The limit of the spectral radius of nonnegative Toeplitz matrices

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ABSTRACT

A bi-infinite sequence $...t_{-2}, t_{-1}, t_0, t_1, t_2, ...$ of nonnegative numbers defines a sequence of nonnegative Toeplitz matrices $T_n = (t_{ik}), n = 1, ...,$, where $t_{ik} = t_{k-i}, i, k = 1, \ldots, n$. We show that the limit of the spectral radius of T_n, as n tends to infinity, is given by $\inf\{\sigma(\xi) : \xi \in [0, \infty]\}$, where $\sigma(\xi) = \sum_{j \in \mathbf{Z}} t_j \xi^j$. A corresponding result holds in the case of block Toeplitz matrices, where the t_i are $p \times p-$nonnegative matrices and $\sigma(\xi)$ is the spectral radius of $\sum_{j \in \mathbf{Z}} \xi^j t_j$. [1]

1 Introduction

Let $t = \{t_i\}_{i \in \mathbf{Z}}$ be a bi-infinite sequence of complex numbers. It defines a sequence of Toeplitz matrices $T_n = (t_{ik}), n = 1, 2, \ldots,$ where $t_{ik} = t_{k-i}, i, k = 1, \ldots, n$. In the case that $t_{-i} = \bar{t}_i, i \in \mathbf{Z}$, i.e. each T_n is a hermitian matrix, the asymptotic distribution of the

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eigenvalues of $T_n, n = 1, ...$, has been studied since the fundamental paper of Szegö [Sze]. See [Hir] for some references on this problem.

Assume now that t is a sequence of nonnegative real numbers. Let $\mu_n(t) = \rho(T_n)$, where $\rho(A)$ is the spectral radius of a square matrix A. As T_n is a principal submatrix of the nonnegative matrix T_{n+1} we deduce that the sequence $\{\mu_n(t)\}_1^\infty$ is monotonously increasing. Let

$$\mu(t) = \lim_{n\to\infty} \mu_n(t) \tag{1.1}$$

and note that $\mu(t) \in [0, \infty]$. We associate with t its symbol

$$\sigma(\xi) = \sum_{-\infty}^{\infty} t_i \xi^i := \lim_{m\to\infty} \sum_{i=-m}^{m} t_i \xi^i, \quad \xi \in [0, \infty]. \tag{1.2}$$

Let

$$\mu_0(t) = \inf_{\xi\in[0,\infty]} \sigma(\xi). \tag{1.3}$$

Again $\mu_0(t) \in [0, \infty]$. The main result of this paper is the equality

$$\mu(t) = \mu_0(t) \tag{1.4}$$

(Theorem 3.4). In the symmetric case, i.e. $t_k = t_{-k}, k = 1, ...$, this result is straightforward, as $\mu_0(t) = \sigma(1) = \max\{|\sigma(\xi)| : |\xi| = 1\}$ [Hir].

We generalize (1.4) to block Toeplitz matrices as follows: Let $t = \{t_i\}_{i\in Z}$ be a sequence of $p \times p$ nonnegative matrices. Then $T_n = (t_{i-j})_{i,j=1}^n$ is an $np \times np$ block Toeplitz matrix and set $\mu_n(t) = \rho(T_n)$. Let $\mu(t)$ be given by (1.1) and let

$$\sigma(\xi) := \lim_{m\to\infty} \rho(\sum_{|i|\le m} \xi^i t_i) \tag{1.5}$$

Define $\mu_0(t)$ by (1.3). Then, under some irreducibility assumptions on the matrices appearing in the sequence t, (1.4) holds.

We summarize briefly the content of our paper. In §2 we show that $\mu(t)$ is a lower bound for the spectral radius of an infinite Toeplitz matrix $T = (t_{i-j})_{i\in\mathbf{Z}}$ with respect to a certain class of absolute monotone norms. We also discuss some needed properties of the lower semi-continuous (l.s.c.) functions. We bring an example of block Toeplitz matrices for which $\mu(t) < \mu_0(t)$. In §3 we prove (1.4) for Toeplitz matrices. In §4 we extend our results to block Toeplitz matrices. In §5 we give another characterization of $\mu(t)$ and an upper bound on $\mu(t)$ for nonnegative tridiagonal block Toeplitz matrices.

2 Preliminaries

Denote the space of all complex, real and nonnegative sequences $t = \{t_i\}_{i\in\mathbf{Z}}$ by $\mathbf{C}^{\mathbf{Z}}, \mathbf{R}^{\mathbf{Z}}$ and $\mathbf{R}_+^{\mathbf{Z}}$ respectively. Let $w = \{w_i\}_{i\in\mathbf{Z}} \in \mathbf{R}_+^{\mathbf{Z}}$ be a sequence with positive coordinates.

