On Generalized Fibonacci Groups with Odd Number of Generators *

Andrzej Szczepański Andrei Vesnin

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Abstract

The geometrical properties of cyclically presented group of Fibonacci type $F(r,m,k)$ and $H(r,m,k)$ are discussed. It is shown that for even r and odd m almost all generalized Fibonacci groups $F(r,m,k)$ can not be fundamental groups of hyperbolic 3–orbifolds of finite volume.

1 Introduction

In the present paper we study geometrical properties of a class of cyclically presented groups of Fibonacci type. In 1965 Conway defined a series of groups $F(2,m)$, $m \geq 3$, with the presentation

$$F(2,m) \;=\; \langle x_1, \ldots, x_m \mid x_i x_{i+1} \;=\; x_{i+2} \quad i = 1, \ldots, m \rangle,$$

where all suffices are by mod m. The natural generalization of these groups was introduced and studied in [8]. The groups $F(r,m)$, $r \geq 2$, $m \geq 3$, with the presentation

$$F(r,m) \;=\; \langle x_1, \ldots, x_m \,|\, x_i x_{i+1} \, \ldots x_{i+r-1} = \; x_{i+r} \quad i = 1, \ldots, m \rangle,$$

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are said to be *Fibonacci groups*. The first question on these groups was: is the group $F(r, m)$ finite? A review of results can be found in [7].

The studding of 3–manifolds uniformized by Fibonacci groups $F(2, m)$ with even number of generators was started by Helling, Kim and Mennicke [5]. We recall that a 3–manifold M is said to be *hyperbolic* if it can be obtained as the quotient-space $M = \mathbb{H}^3/\Gamma$, where $\mathbb{H}^3$ is the Lobachevsky (also called hyperbolic) 3–space, and $\Gamma \subset \mathrm{Isom}(\mathbb{H}^3)$ is a discrete, torsion–free group of isometries. From [5] the group $F(2, 6)$ is the fundamental group of the Hantzsche-Wendt flat 3–manifold, and $F(2, 2n)$, $n \geq 4$, is the fundamental group of a closed orientable hyperbolic 3–manifold.

The groups $F(2, m)$ with odd number of generators have completely different geometrical properties. It is easy to see [7, p. 84] that its have a torsion and so, can not be fundametal groups of Euclidean or hyperbolic 3-manifolds. We recall that a *hyperbolic 3–orbifold* $\mathcal{O}$ is the quotient space $\mathbb{H}^3/\Gamma$, where Γ is a discrete group of isometries (in particular, if Γ is torsion–free we get a manifold). It was shown in [9] that if m is odd, then the group $F(2, m)$ can not be the fundamental group of a hyperbolic 3–orbifold of finite volume. However in [6] is proved that $F(2, 9)$ is automatic. Moreover, if $r \geq 4$ is even, and $m \geq r$ is odd, then the Fibonacci group $F(r, m)$ can not be the fundamental group of a hyperbolic 3–orbifold of finite volume [12].

In the present paper we consider two series of groups of Fibonacci type. The first series are *generalized Fibonacci groups* $F(r, m, k)$, $r \geq 2$, $k \geq 1$, with the presentation

$$F(r, m, k) = \langle x_1, \ldots, x_m \mid x_i x_{i+1} \cdots x_{i+r-1} = x_{i+r-1+k} \quad i = 1, \ldots, m \rangle,$$

where suffices are by mod m.

The second series are groups $H(r, m, k)$, $r \geq 2$, $k \geq 1$, with the presentation

$$H(r, m, k) = \langle x_1, \ldots, x_m \mid x_i x_{i+1} \cdots x_{i+r-1} = x_{i+r} \ldots x_{i+r+k} \quad i = 1, \ldots, m \rangle,$$

where suffices are by mod m. These groups were introduced in [2]. We have $F(r, m, 1) = F(r, m)$ and $H(r, m, 1) = F(r, m)$.

Here we generalize the method from [9] and [12] for the groups $F(r, m, k)$. We show that for triples of integers r, m, and k with r – even, m – odd, and m does not divide $r + 2k - 1$ the groups $F(r, m, k)$ can not be the fundamental groups of hyperbolic 3–orbifolds of finite volume. Moreover, we discuss some extremal cases with $m = r + 2k - 1$. In particular our groups are fundamental groups of Neuwirth manifolds [3], [11], of Sieradski manifolds [4], and of Brieskorn manifolds [10]. We also show that there is the relation between the groups $H(r, m, k)$ and the generalized Sieradski groups $S(k, n)$ (cf. [4]).

2 Groups $F(r, m, k)$

Applying the methods from [9] and [12] for the groups $F(r, m, k)$ with odd number of generators, we get the following result.

