Coxeter Decompositions of Hyperbolic Tetrahedra.

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Introduction

Let P be a polyhedron in a hyperbolic space H^3.

Definition 1. *A polyhedron P is a* **Coxeter polyhedron** *if all dihedral angles of P are the integer parts of π.*

Definition 2. *A polyhedron P admits a* **Coxeter decomposition** *if P can be tiled by N Coxeter polyhedra F_i,...,F_N ($1 < N < \infty$), such that any two tiles F_i and F_j with a common face are symmetric with respect to this face.*

In this paper we classify all Coxeter decompositions of hyperbolic tetrahedra.

Definitions

All polyhedra F_i in a Coxeter decomposition are obviously congruent. We call these polyhedra **fundamental** and denote by F. A plane containing a face of a fundamental polyhedron will be called **mirror** if it contains no face of P.

Remark 1. *A Coxeter decomposition of any polyhedron can be prolonged to the decomposition of the whole hyperbolic space (to define a Coxeter decomposition of the space take definition 2 without the condition $N < \infty$).*

In this paper we deal with polyhedra bounded by the mirrors of a Coxeter decomposition of some polyhedron or the space.

Definition 3. *Fix a Coxeter decomposition of a polyhedron P.*

A **vertex** *A of P is* **fundamental** *if it does not belong to any mirror.*

A **dihedral angle** *of P made up by faces α and β is called* **fundamental** *if $\alpha \cap \beta$ does not belong to any mirror.*

Definition 4. *We say that a point $x \in F_i$* **belongs to a neighborhood** *of a vertex A of P if $A \in F_i$.*

Remark 2. *All the definitions above can be used for polytopes of arbitrary dimension, in particular, for polygons.*

Notation

$N > 1$ is a number of the decomposition tiles F in P.
$Vol(T)$ is a volume of a tetrahedron T.
k-edge is an edge of the fundamental tetrahedron with dihedral angle $\frac{\pi}{k}$.
k-l-m-vertex is a vertex of the fundamental tetrahedron, such that the dihedral angles at the edges intersecting in this vertex are equal to $\frac{\pi}{k}$, $\frac{\pi}{l}$ and $\frac{\pi}{m}$.
A dihedral angle $\frac{k\pi}{q}$ is an angle decomposed into k parts of size $\frac{\pi}{q}$.

The Coxeter tetrahedra are represented by their Coxeter diagrams. Denote any Coxeter hyperbolic tetrahedra by H_i, where i is a number of this tetrahedron in Table 1.

1 Fundamental polyhedron

Lemma 1. *(Obvious).*

1) Let P be a polyhedron admitting a Coxeter decomposition. Then any polyhedron contained in P admits a Coxeter decomposition too.

2) Let P be a polyhedron admitting a Coxeter decomposition. If all dihedral angles of P are fundamental, then P is a Coxeter polyhedron.

Notation. By $\alpha \cap \beta$ we mean the intersection of the sets α and β in the *inner* part of $\mathbf{H}^3$. Let α be a face of a polyhedron, denote by $\overline{\alpha}$ a plane containing α.

Lemma 2. *Let F be a Coxeter polyhedron.*
Then either F is a tetrahedron or F has two faces α and β such that $\overline{\alpha} \cap \overline{\beta} = \emptyset$.

Proof. It is well known that for any polyhedron without obtuse angles

$$\alpha \cap \beta = \emptyset \quad \Rightarrow \quad \overline{\alpha} \cap \overline{\beta} = \emptyset.$$

Coxeter polyhedra have no obtuse angles. Thus, it is enough to prove that F has faces α and β such that $\alpha \cap \beta = \emptyset$.

Suppose that $\alpha \cap \beta \neq \emptyset$ for any faces α and β of F. Let A be an arbitrary vertex of F. Let α_1,...,α_k be a complete set of faces of F such that $A \in \alpha_i$. Let β be any face of P such that $A \notin \beta$. By assumption, $\alpha_i \cap \beta \neq \emptyset \quad \forall i = 1, ..., k$. Therefore, F has no faces except α_i and β, i.e. F is a pyramid.

Suppose that A is not an ideal vertex of F. It is well known that any bounded Coxeter polyhedron is simple. Thus, the pyramid F is a tetrahedron and everything is proved.

Suppose that A is an ideal vertex of F and the pyramid F is not a tetrahedron. Take two faces (say, α_1 and α_3) having no common edges. Since A does not belong to $\mathbf{H}^3$, $\alpha_1 \cap \alpha_3 = \emptyset$.

□

Lemma 3. *Let P be a tetrahedron. Then F is a tetrahedron.*

Proof. By lemma 2, F has two faces α and β such that $\overline{\alpha} \cap \overline{\beta} = \emptyset$. Take any fundamental polyhedron F_0 in P; let α_0 and β_0 be its disjoint faces. Consider a sequence of fundamental tetrahedra $F_i \in P$, $i \in \mathbf{Z}$, such that $\alpha_i = \alpha_{i+1}$, if i is odd, and $\beta_i = \beta_{i+1}$, if i is even (see Fig 1).

