Probabilistic Representations and Hyperbound Estimates for Semigroups

M.Röckner T. S. Zhang

October 19, 1998

Fakultät für Mathematik
Universität Bielefeld
D-33501 Bielefeld, Germany

Faculty of Engineering
HSH,Skåregt 103
5500 Haugesund ,Norway

Abstract

In this paper, we study lower order perturbations of a symmetric second order differential operator generating a hypercontractive semigroup. We give a probabilistic representation of the in general not sub-Markovian semigroup associated with the perturbed operator and prove that the perturbed semigroup is also hypercontractive under some exponential integrability conditions on the coefficients.

1 Introduction

Let X be a locally convex Hausdorff topological vector space which is Souslinean and let $(H, <,>_H)$ be a Hilbert space continuously and densely embedded into X . Let μ be a probability measure on $(X, \mathcal{B}(X))$, where $\mathcal{B}(X)$ stands for the Borel σ-algebra. The starting point of this paper is a second order symmetric differential operator

$$Lu(x) = \frac{1}{2} div(A(x)\nabla u(x)) \tag{1}$$

on $L^2(X, \mu)$ determined by a quadratic form $(\mathcal{E}, D(\mathcal{E}))$, which satisfies a Log-Sobolev inequality. This is equivalent to the associated semigroup being hypercontractive (see e.g. [8]). Consider now a lower order perturbation of the above operator

$$\mathcal{L}_p u(x) = \frac{1}{2} div(A(x)\nabla u(x)) + < B(x), \nabla u(x) > - div(\hat{B}(x)\nabla u(x)) - c(x)u(x). \tag{2}$$

where $B(x) : X \to H$, $\hat{B}(x) : X \to H$ are proper vector fields and $c(x) : X \to R$ is a scale function.

The problem we are concerned with is when the semigroup T_t associated with the perturbed operator $\mathcal{L}_p$ is also hypercontractive , that is, T_t is a bounded operator from

$L^a(X,\mu)$ to $L^b(X,\mu)$ for $b > a$ and big enough t. Our result states that this is the case if any positive multiple of $|B|, |\hat{B}|, c^- := -inf(c, 0)$ are exponentially integrable. We don't impose any smoothness conditions on the coefficients. This means that $\mathcal{L}_p$ is obtained from L by adding some drifts and very singular potentials $div(\hat{B})$, which are usually distributions.

This kind of perturbation theory has been of substantial interest in the past. If $B = \hat{B} = 0$, the problem was studied in [18] under exponential integrability conditions on the potential term c. The case $\hat{B} = 0$ was treated in [9], with the underline measure μ being Gaussian . Perturbations by changing the measure μ via multiplication by suitable density functions was considered in [4] . For other related works, see, for example , [2], [16].

To achieve our results, we combine probabilistic methods with analytic tools. One of the main results is the probabilistic representation of the semigroup associated with the perturbed operator $\mathcal{L}_p$, which involves both forward and also backward stochastic integration. The finite dimensional version of such a representation appeared first in the paper ([12]) by J.Lunt, T.J.Lyons and T.S.Zhang. Some new machinery must be developed in order to get the infinite dimensional version,see Sections 3,4 and 5. The hypercontractivity of the semigroup is then a consequence of this representation.

The paper is organized as follows: In Section 2 we present our framework. The rigorous definition of the perturbed operator via the associated coercive closed form is contained in Section 3. In Section 4, we present a convergence result of semigroups corresponding to a sequence of coercive closed forms. The result is of its own interest and will also be used subsequently. The forward and backward stochastic integrals are discussed in Section 5. In Section 6, we prove the probabilistic representations of the semigroups and identify the quadratic forms for a class of generalized Feynman-Kac semigroups. The hyperbound estimates of the semigroups are given in Section 7.

2 Framework

Let X be a Souslinean locally convex Hausdorff topological vector space with dual X^*. Let $H \subset X$ be a continuously and densely embedded Hilbert space. By $\mathcal{FC}_b^\infty$ we denote the class of all functions f on X of the form

$$f(x) = \varphi(l_1(x), \dots, l_n(x)),\ \varphi \in C_b^\infty(R^n),\ l_i \in X^*, \quad x \in X.$$

For $f \in \mathcal{FC}_b^\infty$, let $\nabla f(x)$ denote the gradient of f in H at $x \in X$, i.e.,

$$< \nabla f(x), h >_H = \frac{df(x+\varepsilon h)}{d\varepsilon}|_{\varepsilon=0}, \quad \text{for all } h \in H.$$

where $<,>_H$ stands for the inner product in H.

By $\mathcal{L}(H)$ we denote the collection of all bounded linear operators on H. Let $A(x) : X \to \mathcal{L}(H)$ be a strongly measurable map such that $A(x)$ is symmetric and

$$c_1 I \le A(x) \le c_2 I \tag{3}$$

where I stands for the identity operator on H.
Suppose that the positive definite symmetric bilinear form

$$\mathcal{E}(f,g) = \frac{1}{2}\int < A(x)\nabla f(x), \nabla g(x) >_H \ \mu(dx), \quad f,g \in \mathcal{F}\mathcal{C}_b^\infty, \tag{4}$$

is closable on $L^2(X,\mu)$.Its closure , denoted by $(\mathcal{E}, D(\mathcal{E}))$, is a Dirichlet form. Let L denote its generator with domain $D(L)$. Note that the closability of the form is equivalent to the closability of the gradient operator ∇ in $L^2(X,\mu)$. For every $f \in D(\mathcal{E})$, the action of the closure ∇ on f is denoted by ∇f.

Throughout this paper, we assume that the above Dirichlet form satisfies a defective Log-Sobolev (DLS) inequality:

$$\int_X |f(x)|^2 log|f(x)|\, d\mu(x) \leq \lambda_1 \mathcal{E}(f,f) + ||f||_2^2 log||f||_2 + \lambda_2 ||f||_2^2, \qquad \forall f \in D(\mathcal{E}) \tag{5}$$

where $||f||_2$ denotes the $L^2(X,\mu)$-norm of f and λ_1 and λ_2 are constants.

Remark 2.1. A typical measure satisfying (DLS) is the centred Gaussian measure μ_0 on X with reproducing kernel Hilbert space H. But there exist many other examples for which (DLS) holds, for instance, the measure $\nu(dx) = exp(F(x))\mu_0(dx)$ satisfies (DLS) if $exp(|\nabla F|_H^2(x)) \in L^p(X,\mu_0)$, for any $p > 1$. See [4], and for examples for μ satisfying (5) which are not absolutely continuous w.r.t. a Gaussian measure also [1].

3 Lower order perturbation and coercivity

In this section, we introduce lower order perturbations of the operator L in Section 2.

Let $B(x), \hat{B}(x) : X \to H, c(x) : X \to R$, be measurable mappings such that for some $p_1, p_2, q \in [1,\infty)$

$$\int_X exp(p_1|B(x)|_H^2)\mu(dx) < \infty, \qquad \int_X exp(p_2|\hat{B}(x)|_H^2)\mu(dx) < \infty,$$

$$\int_X exp(qc^-(x))\mu(dx) < \infty, \tag{6}$$

where $c^- := -inf(c,0)$. Consider the following bilinear form:

$$Q(f,g) = \frac{1}{2}\int_X < A(x)\nabla f, \nabla g >_H (x)\mu(dx) - \int_X < B(x), \nabla f(x) >_H g(x)\, \mu(dx)$$

$$- \int_X < \hat{B}(x), \nabla g(x) >_H f(x)\, \mu(dx) + \int_X c(x)f(x)g(x)\mu(dx). \tag{7}$$

$$f,g \in D(Q) = D(\mathcal{E}) \tag{8}$$

Proposition 3.1. Assume (DLS) holds. If $(\frac{1}{q} + \frac{1}{c_1p_1} + \frac{1}{c_1p_2})2\lambda_1 < \frac{1}{2}$, then there exists a $\alpha > 0$ such that

$$Q_\alpha(f,g) = Q(f,g) + \alpha(f,g); \quad f,g \in D(Q)$$

is a coercive closed form on $L^2(X,\mu)$, i.e., it is a closed form satisfying the following sector condition: there exists $K \in (0,\infty)$ such that

$$Q_\alpha(f,g) \le K Q_\alpha(f,f)^{\frac{1}{2}} Q_\alpha(g,g)^{\frac{1}{2}} \quad \text{for all} \quad f,g \in D(Q). \tag{9}$$

Proof. Since $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form, to prove the proposition, by [15,C hap.I,Sect.2] it suffices to show that there exists constants $\alpha_0 > 0, \gamma_0 < 1, C > 0$ such that
(I)

$$\int_X < B(x), \nabla f(x) >_H f(x)\mu(dx) + \int_X < \hat{B}(x), \nabla f(x) >_H f(x)\mu(dx) + \int_X c^-(x) f^2(x)\mu(dx)$$
$$\le \gamma_0 \mathcal{E}(f,f) + \alpha_0(f,f), \quad \forall f \in D(\mathcal{E}). \tag{10}$$

and
(II)

$$|\int_X < B(x) - \hat{B}(x), \nabla f(x) >_H g(x)\mu(dx)| \le C\mathcal{E}_1(f,f)^{\frac{1}{2}} \mathcal{E}_1(g,g)^{\frac{1}{2}} \quad \text{for all} \quad f,g \in \mathcal{F}C_b^\infty.$$