Theorem. *Suppose $r \geq 4$ is even, $m \geq 3$ is odd and m does not divide $r + 2k - 1$. Then the group $F(r, m, k)$ can not be the fundamental group of a hyperbolic 3–orbifold of finite volume.*

Proof: Let $\rho : F(r, m, k) \to \mathrm{Isom}(\mathbb{H}^3)$ be a faithful representation such that $\overline{F}(r, m, k) = \rho(F(r, m, k))$ is a discrete group of finite covolume. Obviously, the group $\overline{F}(r, m, k)$ admits an automorphism α of order m permuting the generators cyclically. By Mostow rigidity theorem there exists the isometry $t \in \mathrm{Isom}(\mathbb{H}^3)$ such that $\alpha(\gamma) = t\gamma t^{-1}$ for each $\gamma \in \overline{F}(r, m, k)$. Let us consider the natural extension $\Gamma(r, m, k) = \langle \overline{F}(r, m, k),\ t \rangle$. That is the fundamental group of a hyperbolic 3–orbifold of finite volume. Since t^m does commutate with all elements of the non-elementary group $\overline{F}(r, m, k))$, we get $t^m = 1$ (cf. [1]). Using $x_{i+1} = tx_it^{-1}$, from the relation $x_1x_2 \ldots x_r = x_{r+k}$, we get

$$(x_1t)^r \ = \ t^r \ (t^{k-1} \ x_1 \ t^{-(k-1)}). \tag{1}$$

Obviously,

$$\Gamma(r, m, k) \ = \ \langle x_1, t \mid t^m = 1,\ (x_1t)^r \ = \ t^r(t^{k-1} \ x_1 \ t^{-(k-1)}) \rangle.$$

Consider the group

$$\Gamma^{(2)}(r, m, k) \ = \ \langle \gamma^2 \mid \gamma \in \Gamma(r, m, k) \rangle.$$

Since m – odd, $t \in \Gamma^{(2)}(r, m, k)$. Moreover, since r – even, we get $x_1 \in \Gamma^{(2)}(r, m, k)$ and $\Gamma^{(2)}(r, m, k) = \Gamma(r, m, k)$. Therefore $\Gamma(r, m, k)$ is a group of orientation preserving isometries of $\mathbb{H}^3$, and $\Gamma(r, m, k) \subset \mathrm{PSL}(2, \mathbb{C})$. Without loss of generality we can assume that $t = P \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^-1 \end{pmatrix}$ where ζ is a $2m$'th root of unity, and $P\begin{pmatrix} \star & \star \\ \star & \star \end{pmatrix}$ denotes the image in $\mathrm{PSL}(2,\mathbb{C})$ of the matrix $\begin{pmatrix} \star & \star \\ \star & \star \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$.

Let $x_1 = P\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ with $xw - yz = 1$. Remark that $yz \neq 0$, because $\overline{F}(r,m,k)$ has finite covolume. From (1) we get

$$\begin{pmatrix} x\zeta & y\zeta^{-1} \\ z\zeta & w\zeta^{-1} \end{pmatrix}^r = \begin{pmatrix} \zeta^r & 0 \\ 0 & \zeta^{-r} \end{pmatrix} \begin{pmatrix} x & y\zeta^{2(k-1)} \\ z\zeta^{-2(k-1)} & w \end{pmatrix}. \tag{2}$$

Note the following property.

Lemma. *For any $j \geq 1$ we have* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^j = \begin{pmatrix} S_j & bR_j \\ cR_j & T_j \end{pmatrix}$, *where* $\begin{pmatrix} S_{j+1} \\ T_{j+1} \\ R_{j+1} \end{pmatrix} = \begin{pmatrix} a & 0 & bc \\ 0 & d & bc \\ 1 & 0 & d \end{pmatrix} \begin{pmatrix} S_j \\ T_j \\ R_j \end{pmatrix}$ *with* $\begin{pmatrix} S_1 \\ T_1 \\ R_1 \end{pmatrix} = \begin{pmatrix} a \\ d \\ 1 \end{pmatrix}$.

Proof: We proceed by induction on j.

□

From (2), using the lemma, we get

$$\begin{pmatrix} S_r & y\zeta^{-1}R_r \\ z\zeta R_r & T_r \end{pmatrix} = \begin{pmatrix} x\zeta^r & y\zeta^{r+2(k-1)} \\ z\zeta^{-r-2(k-1)} & w\zeta^{-r} \end{pmatrix}.$$

Therefore

$$\begin{cases} y\zeta^{-1}R_r = y\zeta^{r+2(k-1)} \\ z\zeta R_r = z\zeta^{-r-2(k-1)} \end{cases}.$$

Since $yz \neq 0$, we have

$$\zeta^{r+2(k-1)+1} = \zeta^{-r-2(k-1)-1},$$

and so,

$$\zeta^{2r+4k-2} = 1.$$

But ζ is the primitive root of degree $2m$ and m does not divide $r + 2k - 1$. Therefore, the group $F(r,m,k)$ can not be the fundamental group of a finite volume hyperbolic 3–orbifold.