The sequence is finite, since P contains only finitely many fundamental tetrahedra. Let F_k and F_s be the endpoints of the sequence. Then α_k or β_k belongs to some face γ of P. Similarly, α_s or β_s belongs to some face δ of P. Obviously, $\overline{\alpha}_i \cap \overline{\beta}_j = \emptyset \quad \forall i, j$. But γ intersects δ, since P is a tetrahedron. Contradiction.

□

Lemma 4. *Let F be a hyperbolic Coxeter tetrahedron. Then there are only finitely many tetrahedra P admitting a Coxeter decomposition with fundamental tetrahedron F.*

Proof. Any hyperbolic tetrahedron is completely determined by its dihedral angles. The dihedral angles of P should be multiples of the dihedral angles of F. Therefore, there are finitely many admissible sets of dihedral angles of P.

□

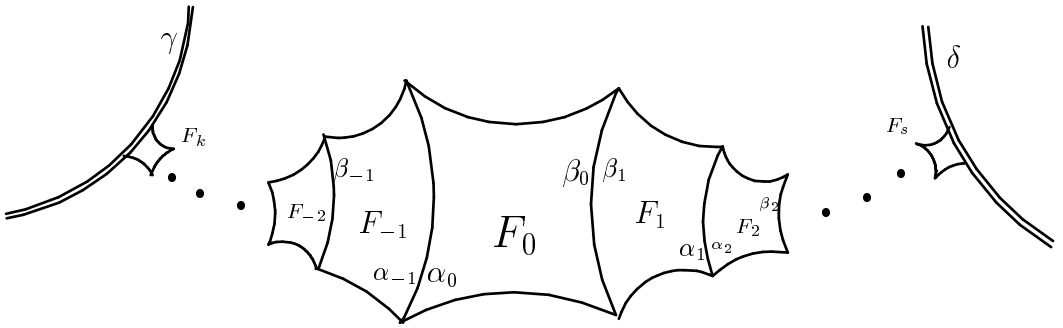

Figure 1.

1.1 Volumes

One of the most important properties of the fundamental polyhedron is the following **volume property**: a volume of P should be a multiple of a volume of F, i.e. $\frac{Vol(P)}{Vol(F)} \in \mathbf{Z}$.

This condition gives a way to conclude that some tetrahedra cannot tessellate some other tetrahedra. For instance, the Coxeter diagrams of F and P should be different (since volumes of F and P are completely determined by the dihedral angles of F and P).

The volumes of the hyperbolic Coxeter tetrahedra are calculated in paper [1]. We reprint the list of volumes of hyperbolic Coxeter tetrahedra in Table 1.

2 Types of Coxeter decompositions

There are three types of Coxeter decompositions:

Definition 5. *If P and any non-fundamental tetrahedron in P has a non-fundamental dihedral angle, then the decomposition is of a* **first type**,
if all dihedral angles of P are fundamental, then the decomposition is of a **second type**,
if P has a non-fundamental dihedral angle, but there is a non-fundamental tetrahedron $T \neq P$ in P without non-fundamental dihedral angles, then the decomposition is of a **third type**.

The decompositions of the first and the second types will be studied separately. The decompositions of the third type will be constructed from the decompositions of the first and the second types.

2.1 Decompositions of the first type

Consider a Coxeter decomposition of a tetrahedron P of the first type. By the definition P has a non-fundamental dihedral angle. Any mirror subdividing this dihedral angle cut P into two smaller tetrahedra P_1 and P_2.

We can find the decomposition of P, if we know the decompositions of P_1 and P_2. By the definition each of these tetrahedra either fundamental or has a non-fundamental dihedral angle. In the last case it can be subdivided into two smaller tetrahedra. We get a process of subdividing the tetrahedra into smaller and smaller ones. This process terminates only when P is decomposed into fundamental tetrahedra.

Inverting the process we can construct the larger tetrahedra from smaller ones. This can be done by the following inductive procedure.

2.1.1 Inductive procedure

Step 0. Take a fundamental tetrahedron F, let $P_0 = F$.

Step 1. Construct all tetrahedra $P_1, ..., P_l$ admitting a Coxeter decomposition which consist of two copies of P_0.

Step 2. Construct all tetrahedra $P_{l+1}, ..., P_k$ from P_1 and P_0 and all tetrahedra $P_{k+1}, ..., P_s$ from two copies of P_1.

. . . . . .

Step m. Suppose that after $m-1$ steps we have a sequence of tetrahedra $F = P_0, P_1, ..., P_m, ..., P_n$. Let $T_1^{i,j}$,...,$T_{k_{i,j}}^{i,j}$ be all the tetrahedra which we can obtain glueing P_i and P_j together. Suppose that for any $i, j < m-1, 1 \leq k \leq k_{i,j}$ any of $T_k^{i,j}$ is in the list P_0,...,P_n. Note that these conditions are satisfied after the steps 0, 1 and 2. Then at the m-th step we construct all the tetrahedra from P_{m-1} and P_i, for all $i \leq m-1$ (we do not put a new tetrahedron P_r in the sequence if $P_r = P_t$, $t < r$, and P_r and P_t have the same decomposition).

