MATCHING FOURIER ORBITAL INTEGRALS ON A UNITARY GROUP VIA PERVERSE SHEAVES

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1. The purpose of this note is to show that the proof of the fundamental lemma by Ngô [N2], [N1] in the context of Fourier orbital integrals on the general linear group can be adapted – on making only minor changes – to prove the analogous fundamental lemma in the context of such integrals on the unitary group, as conjectured in [F], for local fields of positive characteristic and for regular elements. That such a simple adaptation is possible is yet another manifestation of the power of the intersection cohomological techniques. This note was written in May-June 1998, and served as a basis for the author's talk at the August 1998 ICM Satellite conference "Algebraic Groups and Related Structures" at Bielefeld. An Appendix below (May 1998) deals separately with the case of the unit element of the Hecke algebra.

Let us recall (part of) the conjecture of [F]. Let F be a local non archimedean field with ring $\mathcal{O}$ of integers and char $F \neq 2$. Let F'/F be a quadratic unramified field extension and $\mathcal{O}'$ the ring of integers in F'. Denote by $x \mapsto \mathrm{Fr}(x)$ the action of $\mathrm{Gal}(F'/F)$. Write $G' = \mathrm{GL}(n, F')$, and signify by w the matrix $(\delta_{i,n+1-j})$ in G', whose non zero entries are 1 placed on the anti diagonal. For $g = (g_{i,j})$ in G' write $\mathrm{Fr}(g) = (\mathrm{Fr}(g_{i,j}))$, $\tilde{\mathrm{Fr}}(g) = {}^t\,\mathrm{Fr}(g)^{-1}$, where t signifies transpose, and $\tilde{\mathrm{Fr}}^w(g) = w\,\tilde{\mathrm{Fr}}(g)w$.

Put $\tilde{\mathrm{G}}' = G' \times F'^{\times}$. The unitary group $U = U(F)$ is $\{g \in G'; g = \tilde{\mathrm{Fr}}^w(g)\}$, and the group $\tilde{\mathrm{U}} = \tilde{\mathrm{U}}(F)$ of unitary similitudes is $\{(g,\mu) \in \tilde{\mathrm{G}}'; (g,\mu) = \tilde{\mathrm{Fr}}^w(g,\mu)\}$, where $\tilde{\mathrm{Fr}}^w(g,\mu) = (\mu\,\tilde{\mathrm{Fr}}^w(g), \mathrm{Fr}(\mu))$. Write N for the unipotent upper triangular subgroup of U, and N' for that of G'. Write A for the diagonal subgroup of U, and A' for that of G'. Let $K' = \mathrm{GL}(n, \mathcal{O}')$ be the standard maximal compact subgroup of G', $\tilde{\mathrm{K}}' = K' \times \mathcal{O}'^{\times}$ of $\tilde{\mathrm{G}}'$, $K = K' \cap U$ that of U, $\tilde{\mathrm{K}} = \tilde{\mathrm{U}} \cap \tilde{\mathrm{K}}'$ that of $\tilde{\mathrm{U}}$. We shall use the Haar measure dx on N normalized to assign $K \cap N$ the volume one, as well as the Haar measure on N' which assigns $K' \cap N'$ the volume one.

Let $\mathcal{H}'$ denote the (Hecke) convolution algebra of complex ($\mathbb{C}$)-valued bi-$\tilde{\mathrm{K}}'$-invariant compactly supported functions on $\tilde{\mathrm{G}}' \cap (\mathfrak{gl}(n, \mathcal{O}') \times \mathcal{O}')$, where $\mathfrak{gl}(n)$ signifies the algebra of $n \times n$ matrices. The functions f in $\mathcal{H}'$ are uniquely determined by their constant terms

$$f^B(a,\mu) = \delta'_B(a) \int_{N'} f(ax,\mu)dx,$$

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where $a = \mathrm{diag}(a_1, \dots, a_n)$ and

$$\delta'_B(a) = \prod_{i<j} |a_i/a_j|_{F'}^{1/2} (= |(I - \mathrm{Ad}(a))_{\mathrm{Lie}(G'/A')}|^{1/2} \cdot |\operatorname{Jac}_{N'}(x \mapsto a^{-1}xax)|^{-1}).$$

Here $|\cdot|_{F'}$ is the absolute value on F' normalized to assign a generator π of the maximal ideal in $\mathcal{O}'$ the value q^{-2}, where q^2 is the cardinality of $\mathcal{O}'/(\pi)$. Also Lie is for the Lie algebra, and Jac is for the Jacobian.

Let $\mathcal{H}$ denote the convolution algebra of $\mathbb{C}$-valued bi-$\tilde{\mathrm{K}}$-invariant compactly supported functions on $\tilde{\mathrm{U}} \cap (\mathfrak{gl}(n, \mathcal{O}') \times \mathcal{O}')$. The functions ϕ in $\mathcal{H}$ are uniquely determined by their constant terms

$$\phi^B(a, \mu) = \delta_B(a) \int_N \phi(ax, \mu) dx,$$

where $a = \mathrm{diag}(a_1, \dots, a_m, \mu/\operatorname{Fr}(a_m), \dots, \mu/\operatorname{Fr}(a_1))$ if $n = 2m$ or

$$a = \mathrm{diag}(a_1, \dots, a_m, a_{m+1}, \mu/\operatorname{Fr}(a_m), \dots, \mu/\operatorname{Fr}(a_1))$$

if $n = 2m+1$, and then $a_{m+1}\operatorname{Fr}(a_{m+1}) = \mu$; a_i, μ, and $\mu/\operatorname{Fr}(a_i)$ lie in $\mathcal{O}' - \{0\}$ if $\phi^B(a, \mu)$ is non zero. Here $\delta_B(a) = \prod_{i \le m} |a_i|_{F'}^{m-i+v} \cdot \prod_{i \le m} |\mu|_F^{-(m-i)-v}$, where v is $1/2$ when n is even and 1 when n is odd.