□

3 Some non-hyperbolic manifolds

In this section we discuss groups $F(r,m,k)$ which do not satisfy to the conditions of the theorem. Suppose,

$$m = r + 2k - 1. \tag{3}$$

If $k = 1$ then from (3) we get $m = r + 1$. The corresponding groups $F(m-1,m,1)$ are Fibonacci groups $F(m-1,m)$ which were discussed in the Neuwirth paper [11]. From [11] and [3] the following property holds.

Proposition 1. *For a given m the group $F(m-1,m,1)$ is the fundamental group of the Siefert fibered space*

$$\Sigma_m \; = \; (0 \; o \; 0 \mid \; -1 \; \underbrace{(2,1), \; (2,1), \; \ldots, (2,1)}_{m \text{ times}} \;).$$

In other particular case, $r = 2$, according to (3) we get groups

$$F(2, 2k+1, k) = \langle x_1, \ldots, x_{2k+1} \mid x_i x_{i+1} = x_{i+1+k}, \quad i = 1, \ldots, 2k+1 \rangle.$$

Let us consider the *Sieradski groups* $S(n)$ with the cyclic presentation

$$S(n) = \langle a_1, \ldots, a_n \mid a_i a_{i+2} = a_{i+1}, \quad i = 1, \ldots n \rangle,$$

where all suffices by mod n. The topological properties of groups $S(n)$ were studied in [4]. It is easy to see that groups $F(2, 2k+1, k)$ and $S(2k+1)$ are isomorphic under the correspondence

$$\begin{pmatrix} x_1 & x_2 & x_3 & \ldots & x_k & x_{k+1} & x_{k+2} & \ldots & x_{2k} & x_{2k+1} \\ a_1 & a_3 & a_5 & \ldots & a_{2k-1} & a_{2k+1} & a_2 & \ldots & a_{2k-2} & a_{2k} \end{pmatrix}.$$

Due to results on the Sieradski groups [4], we get

Proposition 2. *For $k \geq 1$ the Sieradski group $F(2, 2k+1, k) \cong S(2k+1)$ is the fundamental groups of a closed 3–manifold which can be obtained as $(2k+1)$–fold cyclic covering of the 3–sphere branched over the trefoil knot.*

We should mention (cf. [10]) that the group $F(2, 2k+1, k)$ is the fundamental group of the Brieskorn manifold $B(3, 2, 2k+1)$ which is spherical for $2k+1 < 6$ and $\widetilde{\mathrm{SL}}(2, \mathbb{R})$–manifold for $2k+1 > 6$.

Summing up, we observe in both cases described in propositions 1 and 2 that the groups $F(r,m,k)$ can not be the fundamental groups of hyperbolic 3–orbifolds of finite volume.

4 Groups $H(r, m, k)$

In this section we discuss properties of the groups $H(r, m, k)$ in some particular cases. The *generalized Sieradski groups* $S(k, n)$ were defined in [4]:

$$S(k,n) = \langle a_1, \ldots, a_n \mid a_i a_{i+2} \ \ldots \ a_{i+2k-2} = a_{i+1} a_{i+3} \ \ldots \ a_{i+2k-3}, \quad i = 1 \ldots n \rangle.$$

It is easy to see that groups $H(k, 2k-1, k-1)$ and $S(k, 2k-1)$ are isomorphic under the correspondence

$$\begin{pmatrix} x_1 & x_2 & x_3 & \ldots & x_k & x_{k+1} & x_{k+2} & \ldots & x_{2k-1} \\ a_1 & a_3 & a_5 & \ldots & a_{2k-1} & a_2 & a_4 & \ldots & a_{2k-2} \end{pmatrix}.$$

Due to results on the generalized Sieradski groups [4], we get

Proposition 3. *For $k \geq 2$ the group $H(k, 2k-1, k-1) \cong S(k, 2k-1)$ is the fundamental groups of a closed 3–manifold. This manifold can be obtained as a $(2k-1)$–fold cyclic covering of the 3–sphere branched over the torus $(2k-1, 2)$–knot.*

For $k = 2$ we get the 3–fold cyclic covering of the 3-sphere branched over the trefoil knot. We remark that in term of [10] the group $H(k, 2k-1, k-1)$ is the fundamental group of the Brieskorn manifold $B(2, 2k-1, 2k-1)$ which is spherical for $k = 2$ and $\widetilde{\mathrm{SL}}(2, \mathbb{R})$–manifold for $k \geq 3$.

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Andrzej Szczepański, Institute of Mathematics, University of Gdańsk, ul.Wita Stwosza 57, 80-952, Gdańsk, Poland. E-mail: *matas@paula.univ.gda.pl*

Andrei Vesnin, Sobolev Institute of Mathematics, pr. Koptyuga 4, Novosibirsk 630090, Russia. E-mail: *vesnin@math.nsc.ru*