By lemma 4 any Coxeter tetrahedron can be fundamental for only finitely many decompositions of tetrahedra. Hence, the procedure is finite (that is we obtain $m = n$ after finitely many steps). Clearly, we get by this procedure the complete list of the Coxeter decompositions of the fist type with fixed fundamental tetrahedron F.

Since there are finitely many Coxeter tetrahedra in H^3, we can do this procedure for all Coxeter tetrahedra. Thus, we get *all* Coxeter decompositions of the first type.

This procedure is realized by a program. The result of the procedure obtained via computer is represented in Fig. 6–7.

2.2 Decompositions of the second type

Suppose that P admits a Coxeter decomposition without non-fundamental dihedral angles. It is evident that P is a Coxeter tetrahedron in this case.

Such decompositions can be of two types: either all vertices of P are fundamental or some vertices of P are non-fundamental.

2.2.1 Decompositions of the tetrahedra without non-fundamental vertices

Let P be a tetrahedron admitting a Coxeter decomposition without non-fundamental vertices. Then any vertex of P belongs to a unique fundamental tetrahedron. Thus, we have a following:

Lemma 5. *(Subdiagram property).*

Let P be a tetrahedron admitting a Coxeter decomposition without non-fundamental vertices, F be a fundamental tetrahedron, $\overline{P}$ and $\overline{F}$ be the Coxeter diagrams of P and F. Then for any node v in $\overline{P}$ there is a node w in $\overline{F}$ such that

$$\overline{P} \setminus v = \overline{F} \setminus w$$

(deleting the node from the Coxeter diagram we delete all the edges containing this node).

Proof. Any vertex of P belongs to a unique fundamental tetrahedron. Hence, three faces of F belong to the faces of P. Thus, three of dihedral angles of F coincide with the dihedral angles of P.

□

Lemma 6. *If all the vertices of P are fundamental, then $N \geq 8$.*

Proof. Let P be a tetrahedron $A_1A_2A_3A_4$. Consider any vertex A_i of the tetrahedron P. A_i belongs to a unique fundamental tetrahedron $A_iK_iL_iM_i$ (see Fig.2). Let $K_iL_iM_iO_i$ be a fundamental tetrahedron symmetric to $A_iK_iL_iM_i$ with respect to the plane $K_iL_iM_i$. It is easy to see that the tetrahedra $A_iK_iL_iM_i$ and $K_iL_iM_iO_i$ cannot coincide with $A_jK_jL_jM_j$ and $K_jL_jM_jO_j$ if $i \neq j$ (look at the plane $K_iL_iM_i$ to find a vertex A_i). Therefore, $N \geq 2 \cdot 4 = 8$.

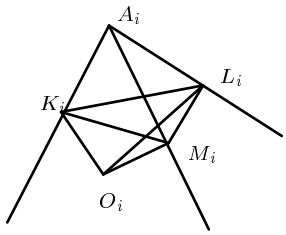

Figure 2.

□

Remark 3. *There is a Coxeter decomposition of an Euclidean tetrahedron without non-fundamental vertices and $N = 8$.*

2.2.2 Decompositions of the second type with non-fundamental vertices

Lemma 7. *(Wide subdiagram property).*

Let P be a tetrahedron admitting a Coxeter decomposition of the second type with some non-fundamental vertices, F be a fundamental tetrahedron, $\overline{P}$ and $\overline{F}$ be the Coxeter diagrams of P and F. Let p_v and F_w be a triangles corresponding to the subdiagrams $\overline{P} \setminus v$ and $\overline{F} \setminus w$. Then for any node v in $\overline{P}$ there exists a node w in F such that

either $\overline{P} \setminus v = \overline{F} \setminus w$,

or p_v admits a Coxeter decomposition without non-fundamental vertices and f_v is a fundamental triangle for this decomposition.

Proof. Let α be a face of P corresponding to a node v in $\overline{P}$, let A be a vertex of P opposite to α. If A is fundamental, then $\overline{P} \setminus v = \overline{F} \setminus w$ (see proof of lemma 5).

Suppose that A is non-fundamental. Consider a small sphere s centered in A such that any mirror intersecting s contains A (if A is an ideal vertex, consider a horosphere centered in A). A trace of the decomposition of P on this sphere (horosphere) is a Coxeter decomposition of a spherical (Euclidean) triangle p corresponding to a subdiagram of $\overline{P}$. Since all dihedral angles of P are fundamental, all angles of the triangle p are fundamental too.

□

2.3 Decompositions of the third type

Suppose that all decompositions of the first and the second types are known. Then it is possible to classify the tetrahedra of the third type using a **wide inductive procedure**.

Let P be a tetrahedron admitting a decomposition of the third type, F be a fundamental tetrahedron and T_1,...,T_k be all distinct tetrahedra of the second type contained in P. The decompositions of the first type are constructed by the inductive procedure from the fundamental tetrahedron F. The same inductive procedure can be used to construct the decompositions of the third type from F, T_1,...,T_k. The only change we need is to add the tetrahedra $P_1 = T_1$,...,$P_k = T_k$ at the Step 0.