There is a base change homomorphism $b : \mathcal{H}' \to \mathcal{H}$, $b(f) = \phi$, which can be defined by: $\phi^B((\tilde{a}, \tilde{\mu})) = \sum f^B(a, \mu)$, the sum ranging over the (a, μ) with $(a, \mu)\tilde{\operatorname{Fr}}^w(a, \mu) = (\tilde{a}, \tilde{\mu})$, and $\phi^B(\tilde{a}, \tilde{\mu}) = 0$ if $(\tilde{a}, \tilde{\mu})$ is not of the form $(a, \mu)\tilde{\operatorname{Fr}}^w(a, \mu)$ for any (a, μ) (in our case where F'/F is unramified this means that $\tilde{\mu}$ has an odd valuation). Note that the norm map

$$N(a, \mu) = (a, \mu)\tilde{\operatorname{Fr}}^w(a, \mu) = (\mu a w \operatorname{Fr}(a)^{-1} w, \mu \operatorname{Fr}(\mu))$$

maps $\tilde{A}' = A' \times F'^{\times}$ onto the group $\{(\mu A', \mu \operatorname{Fr}\mu); \mu \in F'^{\times}\}$, and that $\delta'_B(a) = \delta_B(\mu a \tilde{\operatorname{Fr}}^w(a))$ $(a \in A',\ \mu \in F^{\times})$.

Fix a character $\psi_0 : F \to \mathbb{C}^{\times}$ with conductor $\mathcal{O}$, and note that the above-diagonal entries $x_i = x_{i,i+1}$ of $x \in N$ satisfy $\overline{x_i} = -x_{n-i}$. Then $\psi(x) = \psi_0(\iota^{-1}\sum_{1 \le i < n} x_i)$ is a $\mathbb{C}^{\times}$-valued character on N. For $x \in N'$ put

$$\psi'(x) = \psi(x \tilde{\operatorname{Fr}}^w(x)) (= \psi_0(\iota^{-1} \sum_{1 \le i < n} (x_i - \overline{x}_{n-i})));$$

then ψ' is a $\mathbb{C}^{\times}$-valued character on N'.

The conjecture of [F] relates the Fourier orbital integral

$$I((\mu a, \mu \operatorname{Fr}(\mu)), \phi) = \int_N \int_N \phi(x^{-1} w \mu a x', \mu \operatorname{Fr}(\mu)) \psi(x^{-1}x')\, dx\, dx'$$

$$= \int_N \int_N \phi({}^t x \mu a x', \mu \operatorname{Fr}(\mu)) \psi(x x')\, dx\, dx'$$

at the diagonal element $(\mu a, \mu\,\mathrm{Fr}(\mu)) \in \tilde{A} \subset \tilde{\mathrm{U}}$ (for the last displayed equality note that $\sum_{1\le i<n} x_i$ is the same for $w^t x w$ and for x, and that $w \in K$), for $\phi = b(f)$, with the twisted Fourier orbital integral

$$J((\mu a, \mu\,\mathrm{Fr}(\mu)), \phi') = \int_{N'} \phi'(w\,\mathrm{Fr}(x)^{-1} w(\mu a, \mu\,\mathrm{Fr}(\mu))x)\psi'(x)\,dx$$

$$= \int_{N'} \phi'({}^t\,\tilde{\mathrm{Fr}}^w(x)(\mu a, \mu\,\mathrm{Fr}(\mu))x)\psi(\tilde{\mathrm{Fr}}^w(x)x)\,dx,$$

where $\phi' = b'(f)$ is defined to be

$$\phi'((g,\mu)\,\tilde{\mathrm{Fr}}^w(g,\mu)) = \int_{G^w} f(gh,\mu)\,dh, \qquad G^w = \{g \in G';\, g = w\,\mathrm{Fr}(g)w\}.$$

Theorem. *Suppose that* $\mathrm{char}\, F > 2$. *Denote by* $|a|$ *the product of the first* $[n/2]$ *entries down the diagonal of* $a \in A$. *Then*

$$J((\mu a, \mu\,\mathrm{Fr}(\mu)), b'(f)) = (-1)^{\mathrm{val}_{F'}(\mu) n(n-1)/2}(-1)^{(n-1)\,\mathrm{val}_{F'}(|a|)} I((\mu a, \mu\,\mathrm{Fr}(\mu)), b(f))$$

for every spherical function f *on* $\tilde{\mathrm{G}}'$.