Denote by $l_{p,w}$ the Banach space with respect to the norm

$$||x||_{p,w} = (\sum_{i\in\mathbf{Z}} |x_i|^p w_i)^{\frac{1}{p}}, \quad 1 \le p \le \infty, \quad x \in \mathbf{C}^{\mathbf{Z}}.$$

Then any bounded linear operator $A : l_{p,w} \to l_{p,w}$ is given by a bi-infinite matrix $A = (a_{ij})_{i,j\in\mathbf{Z}}$. Recall that the spectral radius of A is given by the formula

$$\lim_{m\to\infty} ||A^m||_{p,w}^{\frac{1}{m}}. \tag{2.1}$$

A is called nonnegative and denoted by $A \ge 0$ if $a_{ij} \ge 0, i, j \in \mathbf{Z}$.

Lemma 2.2 *Let $A : l_{p,w} \to l_{p,w}, 1 \le p \le \infty$ be a bounded nonnegative operator. Let $A_{m,n} := (a_{ij})_{i,j=m}^n, m \le n$. Then $\rho(A) \ge \rho(A_{m,n})$.*

Proof. For $x = \{x_i\} \in \mathbf{C}^{\mathbf{Z}}$ let $|x| := \{|x_i|\}_{i\in\mathbf{Z}}$. Then $||\cdot||_{p,w}$ is an absolute monotone norm:

$$||x||_{p,w} = ||\ |x|\ ||_{p,w}, \quad |y| - |x| \in \mathbf{R}_+^{\mathbf{Z}} \Rightarrow ||y||_{p,w} \ge ||x||_{p,w}.$$

Suppose that $B : l_{p,w} \to l_{p,w}$ is a bounded nonnegative linear operator. Assume that $A - B \ge 0$. The absolute monotonicity of the norm and the characterization (2.1) yields $\rho(A) \ge \rho(B)$. Let $B = (b_{ij})_{i,j\in\mathbf{Z}} \ge 0$ be the trivial extension of the matrix $A_{m,n}$. That is $b_{ij} = 0$ except for $m \le i, j \le n$. Furthermore, $B_{m,n} = A_{m,n}$. Clearly, B is a bounded nonnegative linear operator on $l_{p,w}$. Moreover, $\rho(A) \ge \rho(B) = \rho(A_{m,n})$. □

Corollary 2.3 *Let $t = (t_i)_{i\in\mathbf{Z}} \in \mathbf{R}_+^{\mathbf{Z}}$. Assume that $T = (t_{i-j})_{i,j\in\mathbf{Z}}$ is a bounded linear operator on some $l_{p,w}$. Then $\rho(T) \ge \mu(t)$.*

Let $H^2(r, R)$ be the Hardy space of holomorphic functions $f(z)$ in the annulus $0 < r < |z| < R$ given by the norm

$$||f|| := (\frac{1}{\pi(R^2 - r^2)} \int_{r<|z|<R} |f(z)|^2 dxdy)^{\frac{1}{2}}.$$

Recall that f has the convergent Laurent series $f = \sum_{-\infty}^{\infty} f_i z^i$ in $r < |z| < R$. Let $\phi := \{f_i\}_{i\in\mathbf{Z}}$. Then $||f|| = ||\phi||_{2,w}$ for some $w \in \mathbf{R}_+^{\mathbf{Z}}$ with positive entries. Let

$$\sigma(z) := \sum_{-\infty}^{\infty} t_i z^i$$

be a rational function ($t_i = 0, |i| > N$). Then the multiplication operator

$$T(t) : H^2(r, R) \to H^2(r, R), \quad T(t)f := \sigma f, \quad f(z) \in H^2(r, R)$$

is a bounded linear operator. Denote by $\rho(t)$ the spectral radius of $T(t)$. Assume that $t := (t_i)_{-\infty}^{\infty}$ is a real nonnegative vector. Then Corollary 2.3 yields that $\rho(t) \ge \mu(t)$.

In what follows we view $[0, \infty]$ as a compact metric space.

Lemma 2.4 *Let $\sigma_m : [0,\infty] \to [0,\infty], m = 0, ...,$ be a sequence of continuous functions. Assume furthermore that for each $\xi \in [0,\infty]$ the sequence $\{\sigma_m(\xi)\}_0^\infty$ is an increasing sequence which converge to $\sigma(\xi) \in [0,\infty]$. Then $\sigma : [0,\infty] \to [0,\infty]$ is a lower semi-continuous (l.s.c.) function. Furthermore, σ achieves it minimum on $[0,\infty]$ and*

$$\min_{\xi\in[0,\infty]} \sigma(\xi) = \lim_{m\to\infty} \min_{\xi\in[0,\infty]} \sigma_m(\xi).$$

Proof. Let $\alpha \in \mathbf{R}$. Define the closed sets $L_m = \{\xi : \sigma_m(\xi) \le \alpha\}$. Then $\{\xi : \sigma(\xi) \le \alpha\} = \bigcap_m L_m$ is closed too, so σ is l.s.c.. It is a well known fact that a l.s.c. function achieves its minimum on a compact set. Let

$$\nu = \min_{\xi\in[0,\infty]} \sigma(\xi) = \sigma(\zeta), \quad \zeta \in [0,\infty]$$

$$\nu_m = \min_{\xi\in[0,\infty]} \sigma_m(\xi) = \sigma_m(\zeta_m), \quad \zeta_m \in [0,\infty], m = 0, ...,.$$