Let us first show (I). Since (10) is homogeneous, we only need to verify it in the case $||f||_2 = 1$. Applying Youngs inequality, $st \le slogs - s + e^t, s \ge 0$, we have

$$\int_X < B(x), \nabla f(x) >_H f(x)\mu(dx) + \int_X < \hat{B}(x), \nabla f(x) >_H f(x)\mu(dx) + \int_X c^-(x) f^2(x)\mu(dx)$$

$$\le (\int_X |B(x)|_H^2 f^2(x)\mu(dx))^{\frac{1}{2}} (\int_X |\nabla f(x)|_H^2 \mu(dx))^{\frac{1}{2}} + \frac{1}{q}\int_X (qc^-(x)) f^2(x)\mu(dx)$$
$$+ (\int_X |\hat{B}(x)|_H^2 f^2(x)\mu(dx))^{\frac{1}{2}} (\int_X |\nabla f(x)|^2 \mu(dx))^{\frac{1}{2}}$$
$$\le \frac{1}{c_1}\int_X |B(x)|_H^2 f^2(x)\mu(dx) + \frac{1}{2}c_1 \int_X |\nabla f(x)|^2\mu(dx) + \frac{1}{q}\int_X (qc^-(x)) f^2(x)\mu(dx)$$
$$+ \frac{1}{c_1}\int_X |\hat{B}(x)|_H^2 f^2(x)\mu(dx)$$
$$\le \frac{1}{c_1 p_1}[\int_X exp(p_1 |B(x)|_H^2)\mu(dx) + \int_X f^2 log f^2 \mu(dx) - 1]$$
$$+ \frac{1}{c_1 p_2}[\int_X exp(p_2 |\hat{B}(x)|_H^2)\mu(dx) + \int_X f^2 log f^2 \mu(dx) - 1]$$
$$+ \frac{1}{q}[\int_X exp(qc^-(x))\mu(dx) + 2\int_X f^2 log|f|\mu(dx) - 1] + \frac{1}{2}\mathcal{E}(f,f)$$
$$\le \alpha_0 + \gamma_0 \mathcal{E}(f,f) \tag{11}$$

where the last inequality follows by the defective Log-Sobolev inequality (DLS) and where

$$\alpha_0 = \frac{1}{c_1p_1}\int_X exp(p_1|B(x)|_H^2)\mu(dx) + \frac{1}{q}\int_X exp(qc^-(x))\mu(dx) - \frac{1}{c_1p_1} - \frac{1}{q}$$

$$+\frac{1}{c_1p_2}\int_X exp(p_2|\hat{B}(x)|_H^2)\mu(dx) - \frac{1}{c_1p_2} + (\frac{1}{q} + \frac{1}{c_1p_1} + \frac{1}{c_1p_2})2\lambda_2$$

$$\gamma_0 = (\frac{1}{q} + \frac{1}{c_1p_1} + \frac{1}{c_1p_2})2\lambda_1 + \frac{1}{2} < 1$$

Now we prove (II). Let $f, g \in \mathcal{F}C_b^\infty$. We may again assume that $||f||_2 = ||g||_2 = 1$. Then by Young's inequality and (DLS) for $p < \frac{1}{2}p_1 \wedge p_2$,

$$|\int_X < B - \hat{B}, \nabla f >_H g\mu(dx)|$$

$$\leq \frac{1}{\sqrt{p}}(\int_X (p|B-\hat{B}|_H^2)g^2\mu(dx))^{\frac{1}{2}}(\int_X |\nabla f|_H^2\mu(dx))^{\frac{1}{2}}$$

$$\leq \frac{1}{\sqrt{p}}[\int_X g^2logg^2\mu(dx) + \int_X exp(p|B-\hat{B}|_H^2)\mu(dx) - 1]^{\frac{1}{2}}\frac{1}{\sqrt{c_1}}\mathcal{E}(f,f)^{\frac{1}{2}}$$

$$\leq \frac{1}{\sqrt{p}}[2\lambda_1\mathcal{E}(g,g) + 2\lambda_2 + \int_X exp(p|B-\hat{B}|_H^2\mu(dx) - 1]^{\frac{1}{2}}\frac{1}{\sqrt{c_1}}\mathcal{E}(f,f)^{\frac{1}{2}}$$

$$\leq C\mathcal{E}_1(f,f)^{\frac{1}{2}}\mathcal{E}_1(g,g)^{\frac{1}{2}}$$

for some constant C.

The generator $\mathcal{L}_p$ of the quadratic form $(Q, D(Q))$, which can henristically be written as (2), is the lower order perturbation of the generator L

4 Convergence of semigroups

In this section, we study a sequence of quadratic forms and prove that the associated semigroups converge if the corresponding coefficients converge. The result has its own interest and will be also used in section 6. Let $A_n(x), A(x) : X \to \mathcal{L}(H)$, $B_n(x), B(x), \hat{B}_n(x), \hat{B}(x) : X \to H$, $c_n(x), c(x) : X \to R$ be measurable mappings. Introduce the following conditions.
(4.1)

$$c_1I \leq A_n(x) \leq c_2I$$

where I stands for the identity operator on H, c_1, c_2 are constants independent of n.

$$A_n(x) \to A(x) \quad \text{in} \quad \mathcal{L}(H), \quad \mu - a.e. \quad x \in X \quad \text{as} \quad n \to \infty$$

(4.2)
$$\sup_n \int_X exp(p_1|B_n|_H^2)\mu(dx) < \infty, \quad \sup_n \int_X exp(p_2|\hat{B}_n|_H^2)\mu(dx) < \infty$$
$$\int_X |B_n(x) - B(x)|_H^2 \, \mu(dx) \to 0, \quad \int_X |\hat{B}_n(x) - \hat{B}(x)|_H^2 \, \mu(dx) \to 0$$

$$\sup_n \int_X exp(qc_n^-(x))\mu(dx) < \infty, \quad \int_X |c_n(x) - c(x)|^2\mu(dx) \to 0, \qquad \text{as} \quad n \to \infty.$$

(4.3):
$$\int_X exp(\delta|B_n - B|_H^2(x))\mu(dx) \to 1, \quad \int_X exp(\delta|\hat{B}_n - \hat{B}|_H^2(x))\mu(dx) \to 1,$$
$$\int_X exp(\delta|c_n - c|(x))\mu(dx) \to 1,$$
$$\text{as} \quad n \to \infty \quad \text{for any} \quad \delta > 0$$

Define
$$Q_n(f,g) = \frac{1}{2}\int_X < A_n\nabla f, \nabla g >_H (x)\mu(dx) - \int_X < B_n(x), \nabla f(x) >_H g(x)\,\mu(dx)$$
$$- \int_X < \hat{B}_n(x), \nabla g(x) >_H f(x)\,\mu(dx) + \int_X c_n(x)f(x)g(x)\mu(dx). \tag{12}$$
$$D(Q_n) = D(\mathcal{E}) \tag{13}$$
Let $Q(f,g)$ be defined as in (7). Let $(T_t^n)_{t>0}, (T_t)_{t>0}$ be the strongly continuous semigroups associated with Q_n and Q. The associated resolvents are denoted by G_α^n and G_α.The adjoint objects will be denoted by the corresponding symbols with a hat.

Theorem 4.1 . Suppose that conditions (4.1),(4.2) hold and $(\frac{1}{q} + \frac{1}{c_1p_1} + \frac{1}{c_1p_2})2\lambda_1 < \frac{1}{2}$. If $f_n \to f$ weakly in $L^2(X,\mu)$, then there exists a $\alpha_0 > 0$ such that for $\alpha > \alpha_0$,
$$G_\alpha^n(f_n) \to G_\alpha(f) \qquad \text{weakly in} \quad (D(\mathcal{E}), \mathcal{E}_1) \tag{14}$$
as $n \to \infty$. The conclusion also holds for the adjoint resolvents.

Proof. By our assumption, it follows from the proof of Proposition 3.1, that there exist $\alpha_0 > 0$ such that if $\alpha > \alpha_0$,
$$c_\alpha^1 \mathcal{E}_1(f,f) \le Q_{n,\alpha}(f,f) \le c_\alpha^2 \mathcal{E}_1(f,f) \tag{15}$$
where $Q_{n,\alpha}(f,g) = Q_n(f,g) + \alpha(f,g)$ and c_α^1, c_α^2 are constants independent of n. This implies that, for all $\alpha > \alpha_0$,
$$\sup_n ||\alpha G_\alpha^n||_{L^2\to L^2} = C_\alpha < \infty \tag{16}$$

Fix $\alpha > \alpha_0$, we have

$$Q_{n,\alpha}(G_\alpha^n(f_n), G_\alpha^n(f_n)) = (f_n, G_\alpha^n(f_n)) \leq \alpha^{-1} C_\alpha \sup_n ||f_n||_2^2 < \infty \tag{17}$$

which yields by (15) that

$$\sup_n \mathcal{E}_1(G_\alpha^n(f_n), G_\alpha^n(f_n)) < \infty \tag{18}$$

Then by the Banach-Alaoglu theorem there exists a subsequence $(n_k)_{k\in N}$ and $h \in D(\mathcal{E})$ such that

$$G_\alpha^{n_k}(f_{n_k}) \to h \quad \text{weakly in} \quad (D(\mathcal{E}), \mathcal{E}_1)$$

So, it remains to show that $h = G_\alpha(f)$. For this end, we may replace the subsequence $(n_k)_{k\in N}$ by the whole sequence .