In fact the conjecture of [F] concerns only spherical functions ϕ_1 on U and f_1 on G'/Z' where Z' is the center of G', but this case is obtained from our Theorem on putting $\phi_1(u) = \int \phi(\mu u, \mu\,\mathrm{Fr}(\mu))\,d\mu$ ($\mu \in F'^{\times}$, $d\mu$ assigns $\mathcal{O}'^{\times}$ the volume one) and

$$f_1(g) = \int_{F'^{\times}} (-1)^{n(n-1)/2\cdot \mathrm{val}(\mu)} f(g,\mu)\,d\mu,$$

as then $\phi_1'(g) = \int (-1)^{n(n-1)/2\cdot\mathrm{val}(\mu)}\phi'(\mu g, \mu\,\mathrm{Fr}(\mu))\,d\mu$.

The group of similitudes $\tilde{\mathrm{U}}$ (and $\tilde{\mathrm{G}}$) is introduced in the proof in order to be able to work with matrices with integral entries, in $\mathfrak{gl}(n, \mathcal{O}')$; $U \cap \mathfrak{gl}(n,\mathcal{O}') = K$ does not support any spherical (K-biinvariant) functions on U except the characteristic function of K.

We deal here only with the open set of elements of the form ${}^t x a x'$ in U and $\overline{x}^{-1} w a x$ of G', while [F] considers also singular elements (or other elements of order two in the Weyl group). To compare with [F] note that we define here the unitary group using the form w, not J of [F] whose anti-diagonal entries alternate from 1 to -1, as this simplifies the notations. Then for the entries of x we have $x_{i,i+1} = -\overline{x}_{n-i,n-i+1}$. For applications of the conjectures of [F] to liftings of automorphic forms from the unitary group $U(n, \mathbb{A}'/\mathbb{A})$ to $\mathrm{GL}(n, \mathbb{A}')$, and for local applications, see [F] and [F1].

Further, the conjecture of [F] takes $\mathrm{char}\, F$ to include zero, but this case is not treated here. It will be very interesting to examine whether the results in positive characteristic could be used to imply the conjecture in characteristic zero (cf. Kazhdan [K]), or to develop an analogue. Dealing with non regular orbits (not of the form ${}^t x a x'$, or $\overline{x}^{-1} w a x$), is a question concerning asymptotic expansion of orbital integrals; perhaps it can also be dealt with directly.

Soon after the conjecture of [F] (submitted May 1990) was formulated, circulated and proven for $n = 2$, an analogous conjecture – where U is replaced by $G = \mathrm{GL}(n, F)$ – was formulated by Jacquet and Ye [JY]. This analogous conjecture is relevant to base change of automorphic forms from G to G'. However, base change from G to G' is already known using the trace formula, while the lifting from U to G' is not yet proven, except for the cases of $n = 2, 3$ ([F2]). Jacquet and Ye [JY1] discussed their conjecture (for the unit element of the Hecke algebra and regular elements) for local F of any characteristic for GL(3) by brute force computations of the Mellin transforms of both sides of the equality. For F of positive characteristic and for the regular elements, the conjecture of [JY] has been proven by Ngô [N2], [N1] for $\mathrm{GL}(n)$ on translating it to a question on Kloosterman sums, and reducing it to a result of Deligne [D] by means of perverse sheaves.

We shall follow Ngô in translating our conjecture to a geometric statement, where the equality amounts to a description of the action of a certain involution. Our observation is that this involution is already described in Ngô [N2], [N1], and the conjecture of [F] follows. In contrast, the technique of brute force computations would require working out the Mellin transforms in the situation of [F] anew, and no simple reduction to the computations of [JY1] is known.

2. The proof – for local fields of positive characteristic – depends on a geometric interpretation of the spherical functions as the trace of a Frobenius on some perverse sheaves over a suitable variety. Our Fourier orbital integrals have a natural interpretation in this context, and the asserted comparison reduces to a computation of the action of one operator. The interpretation and the comparison are due to Ngô [N2] in the analogous situation of the base group $\mathrm{GL}(n)$. Our observation is that the same geometric construction can be used in our case where the base group is the unitary group U, simply on using another Frobenius, $\tilde{\mathrm{Fr}}^w$ rather than Fr of [N2]. Since the definition of $\tilde{\mathrm{Fr}}^w$ involves taking the inverse, and the geometric construction requires using integral matrices, we are led to considering the group of unitary similitudes instead of the unitary group. We then start by reviewing the geometric situation from Ngô [N2].

Let k be the finite field $\mathbb{F}_q$, k' its quadratic extension $\mathbb{F}_{q^2}$, and $\hat{k}$ an algebraic closure. Let $\mathcal{O} = k[\pi]$ be the ring of polynomials in one variable π over k, and $F = k(\pi)$ its fraction field. Our local field is the completion $F_\pi = k((\pi))$ of F at π; its ring of integers is $\mathcal{O}_\pi = k[[\pi]]$, and we have $F' = k'(\pi)$, $\mathcal{O}' = k'[\pi]$, $F'_\pi = k'((\pi))$, $\mathcal{O}'_\pi = k'[[\pi]]$, $\hat{F} = \hat{k}(\pi)$, $\hat{F}_\pi = \hat{k}((\pi))$, etc. Fix a positive integer n. Write G' for $\mathrm{GL}(n, F')$, $\tilde{\mathrm{G}}' = G' \times F'^\times$, $K' = \mathrm{GL}(n, \mathcal{O}')$, $\tilde{\mathrm{K}}' = K' \times \mathcal{O}'^\times$, $G'^{,+} = G' \cap \mathfrak{gl}(n, \mathcal{O}')$, $\tilde{\mathrm{G}}'^{,+} = G'^{,+} \times (\mathcal{O}' - \{0\})$. We add a subscript π for the group of points over $\mathcal{O}'_\pi$ or F'_π.