Observe first

$$\nu = \sigma(\zeta) \ge \sigma_{m+1}(\zeta) \ge \nu_{m+1} = \sigma_{m+1}(\zeta_{m+1}) \ge \sigma_m(\zeta_{m+1}) \ge \nu_m = \sigma_m(\zeta_m), m = 0, ...,.$$

Therefore

$$\nu \ge \tilde{\nu} = \lim_{m\to\infty} \nu_m.$$

It is left to show the reversed inequality $\nu \le \tilde{\nu}$. Pick up a convergent subsequence $\zeta_{m_i}, i = 1, ...,$ which converges to $\eta \in [0,\infty]$. Fix $k \ge 0$. Then

$$\nu_{m_i} = \sigma_{m_i}(\zeta_{m_i}) \ge \sigma_k(\zeta_{m_i}), \quad m_i \ge k.$$

Letting $i \to \infty$ and using the continuity of σ_k we deduce $\tilde{\nu} \ge \sigma_k(\eta)$. Now let $k \to \infty$ to get $\tilde{\nu} \ge \sigma(\eta) \ge \nu$.
$\square$

For $p \ge 1$ let $\mathbf{R}_+^{pp}$ denote the cone of $p \times p$ nonnegative matrices. Let $(\mathbf{R}_+^{pp})^{\mathbf{Z}}$ be the space of the sequences $\{t_i\}_{i\in\mathbf{Z}}$ of $p \times p$ nonnegative matrices. Assume that $t \in (\mathbf{R}_+^{pp})^{\mathbf{Z}}$. Let

$$s_m(\xi) = \sum_{i=-m}^{i=m} t_i \xi^i, \quad 0 < \xi < \infty. \tag{2.5}$$

Then $\sigma_m(\xi) = \rho(s_m(\xi))$ is the spectral radius of the nonnegative $p \times p$ matrix $s_m(\xi)$. Clearly, σ_m extends to the continuous function $\sigma_m : [0,\infty] \to [0,\infty]$. Then the sequence $\sigma_m, m = 0, ...,$ satisfies the assumptions of Lemma 2.4. Let

$$\mu_{0,m}(t) := \min_{\xi\in[0,\infty]} \sigma_m(\xi). \tag{2.6}$$

By Lemma 2.4

$$\lim_{m\to\infty} \mu_{0,m}(t) = \mu_0(t). \tag{2.7}$$

Note that $\mu_{0,m}(t)$ depends only on the vector $(t_{-m}, ..., t_m)$. It is straightforward to show that $\mu_{0,m}(\cdot) : (\mathbf{R}_+^{pp})^{2m+1} \to \mathbf{R}_+$ is a continuous function.

Lemma 2.8 *Let $t = \{t_i\}_{i \in \mathbf{Z}} \in (\mathbf{R}_+^{22})^{\mathbf{Z}}$ be the following sequence:*

$$t_1 = diag(1,2), \quad t_0 = 0, \quad t_{-1} = t_1^{-1}, \quad t_i = 0, \quad |i| > 1.$$

Then

$$\mu_0(t) = \frac{3}{\sqrt{2}} \;>\; \mu(t) = 2.$$

Proof. As $s_1(\xi)$ is a diagonal matrix we deduce that

$$\sigma(\xi) = \max(\xi + \xi^{-1}, 2\xi + (2\xi)^{-1}).$$

The minimim of $\sigma(\xi)$ is achieved when the two diagonal entries are equal, i.e. $\xi = \frac{1}{\sqrt{2}}, \mu_0(t) = \frac{3}{\sqrt{2}}$. It is straightforward to show that $T_n(t)$ is permutationally similar to $J_n \oplus K_n$, where J_n, K_n are tridiagonal matrices with zero diagonal. The entries on the upper and the lower diagonals of J_n are equal to 1. The entries on the upper diagonal of K_n are equal to 2 and the entries on the lower diagonal of K_n are equal to $\frac{1}{2}$. Hence K_n is similar to J_n by a diagonal matrix. Thus $\mu_n(t) = \rho(J_n)$. It is well known that the eigenvalues of J_n are the roots of the Chebyshev polynomial of degree n. Hence $\mu_n(t) = 2\cos\frac{\pi}{n+1}, n = 2, ...,$ and $\mu(t) = 2$. (The equality $\lim_{n\to\infty} \rho(J_n) = 2$ follows also from (1.4).)

$\square$

3 Toeplitz matrices

In what follows we will sometimes suppress in the functions $\mu_m(t), \mu(t), \mu_{0,m}(t), \mu_0(t)$ the dependence on the parameter t and no ambiguity will arise.