Claim. Let $g \in \mathcal{FC}^\infty$. Then

$$\lim_{n\to\infty}[Q_\alpha(G_\alpha^n(f_n), g) - Q_\alpha^n(G_\alpha^n(f_n), g)] = 0$$

Suppose that the claim has been proved. Then it follows that

$$Q_\alpha(h, g) = \lim_{n\to\infty} Q_\alpha(G_\alpha^n(f_n), g) = \lim_{n\to\infty} Q_\alpha^n(G_\alpha^n(f_n), g)$$

$$= \lim_{n\to\infty}(f_n, g) = (f, g) = Q_\alpha(G_\alpha(f), g)$$

for all $g \in \mathcal{FC}^\infty$. This gives $h = G_\alpha(f)$ and the proof is complete.
We now prove the claim. Observe that

$$Q_\alpha(G_\alpha^n(f_n), g) - Q_\alpha^n(G_\alpha^n(f_n), g)$$

$$= \frac{1}{2}\int_X < (A - A_n)\nabla G_\alpha^n(f_n), \nabla g >_H (x)\mu(dx) + \int_X < B_n(x) - B(x), \nabla G_\alpha^n(f_n)(x) >_H g(x)\, \mu(dx)$$

$$+ \int_X < \hat{B}_n(x) - \hat{B}(x), \nabla g(x) >_H G_\alpha^n(f_n)(x)\, \mu(dx) + \int_X (c(x) - c_n(x)) G_\alpha^n(f_n)(x) g(x) \mu(dx). \tag{19}$$

Using the facts that

$$\sup_n [\int_X |\nabla G_\alpha^n(f_n)|_H^2 \mu(dx) + \int_X (G_\alpha^n(f_n))^2 \mu(dx)] < \infty$$

and $\nabla g(x), g(x)$ are bounded, it is easy to see that every term of the right hand side of (19) goes to zero as $n \to \infty$. This completes the proof.

Corollary 4.2 : Keep the same assumptions as in Theorem 4.1 . There exists $\alpha_0 > 0$ such that for $\alpha > \alpha_0$, *as* $n \to \infty$,$G_\alpha^n \to G_\alpha$ strongly in $L^2(X, \mu)$, and hence $T_t^n \to T_t$ also strongly in $L^2(X, \mu)$ for all $t > 0$.

Proof. Fix any $f \in L^2(X,\mu)$, we know from Theorem 4.1 that $G_\alpha^n(f) \to G_\alpha(f)$,$\hat{G}_\alpha^n(f) \to \hat{G}_\alpha(f)$ weakly in $L^2(X,\mu)$. Applying Theorem 4.1 again, we get that $\hat{G}_\alpha^n(G_\alpha^n(f)) \to \hat{G}_\alpha(G_\alpha(f))$ weakly in $L^2(X,\mu)$.This yields

$$\lim_{n\to\infty}\int_X (G_\alpha^n(f))^2(x)\mu(dx) = \lim_{n\to\infty}\int_X f(x)\hat{G}_\alpha^n(G_\alpha^n(f))\mu(dx)$$

$$= \int_X f(x)\hat{G}_\alpha(G_\alpha(f))(x)\mu(dx) = \int_X (G_\alpha(f))^2\mu(dx) \tag{20}$$

which together with the weak convergence implies the strong convergence.

Theorem 4.3. Suppose the conditions (4.1), (4.2) and (4.3) are satisfied. Then there exists $\alpha_0 > 0$ such that for $\alpha > \alpha_0$, $f \in L^2(X,\mu)$,

$$G_\alpha^n(f) \to G_\alpha(f) \quad \text{in} \quad (D(\mathcal{E}), \mathcal{E}_1) \tag{21}$$

Proof. By Theorem 4.1 ,

$$G_\alpha^n(f) \to G_\alpha(f) \quad \text{weakly in} \quad (D(\mathcal{E}), \mathcal{E}_1) \tag{22}$$

By (15), we only need to prove

$$\lim_{n\to\infty} Q_{n,\alpha}(G_\alpha^n(f) - G_\alpha(f), G_\alpha^n(f) - G_\alpha(f)) = 0 \tag{23}$$

Note that

$$Q_{n,\alpha}(G_\alpha^n(f)-G_\alpha(f), G_\alpha^n(f)-G_\alpha(f)) = Q_\alpha(G_\alpha(f), G_\alpha^n(f)-G_\alpha(f)) - Q_{n,\alpha}(G_\alpha(f), G_\alpha^n(f)-G_\alpha(f))$$

$$= \frac{1}{2}\int_X < (A - A_n)\nabla G_\alpha(f), \nabla(G_\alpha^n(f) - G_\alpha(f)) >_H (x)\mu(dx)$$

$$+ \int_X < B_n(x) - B(x), \nabla G_\alpha(f)(x) >_H (G_\alpha^n(f)(x) - G_\alpha(f)(x))\,\mu(dx)$$

$$+ \int_X < \hat{B}_n(x) - \hat{B}(x), \nabla(G_\alpha^n(f) - G_\alpha(f))(x) >_H G_\alpha(f)(x)\,\mu(dx)$$

$$+ \int_X (c(x) - c_n(x))G_\alpha(f)(x)(G_\alpha^n(f) - G_\alpha(f))(x)\mu(dx) \tag{24}$$

The first term converges to zero by the dominated convergence theorem. Because of the similarity of the treatment for the remaining terms, we, for example , consider the second one.

$$|\int_X < B_n(x) - B(x), \nabla G_\alpha(f)(x) >_H (G_\alpha^n(f)(x) - G_\alpha(f)(x))\,\mu(dx)|$$

$$\leq (\int_X |B_n(x) - B(x)|_H^2 (G_\alpha^n(f)(x) - G_\alpha(f)(x))^2\mu(dx))^{\frac{1}{2}} \times (\int_X |\nabla G_\alpha(f)(x)|_H^2\mu(dx))^{\frac{1}{2}}$$

$$\leq C(\int_X |B_n(x) - B(x)|_H^2 (G_\alpha^n(f_n)(x) - G_\alpha(f)(x))^2\mu(dx))^{\frac{1}{2}} \tag{25}$$

where

$$C = \int_X |\nabla G_\alpha(f)|_H^2(x)\mu(dx))^{\frac{1}{2}} < \infty$$

Applying Young's inequality , the (DLS) inequality and Corollary 4.2, for any $\epsilon > 0$, it follows that

$$\lim_{n\to\infty} \int_X |B_n(x) - B(x)|_H^2 (G_\alpha^n(f)(x) - G_\alpha(f)(x))^2 \mu(dx)$$

$$\leq \lim_{n\to\infty} \epsilon[\int_X exp(\epsilon^{-1}|B_n(x) - B(x)|_H^2)\mu(dx) + 2\lambda_1 \sup_n \mathcal{E}(G_\alpha^n(f) - G_\alpha(f), G_\alpha^n(f) - G_\alpha(f))$$

$$+2||G_\alpha^n(f) - G_\alpha(f)||_2^2 log||G_\alpha^n(f) - G_\alpha(f)||_2 + 2\lambda_2||G_\alpha^n(f) - G_\alpha(f)||_2^2]$$

$$\leq \epsilon[2\lambda_1 \sup_n \{\mathcal{E}(G_\alpha^n(f) - G_\alpha(f), G_\alpha^n(f) - G_\alpha(f))\} + 1]$$

Letting ϵ go to zero, we obtain that the second term of (24) tends to zero. The proof is complete.