The set of $(g, \mu)\,\tilde{\mathrm{K}}'$ in $\tilde{\mathrm{G}}'^{,+} / \tilde{\mathrm{K}}'$ with $\mu\,\tilde{\mathrm{Fr}}^w(g) \in \mathfrak{gl}(n, \mathcal{O}')$ can naturally be identified with the set of k'-points of a k-scheme X which is locally of finite type, and whose k-points are the $(g, \mu)\,\tilde{\mathrm{K}}$, where $(g, \mu)\,\tilde{\mathrm{K}} = \tilde{\mathrm{Fr}}^w(g, \mu)\,\tilde{\mathrm{K}}$ $(=(\mu\,\tilde{\mathrm{Fr}}^w(g), \mathrm{Fr}(\mu))\,\tilde{\mathrm{K}}$, and $\tilde{\mathrm{Fr}}^w(g) = w^t\,\mathrm{Fr}(g)^{-1} w)$, and $\tilde{\mathrm{K}}$ is the group $\tilde{\mathrm{K}}'^{\,\tilde{\mathrm{Fr}}^w}$ of fixed points of $\tilde{\mathrm{Fr}}^w$ in $\tilde{\mathrm{K}}'$. Indeed, X is the infinite disjoint union of the schemes $X_{d,\nu}$ $(d,\, \nu \geq 0)$, of finite type over k, whose sets of k'-points are

$$X_{d,\nu}(k') = \{(g, \mu)\,\tilde{\mathrm{K}}' \in \tilde{\mathrm{G}}'^{,+} / \tilde{\mathrm{K}}';\ \deg(\det g) = d,\ \deg(\mu) = \nu,\ \mu\,\tilde{\mathrm{Fr}}^w(g) \in \mathfrak{gl}(n, \mathcal{O}')\},$$

and $X_{d,\nu}(k) = X_{d,\nu}(\hat{k})^{\tilde{\mathrm{Fr}}^w}$; the last set is empty unless $2d = n\nu$. The union over d, $\nu \geq 0$ of $X_{d,\nu}(k)$ is $\tilde{\mathrm{U}}^+ = \tilde{\mathrm{U}} \cap (\mathfrak{gl}(n, \mathcal{O}') \times \mathcal{O}')$, where $\tilde{\mathrm{U}}$ is the group of fixed points of $\tilde{\mathrm{Fr}}^w$ in $\tilde{\mathrm{G}}'$.

The variety $X_{d,\nu}(k')$ can also be described in terms of lattices: $gK' \in G^{\prime,+}/K'$ defines the lattice $\mathcal{R} = g\mathcal{O}'^n \subset \mathcal{O}'^n$, and $\deg(\det g) = d$ means $\dim_{k'}(\mathcal{O}'^n/\mathcal{R}) = d$. Consider the non degenerate Hermitian form $(x, y) \mapsto \langle x, y\rangle = {}^t x w \,\mathrm{Fr}(y)$ on $\mathcal{O}'^n \times \mathcal{O}'^n$. The dual of the lattice $\mathcal{R} \subset \mathcal{O}'^n$ is the lattice $\mathcal{R}' = \{x \in F'^n;\ \langle x, y\rangle \in \mathcal{O}'\}$ in $\mathcal{O}'^n$. The set $X_{d,\nu}(k)$ of k-points of $X_{d,\nu}$ can be identified with the set of self dual lattices $\mathcal{R}$ with $\dim_{k'}(\mathcal{O}'^n/\mathcal{R}) = d$. Thus ${}^t g w \,\mathrm{Fr}(g) = \mu w$ translates to $\langle x, y\rangle \in \mu\mathcal{O}'$, $\mu = \mu(\mathcal{R})$, to each pair x, $y \in \mathcal{R}$.

By Cramer's rule $X_{d,d'}(k')$ is non empty if $d' \geq d$. The polynomial $\det g$ is well defined only up to an invertible scalar, and we write $\phi_d(g)$ for the monic polynomial in the class $\det g$. We obtain a morphism $\phi_d : X_{d,\nu} \to Q_d$, where Q_d is the affine space of dimension d over k' whose set of k'-points are the monic polynomials of degree d in π over k'.

Write $X_{d,\nu}(P)$ for the fiber $\phi_d^{-1}(P)$ of ϕ_d. Note that the group $K' = \mathrm{GL}(n, \mathcal{O}')$ acts on $X_{d,\nu}(k')$ ($h : g \mapsto hg$), and its subgroup $1 + P \cdot \mathfrak{gl}(n, \mathcal{O}')$ acts trivially on $X_{d,\nu}(P)$ (since by Cramer's rule, if $P = \det g$ then $Pg^{-1} \in \mathfrak{gl}(n, \mathcal{O}')$). Hence the group scheme G_d over Q_d, whose fiber at $P \in Q_d(k')$ is $G_d(P)(k') = \mathrm{GL}(n, \mathcal{O}'/(P)) = \mathrm{GL}(n, \mathcal{O}')/(1 + P \cdot \mathfrak{gl}(n, \mathcal{O}'))$, acts on $X_{d,\nu}$. It is in fact a k-group, its k-points are $G_d(P)(k) = G_d(P)(k')^{\tilde{\mathrm{Fr}}^w}$.