Lemma 3.1 *Let A be an $n \times n$ real matrix. Assume that there exists a nonzero real vector p such that*

$$A^r p \geq 0 \quad \text{for} \quad r = 0, 1, \ldots, . \tag{3.2}$$

holds. Then A has a nonnegative real eigenvalue with an associated nonnegative eigenvector.

Proof. Let K be the smallest closed cone spanned by all finite convex combinations of the vectors $A^r p, \quad r \in \mathbf{Z}_+$. Obviously $AK \subset K$. (3.2) implies that K is contained in the standard cone of nonnegative vectors. Hence $K \cap -K = \{0\}$, i.e. K is a pointed cone. Let $L = L(K) = span(K)$ be the invariant subspace of A spanned by K. Then K has inner points relative to L. By the theory of Krein-Rutman, see e.g. [B-P, p.6], $\rho(A|_L)$ is a real eigenvalue of A. An associated eigenvector can be chosen in K and is hence nonnegative.

$\square$

We remark that a similar construction was used in [Fri][Theorem 9, p. 59].

Lemma 3.3 *Let $t \in \mathbf{R}_+^{\mathbf{Z}}$. Then*

$$\mu \le \mu_0.$$

Proof. If $\mu_0 = \infty$ then the lemma is trivial. Assume now that $\mu_0 < \infty$. If either $\mu_0 = \sigma(0)$ or $\mu_0 = \sigma(\infty)$ then $\mu_n = t_0 = \mu_0, n = 1, ...,$ and the lemma holds.

Assume that $\mu_0 = \sigma(\xi), \xi \in (0, \infty)$. Define the positive n-vector $x^n = x^n(\xi)$ with components $x_i = \xi^{i-1}, i = 1, ..., n$. It then follows that $T_n x^n \le \sigma(\xi) x^n$. Wielandt's characterization yields $\mu_n \le \sigma(\xi)$. (See for example [B-P].)

□

Theorem 3.4 *Let $t \in \mathbf{R}_+^{\mathbf{Z}}$. Denote by $T_n = (t_{j-i})_{i,j=1}^n, n = 1, ...,$ the corresponding $n \times n$ Toeplitz matrices. Then*

$$\lim_{n\to\infty} \rho(T_n) = \inf\{\sigma(\xi), \xi \in [0, \infty]\}. \tag{3.5}$$

Proof. We first consider the case where $t = \{t_i\}_{i\in\mathbf{Z}}$ has a finite number of positive terms. Let k be a nonnegative integer so that $t_r = 0$ for $|r| > k$. The theorem trivially holds if either $t_s = 0$ for $s \ge 1$ or $t_s = 0$ for $s \le -1$. Hence we can assume without loss of generality that $t_k > 0, t_{-l} > 0$ where $0 < l \le k$ and $t_{-\nu} = 0$ for all $\nu > l$. For $n > k$ consider the equation

$$(T_n - \mu_n I)x^{(n)} = 0, \quad x^{(n)} = (x_{1,n}, \ldots, x_{n,n})^T \ge 0, \quad x^{(n)} \ne 0. \tag{3.6}$$

Solving the i-th equation of (3.6) for $x_{i+k,n}$ we obtain

$$x_{i+k,n} = \sum_{r=-l}^{k-1} s_r^n x_{i+r,n} \quad i = 1, \ldots, n-k, \tag{3.7}$$

where

$$s_r^n = \begin{cases} -t_k^{-1} t_r, & \text{if } r \ne 0; \\ t_k^{-1}(\mu_n - t_0), & \text{if } r = 0. \end{cases} \tag{3.8}$$

Introducing the $k + l-$ vector

$$x_n^{(\nu)} = (x_{\nu-k-l+1,n}, \ldots, x_{\nu,n})^T, \quad \nu = k, \ldots, n-1,$$

where it is understood that $x_{r,n} = 0$ for $r \le 0$, (3.7) can be rewritten as

$$S_n x_n^{(\nu)} = x_n^{(\nu+1)} \quad \text{for} \quad k \le \nu \le n-1. \tag{3.9}$$

Here S_n is the following companion matrix.

$$S_n = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & \ldots & 0 & 1 \\ s_{-l}^n & \ldots & \ldots & s_{k-2}^n & s_{k-1}^n \end{pmatrix}.$$

Fix $x^{(n)}, n \geq k$ by the condition

$$||x_n^{(k)}||_1 = \sum_{i=-l+1}^{k} x_{i,n} = 1, n \geq k.$$

This is well defined as by (3.9) necessarily $x_n^{(k)} \neq 0$. We now let $n \to \infty$ in (3.9). The above normalization yields a subsequence $x_{n_i}^{(k)}$ which converges to some nonzero nonnegative vector $\tilde{x}^{(k)}, ||\tilde{x}^{(k)}||_1 = 1$. Observe next that the entries of S_n except that in position $(k+l, l+1)$ do not depend on n. It then follows that $S_n \to S_\infty$ where the $(k+l, l+1)$ entry of S_∞ is equal to $t_k^{-1}(\mu - t_0)$. Use (3.9) to deduce for any positive integer r

$$S_\infty^r \tilde{x}^{(k)} = \lim_{i \to \infty} S_{n_i}^r x_{n_i}^{(k)} = \lim_{i \to \infty} x_{n_i}^{(k+r)} \geq 0.$$

By Lemma 3.1 the matrix S_∞ has a nonnegative eigenvalue η with corresponding eigenvector $x(\eta) = (1, \eta, \ldots, \eta^{l+k-1})^T$.