5 Stochastic Integration

Let $\chi = \{\Omega, \mathcal{F}, \mathcal{F}_t = \sigma(X_s, s \leq t), X_t, P_x, x \in X\}$ denote the diffusion process associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. ¿From now on, we assume that

$$\int_X l^2(x)\mu(dx) < \infty, \qquad \forall \quad l \in X^* \tag{26}$$

In this case, it is not difficult to see that l belongs to the domain $D(\mathcal{E})$ of the Dirichlet form. Hence, the following Fukushima s decomposition holds (see [6])

$$l(X_t) - l(X_0) = M_t^l + N_t^l, \qquad P_x - a.e, \text{ for } \quad q.e \quad x \in X. \tag{27}$$

where M_t^l is an $\mathcal{F}_t$ -square integrable martingale additive functional with

$$< M^{l_1}, M^{l_2} >_t = \int_0^t < A(X_s)\nabla l_1, \nabla l_2 >_H ds, \tag{28}$$

N_t^l is a continuous process of zero energy. Here q.e means quasi-everywhere. Set

$$\mathcal{H} = \{h : \Omega \times [0, \infty) \to H; h \quad \text{is} \quad \mathcal{F}. - \text{progressively}$$

$$\text{measurable and} \quad E_x[\int_0^t |h(s,\omega)|_H^2 ds] < \infty, \quad \mu - a.e, \text{ for any} \quad t > 0\}$$

We are gong to define forward and backward stochastic integrals with respect to the martingale part M for a class of integrands .(A similar construction is also carried out in [9] and [19], but only for the forward integral)

Step 1: Assume $h \in \mathcal{H}$ has the special form

$$h(t) = \sum_{i=1}^{m} h_i(t) l_i \tag{29}$$

where $l_i \in X^* \quad (\subset H)$ and h_i is progressively measurable with $E_x[\int_0^t h_i^2(s)ds] < \infty$, for $t > 0$ and $1 \leq i \leq m$.
The stochastic integral of h with respect to the martingale part M, denoted by $\int_0^t < h(s), dM_s >$, is defined by

$$\int_0^t < h(s), dM_s > := \sum_{i=1}^{m} \int_0^t h_i(s) dM_s^{l_i} \tag{30}$$

where $M_s^{l_i}$ is the martingale part of $l_i(X_t) - l_i(X_0)$ appearing in (27) .

It follows immediately from the definition that the integral is well-defined and enjoys the following properties

(i): $\int_0^t < h(s), dM_s >$ is an $\mathcal{F}_t$-square integrable, continuous martingale.
(ii): $< \int_0^{\cdot} < h(s), dM_s >, \int_0^{\cdot} < k(s), dM_s >>_t = \int_0^t < A(X_s)h(s), k(s) >_H ds$
In particular,

$$E_x[(\int_0^t < h(s), dM_s >)^2] = E_x[\int_0^t < A(X_s)h(s), h(s) >_H ds], \quad \mu - a.e. \quad x \in X$$

Step 2: General $h \in \mathcal{H}$.

Since the imbedding $J_H : X^* \hookrightarrow H$ is dense, there exists a family $\{l_n, n \geq 1\} \subset X^*$ such that $\{e_n = J_H(l_n), n \geq 1\}$ forms an orthonormal basis for H. Thus , for any $h \in \mathcal{H}$, we can find a sequence h^n, being of the form (29) , such that

$$E_x[\int_0^t |h^n(s) - h(s)|_H^2 ds] \to 0, \forall t > 0, \mu - a.e. \quad x \in X \tag{31}$$

Define

$$\int_0^t < h(s), dM_s > := \lim_{n \to \infty} \int_0^t < h^n(s), dM_s > \tag{32}$$

Then $\int_0^t < h(s), dM_s >$ is an $\mathcal{F}_t$-square integrable, continuous martingale under P_x for μ almost all $x \in X$ and enjoys the property (ii).

Remark 5.1: If $h \in \mathcal{H}$ is a functional of χ, i.e, $h(s) = f(X_s)$ for some f, then the integral $\int_0^t < h(s), dM_s >$ can be made to be an additive functional of χ. See [9] and [19].

Let P_μ be the probability measure on Ω defined by

$$P_\mu(\cdot) = \int_X P_x(\cdot)\mu(dx) \tag{33}$$

The reverse operator ,γ_t, is determined by

$$X_s(\gamma_t(\omega)) = X_{t-s}(\omega), \qquad s \le t. \tag{34}$$

Set, for $s \le t$,

$$\bar{\mathcal{F}}_s^t = \sigma(X_{t-u}, u \le s)$$

Define, for $t > 0$,

$$\bar{\mathcal{H}}^t = \{\bar{h} : \Omega \times [0,t] \to H; \bar{h}(s,\omega) \quad \text{is} \quad \bar{\mathcal{F}}_\cdot^t$$
$$\text{progressively measurable and} \quad E_\mu[\int_0^t |\bar{h}(s)|_H^2 ds] < \infty\}$$

E_μ stands for the expectation with respect to P_μ.

Since the diffusion process X_t is reversible under P_μ, for any $t > 0$, $l \in X^*$, $M_s^l(\gamma_t)$ is an $\bar{\mathcal{F}}_s^t$-square integrable continuous martingale under P_μ with

$$< M_\cdot^{l_1}(\gamma_t), M_\cdot^{l_2}(\gamma_t) >_s = \int_0^s < A(X_{t-u})\nabla l_1, \nabla l_2 >_H du \tag{35}$$

Therefore, the stochastic integral $\int_0^s < \bar{h}(u), dM_u(\gamma_t) >, s \le t$, for $\bar{h} \in \bar{\mathcal{H}}^t$, can be defined similarly and is an $\bar{\mathcal{F}}_s^t$- square integrable continuous martingale under P_μ with

$$< \int_0^\cdot < \bar{h}(u), dM_u(\gamma_t) >, \int_0^\cdot < \bar{k}(u), dM_u(\gamma_t) >_s = \int_0^s < A(X_{t-u})\bar{h}(u), \bar{k}(u) >_H ds \tag{36}$$

6 Probabilistic Representations of Semigroups

The aim of this section is to give a probabilistic representation for the semigroup associated with the quadratic form $(Q, D(Q))$ defined in section 3. As in the last section, $\{X_t, t \ge 0\}$ denotes the diffusion associated with the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$.

Lemma 6.1. Assume $\int_X exp(2\lambda_1 V(x))\mu(dx) < \infty$. Then

$$E_\mu[exp(\int_0^t V(X_s)ds)] \le exp\{(\frac{1}{2\lambda_1}\int_X exp(2\lambda_1 V(x))\mu(dx) + \frac{2\lambda_2 - 1}{2\lambda_1})t\} \tag{37}$$

where λ_1, λ_2 are the constants appeared in the Log-Sobolev inequality (5).

Proof. Assume first that V is bounded. Let

$$\alpha = \frac{1}{2\lambda_1}\int_X exp(2\lambda_1 V(x))\mu(dx) + \frac{2\lambda_2 - 1}{2\lambda_1}$$

Define

$$Q_V(f,g) = \mathcal{E}(f,g) - \int_X V(x)f(x)g(x)\mu(dx) + \alpha \int_X f(x)g(x)\mu(dx)$$

$$D(Q_V) = D(\mathcal{E})$$

Then $(Q_V, D(Q_V))$ is a coercive closed form on $L^2(X, \mu)$. To see this , since $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form we only need to prove that $(Q_V, D(Q_V))$ is positive definite (cf.[8]). Apply Young's inequality $st \le slogs - s + e^t$ to $t = 2\lambda_1 V(x)$ and $s = f^2(x)$ and the defective Log-Sobolev inequality (5) to get

$$\int_X V(x)f^2(x)\mu(dx) = \frac{1}{2\lambda_1}\int_X (2\lambda_1 V(x))f^2(x)\mu(dx)$$

$$\le \frac{1}{2\lambda_1}[\int_X (exp(2\lambda_1 V(x)) + f^2(x)logf^2(x) - f^2(x))\mu(dx)]$$

$$\le \frac{1}{2\lambda_1}[\int_X (exp(2\lambda_1 V(x))\mu(dx) + 2\lambda_1\mathcal{E}(f,f) + 2||f||_2^2 log||f||_2 + (2\lambda_2 - 1)||f||_2^2]$$

If $||f||_2 = 1$, then

$$\int_X V(x)f^2(x)\mu(dx) \le \mathcal{E}(f,f) + (\frac{1}{2\lambda_1}\int_X exp(2\lambda_1 V(x))\mu(dx) + \frac{2\lambda_2 - 1}{2\lambda_1})$$

Replacing a general f by $\frac{f}{||f||_2}$, we see that $(Q_V, D(Q_V))$ is positive definite. Let T_t^V denote the contraction semigroup on $L^2(X, \mu)$ associated with $(Q_V, D(Q_V))$. Since $V(x)$ is bounded, it is well-known that T_t^V admits the following Feynman-Kac representation

$$T_t^V f(x) = E_x[f(X_t)exp(\int_0^t V(X_s)ds)]exp(-\alpha t)$$

In particular, we have

$$exp(-\alpha t)E_\mu[exp(\int_0^t V(X_s)ds)] = \int_X T_t^V 1(x)\mu(dx)$$

$$\le (\int_X (T_t^V 1)^2(x)\mu(dx))^{\frac{1}{2}} \le (\int_X 1\mu(dx))^{\frac{1}{2}} = 1$$

This proves (37) for bounded $V(x)$. Now if $V(x)$ is not necessarily bounded, put $V_n(x) = V(x)\chi_{\{|V(x)|\le n\}}$. Apply (37) to V_n and let $n \to \infty$ to get the desired result by Fatou's lemma.

Let (T_t) denote the semigroup on $L^2(X, \mu)$ associated with the quadratic form (7) and $\mathcal{L}_p$ denote its generator.