We are concerned with the local situation: the set of $(g, \mu)\,\tilde{\mathrm{K}}'_\pi$ in $\tilde{\mathrm{G}}^{\prime,+}_\pi / \tilde{\mathrm{K}}'_\pi$, where $\tilde{\mathrm{G}}^{\prime,+}_\pi = G^{\prime,+}_\pi \times (\mathcal{O}'_\pi - \{0\})$ and $G^{\prime,+}_\pi = \mathrm{GL}(n, F'_\pi) \cap \mathfrak{gl}(n, \mathcal{O}'_\pi)$, and $\tilde{\mathrm{K}}'_\pi = \mathrm{GL}(n, \mathcal{O}'_\pi) \times \mathcal{O}'^{\times}_\pi$, with $\deg(\det g) = d$, $\deg \mu = \nu$ and $\mu\, \tilde{\mathrm{Fr}}^w(g) \in \mathfrak{gl}(n, \mathcal{O}'_\pi)$, is naturally isomorphic to the set of k'-points of the fiber $X_{d,\nu,\pi}$ of the morphism ϕ_d above the k'-point $P = \pi^d \in Q_d(k')$ and $\mu = \pi^d$.

The complexes $\mathcal{A}_\lambda$ which appear in the geometric definition of the spherical functions are defined by means of the k-scheme $\tilde{X}_{d,\nu}$ whose set of k'-points is

$$\tilde{X}_{d,\nu}(k') = \{(\{K' = g_0K' \supset g_1K' \supset \cdots \supset g_{d-1}K' \supset g_dK' = gK'\}, \mu);$$

$$g_iK' \in G^{\prime,+}/K',\ \mu \in \mathcal{O}' - \{0\},\ \mu\, \tilde{\mathrm{Fr}}^w(g_i) \in \mathfrak{gl}(n, \mathcal{O}'),\ g_{i-1}^{-1}g_i \in \mathfrak{gl}(n, \mathcal{O}')$$

$$\text{has determinant of degree 1 with monic form } \pi - x_i; \deg \mu = \nu\}.$$

There is a morphism $\pi_d : \tilde{\mathrm{X}}_{d,\nu} \to X_{d,\nu}$ with $\pi_d(K' \supset \cdots \supset gK', \mu) = (gK', \mu)$, and a morphism $\tilde{\phi}_d : \tilde{\mathrm{X}}_{d,\nu} \to \mathbb{A}^d$, $(K' \supset \cdots \supset gK', \mu) \mapsto (x_1, \ldots, x_d)$. Write $\mathbb{A}^d_{rss}$ for the open dense subset of $(x_1, \ldots, x_d)$ in $\mathbb{A}^d$ with $x_i \neq x_j$ ($i \neq j$), $Q_{d,rss} = \mathbb{A}^d_{rss}/S_d$, where S_d is the symmetric group on d letters, $X_{d,\nu,rss} = \phi_d^{-1}(Q_{d,\nu,rss})$ and $\tilde{\mathrm{X}}_{d,\nu,rss} = \tilde{\phi}_d^{-1}(\mathbb{A}^d_{rss})$. Then $\tilde{\mathrm{X}}_{d,\nu,rss}$ is an S_d-torsor above $X_{d,\nu,rss}$.

Given a finite dimensional ℓ-adic representation $\rho : S_d \to \mathrm{GL}(V_\rho)$ we obtain a representation of the fundamental group of $X_{d,\nu,rss}$, hence a local system $\mathcal{L}_{\rho,\nu}$ on $X_{d,\nu,rss}$. Further, $\mathcal{L}_{\rho,\nu} = (\pi_{rss,*}\hat{\mathbb{Q}}_\ell \boxtimes V_\rho)^{S_d}$, where S_d acts diagonally. Note that the dimension of $X_{d,\nu}$ is even, and define (following [N2]) $\mathcal{A}_{\rho,\nu}$ to be the perverse sheaf which is the intermediate extension of $\mathcal{L}_{\rho,\nu}[\dim X_{d,\nu}](\frac{1}{2}\dim X_{d,\nu})$ to $X_{d,\nu}$. Thus $\mathcal{A}_{\rho,\nu} \simeq (R\pi_*\hat{\mathbb{Q}}_\ell \boxtimes V_\rho)^{S_d}[\dim X_{d,\nu}](\frac{1}{2}\dim X_{d,\nu})$, and for each geometric point $P \in Q_d(\hat{k})$ (i.e. $P = \prod_i(\pi - x_i)$), the restriction of $\mathcal{A}_{\rho,\nu}[-d](-\frac{1}{2}d)$ to the fiber $\phi_d^{-1}(P)$ remains a perverse sheaf.