The last equation of

$$S_\infty x(\eta) = \eta x(\eta)$$

gives after simple manipulations

$$t_k \eta^{l+k} + \ldots + (t_0 - \mu)\eta^l + \ldots + t_{-l} = 0. \tag{3.10}$$

As $t_{-l} \neq 0$ we get that $\eta \neq 0$. Hence

$$\mu = \sum_{\nu=-l}^{k} t_\nu \eta^\nu = \sigma(\eta).$$

Combining the above equality with Lemma 3.3 we deduce the theorem in this case.

Assume finally that $t \in \mathbf{R}_+^{\mathbf{Z}}$ has an infinite number of positive terms. We now use the results and the notation of section 2. That is we let $\mu^{(m)}, \mu_{0,m}$ to the corresponding limits of the $2m+1$ band Toeplitz matrices induced by the symbol $\sigma_m(\xi)$. By the previous arguments we have

$$\mu^{(m)} = \mu_{0,m}, m = 0, 1, ..., . \tag{3.11}$$

Clearly $\mu \geq \mu^{(m)}, m = 1, ...,$. Hence $\mu \geq \lim_{m \to \infty} \mu^{(m)} = \lim_{m \to \infty} \mu_{0,m} = \mu_0$, where the last equation follows from Lemma 2.4. Use Lemma 3.3 to deduce the theorem. □

4 Block Toeplitz matrices

We first prove the analogue of Lemma 3.3 for block Toeplitz matrices.

Lemma 4.1 *Let $t \in (\mathbf{R}_+^{pp})^{\mathbf{Z}}$. Then $\mu \leq \mu_0$.*

Proof. We first show that $\mu_m \leq \mu_{0,m}$. Assume first that t_m is an irreducible matrix. Then $s_m(\xi), 0 < \xi < \infty$, is an irreducible matrix. Denote by $T_n^{(m)}, n = 1, ...,$ the $2m+1$ banded block Toeplitz matrices induced by s_m. Let $x(\xi) \in R^p$ be a Perron vector of $s_m(\xi)$, i.e. $x(\xi) > 0$ and $s_m(\xi)x(\xi) = \sigma_m(\xi)x(\xi)$. Then it is easy to check that $y^T(\xi) = (x(\xi)^T, \xi x(\xi)^T, \ldots, \xi^{n-1}x(\xi)^T)$ satisfies

$$T_n^{(m)} y(\xi) \leq \sigma_m(\xi) y(\xi), \quad n = 1, ...,.$$

Hence $\mu_m \leq \sigma_m(\xi), \xi \in (0, \infty)$ and $\mu_m \leq \mu_{0,m}$. Suppose now that t_m is reducible. For $\epsilon > 0$ let $t(\epsilon) = \{t_i(\epsilon)\}_{i \in \mathbf{Z}} \in (\mathbf{R}_+^{pp})^{\mathbf{Z}}$ be different from t only at $i = m$. Furthermore $t_m(\epsilon) = t_m + \epsilon J$ where J is a $p \times p$ matrix whose all entries are equal to one. Denote by $\mu_m(\epsilon), \mu_{0,m}(\epsilon)$ the corresponding functions defined for $t(\epsilon)$. As $t_m(\epsilon)$ is irreducible $\mu_m(\epsilon) \leq \mu_{0,m}(\epsilon)$. Since $t(\epsilon) \geq t$ we have that

$$\mu_m \leq \mu_m(\epsilon) \leq \mu_{0,m}(\epsilon).$$

Let $\epsilon \to 0^+$ and use the continuity of the function $\mu_{0,m}(t)$ to deduce that $\mu_m \leq \mu_{0,m}$. Then (1.1) and (2.7) yield $\mu \leq \mu_0$.
$\square$.

Let $A \in \mathbf{R}_+^{pp}$. Then A is called combinatorially singular if $\det(B) = 0$ for any $B \in \mathbf{R}^{pp}$ having zero entries at locations where A has. Otherwise A is called combinatorially nonsingular. There exist irreducible A which are combinatorially singular, e.g. if the digraph $\Gamma(A)$ of A is a star and $p > 2$.