Proposition 6.2. Let $\hat{B}(x) = 0$. Suppose $\int_X exp(p_1|B|_H^2(x))\mu(dx) < \infty$ and $\int_X exp(-qc(x))\mu(dx) < \infty$, for some $p_1 > 12\lambda_1\frac{1}{c_1}$ and $q > 8\lambda_1$. Then T_t is given by

$$T_t f(x) = E_x[f(X_t)exp\{\int_0^t < A^{-1}B(X_s), dM_s > -\frac{1}{2}\int_0^t < A^{-1}B(X_s), B(X_s) >_H ds$$

$$-\int_0^t c(X_s)ds\}] \quad \forall f \in \mathcal{B}_b(X), \quad \mu - a.e. \quad x \in X \tag{38}$$

In particular, if $c = 0$, the diffusion $\bar{\chi} = \{\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \bar{P}_x, x \in X\}$ generated by $\mathcal{L}_p$ is given by the Girsanov transform

$$d\bar{P}_x|_{\mathcal{F}_t} = exp(\int_0^t < A^{-1}B(X_s), dM_s > -\frac{1}{2}\int_0^t < A^{-1}B(X_s), B(X_s) >_H ds)dP_x|_{\mathcal{F}_t}$$

Proof. Let us first assume that $B(x), c(x)$ are bounded. Let $\hat{T}_t f(x)$ denote the right hand side of (38). Then

$$\int_X (\hat{T}_t f(x))^2 \mu(dx)$$

$$\leq \int_X E_x[exp\{2\int_0^t < A^{-1}B(X_s), dM_s > -2\int_0^t < A^{-1}B(X_s), B(X_s) >_H ds\}]$$

$$\times E_x[exp\{\int_0^t < A^{-1}B(X_s), B(X_s) >_H ds - 2\int_0^t c(X_s)ds\}f^2(X_t)]\mu(dx)$$

$$\leq e^{Mt}\int_X f^2(x)\mu(dx).$$

where M is a constant. This shows that $\hat{T}_t$ extends to a semigroup of bounded linear operators on $L^2(X, \mu)$,which is clearly strongly continuous.
Let $\hat{L}$ denote the generator of $\hat{T}_t$ on $L^2(X, \mu)$. To prove (38), it suffices to verify $(\hat{L}, D(\hat{L})) = (\mathcal{L}_p, D(\mathcal{L}_p))$. This already follows if we show $(\mathcal{L}_p, D(\mathcal{L}_p)) \subset (\hat{L}, D(\hat{L}))$, since for generators of strongly continuous semigroups, one can not extend the other unless they are equal.

Let $u \in D(\mathcal{L}_p)$ be bounded. We may assume that u is quasi-continuous. For $v \in D(\mathcal{E})$,we have

$$\mathcal{E}(u, v) = Q(u, v) + \int_X < B(x), \nabla u(x) >_H v(x)\mu(dx) - \int_X c(x)u(x)v(x)\mu(dx)$$

$$= \int_X (-\mathcal{L}_p u(x)+ < B(x), \nabla u(x) >_H -c(x)u(x))v(x)\mu(dx) \tag{39}$$

By Theorem 5.3.2 in [6],this implies that $u(X_t) - u(X_0)$ is a semimartingale with the following decomposition

$$u(X_t) - u(X_0) = M_t^u + \int_0^t (\mathcal{L}_p u(X_s) - < B, \nabla u >_H (X_s) + c(X_s)u(X_s))ds \tag{40}$$

where M_t^u is a square integrable martingale with the representation

$$M_t^u = \int_0^t < \nabla u(X_s), dM_s >, \quad P_x - a.e. \quad \text{for almost all} \quad x \in X$$

Set

$$\hat{Z}_t = exp\{\int_0^t < A^{-1}B(X_s), dM_s > -\frac{1}{2}\int_0^t < A^{-1}B(X_s), B(X_s) >_H ds - \int_0^t c(X_s)ds\}$$

By Ito's formula,

$$u(X_t)\hat{Z}_t = u(X_0) + \bar{M}_t + \int_0^t \hat{Z}_s \mathcal{L}_p u(X_s) ds \tag{41}$$

where

$$\bar{M}_t = \int_0^t \hat{Z}_s dM_s^u + \int_0^t u(X_s)\hat{Z}_s < A^{-1}B(X_s), dM_s >$$

is a local martingale . Hence, there exists a sequence τ_n of stopping times such that $\tau_n \uparrow \infty$ and $\bar{M}_{t\wedge\tau_n}$ is a martingale.
It follows from (41) that

$$E_x[u(X_{t\wedge\tau_n})\hat{Z}_{t\wedge\tau_n}] = u(x) + E_x[\int_0^{t\wedge\tau_n} \hat{Z}_s \mathcal{L}_p u(X_s) ds] \tag{42}$$

Observe that

$$E_x[\int_0^t \hat{Z}_s |\mathcal{L}_p u|(X_s)\, ds] \leq \int_0^t \hat{T}_s(|\mathcal{L}_p u|)(x)\, ds < \infty \tag{43}$$

for $\mu - a.e. \quad x \in X$.
Since u is bounded and $(\hat{Z}_{t\wedge\tau_n})_{n\in N}$ is uniformly integrable ,by the dominated convergence theorem, letting $n \to \infty$ in (42) we obtain that

$$\hat{T}_t u(x) - u(x) = \int_0^t \hat{T}_t \mathcal{L}_p u(x)\, ds \quad \text{for almost all} \quad x \in X \tag{44}$$

which implies that $u \in D(\hat{L})$ and that $\mathcal{L}_p u = \hat{L}u$. But any $u \in D(\mathcal{L}_p)$ can be approximated in graph norm by bounded $u_n \in D(\mathcal{L}_p)$. Hence

$$D(\mathcal{L}_p) \subset D(\hat{L}) \quad \text{and} \quad \mathcal{L}_p = \hat{L}|_{D(L)}. \tag{45}$$

Now we remove the boundness assumptions on B and c. Let $B_n(x) = B(x)\chi_{\{|B(x)|\leq n\}}$ and $c_n(x) = c(x)\chi_{\{|c(x)|\leq n\}}$. Define the quadratic forms $(Q^n, D(Q^n))$ as in (7) by replacing B and c by B_n and c_n. Let $\{T_t^n, t \geq 0\}$ be the corresponding semigroups. Then, for $f, g \in \mathcal{B}_b(X)$,

$$\int_X T_t^n f(x)g(x)d\mu(x) = E_\mu[f(X_t)g(X_0)exp\{\int_0^t < A^{-1}B_n(X_s), dM_s >$$

$$-\frac{1}{2}\int_0^t < A^{-1}B_n(X_s), B_n(X_s) >_H ds - \int_0^t c_n(X_s)ds\}] \tag{46}$$

Set,

$$Y_t^n = \int_0^t < A^{-1}B_n(X_s), dM_s > -\frac{1}{2}\int_0^t < A^{-1}B_n(X_s), B_n(X_s) >_H ds - \int_0^t c_n(X_s)ds\}$$

$$Y_t = \int_0^t < A^{-1}B(X_s), dM_s > -\frac{1}{2}\int_0^t < A^{-1}B(X_s), B(X_s) >_H ds - \int_0^t c(X_s)ds\}$$

It is easy to see that

$$\lim_{n\to\infty} E_\mu[(Y_t^n - Y_t)^2] = 0 \tag{47}$$

On the other hand,

$$\sup_n E_\mu[exp(2Y_t^n)] = E_\mu[exp\{2\int_0^t < A^{-1}B_n(X_s), dM_s > -4\int_0^t < A^{-1}B_n(X_s), B_n(X_s) >_H ds\}$$

$$\times exp\{3\int_0^t < A^{-1}B_n(X_s), B_n(X_s) >_H ds - 2\int_0^t c_n(X_s)ds\}]$$

$$\leq \sup_n (E_\mu[exp\{4\int_0^t < A^{-1}B_n(X_s), dM_s > -8\int_0^t < A^{-1}B_n(X_s), B_n(X_s) >_H ds\}])^{\frac{1}{2}}$$

$$\times (E_\mu[exp\{6\int_0^t < A^{-1}B_n(X_s), B_n(X_s) >_H ds - 4\int_0^t c_n(X_s)ds\}])^{\frac{1}{2}}$$

$$\leq exp\{(\frac{1}{4\lambda_1}\int_X exp(2\lambda_1(6 < A^{-1}B(x), B(x) >_H -4c(x)))\mu(dx) + \frac{2\lambda_2 - 1}{4\lambda_1})t\} < \infty$$

where we used Lemma 6.1. This shows that $exp(Y_t^n), n \geq 1$, is uniformly integrable. Combining this with (47),applying Corollary 4.2 and letting $n \to \infty$ in (46) we get

$$\int_X T_t f(x)g(x)d\mu(x) = E_\mu[f(X_t)g(X_0)exp\{\int_0^t < A^{-1}B(X_s), dM_s >$$

$$-\frac{1}{2}\int_0^t < A^{-1}B(X_s), B(X_s) >_H ds - \int_0^t c(X_s)ds\}] \tag{48}$$

which completes the proof.

