The Ellis semigroup of the full group of isometries of a locally compact metric space

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Abstract

If the space $S(X)$ of connected components of a locally compact metric space X is compact, with respect to the quotient topology, then the full group of isometries of X is a locally compact, closed subgroup of the group of homeomorphisms of X with respect to the pointwise topology.

Let (X, d) be a locally compact compact metric space and X^X be the set of all maps in X with the pointwise convergence topology. If G is a group of homeomorpisms of X acting on X via the evaluation map then the Ellis semigroup of G, $E(G)$ is the closure of G in X^X.

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If $Iso(X)$ is the full group of isometries of X, we are interested in finding conditions for the space X that ensure that $E(Iso(X))$ is a (*locally compact* topological) group, that is to try to answer the question "when does $E(Iso(X)) = Iso(X)$ hold?" Note that since the pointwise closure of $Iso(X)$ consists of maps that preserve the distance but are not necessarily onto, whenever $E(Iso(X))$ is a group then $E(Iso(X)) = Iso(X)$, so in this case $E(Iso(X))$ is a topological group. If X is compact then every map that preserves the distance is onto, so $E(Iso(X))$ $(= Iso(X))$ is a (compact) topological group. In 1928 D. van Dantzig and B.L. van der Waerden proved that if X is *connected* and locally compact, then $E(Iso(X))$ $(= Iso(X))$ is a *locally compact* topological group [1]. Their methods work only if the space X is connected. The following example shows that the result is not true if X is not connected.

Example. Let $\mathbb{Z}$ be endowed with the Euclidean metric d_E. We modify d_E by letting $d = \min\{1, d_E\}$. Let

$$f_n(x) = \begin{cases} x+1, \text{ if } x \geq 0 \text{ or } x < -n \\ 0, \text{ if } x = -n \\ x, \text{ if } -n < x < 0. \end{cases}$$

It is easy to see that for every $x \in X$ the following equations hold eventually, i.e. for $n >> 0$:

$$f_n(x) = \begin{cases} x+1, \text{ if } x \geq 0 \\ x, \text{ if } x < 0. \end{cases}$$

So, if $f : \mathbb{Z} \to \mathbb{Z}$ is defined correspondingly:

$$f(x) = \begin{cases} x+1, \text{ if } x \geq 0 \\ x, \text{ if } x < 0 \end{cases}$$

then $f_n \to f$, pointwise but f is not onto.

Note that in the example above, the space of connected components of $\mathbb{Z}$ is not compact. The example is typical, since we will prove that

Proposition 1 *If the space $S(X)$ of connected components of a locally compact metric space X is compact, with respect to the quotient topology, then the group $Iso(X)$ is closed in X^X with respect to the pointwise topology.*

Before we proceed with the proof of the proposition, we give a proof of a lemma from general topology.

Lemma 2 *If the space $S(X)$ of connected components of a metric space X is compact, with respect to the quotient topology, then X is second countable and $S(X)$ is a first countable compact space.*

Proof. The space X is second countable [3] (The proof is similar to the proof of the special case where X is a connected, locally separable metric space [5]). Now, take $\{V_i,\, i \in I\}$ an open neighborhood base of S in $S(X)$. The space $S(X)$ is a T_1 space because if S_1 and S_2 are two distinct connected components, then $U = X \setminus S_1$ is an open subset of X that contains S_2 and not contains S_1. If $p : X \to S(X)$ is the quotient map, then $\{p^{-1}(S(X) \setminus V_i),\, i \in \ I\}$ is an open cover of the Lindelöf space $X \setminus S$, so there exists a countable subcover $\{p^{-1}(S(X) \setminus V_n),\, n \in \mathbb{N}\}$. Since $S(X)$ is a T_1 space, then $S(X) \setminus \{S\} = \cup_{n=1}^{\infty} \{S(X) \setminus V_n\}$. We can modify the family $\{V_n,\, n \in \mathbb{N}\}$ in such a way that the family is decreasing, as follows. Set $U_1 = V_1$ and $U_n = \cap_{k=1}^{n} V_k$, then $S(X) \setminus \{S\} = \cup_{n=1}^{\infty} \{S(X) \setminus U_n\}$ and $\{U_n,\, n \in \mathbb{N}\}$ is a neighborhood base of

S in $S(X)$. Indeed, if we take an open neighborhood W of S in $S(X)$, then $S(X)\setminus W$ is a compact subset of $S(X)\setminus\{S\}=\cup_{n=1}^{\infty}\{S(X)\setminus U_n\}$, so $S(X)\setminus W\subseteq\cup_{n=1}^{n_0}\{S(X)\setminus U_{k_n}\}$. Hence, $S(X)\setminus W\subseteq S(X)\setminus U_m$ for some $k_1\leq m\leq k_{n_0}$. Therefore, $U_m\subseteq W$.