Lemma 4.2 *Let $l < 0 < k \in \mathbf{Z}_+$. Assume that $(t_i)_l^k$ be a sequence of nonnegative $p \times p$ matrices. Denote by $T_n = (t_{j-i})_{i,j=1}^n, n = 1, ...,,$ where $t_i = 0$ for $i < l$ or $i < k$, the corresponding $np \times np$ block Toeplitz matrices. Set*

$$s(\xi) = \sum_{i=l}^{k} t_i \xi^i, 0 < \xi < \infty.$$

Denote by $\sigma(\xi)$ the spectral radius of $s(\xi)$. Assume that t_l is irreducible. Suppose furthermore that t_k is not combinatorially singular. Then

$$\lim_{n \to \infty} \rho(T_n) = \inf\{s(\xi), \xi \in (0, \infty)\}.$$

Proof. Our assumptions imply that $s(\xi), \xi > 0$ is irreducible. Suppose first that t_k is invertible. Then we can repeat the arguments of the proof of Theorem 3.4. Let $m = k - l$. Note that the matrix S_n is $mp \times mp$ a block matrix where 1 stands for $p \times p$ identity matrix and s_r^n given by (3.8) are $p \times p$ corresponding matrices. Consider the $mp \times mp$ S_∞ matrix which has a nonnegative eigenvalue η. It has a corresponding nonnegative eigenvector

$$x(\eta) = (y^T, \eta y^T, ..., \eta^{m-1} y^T)^T, \quad y \in \mathbf{R}_+^p.$$

This follows from the special form of S_∞. From the last block of the equation $S_\infty x(\eta) = \eta x(\eta)$ we deduce an identity similar to (3.11).

$$(\sum_{i=l}^{k} t_i \eta^{-l+i}) y = \mu \eta^{-l} y.$$

As y is a nonzero nonnegative vector and t_l is irreducible we deduce that $\eta > 0$. Thus $s(\eta)y = \mu y$. Since $s(\eta)$ is irreducible we obtain that $\mu = \sigma(\eta), y = x(\eta)$. Combine that with Lemma 4.1 to obtain our lemma.

Assume now that t_k is singular and combinatorially nonsingular. Then we can find a sequence $t_k^{(j)} \in \mathbf{R}_+^{pp}, j = 1, ...,$ of irreducible nonsingular matrices which increase monotonically and converge to t_k. Let $t^{(j)} = (t_i^{(j)})_l^k \in \mathbf{R}_+^{pp}$ be distinct from $(t_i)_l^k$ only at the entry $i = k$ for $j = 1, ...,$. Denote by $\mu^{(j)} = \mu_0^{(j)}$ the induced value by $t^{(j)}, j = 1, ...,$. As $t^{(j)} \le t$ we obtain

$$\mu^{(j)} \le \mu, \quad \mu_0^{(j)} \le \mu_0.$$

Let $j \to \infty$ and use (2.6) (with $m = \max(k, l)$) to deduce

$$\mu_0 = \lim_{j \to \infty} \mu_0^{(j)} = \lim_{\to \infty} \mu^{(j)} \le \mu.$$

Lemma 4.1 yields $\mu = \mu_0$.

$\square$

We remark that the same result holds also for t_k irreducible and t_l not combinatorially singular. Just apply Lemma 4.2 to the reversed sequence $\hat{t} = \{t_{-i}\}_{i \in \mathbf{Z}}$.

Combine the above lemmas with the definitions and the results of §2 to deduce

Theorem 4.3 *Let $p > 1$ and assume that $t = \{t_i\}_{i \in \mathbf{Z}} \in (\mathbf{R}_+^{pp})^{\mathbf{Z}}$. Then*

$$\mu = \min_{\xi \in [0, \infty]} \sigma(\xi)$$

if one of the following conditions hold.

1. *There exist $l < 0 < k$ so that*
$$t_i = 0 \begin{cases} i < l; \\ i > k. \end{cases}$$
Furthermore t_l, t_k are irreducible and at least one of these matrices is not combinatorially singular.

2. *There exists $l < 0$ so that $t_i = 0, i < l$ and t_l is irreducible. There exists an increasing subsequence of positive integers $m_i, i = 1, ...,$ so that $t_{m_i}, i = 1, ...,$ are irreducible. Furthermore, either t_l is combinatorially nonsingular or each $t_{m_i}, i = 1, ...,$ is combinatorially nonsingular.*

3. *The reversed sequence* $\hat{t} = \{t_{-i}\}_{i \in \mathbf{Z}}$ *satisfies the conditions of (2).*

4. *There exist two subsequences of increasing positive integers* $\{m_i\}_1^\infty, \{q_i\}_1^\infty$ *so that* t_{m_i}, t_{-q_i} *are irreducible for* $i = 1, ...,$. *Furthermore either* t_{m_i} *is combinatorially nonsingular for* $i = 1, ...,$ *or* t_{-q_i} *is combinatorially nonsingular for* $i = 1, ...,$.

Lemma 2.8 shows that the assumption that $s(\xi)$ is irreducible cannot be dropped.