The perverse sheaf $\mathcal{A}_{\rho,\nu}$ is naturally G_d-equivariant. Put $X_{d,\nu,\pi} = X_{d,\nu}(\pi^d)$. Recall that the irreducible representations of the symmetric group S_d are parametrized by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$ of d (thus $\sum_i \lambda_i = d$, and λ is called an n-partition of d). If $\rho = \rho_\lambda$, we write $\mathcal{A}_{\lambda,\nu}$ for $\mathcal{A}_{\rho_\lambda,\nu}$, $\mathcal{A}_{\lambda,\nu,\pi}$ for the restriction of $\mathcal{A}_{\rho_\lambda,\nu}[-d](-\frac{1}{2}d)$ to $X_{d,\nu,\pi}$. A convolution product $\mathcal{A}_{\lambda,\nu} * \mathcal{A}_{\lambda',\nu'}$ is defined in [N2]; it is commutative, associative, and satisfies $\mathcal{A}_{\rho',\nu'} * \mathcal{A}_{\rho'',\nu''} = \mathcal{A}_{\rho,\nu'+\nu''}$, $\rho = \mathrm{Ind}_{S_{d'}\times S_{d''}}^{S_d}(\rho'\otimes\rho'')$.

An alternative description of the perverse sheaves $\mathcal{A}_{\lambda,\nu}$ is given via the Cartan decomposition $\tilde{\mathrm{G}}' = \cup_{\lambda,\nu}\tilde{\mathrm{K}}'(\pi^\lambda,\pi^\nu)\tilde{\mathrm{K}}'$ where $\tilde{\mathrm{K}}' = \mathrm{GL}(n,\mathcal{O}')\times\mathcal{O}'^\times$ and $\pi^\lambda = \mathrm{diag}(\pi^{\lambda_1},\dots,\pi^{\lambda_n})$, $\lambda = (\lambda \geq \cdots \geq \lambda_n \geq 0)$ is an n-partition of d. Then $X_{d,\nu} = \cup_{|\lambda|=d} X_{\lambda,\nu}$ is a decomposition of $X_{d,\nu}$, where $X_{\lambda,\nu}$ is the variety whose k' points are $\{kg\tilde{\mathrm{K}}';\, k\in\tilde{\mathrm{K}}'\}$. Since the closure $\overline{X}_{\lambda,\nu}$ of $X_{\lambda,\nu}$ in $X_{d,\nu}$ is in general singular, one considers its ℓ-adic intersection complex $\mathcal{A}_\lambda = \mathrm{IC}(\overline{X}_{\lambda,\nu},\hat{\mathbb{Q}}_\ell)$. It is defined over k and is G_d-equivariant.

3. Following Lusztig [L], one defines the functions

$$a_{\lambda,\nu} : X_{d,\nu,\pi}(k') \to \hat{\mathbb{Q}}_\ell, \qquad a_{\lambda,\nu}(x) = \mathrm{tr}\{\mathrm{Fr}_{q^2};(\mathcal{A}_{\lambda,\nu,\pi})_x\}.$$

Choosing an isomorphism $\hat{\mathbb{Q}}_\ell \simeq \mathbb{C}$, the functions $a_{\lambda,\nu}$ can be viewed as elements of $\mathcal{H}' = C_c(\tilde{\mathrm{G}}'^{,+}//\tilde{\mathrm{K}}',\hat{\mathbb{Q}}_\ell)$, since $\mathcal{A}_{\lambda,\nu}$ is G_d-equivariant. As the transition matrix from the $a_{\lambda,\nu}$ to the characteristic functions of $\tilde{\mathrm{K}}'(\pi^\lambda,\pi^\nu)\tilde{\mathrm{K}}'$ is upper triangular, the $a_{\lambda,\nu}$ make a basis of $\mathcal{H}'$.

It was observed by Ngô [N2] that for comparison purposes it is advantageous to redefine the function $a_{\lambda,\nu}$ as a function on the set of k-points of the k-variety $\mathrm{Res}_{k'/k} X_{d,\nu,\pi}$. Thus let σ be the endomorphism $\sigma(x,x') = (x',x)$ of $X^2_{d,\nu,\pi}$, and note that the set of fixed points $(X_{d,\nu,\pi}\times X_{d,\nu,\pi})(\hat{k})^{\mathrm{Fr}\circ\sigma}$ is $\{(x,\mathrm{Fr}(x));\, x\in X_{d,\nu,\pi}(k')\}$, isomorphic to $X_{d,\nu,\pi}(k')$. Recall again that for any n-partition λ of d, there is a perverse sheaf $\mathcal{A}_{\lambda,\nu,\pi}$, the restriction of $\mathcal{A}_{\lambda,\nu}[-d](-\frac{1}{2}d)$ to $X_{d,\nu,\pi}$. In fact $\mathcal{A}_{\lambda,\nu,\pi} = \mathrm{IC}(\overline{G_{d,\pi}\pi^\lambda}\times\pi^\nu,\hat{\mathbb{Q}}_\ell)$, where the closure is in $X_{d,\nu,\pi}$. The endomorphism σ lifts to an isomorphism $\tilde{\sigma} : \mathcal{A}_{\lambda,\nu,\pi}\boxtimes\mathcal{A}_{\lambda,\nu,\pi} \to \mathcal{A}_{\lambda,\nu,\pi}\boxtimes\mathcal{A}_{\lambda,\nu,\pi}$ which permutes the two factors, and [N2] defines the function

$$f_{\lambda,\nu} : (X_{d,\nu,\pi}\times X_{d,\nu,\pi})^{\mathrm{Fr}\circ\sigma} \to \hat{\mathbb{Q}}_\ell, \qquad f_{\lambda,\nu}(x,\mathrm{Fr}(x)) = \mathrm{tr}\{\mathrm{Fr}\circ\tilde{\sigma};\,(\mathcal{A}_{\lambda,\pi}\boxtimes\mathcal{A}_{\lambda,\pi})_{(x,\mathrm{Fr}(x))}\}.$$

This can be viewed as a function on $X_{d,\nu,\pi}(k')$, hence on G'/K'. Of course in the case of [N2] the parameter ν plays no role and was taken to be 0, but it does play a significant role in our case.

