd}$$

with the inclusion $i : Y \hookrightarrow \Omega^2_M\Sigma^2 Y$ given by

$$(i(y))\,(s,t) = y \wedge s \wedge t,$$

commutes. Therefore The homomorphism $\varphi = \Omega^2_M\psi$ is an extension of the identity and hence surjective.

Since $J^{(2)}Y$ is the free $J^{(2)}$-algebra of the space Y, it is an abelian monoid. The surjectivity of φ now shows that $Y = \Omega^2_M X$ is strictly commutative, too. But obviously this is wrong. Therefore $J^{(2)}$ can not be an operad whose associated monad is the Milgram construction.

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