5 Additional results

Theorem 3.4 shows that $\mu(t)$ of $2m+1$ banded nonnegative Toeplitz matrices is a continuous function in its parameters. The proof is an easy consequence of our characterization (3.5). Fix $p, m > 1$ and consider the space of all $2m+1$ banded nonnegative $p \times p$ block Toeplitz matrices. It is straightforward to show that $\mu_0(t)$ is a continuous function in its parameters. In view of Theorem 4.3, $\mu(t)$ is continuous at any τ which satisfies the first condition of Theorem 4.3. Lemma 2.8 shows that $\mu(t)$ is not continuous for certain values of t. It is an interesting problem to characterize the values of t for which $\mu(t)$ is continuous (discontinuous).

By considering bigger $q \times q$ blocks it is enough to consider the above mentioned problem for 3 banded $q \times q$ nonnegative block Toeplitz matrices, i.e. tridiagonal block Toeplitz matrices. It seems that the following lemma can be used for this purpose.

Lemma 5.1 *Let* $t_{-1}, t_0, t_1 \in \mathbf{R}_+^{pp}$. *Denote by* $T_n \in \mathbf{R}_+^{(np)(np)}, n = 1, ...,$ *the induced sequence of block tridiagonal Toeplitz matrices. Fix* $k \geq 1$. *Then there exists a polynomial* $P_k(x, y, z)$ *over* $\mathbf{R}$ *in noncommutative variables* x, y, z *of degree* k *with nonnegative coefficients (equal to* 0 *or* 1*) so that any block diagonal entry* $D \in \mathbf{R}_+^{pp}$ *of* $T_n^k, n = 1, ...,$ *satisfies* $D \leq P_k(t_{-1}, t_0, t_1)$. *For* $n > k$ *at most* $n - k$ *of the diagonal blocks of* T_n^k *are different from* $P_k(t_{-1}, t_0, t_1)$. *Moreover*

$$\mu = \limsup_{k \to \infty} (trace P_k(t_{-1}, t_0, t_1))^{\frac{1}{k}}.$$

Proof. Consider a graph G on the vertices $-k, ..., -1, 0, ..., k$ arranged linearly on the real line. From each vertex you can loop to the vertex, or go forward or backward one vertex when possible. We associate with the loop the variable y, with the right arrow the variable z and with the left arrow the variable x. Consider a closed path in G of length k starting at 0. It induces a unique noncommutative monomial in x, y, z of degree k. For example the monomial yx^2z^2y represents the path $0 \to 0 \to -1 \to -2 \to -1 \to 0 \to 0$. Then $P_k(x, y, z)$ is the sum of all monomials representing all closed path of length k starting at 0. Note that all such closed paths are containing only the vertices in the interval $[\frac{-k}{2}, \frac{k}{2}]$.

It then follows that for $n > k$ at least $n - k$ (the middle) block diagonal entries of T_n^k are equal to $P_k(t_{-1}, t_0, t_1)$. Other diagonal entries are polynomials in t_{-1}, t_0, t_1 which may have not some of the monomials that in P_k. Hence $D \leq P_n(t_{-1}, t_0, t_1)$. Set

$$\tilde{\mu} = \limsup_{k \to \infty}(trace(P_k(t_{-1}, t_0, t_1))^{\frac{1}{k}}.$$

We now show that $\mu = \tilde{\mu}$. By the previous observation

$$trace(T_n^k) \leq n \;\; trace(P_k(t_{-1}, t_0, t_1)), n, k \geq 1.$$

Hence

$$\mu_n = \limsup_{k \to \infty}(trace(T_n^k))^{\frac{1}{k}} \leq \tilde{\mu}.$$

Thus $\mu \leq \tilde{\mu}$. Consider T_{k+1}. As μ_{k+1} is its spectral radius and $P_k(t_{-1}, t_0, t_1)$ is one of the block diagonal entries of T_{k+1}^k we deduce that

$$trace(P_k(t_{-1}, t_0, t_1)) \leq trace(T_{k+1}^k) \leq (k+1)\mu_{k+1}^k.$$

Hence $\tilde{\mu} \leq \mu$.
$\square$

We now show that the limit spectral radius of an infinite tridiagonal nonnegative block Toeplitz matrix can be bounded from above by using some special solutions of a certain matrix Riccati equation.

Let $A, B, C \in \mathbf{C}^{pp}$. Consider the following Riccati equation

$$(I - zA - z^2CX)X = B, X = X(z) \in \mathbf{C}^{nn}, z \in \mathbf{C} \tag{5.2}$$

We first observe that there exists $r > 0$ so that (5.2) has a unique power series solution

$$X(z) = \sum_{k=0}^{\infty} z^k X_k, \quad X_k \in \mathbf{C}^{pp}, \quad k = 0, ..., \tag{5.3}$$

where

$$X_0 = B, \quad X_1 = AB,$$
$$X_k = AX_{k-1} + \textstyle\sum_{j=0}^{k-2} CX_jX_{k-2-j}, \quad k = 2, ...,$$
$$\textstyle\sum_0^{\infty} |z|^k ||X_k|| < \infty, |z| < r.$$