To prove the next theorem, we assume that μ satisfies an integration by parts formula, that is, for any $k \in X^*$ there exists a function β_k such that $exp(p\beta_k) \in L^1(X, \mu)$ for some $p > 2\lambda_1$ and

$$\int_X \frac{\partial f}{\partial k}\mu(dx) = -\int_X f\beta_k\mu(dx), \quad \text{for all} \quad f \in \mathcal{F}\mathcal{C}_b^\infty \tag{49}$$

Actually, if (49) holds , it also holds for all $f \in D(\mathcal{E})$. Let us now recall the definition of the divergence in the $L^2(X, \mu)$-setting. Let $u \in L^2(X \to H, \mu)$. We say that u is in the domain of the divergence, $u \in D(div)$, if there exists a measurable function on X , denoted by $div(u)$, satifying

$$div(u) \in L^2(X, \mu), \tag{50}$$

$$\int_X < u, \nabla v > \mu(dx) = -\int_X div(u)(x)v(x)\mu(dx) \qquad \text{for } v \in D(\mathcal{E}). \tag{51}$$

$div(u)$ is called the divergence of the vector field u. As a consequence of the definition, if $u \in D(div), f \in \mathcal{F}\mathcal{C}_b^\infty$ then $fu \in D(div)$ and $div(fu) = f div(u) + < u, \nabla f >$.

Let $(\mathcal{E}, D(\mathcal{E}))$ be the Dirichlet form defined in (4). It is clear that $u \in D(L)$ if and only if

$$A\nabla u \in D(div) \quad \text{and} \quad L = \frac{1}{2}div(A\nabla u).$$

Condition 6.3: X^* is contained in the domain $D(L)$ of the operator.

Remark 6.4: This technical condition is obivously fullfilled if $A(x) : X \to \mathcal{L}(H)$ is smooth, for example, A has the form

$$A(x) = \sum_{i=1}^{m} f_i(x) A_i, \qquad \text{for some} \qquad f_i \in \mathcal{FC}_b^\infty, A_i \in \mathcal{L}(H)$$

Theorem 6.5. Suppose that the integration by parts formula (49) and condition 6.3 hold and $\int_X exp(p(|B|_H^2(x) + |\hat{B}|_H^2(x) - c(x)))\mu(dx) < \infty$, for some $p > 12\lambda_1$. Let $f, g \in L^2(X, \mu)$. T_t is given by

$$\int_X f(x) T_t g(x) \mu(dx) = E_\mu[g(X_t) f(X_0) \exp(\int_0^t < A^{-1}B(X_s), dM_s > + \int_0^t < A^{-1}\hat{B}(X_{t-s}), dM_s(\gamma_t) >$$

$$-\frac{1}{2}\int_0^t < A^{-1}(B - \hat{B})(X_s), (B - \hat{B})(X_s) >_H ds - \int_0^t c(X_s) ds)] \tag{52}$$

Proof. We split the proof into two steps.

Step1: $\hat{B}$ is of the following form

$$\hat{B}(x) = \sum_{i=1}^{m} g_i(x) e_i, \quad \text{for some } g_i \in \mathcal{FC}_b^\infty, e_i \in X^* \tag{53}$$

In this case,

$$Q(f, g) = \frac{1}{2}\int_X < A(x)\nabla f, \nabla g >_H (x)\mu(dx) - \int_X < B(x), \nabla f(x) >_H g(x)\, \mu(dx)$$

$$+ \int_X < \hat{B}(x), \nabla f(x) >_H g(x)\, \mu(dx) + \int_X div(\hat{B})(x) f(x) g(x) \mu(dx) + \int_X c(x) f(x) g(x) \mu(dx). \tag{54}$$

Since $div(\hat{B}) = \sum_{i=1}^m (< \nabla g_i, e_i > + \beta_{e_i} g_i)$, it follows by the exponential integrability condition on β_{e_i} that Theorem 6.2 applies to give that

$$T_t g(x) = E_x[g(X_t) \exp(\int_0^t < A^{-1}B(X_s), dM_s > - \int_0^t < A^{-1}\hat{B}(X_s), dM_s >$$

$$-\frac{1}{2}\int_0^t < A^{-1}(B - \hat{B})(X_s), (B - \hat{B})(X_s) >_H ds - \int_0^t c(X_s) ds - \int_0^t div(\hat{B})(X_s) ds)] \tag{55}$$

Thus to prove the theorem in this case, it is enough to show that

$$\int_0^t < A^{-1}\hat{B}(X_s), dM_s > + \int_0^t < A^{-1}\hat{B}(X_{t-s}), dM_s(\gamma_t) >= - \int_0^t div(\hat{B})(X_s) ds \tag{56}$$

For simplicity, let $t=1$. Define

$$\mathcal{FC}_b^\infty(X\to H)=\{\sum_{i=1}^m g_i e_i;\quad m\in N, g_i\in\mathcal{FC}_b^\infty\}$$

Let $\bar{B}_n=\sum_{i=1}^{m_n} g_i^n e_i\in\mathcal{FC}_b^\infty(X\to H)$,$n\geq 1$, be a sequence of vector fields converging to $A^{-1}\hat{B}$ in $L^2(X\to H,\mu)$. Such a sequence always exists , since $\mathcal{FC}_b^\infty$ is dense in $L^2(X,\mu)$.

It follows from the proof of the first step of Theorem 2.2.1 in [11] that

$$\int_0^1 <\bar{B}_n(X_s),dM_s>+\int_0^1<\bar{B}_n(X_{1-s}),dM_s(\gamma_1)>$$

$$=\sum_{i=1}^{m_n}[\int_0^1 g_i^n(X_s)dM_s^{l_i}+\int_0^1 g_i^n(X_{1-s})dM_s^{l_i}(\gamma_1)]$$

$$=-\sum_{i=1}^{m_n}\int_0^1 div(A(g_i^n\nabla l_i))(X_s)ds=-\int_0^1 div(A\bar{B}_n)(X_s)ds \tag{57}$$

where the divergence $div(A(g_i^n\nabla l_i))$ exists according to the condition 6.3.

As $n\to\infty$,the left hand side of (57) goes to

$$\int_0^1<A^{-1}\hat{B}(X_s),dM_s>+\int_0^1<A^{-1}\hat{B}(X_{1-s}),dM_s(\gamma_1)>\qquad \text{in } L^2(\Omega,P_\mu)$$

This implies that $d=\lim_{n\to\infty}-\int_0^1 div(A\bar{B}_n)(X_s)ds$ exists. We will complete the proof by showing that

$$d=-\int_0^1 div(\hat{B})(X_s)ds \tag{58}$$

We have, for $v\in D(\mathcal{E})$

$$\lim_{n\to\infty}-\int_X div(A\bar{B}_n)(x)v(x)\mu(dx)=\lim_{n\to\infty}\int_X<A\bar{B}_n,\nabla v>_H\mu(dx)$$

$$=\int_X<AA^{-1}\hat{B},\nabla v>_H\mu(dx)=-\int_X div(\hat{B})(x)v(x)\mu(dx) \tag{59}$$

Set $g_n=-div(A\bar{B}_n)$, $g=-div(\hat{B})$. Denote R_1 the resolvent operator associated with $(\mathcal{E},D(\mathcal{E}))$. Recall $\mathcal{E}_1(u,v)=\mathcal{E}(u,v)+(u,v)$. (59) is equivalent to

$$\mathcal{E}_1(R_1g_n,v)=\int_X g_n v\mu(dx)\to\int_X gv\mu(dx)=\mathcal{E}_1(R_1g,v)$$

In other words, $R_1g_n\to R_1g$ weakly in $(D(\mathcal{E}),\mathcal{E}_1)$. By the Banach-Sacks theorem, there exists a subsequence $\{R_1g_{n_k}\}$ so that

$$\frac{1}{m}\sum_{k=1}^m R_1g_{n_k}\to R_1g\qquad\text{strongly in}\quad (D(\mathcal{E}),\mathcal{E}_1)$$

Set $\bar{g}_m=\frac{1}{m}\sum_{k=1}^m g_{n_k}$. It follows from Lemma 5.1.2 and Corollary 1 to Theorem 5.2.2 in [6] that

$$R_1\bar{g}_{m_k}(X_s)\to R_1g(X_s),\quad M_s^{R_1\bar{g}_{m_k}}\to M_s^{R_1g}$$

uniformly on $[0,1]$ for some subsequence $(m_k)_{k\in N}$. We may assume that $(m_k)_{k\in N} = (m)_{m\in N}$.