Since there are finitely many possibilities for the Coxeter tetrahedron F and for the tetrahedra of the second type T_1,...,T_k, we can find all the decompositions of the third type using this procedure finitely many times. In fact, there is no hyperbolic tetrahedron admitting a Coxeter decomposition of the third type.

3 Classification of the decompositions

3.1 Decompositions of the first type

There are finitely many hyperbolic Coxeter tetrahedra (see Table 1 for the list). Therefore, we can classify all the hyperbolic Coxeter decompositions of the first type using the inductive procedure. See the list of these decompositions in Fig 6–7.

3.2 Hyperbolic decompositions of the second type

There are finitely many bounded Coxeter tetrahedra and finitely many unbounded Coxeter tetrahedra in $\mathbf{H}^3$ (unbounded tetrahedron is a tetrahedron having some vertices on absolute; such vertices are called **ideal**). It is evident that F and P should be bounded or unbounded simultaneously.

Lemma 8. *No bounded hyperbolic tetrahedron admits a Coxeter decomposition of the second type.*

Proof. Compare the volumes of the hyperbolic Coxeter tetrahedra (see Table 1).

Note, that there is a unique pair of bounded tetrahedra F and P such that $\frac{Vol(P)}{Vol(F)}$ can be integer. These tetrahedra are $F = H_1$ and $P = H_3$. There exists a unique decomposition with $(F,P)=(H_1,H_3)$ and $N = 2$. This decomposition is of the first type. Thus, the lemma is true. □

Consider unbounded tetrahedra.

Lemma 9. *Let P be an unbounded tetrahedron admitting a Coxeter decomposition. Then P has a non-fundamental vertex.*

Proof. Suppose that there is such a decomposition. Then any vertex of P belongs to a unique fundamental tetrahedron. Since any fundamental tetrahedron have an ideal vertex, $N \leq 4$. This contradicts to lemma 6. □

To prove the following lemma we need a list of the Coxeter decompositions of the Euclidean and spherical triangles p such that all angles of p are fundamental and $N < 24$. See Fig. 3 for the list. It is easy to check that the list is complete.

Decompositions of spherical and Euclidean triangles
without non-fundamental vertices ($N < 24$ in Euclidean case)

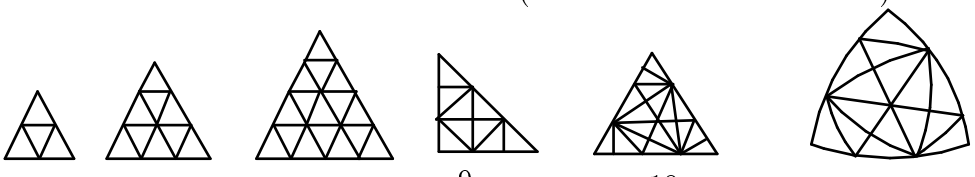

Figure 3.
The decompositions 3a–e are Euclidean, the decomposition 3f is spherical.
The numbers of fundamental triangles are written under the decompositions.

Lemma 10. *Let P be a tetrahedron admitting a Coxeter decomposition without non-fundamental dihedral angles. Then the decomposition is one of two decompositions shown in Fig 8.*

Proof. Let P be a hyperbolic tetrahedron admitting a Coxeter decomposition without non-fundamental dihedral angles. By lemma 8 P is unbounded tetrahedron. By lemma 9 P has a non-fundamental vertex. Let A be a non-fundamental vertex of P.

Suppose that A is not an ideal vertex. Consider a small sphere centered in A. The section of the decomposition by this sphere is a Coxeter decomposition of a spherical triangle p, all angles of p are fundamental. The only such a decomposition is one shown in Fig. 3f. The number of the tiles in p equals to 15. Hence, the number of the tiles in P cannot be less than 15. The fundamental triangle in the decomposition shown in Fig. 3f has an angle $\frac{\pi}{5}$. Hence the fundamental tetrahedron has a dihedral angle equal $\frac{\pi}{5}$. But there is no pair of unbounded hyperbolic tetrahedra F and P such that $\frac{Vol(P)}{Vol(F)} \geq 15$ and F has a dihedral angle equal to $\frac{\pi}{5}$. Therefore, any non-fundamental vertex should be ideal.

Let A be an ideal non-fundamental vertex of P. Consider a small horosphere centered in A. The section of the decomposition by this horosphere is a Coxeter decomposition of a Euclidean triangle p without non-fundamental angles. Since the maximum ratio of volumes of unbounded hyperbolic tetrahedra is 24, the number of triangular tiles f in p cannot be greater than 24. Therefore, the decomposition of the triangle p is one of shown in Fig. 3a–e. Consider these decompositions of p.