In fact r can be given (in the nontrivial case where $B, C \neq 0$) as the smallest positive solution of

$$(1 - r||A||)^2 - 4r^2||B||\;||C|| = 0 \Rightarrow r^{-1} = ||A|| + 2\sqrt{||B||||C||}. \tag{5.4}$$

Indeed, consider the recursive formula for $X_k = 0, ...,$ given by (5.3). Then the norm estimate for $||X_k||$ are

$$||X_0|| \le ||B||, \quad ||X_1|| \le ||A||\,||B||,$$
$$||X_k|| \le ||A||\,||X_{k-1}|| + \sum_{j=0}^{k-2} ||C||\,||X_j||\,||X_{k-2-j}||, \quad k = 2, ..., .$$

These estimates will be maximal if we replace all the inequalities by the equalities. It then follows that the upper estimates for $||X_k||, k = 0, ...,$ will be obtained by the power series of the scalar version of (5.2) given by

$$(1 - r||A|| - r^2||C||x)x = ||B||.$$

The radius of convergence of these scalar series are given by (5.4). ¿From (5.3) it follows that if $A, B, C \ge 0$ then $X_k \ge 0, k = 0, ...,$. Observe next that the radius of convergence of the series (5.3) is given by the standard Cauchy-Hadamard (C-H) formula

$$\rho = (\limsup_{k\to\infty} ||X_k||^{\frac{1}{k}})^{-1}. \tag{5.5}$$

Indeed, since all vector norms on $\mathbf{R}^{pp}$ are equivalent we may take $||(x_{ij})_1^p|| = \max |x_{ij}|$. View $X(z) = (x_{ij}(z))_1^p$. Now use the standard C-H formula for the power series of each $x_{ij}(z)$ to deduce that (5.5) is the minimal radius of convergence of all $x_{ij}(z), i, j = 1, ..., p$. Suppose now that $A, B, C \ge 0$. Then each $x_{ij}(z)$ has nonnegative Taylor coefficients. The Pringsheim theorem yields that ρ is a singular point of $X(z)$. Assume that $\rho < \infty$. It then follows that for $|z| < \rho$ (5.2) has an analytic solution $X(z)$ whose entries are algebraic functions. Hence our power series can be viewed as a branch of a matrix algebraic function given by the Riccati equation (5.2). The point $z = \rho$ is a branch point of this algebraic function.

Theorem 5.6 *Let $A, B, C \in \mathbf{R}_+^{pp}$. Set $t_{-1} = C, t_0 = A, t_1 = B$. Let μ be the limit spectral radius of the tridiagonal block Toeplitz matrices $T_n, n = 1, ...,$ induced by t_{-1}, t_0, t_1. Suppose that B is invertible and $T_n, n > n_0$, is irreducible. Then $\mu \le \frac{1}{\rho}$.*

Proof. Fix $n > n_0$. Let $u^T = (0, ..., 0, v^T) \in \mathbf{R}_+^{np}, v > 0$. If we can show that the equation $T_n x = \eta x - u$ is uniquely solvable for all $\eta \ge r > 0$ it follows that $\mu_n < r$. Assume that $\eta > \rho^{-1}$. Let $z = \eta^{-1}$. Set $x^T = (x_1^T, ..., x_n^T) \in \mathbf{R}^{pn}$. Consider the equation $T_n x = \eta x - u$. The first equation of this block system boils down to

$$x_1 = zP_1(z)x_2, \quad P_1(z) = (I - zA)^{-1}B.$$

Other equations are

$$x_i = zP_i(z)x_{i+1}, \quad P_i(z) = (I - zA - z^2CP_{i-1}(z))^{-1}B, \quad i = 2, ..., n-1.$$

Finally the last equation boils down to

$$x_n = zP_n(z)B^{-1}v, \quad P_n(z) = (I - zA - z^2 CP_n{-}1(z))^{-1}B.$$

We now claim that $P_n(z)$ have convergent power series for $|z| < \rho$. This follows from the fact that when one computes the formal power series of

$$0 \leq P_i(z) = (I - zA - z^2 P_{i-1}(z))^{-1}B = \sum_{k=0}^{\infty} P_{i,k} z^k$$

we have the inequalites

$$P_{i,k} \leq X_k, k = 0, ..., i = 1, ...,. \tag{5.7}$$

The proof goes by induction on i. For $i = 1$ it is clear. Assume it is true for $i = j$. Note that $(I - zA - z^2 CP_j(z))P_{j+1}(z) = B$. Computing the coefficients of $P_{j+1,k}$ from the above equation and using the definitions of X_k given by (5.3) and the induction hypothesis we deduce the inequalities (5.7). Hence $\mu \leq \frac{1}{\rho}$.

□

In the scalar case $p = 1$ the assumptions of Theorem 5.6 yield $B, C > 0, A \geq 0$. We then get that $\rho^{-1} = A + 2\sqrt{BC}$ which is equal to the limit spectral radius of the corresponding tridiagonal Toeplitz matrix. It is an interesting question if the inequality given by Theorem 5.6 is sharp.

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