On the other hand, one knows that

$$R_1\bar{g}_m(X_t) - R_1\bar{g}_m(X_0) = M_t^{R_1\bar{g}_m} + \int_0^t R_1\bar{g}_m(X_s)ds - \int_0^t \bar{g}_m(X_s)ds$$

$$R_1 g(X_t) - R_1 g(X_0) = M_t^{R_1 g} + \int_0^t R_1 g(X_s)ds - \int_0^t g(X_s)ds$$

Hence,

$$\lim_{m\to\infty}\int_0^1 \bar{g}_m(X_s)ds = \int_0^1 g(X_s)ds = -\int_0^1 div(\hat{B})(X_s)ds$$

This implies

$$d = \lim_{m\to\infty}\int_0^1 g_m(X_s)ds = \lim_{m\to\infty}\int_0^1 \bar{g}_m(X_s)ds = -\int_0^1 div(\hat{B})(X_s))ds$$

which ends step 1.

Step 2: The general situation.

Take $\hat{B}_n \in \mathcal{FC}_b^\infty(X\to H)$ such that

$$\int_X exp\{p|\hat{B}_n - \hat{B}|_H^2\}\mu(dx) \to 1, \text{as} \quad n\to\infty$$

The existence of a such sequence can be seen by a two-step argument. The first step is to approximate a bounded $\hat{B}$ by elements in $\mathcal{FC}_b^\infty(X\to H)$. The second step is to approximate $\hat{B}$ by bounded vector fields.

Let (T_t^n) be the semigroup corresponding to the quadratic form Q with $\hat{B}_n$ in place of $\hat{B}$. Then by step 1

$$\int_X fT_t^n g(x)\mu(dx) = E_\mu[g(X_t)f(X_0)\exp(\int_0^t < A^{-1}B(X_s), dM_s > + \int_0^t < A^{-1}\hat{B}_n(X_{t-s}), dM_s(\gamma_t) >$$

$$-\frac{1}{2}\int_0^t < A^{-1}(B-\hat{B}_n), (B-\hat{B}_n) >_H (X_s)ds - \int_0^t c(X_s)ds)] \tag{60}$$

Letting $n\to\infty$ in (60) the left hand side goes to $\int_X f(x)T_t g(x)\mu(dx)$ by Corollary 4.2 , the right hand side tends to the right hand side of (52) as in the proof of Proposition 6.2. This completes the proof.

Corollary 6.6. T_t is given by

$$T_t g(x) = E_x[g(X_t)\exp(\int_0^t < A^{-1}B(X_s), dM_s > + \int_0^t < A^{-1}\hat{B}(X_{t-s}), dM_s(\gamma_t) >$$

$$-\frac{1}{2}\int_0^t < A^{-1}(B-\hat{B})(X_s), (B-\hat{B})(X_s) >_H ds - \int_0^t c(X_s)ds)] \qquad \mu - a.e. \qquad x\in X$$

Before closing this section, we apply the above theorem to identifying the quadratic forms associated with a class of generalized Feynman-Kac semigroups. Recall that $\chi = \{\Omega, \mathcal{F}, \mathcal{F}_t = \sigma(X_s, s \leq t), X_t, P_x, x \in X\}$ is the diffusion process associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Let $u \in D(\mathcal{E})$. The Fukushima decomposition reads as

$$u(X_t) - u(X_0) = M_t^u + N_t^u, \qquad P_x - a.e. \quad \text{for} \quad q.e. \quad x \in X \tag{61}$$

where M_t^u is a square integrable martingale additive functional and N_t^u is a continuous additive functional of zero energy.

Define, for $f \in \mathcal{B}^+(X)$,

$$T_t^u f(x) = E_x[exp(N_t^u) f(X_t)], \qquad \mu - a.e. \quad x \in X \tag{62}$$

If $u \in D(L)$, then $N_t^u = \int_0^t Lu(X_s)ds = \frac{1}{2}\int_0^t div(A(X_s)\nabla u(X_s))ds$. In this case , T_t^u is just the usual Feynman-Kac semigroup. For general $u \in D(\mathcal{E})$, Lu does not exist as an ordinary function, but can be understood as a distribution. Therefore, $T_t^u, t > 0$, is often called generalized Feynman-Kac semigroup.

Proposition 6.7. Suppose that the integration by parts formula (49) and condition 6.3 hold. If $\int_X exp(p|\nabla u|_H^2(x))\mu(dx) < \infty$, for some $p > 12\lambda_1$, then $T_t^u, t > 0$ is the strongly continuous semigroup on $L^2(X, \mu)$ associated with the quadratic form

$$Q(f,g) = \frac{1}{2}\int < A(x)\nabla f(x), \nabla f(x) >_H \ \mu(dx) + \frac{1}{2}\int < A(x)\nabla u(x), \nabla(fg)(x) >_H \ \mu(dx)$$
$$D(Q) = D(\mathcal{E}) \tag{63}$$

Proof. Let P_μ be defined as in Section 5 and $u \in D(\mathcal{E})$. Under P_μ, the Lyons-Zheng decomposition takes the form (See [14]):

$$u(X_s) - u(X_0) = \frac{1}{2}M_s^u - \frac{1}{2}(M_t^u(\gamma_t) - M_{t-s}^u(\ gamma_t)), \quad 0 \leq s \leq t. \tag{64}$$

where γ_t is the reverse operator and M_t^u is the martingale part in (61).

Comparing with (61), we get that

$$N_t^u = -\frac{1}{2}(M_t^u + M_t^u(\gamma_t)), \quad P_x - a.e. \qquad \mu - a.e. \quad x \in X \tag{65}$$

Let (T_t) denote the semigroup associated with the quadratic form (63). By Theorem 6.5, for $f, g \in L^2(X, \mu)$,

$$\int_X f(x)T_t g(x)\mu(dx) = E_\mu[g(X_t)f(X_0)\exp(-\frac{1}{2}\int_0^t < \nabla u(X_s), dM_s >$$

$$-\frac{1}{2}\int_0^t < \nabla u(X_{t-s}), dM_s(\gamma_t) >)] = E_\mu[g(X_t)f(X_0)\exp(-\frac{1}{2}M_t^u - \frac{1}{2}M_s^u(\gamma_t))]$$

$$= E_\mu[g(X_t)f(X_0)\exp(N_t^u)] = \int_X f(x)T_t^u g(x)\mu(dx) \tag{66}$$

which implies the desired conclusion.

7 Hyperbound Estimates of Perturbed Semigroups

In this section, we use the probabilistic representation in last section to obtain a hyperbound estimate for (T_t). Let $P_t, t > 0$, denote the semigroup associated with the Dirichlet form $(\mathcal{E}, D(\mathcal{E})$.

Since $(\mathcal{E}, D(\mathcal{E})$ satisfies the log-Soblev inequality (LS), it follows from Theorem 3.7 in [8] that for any $q > 1$,

$$||P_t||_{q\to p(t)} \le exp(\frac{\lambda_2}{\lambda_1}t) \tag{67}$$

where $p(t) = 1+exp(\frac{2}{\lambda_1}t)(q-1)$ and $||P_t||_{q\to p(t)}$ stands for the operator norm from $L^q(X,\mu)$ into $L^{p(t)}(X,\mu)$.

Lemma 7.1.Suppose

$$\int_X exp(p|B|_H^2(x) + p|\hat{B}|_H^2(x) - pc(x))\mu(dx) < \infty \qquad \text{for any} \quad p > 1$$

Then for $p > 1$,

$$E_\mu[\exp(p\int_0^t < A^{-1}B(X_s), dM_s > +p\int_0^t < A^{-1}\hat{B}(X_{t-s}), dM_s(\gamma_t) >$$

$$-p\frac{1}{2}\int_0^t < A^{-1}(B-\hat{B})(X_s), (B-\hat{B})(X_s) >_H ds - p\int_0^t c(X_s)ds)]$$

$$\le exp\{\frac{1}{3}(\frac{1}{2\lambda_1}\int_X exp(2\lambda_1(\frac{9}{2}p^2 < A^{-1}B, B >_H (x) + \frac{9}{2}p^2 < A^{-1}\hat{B}, \hat{B} >_H (x) - 3pc(x)))\mu(dx)$$

$$+\frac{2\lambda_2 - 1}{2\lambda_1})t\} \tag{68}$$

Proof. Note that

$$E_\mu[\exp(p\int_0^t < A^{-1}B(X_s), dM_s > +p\int_0^t < A^{-1}\hat{B}(X_{t-s}), dM_s(\gamma_t) >$$

$$-p\frac{1}{2}\int_0^t < A^{-1}(B-\hat{B})(X_s), (B-\hat{B})(X_s) >_H ds - p\int_0^t c(X_s)ds)]$$

$$\le E_\mu[\exp(p\int_0^t < A^{-1}B(X_s), dM_s > -\frac{3}{2}p^2\int_0^t < A^{-1}B(X_s), B(X_s) >_H ds)$$

$$\times \exp(p \int_0^t < A^{-1}\hat{B}(X_s), dM_s(\gamma_t) > -\frac{3}{2}p^2 \int_0^t < A^{-1}\hat{B}(X_s), \hat{B}(X_s) >_H ds)$$

$$\times \exp(\frac{3}{2}p^2 \int_0^t < A^{-1}B(X_s), B(X_s) >_H ds + \frac{3}{2}p^2 \int_0^t < A^{-1}\hat{B}(X_s), \hat{B}(X_s) >_H ds - p \int_0^t c(X_s)ds)]$$