1) Suppose that p is decomposed as shown in Fig. 3a–c. Then all angles of the triangles f and p are equal to $\frac{\pi}{3}$. Therefore, each of the Coxeter diagrams $\overline{F}$ and $\overline{P}$ has a subdiagram corresponding to a triangle with angles $(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3})$. All hyperbolic Coxeter tetrahedra satisfying this condition are shown in Fig. 4. The tetrahedra H_{24} and H_{32} cannot be fundamental for any Coxeter decomposition of the second type, because for any Coxeter tetrahedron T $\frac{Vol(T)}{Vol(H_{24})} < 2$ $\frac{Vol(T)}{Vol(H_{32})} < 2$ The maximal ratio of volumes of the tetrahedra shown in Fig. 4a–4c is 12. Therefore, p cannot be decomposed as the triangle shown in Fig.3c. Thus, F is one of the tetrahedra shown in Fig. 4a, P is one of the tetrahedra shown in Fig. 4a–c and p is decomposed as shown in Fig.3a or 3b.

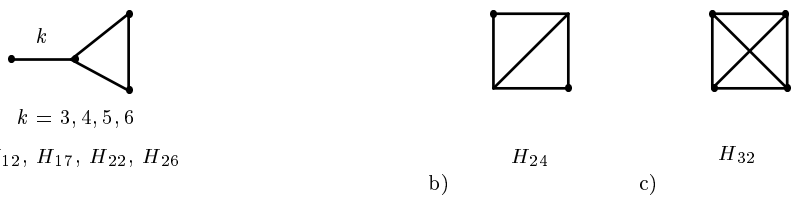

Figure 4.

Prove first that H_{17}, H_{22} and H_{26} cannot be fundamental for a decomposition of the second type. Really, suppose the contrary, i.e. $\overline{F}$ is shown in Fig. 4a and $k > 3$. Then P has no dihedral angle which is equal to $\frac{\pi}{k}$, since $\overline{P} \neq \overline{F}$. Consider the set of the fundamental tetrahedra having a common ideal vertex whose neighborhood is decomposed as shown in Fig. 3a–c. Each of these tetrahedra has its own edge with dihedral angle $\frac{\pi}{k}$. Since P has no such a dihedral angle, each of these edges belongs to k distinct tetrahedra. Therefore, $N \geq n \cdot k$, where n is a number of the tiles f in p, i.e. $n = 4$ or 9.
If $n = 9$, then $n \cdot k \geq 9 \cdot 4 > 24$. This is impossible.
If $n = 4$, then $n \cdot k \geq 4 \cdot 4 = 16$; but there is no Coxeter tetrahedron P having the volume big enough. Therefore, $k = 3$.

The tetrahedron H_{12} can tessellate only the tetrahedra H_{24} and H_{32}, since it has no dihedral angle which is equal to $\frac{\pi}{l}$, l = 4,5,6. These tessellations are the decompositions shown in Fig 8.

2) Suppose that p is decomposed as shown in Fig. 3d. Then the number of fundamental tetrahedra in the decomposition cannot be smaller than nine. Further, each of Coxeter the diagrams $\overline{F}$ and $\overline{P}$ has a subdiagram corresponding to a triangle with angles $(\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{2})$. The unique pair of tetrahedra (F,P) satisfying these conditions is (H_{11},H_{31}). Prove that there is no such a decomposition. Consider nine fundamental tetrahedra F_i, (i = 1,...,9) having a common ideal vertex A. Let f_i be their faces opposite to A. Since F_i has a face which is not orthogonal to f_i, the faces f_i (i = 1,...,9) do not lie in one and the same plane. These faces belong to three different planes (see Fig. 5). Any of these planes contains at most four faces f_i. Therefore, the decomposition contains at list five fundamental tetrahedra additional to F_1,...,F_9. But $9 + 5 = 14 \geq 12$. This is impossible, since $\frac{Vol(P)}{Vol(F)} = 12$.

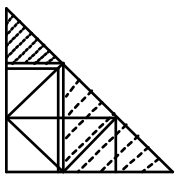

Figure 5.
The triangles shaded by distinct ways belong to the distinct planes.
These planes intersect each other with the dihedral angle $\frac{2\pi}{3}$.

3) Suppose that p is decomposed as shown in Fig. 3e. Then the number of the fundamental tetrahedra in the decomposition of P cannot be smaller than 18. Further, $\overline{F}$ has a subdiagram corresponding to a triangle with angles $(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3})$ and $\overline{P}$ has a subdiagram

corresponding to a triangle with angles $(\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{6})$. A unique pair of unbounded hyperbolic tetrahedra (F,P) satisfying these conditions is (H_{10},H_{32}). Since any dihedral angle of P equals to $\frac{\pi}{3}$, the neighborhood of any vertex of P is decomposed as shown in Fig. 3e. Then $N \geq 18 \cdot 4 > 24$ (F has a unique ideal vertex). Since $\frac{Vol(P)}{Vol(F)} = 24$, such a decomposition is impossible.

All the cases are considered, the lemma is proved.