Proof of the proposition. The space X is second countable so sequences (of points and maps) are adequate to describe the problem [2, Th. 5.2., p.265]. Take a sequence $f_n\in Iso(X)$ and $f\in X^X$ such that $f_n\to f$ with respect to the pointwise topology. It it sufficient to prove that f is onto (it is obvious that f preserves the distance). Let $y\in X$, then by the compactness of the space $S(X)$ there exists a subsequence $S_{f_{n_k}^{-1}y}\to S$ for some $S\in S(X)$. Note that since $S(X)$ is compact it is countably compact [6, 17F p.125]. Moreover, since $S(X)$ is a first countable space and countably compact it is sequentially compact [6, 17G(3) p.125]. If there is a subsequence (denoted again with the index n_k) such that $x_{n_k}\to x\in C$, for some $C\in[S]$, where $[S]=\{C\in S(X)\mid C$ and S cannot be separated by two disjoint neighborhoods in $S(X)\}$, and $x_{n_k}\in S_{f_{n_k}^{-1}y}$, then there exists $y_{n_k}\in S_y$ such that $f_{n_k}^{-1}y_{n_k}=x_{n_k}\to x$ since $S_{f_{n_k}^{-1}y}=f_{n_k}^{-1}S_y$. So

$$\begin{aligned} d(y_{n_k},f(x)) &\leq d(y_{n_k},f_{n_k}x)+d(f_{n_k}x,f(x)) \\ &= d(f_{n_k}^{-1}y_{n_k},x)+d(f_{n_k}x,f(x))\to 0. \end{aligned}$$

But $y_{n_k}\in S_y$ and S_y is closed hence $f(x)\in S_y$. Now $f(X)$ is clopen in X by [4, Th. 4.2] so $y\in S_y\subseteq f(X)$, since $S_y\cap f(X)\neq\emptyset$.

The only thing that remains to be proved is that if a non constant sequence S_n of connected components converges to a component S then there exists a subsequence $x_{n_k}\in S_{n_k}$ and a point $x\in C$, for some $C\in[S]$, such that $x_{n_k}\to x$. Assume the contrary, then $\cup_{n=1}^{\infty}S_n$ is a closed set. Indeed, if $y_m \in \cup_{n=1}^{\infty} S_n$ is a new sequence and $y_m \to y \in X$, then either there exist n_0 and m_0 such that $y_m \in \cup_{n=1}^{n_0} S_n$ for every $m \geq m_0$ or not. In the first case it is easy to see that there exists a component S_p and a subsequence $y_{m_n} \in S_p$, so $y \in S_p$. In the latter case for S_1 we can find $y_{m_1} \in S_{n_1}$ such that $n_1 > 1$ and $d(y_{m_1}, y) < 1$. For $\cup_{n=1}^{n_1} S_n$ we can find $y_{m_2} \in S_{n_2}$ such that $n_2 > n_1$ and $d(y_{m_2}, y) < \frac{1}{2}$. If we proceed by induction, we can find a subsequence S_{n_k} and points $y_{n_k} \in S_{n_k}$ such that $d(y_{n_k}, y) < \frac{1}{k}$. So $y_{n_k} \to y$ and $y_{n_k} \in S_{n_k}$. Hence, $y \in S_y$, $S_y \in [S]$ and $y_{n_k} \to y$, a contradiction. Now, since $\cup_{n=1}^{\infty} S_n$ is a closed set then $X \setminus \cup_{n=1}^{\infty} S_n$ is an open set that contains entire connected components and $S \subseteq X \setminus \cup_{n=1}^{\infty} S_n$, a new contradiction since S_n converges to S, with respect to the quotient topology.

Corollary 3 *If the space $S(X)$ of connected components of a locally compact metric space X is compact, with respect to the quotient topology, then the group $Iso(X)$ is a locally compact topological group with respect to the pointwise (equiv. compact-open) topology.*

Proof. [4, Prop. 2.4].

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