By the Hölder inequality,

$$\leq (E_\mu[\exp(3p \int_0^t < A^{-1}B(X_s), dM_s > -\frac{9}{2}p^2 \int_0^t < A^{-1}B(X_s), B(X_s) >_H)ds])^{\frac{1}{3}}$$

$$\times (E_\mu[\exp(3p \int_0^t < A^{-1}\hat{B}(X_{t-s}), dM_s(\gamma_t) > -\frac{9}{2}p^2 \int_0^t < A^{-1}\hat{B}(X_s), \hat{B}(X_s) >_H)ds])^{\frac{1}{3}}$$

$$\times (E_\mu[\exp(\frac{9}{2}p^2 \int_0^t < A^{-1}B(X_s), B(X_s) >_H ds + \frac{9}{2}p^2 \int_0^t < A^{-1}\hat{B}(X_s), \hat{B}(X_s) >_H ds - 3p \int_0^t c(X_s)ds)])^{\frac{1}{3}}$$

Using the property of exponential martingales and lemma 6.1,

$$\leq exp\{\frac{1}{3}(\frac{1}{2\lambda_1} \int_X exp(2\lambda_1(\frac{9}{2}p^2 < A^{-1}B, B >_H (x) + \frac{9}{2}p^2 < A^{-1}\hat{B}, \hat{B} >_H (x) - 3pc(x)))\mu(dx)$$

$$+ \frac{2\lambda_2 - 1}{2\lambda_1})t\} \tag{69}$$

Theorem 7.2. Suppose that the integration by parts formula (49) , condition 6.3 hold and

$$\int_X exp(q|B|_H^2(x) + q|\hat{B}|_H^2(x) - qc(x))\mu(dx) < \infty \qquad \text{for any} \quad q > 1$$

Let $b > a > 1$. If $t \geq \frac{\lambda_1}{2} log(\frac{p^2}{(\frac{b}{b-1} - p)(a-p)})$, then

$$||T_t||_{a \to b} \leq exp\{\frac{1}{3}\frac{p-1}{p}(\frac{1}{2\lambda_1} \int_X exp(2\lambda_1(\frac{9}{2}(\frac{p}{p-1})^2 < A^{-1}B, B >_H (x) + \frac{9}{2}(\frac{p}{p-1})^2 < A^{-1}\hat{B}, \hat{B} >_H (x)$$

$$-3\frac{p}{p-1}c(x)))\mu(dx) + \frac{2\lambda_2 - 1}{2\lambda_1})t\} \times exp(\frac{1}{p}\frac{\lambda_2}{\lambda_1}t) \tag{70}$$

where p is any number belonging to the interval $(1, min(a, \frac{b}{b-1}))$

Proof. Let a, b, p be as in the statement of the theorem. For any $d > 1$, d' denotes the number such that $\frac{1}{d} + \frac{1}{d'} = 1$. Let $f, g \in \mathcal{B}_b(X)$. By Theorem 6.5,

$$\int_X f(x)T_t g(x)\mu(dx) = E_\mu[g(X_t)f(X_0)\exp(\int_0^t < A^{-1}B(X_s), dM_s > + \int_0^t < A^{-1}\hat{B}(X_{t-s}), dM_s(\gamma_t) >$$

$$-\frac{1}{2}\int_0^t < A^{-1}(B-\hat{B})(X_s), (B-\hat{B})(X_s) >_H ds - \int_0^t c(X_s)ds)] \tag{71}$$

It then follows from the Hölder inequality that

$$|\int_X f(x)T_t g(x)\mu(dx)|$$

$$\leq (E_\mu[g^p(X_t)f^p(X_0)])^{\frac{1}{p}}(E_\mu[\exp(p'\int_0^t < A^{-1}B(X_s), dM_s > + p'\int_0^t < A^{-1}\hat{B}(X_{t-s}), dM_s(\gamma_t) >$$

$$-p'\frac{1}{2}\int_0^t < A^{-1}(B-\hat{B})(X_s), (B-\hat{B})(X_s) >_H ds - p'\int_0^t c(X_s)ds)])^{\frac{1}{p'}} \tag{72}$$

By Lemma 7.1,

$$\leq (f^p, P_t(g^p))^{\frac{1}{p}} C_t \tag{73}$$

where

$$C_t = exp\{\frac{1}{3}\frac{1}{p'}(\frac{1}{2\lambda_1}\int_X exp(2\lambda_1(\frac{9}{2}p'^2 < A^{-1}B, B >_H (x) + \frac{9}{2}p'^2 < A^{-1}\hat{B}, \hat{B} >_H (x) - 3p'c(x)))\mu(dx)$$

$$+\frac{2\lambda_2 - 1}{2\lambda_1})t\}$$

Set $A = \frac{b'}{p}$. (72) is less than

$$C_t|f|_{b'}(\int_X (P_t(g^p)(x))^{A'}\mu(dx))^{\frac{1}{A'p}} \leq |f|_{b'}||P_t||^{\frac{1}{p}}_{\frac{a}{p}\to A'}|g|_a \tag{74}$$

Applying (67) for $q = \frac{a}{p}$, $p(t) = A'$, we have that if $t \geq \frac{\lambda_1}{2}log(\frac{p^2}{(\frac{b}{b-1}-p)(a-p)})$, (72) is less than

$$C_t exp(\frac{\lambda_2}{\lambda_1}\frac{1}{p}t)|f|_{b'}|g|_a$$

which yields the conclusion of the theorem.

References

[1] S.Albeverio,Y.G.Kondratiev, M.Röckner, Dirichlet operators via stochastic analysis, J.Funct. Anal.128 (1995), 102-138.

[2] S.Albeverio, M.Röckner, Stochastic differential equations in infinite dimensions: Solutions via Dirichlet forms. Probab. Theory Related Fields 89 (1991) 347-386.

[3] S.Albeverio, M.Röckner, T.S.Zhang, Girsanov transform for symmetric diffusions with infinite dimensional state space, The Annals of Probability 21:2 (1993) 961-978.

[4] S.Aida, I.Shigekawa, Logarithmic Sobolev Inequalities and Spectal Gaps: Perturbation Theory, J.Funct.Anal. (1994) 448-475.

[5] V.Bogachev, M.Röckner and T.S.Zhang, Existence of invariant measures: A Quadratic form approach. Preprint, Bielefeld,1997.

[6] M.Fukushima, Dirichlet forms and Markov processes. Amsterdam-Oxford-New York: North Holland. 1980.

[7] J.Glover, M.Rao, H.Sikic and R.Song, Quadratic forms corresponding to the generalized Schödinger semigroups, J.Funct.Anal. 125:2(1994) 358-379.

[8] L.Gross,Logarithmic Sobolev Inequalities and Contractive Properties of Semigroups, in (Editors) G.Dell ntonio, U.Mosco, Dirichlet Forms, Springer-Verlag Berlin Heidelberg 1993.

[9] , M.Hino, Existence of invariant measures for diffusion processes on a Wiener space, preprint 1997.

[10] N.Ikeda and S.Watanable, Stochastic Differential Equations and Diffusion Processes, North-Holland and Kodansha,Amsterdam and Tokyo, 1981.

[11] T.J.Lyons and T.S.Zhang: Decomposition of Dirichlet processes and Its Applications, The Annals of Probability 22:1(1994)494-524.

[12] T.J.Lyons and T.S.Zhang: Convergence of non-symmetric Dirichlet processes, Stochastics and Stochastics Reports 57 (1996) 159-167.

[13] T.J.Lyons,J.Lunt and T.S.Zhang, Integrability of Functionals of Dirichlet Processes ,Probabilistic Representations of Semigroups and Estimates of Heat Kernels, J.Funct. Anal. 153:2 (1998) 320-342.

[14] T.J.Lyons, W.A. Zheng, Acrossing estimate for the canonical process on a Dirichlet space and tightness result, Colloque Paul Levy sur les processes stochastoque, Asterisque (1988) 157-158, 248-272.

[15] Z.M.Ma and M.Röckner, Introduction to the Theory of (Non.Symmetric) Dirichlet Forms, Springer-Verlag Berlin Heidelberg, 1992.

[16] M.Röckner and T.S. Zhang: Uniqueness of generalized Schrödinger operators and applications, Journal of Fuctional Analysis 105(1992) 187-231.

[17] M.Röckner and T.S.Zhang, Convergence of Operators Semigroups Generated by Elliptic Operators, Osaka J.Math. 34(1997) 923-932.

[18] B.Simon and R.Høegh-Krohn, Hypercontractive semigroups and two dimensional self-coupled bose fields, J.Funct.Anal. 9(1972) 121-180.

[19] W.Stannat, The Theory of Generalized Dirichlet Forms and its Applications in Analysis and Stochastics. Ph.D thesis, Bielefeld, 1997.

[20] T.S.Zhang, Generalized Feynman-Kac Semigroups, Associated Quadratic Forms and Asymptotic Properties, Preprint 1998, Haugesund.