□

3.3 Hyperbolic decompositions of the third type

Lemma 11. *There is no Coxeter decomposition of hyperbolic tetrahedron of the third type.*

Proof. Suppose that a hyperbolic tetrahedron P admits a decomposition of the third type. Then P has a non-fundamental dihedral angle and P contains a tetrahedron T of the second type. By lemma 10 the decomposition of T is one of two decompositions shown in Fig 8.

The decompositions shown in Fig 8b cannot be a part of a greater decomposition, since P already has four ideal vertices (otherwise all these vertices should be vertices of the greater tetrahedron; this is impossible).

Let T be a tetrahedron shown in Fig 8a. Let F be a tetrahedron fundamental for this decomposition. We prove using the inductive procedure (see subsection 2.3) that T cannot be a part of greater tetrahedron.

Step 0. $P_0 = F$, $P_1 = T$.

Step 1. Gluing two copies of P_0 we get a tetrahedron P_2 admitting a Coxeter decomposition (there is only way to obtain a tetrahedron glueing two copies of P_0).

Step 2. It is impossible to get a tetrahedron glueing P_1 with P_0 or with P_1.

Step 3. It is impossible to get a tetrahedron glueing P_2 with P_0, P_1 or P_2.

The procedure terminated. No decomposition of the third type was found. Therefore, there are no hyperbolic decomposition of the third type.

□

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References

[1] N. W. Johnson, R. Kellerhals, J. G. Ratcliffe, S. T. Tschantz. The size of a hyperbolic Coxeter simplex, preprint, 1998.

Tables

Notation

The tetrahedra with the dihedral angles $\frac{k_i\pi}{q_i}$ i = 1,...,6 are represented by the following diagrams: take a Coxeter diagram of the tetrahedron whose dihedral angles equals to $\frac{\pi}{q_i}$ i = 1,...,6 and subdivide the edges corresponding to the dihedral angles of the size $\frac{k_i\pi}{q_i}$ into k_i parts.

The list of the decompositions of the first type was obtained by an inductive procedure. The non-trivial decompositions of the first type are listed here for each of the Coxeter tetrahedra. The Coxeter diagram of the fundamental tetrahedra is presented before the list of the decompositions with this tetrahedron fundamental.
The numbers (k,l ; m,n,p,q) under the decompositions describe the decompositions.
k is the number of fundamental tetrahedra in the decomposition,
l is a number of glueing necessary for construction of the decomposition
(from the tetrahedra with l_1 and l_2 we obtain a tetrahedron with $l = 1 + max\{l_1, l_2\}$),
m and n are the numbers of the tetrahedra which should be glued together to obtain the decomposition,
p and q are the numbers of the glued faces of the tetrahedra m and n respectively,
the nodes of the diagrams are numbered from left to right by the numbers 0,1,2,3.

Volumes of hyperbolic Coxeter tetrahedra

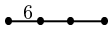
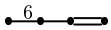
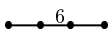
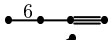
bounded tetrahedra			unbounded tetrahedra			unbounded tetrahedra		
n	Coxeter diagram	volume	n	Coxeter diagram	volume	n	Coxeter diagram	volume
H_1		0.0358850633	H_{10}		0.0422892336	H_{23}		0.3641071004
H_2		0.0390502856	H_{11}		0.0763304662	H_{24}		0.4228923360
H_3		0.0717701267	H_{12}		0.0845784672	H_{25}		0.4579827971
H_4		0.0857701820	H_{13}		0.1057230840	H_{26}		0.5074708032
H_5		0.0933255395	H_{14}		0.1526609324	H_{27}		0.5258402692
H_6		0.2052887885	H_{15}		0.1691569344	H_{28}		0.5562821156
H_7		0.2222287320	H_{16}		0.1715016613	H_{29}		0.6729858045
H_8		0.3586534401	H_{17}		0.2114461680	H_{30}		0.8457846720
H_9		0.5021308905	H_{18}		0.2114461680	H_{31}		0.9159655942
			H_{19}		0.2289913985	H_{32}		1.014916064
			H_{20}		0.2537354016			
			H_{21}		0.3053218647			
			H_{22}		0.3430033226			

Table 1.

The table is reprinted from [1]