To describe geometrically the constant terms of our spherical functions, for any n-tuple $\underline{d} = (d_1,\dots,d_n)$, $d_i \geq 0$, $\sum d_i = d$, consider the locally closed subscheme $S_{\underline{d},\nu,\pi}$ of $X_{d,\nu,\pi}$, whose set of k'-points consists of the points of $X_{d,\nu,\pi}(k')$ of the form $(\pi^{\underline{d}}x,\pi^\nu)\tilde{\mathrm{K}}'$, where $x \in N(F'_\pi)/N(\mathcal{O}'_\pi)$ and $\pi^{\underline{d}}x \in \mathfrak{gl}(n,\mathcal{O}'_\pi)$. The k-points are those defined by the x with $x = \tilde{\mathrm{Fr}}^w(x)$, and $S_{\underline{d},\nu,\pi}(k)$ is empty unless $\underline{d}+w(\underline{d})$ is constant, equal to ν.

In terms of lattices, $S_{\underline{d},\nu,\pi}(k')$ consists of the lattices $\mathcal{R}_\pi \subset \mathcal{O}^n_\pi$ such that, putting $E_{i,\pi} = \oplus_{1\leq j\leq i}\mathcal{O}_\pi e_j$, for each $i (1\leq i\leq n)$ we have

$$\mathcal{R}_\pi \cap E_{i,\pi}/\mathcal{R}_\pi\cap E_{i-1,\pi} = \mathcal{O}_\pi\pi^{d_i}e_i \oplus E_{i-1,\pi}/E_{i-1,\pi};$$

here $e_1, \dots, e_n$ is the standard basis of $\mathcal{O}_\pi^n$. Indeed, the map is given by $x \mapsto \mathcal{R}_\pi = \pi^{\underline{d}} x \mathcal{O}_\pi^n$. The set $S_{\underline{d},\nu,\pi}(k)$ consists of the self dual lattices of the form $\mathcal{R}_\pi = \pi^{\underline{d}} x \mathcal{O}_\pi^n$.

The constant terms of the spherical function $a_{\lambda,\nu}(g,\mu) = \mathrm{tr}\{\mathrm{Fr}^2; (\mathcal{A}_{\lambda,\nu,\pi})_{(g,\mu)}\}$, where $\mathcal{A}_{\lambda,\nu,\pi}$ is a complex of ℓ-adic sheaves on $X_{d,\nu,\pi}$ which is $G_{d,\pi}$-equivariant, are given by

$$a^B_{\lambda,\nu}(\pi^{\underline{d}}, \mu) = \delta'_B(\pi^{\underline{d}}) \sum_{x \in N(F'_\pi)/N(\mathcal{O}'_\pi)} a_{\lambda,\nu}(\pi^{\underline{d}} x, \mu) = \delta'_B(\pi^{\underline{d}}) \sum_{x \in S_{\underline{d},\nu,\pi}(k')} a_{\lambda,\nu}(\pi^{\underline{d}} x, \mu)$$

$$= \delta'_B(\pi^{\underline{d}}) \sum_{x \in S_{\underline{d},\nu,\pi}(\hat{k})^{\mathrm{Fr}^2}} \mathrm{tr}\{\mathrm{Fr}^2; (\mathcal{A}_{\rho,\nu,\pi})_x\} = \delta'_B(\pi^{\underline{d}})\, \mathrm{tr}\{\mathrm{Fr}^2;\, R\Gamma_c(\hat{S}_{\underline{d},\nu,\pi}, \mathcal{A}_{\rho,\nu,\pi})\},$$

where $\hat{X} = X \otimes_k \hat{k} (= X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \hat{k})$ for any k-scheme X, and where the last equality follows from the Grothendieck-Lefschetz fixed point formula. Of course $a_{\lambda,\nu}(g,\mu) \neq 0$ implies that $\mu = \pi^\nu$ and that $|\lambda| = \deg(\det g)$.