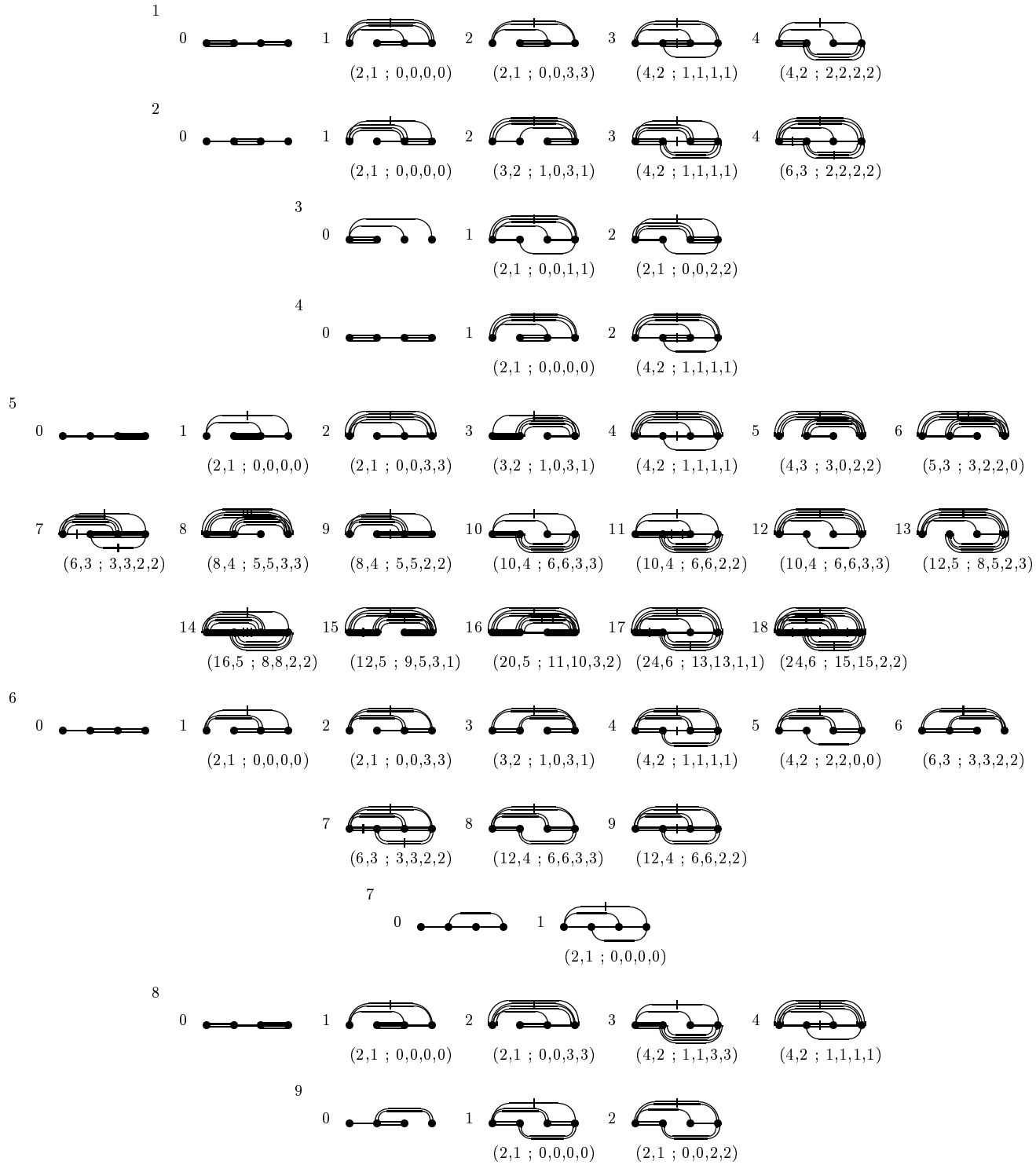

1
0
1 (2,1 ; 0,0,0,0)
2 (2,1 ; 0,0,3,3)
3 (4,2 ; 1,1,1,1)
4 (4,2 ; 2,2,2,2)
2
0
1 (2,1 ; 0,0,0,0)
2 (3,2 ; 1,0,3,1)
3 (4,2 ; 1,1,1,1)
4 (6,3 ; 2,2,2,2)
3
0
1 (2,1 ; 0,0,1,1)
2 (2,1 ; 0,0,2,2)
4
0
1 (2,1 ; 0,0,0,0)
2 (4,2 ; 1,1,1,1)
5
0
1 (2,1 ; 0,0,0,0)
2 (2,1 ; 0,0,3,3)
3 (3,2 ; 1,0,3,1)
4 (4,2 ; 1,1,1,1)
5 (4,3 ; 3,0,2,2)
6 (5,3 ; 3,2,2,0)
7 (6,3 ; 3,3,2,2)
8 (8,4 ; 5,5,3,3)
9 (8,4 ; 5,5,2,2)
10 (10,4 ; 6,6,3,3)
11 (10,4 ; 6,6,2,2)
12 (10,4 ; 6,6,3,3)
13 (12,5 ; 8,5,2,3)
14 (16,5 ; 8,8,2,2)
15 (12,5 ; 9,5,3,1)
16 (20,5 ; 11,10,3,2)
17 (24,6 ; 13,13,1,1)
18 (24,6 ; 15,15,2,2)
6
0
1 (2,1 ; 0,0,0,0)
2 (2,1 ; 0,0,3,3)
3 (3,2 ; 1,0,3,1)
4 (4,2 ; 1,1,1,1)
5 (4,2 ; 2,2,0,0)
6 (6,3 ; 3,3,2,2)
7 (6,3 ; 3,3,2,2)
8 (12,4 ; 6,6,3,3)
9 (12,4 ; 6,6,2,2)
7
0
1 (2,1 ; 0,0,0,0)
8
0
1 (2,1 ; 0,0,0,0)
2 (2,1 ; 0,0,3,3)
3 (4,2 ; 1,1,3,3)
4 (4,2 ; 1,1,1,1)
9
0
1 (2,1 ; 0,0,0,0)
2 (2,1 ; 0,0,2,2)

Figure 6.

Hyperbolic tetrahedra of the first type (continuation)

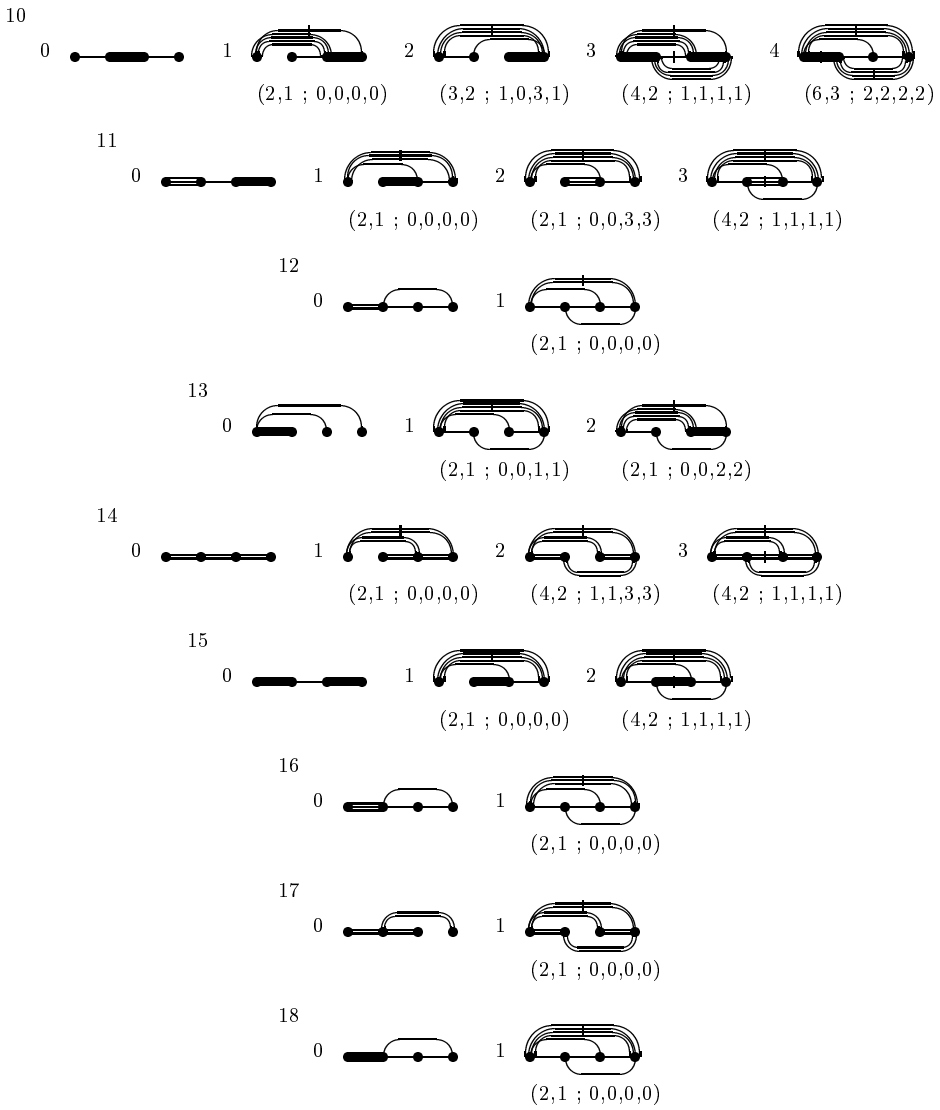


Figure 7.

Hyperbolic tetrahedra of the second type.

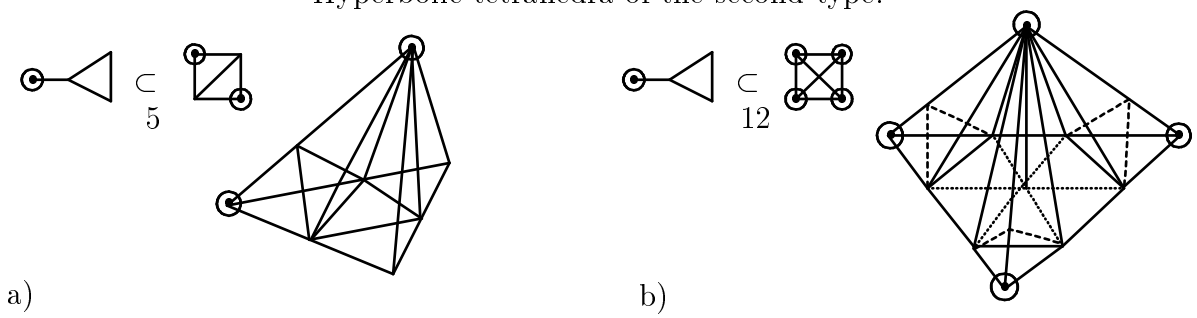


Figure 8.

The ideal vertices are marked by the small circles.