Similarly, the constant term $f^B_{\lambda,\nu}(\pi^{\underline{d}}, \mu)$ of the function

$$f_{\lambda,\nu}(g,\mu) = f_{\lambda,\nu}(g,\mu; \mathrm{Fr}(g,\mu)) = \mathrm{tr}\{\tilde{\sigma} \circ \mathrm{Fr}; (\mathcal{A}_{\lambda,\nu,\pi} \boxtimes \mathcal{A}_{\lambda,\nu,\pi})_{(g,\mu;\mathrm{Fr}(g,\mu))}\}$$

at $(\pi^{\underline{d}}, \mu) = (\pi^{\underline{d}}, \mu; \pi^{\underline{d}}, \mu)$ is given by

$$\delta'_B(\pi^{\underline{d}}) \sum_{x \in N(F'_\pi)/N(\mathcal{O}'_\pi)} f_{\lambda,\nu}(\pi^{\underline{d}} x, \mu; \pi^{\underline{d}} \mathrm{Fr}(x), \mu)$$

(non zero contributions are indexed only by the x in $\mathfrak{gl}(n, \mathcal{O}'_\pi)$, thus x lies in $(S_{\underline{d},\nu,\pi} \times S_{\underline{d},\nu,\pi})(\hat{k})^{\mathrm{Fr} \circ \sigma} \simeq S_{\underline{d},\nu,\pi}(k')$),

$$= \delta'_B(\pi^{\underline{d}}) \sum_{x \in S_{\underline{d},\nu,\pi}(k')} \mathrm{tr}\{\tilde{\sigma} \circ \mathrm{Fr}; (\mathcal{A}_{\lambda,\nu,\pi} \boxtimes \mathcal{A}_{\lambda,\nu,\pi})_{(\pi^{\underline{d}} x, \mu; \mathrm{Fr}(\pi^{\underline{d}} x), \mathrm{Fr}(\mu))}\}$$

$$= \delta'_B(\pi^{\underline{d}})\, \mathrm{tr}\{\mathrm{Fr} \circ \tilde{\sigma}; R\Gamma_c(\hat{S}_{\underline{d},\nu,\pi} \times \hat{S}_{\underline{d},\nu,\pi}, \mathcal{A}_{\lambda,\nu,\pi} \boxtimes \mathcal{A}_{\lambda,\nu,\pi})\}.$$

The function $f_{\lambda,\nu}(g,\mu)$ is equal to $a_{\lambda,\nu}(g,\mu)$ since spherical functions are determined by their constant terms (by the Satake isomorphism), and these are equal. Indeed, by [N2, 5.2.3], the complex $R\Gamma_c(\hat{S}_{\underline{d},\nu,\pi}, \mathcal{A}_{\lambda,\nu,\pi})$ is concentrated in degree $2\langle \underline{d}, \delta \rangle = 2\sum_{1 \le i \le n} d_i(n - 2i + 1)$, and Fr acts on the cohomology space $V = H_c^{2\langle \underline{d}, \delta \rangle}(\hat{S}_{\underline{d},\nu,\pi}, \mathcal{A}_{\lambda,\nu,\pi})$ as multiplication by $q^{\langle \underline{d}, \delta \rangle}$. Choosing a basis $e_1, \dots, e_t$ for V, we may identify V with the span of $e_i \otimes e_i$ in $V \otimes V$, and note that $V \otimes V = V \oplus V'$, where V' is the span of $e_i \otimes e_j$ $(i \neq j)$. Then the trace of $\tilde{\sigma}$ on $V \otimes V$ is equal to the trace of the identity on V, and $f^B_{\lambda,\nu}(\pi^{\underline{d}}, \mu) = \delta'_B(\pi^{\underline{d}})\, \mathrm{tr}\{\mathrm{Fr}^2; R\Gamma_c(\hat{S}_{\underline{d},\nu,\pi}, \mathcal{A}_{\lambda,\nu,\pi})\} = a^B_{\lambda,\nu}(\pi^{\underline{d}}, \mu)$.

4. We shall in fact need another realization of this spherical function, namely

$$\tilde{f}_{\lambda,\nu}(g,\mu) = \tilde{f}_{\lambda,\nu}(g,\mu; \tilde{\mathrm{Fr}}^w(g,\mu)) = \mathrm{tr}\{\tilde{\sigma} \circ \tilde{\mathrm{Fr}}^w; (\mathcal{A}_{\lambda,\nu,\pi} \boxtimes \mathcal{A}_{\lambda^\vee,\nu,\pi})_{(g,\mu;\tilde{\mathrm{Fr}}^w(g,\mu))}\}.$$

Here, if λ is the n-partition of d parametrizing the representation ρ of S_d, then $\lambda^\vee$ parametrizes the dual representation $\rho^\vee$ of ρ. The constant terms $\tilde{f}^B_{\lambda,\nu}(\pi^{\underline{d}},\mu)$ of $\tilde{f}_{\lambda,\nu}$ are given by

$$\delta'_B(\pi^{\underline{d}}) \sum_{x\in N(F'_\pi)/N(\mathcal{O}'_\pi)} \tilde{f}_{\lambda,\nu}(\pi^{\underline{d}}x,\mu;\tilde{\mathrm{Fr}}^w(\pi^{\underline{d}}x,\mu)).$$

The sum ranges over x such that $(\pi^{\underline{d}}x,\mu;\tilde{\mathrm{Fr}}^w(\pi^{\underline{d}}x,\mu))$ lies in $(\mathfrak{gl}(n,\mathcal{O}'_\pi)\times\mathcal{O}'_\pi)^2$, thus $(x,\tilde{\mathrm{Fr}}^w(x))$ lies in $(S_{\underline{d},\nu,\pi}\times S_{\nu-w(\underline{d}),\nu,\pi})(\hat{k})^{\tilde{\mathrm{Fr}}^w\circ\sigma}$. Recall that

$$S_{\underline{d},\nu,\pi}(k') = \{x\in N(F'_\pi)/N(\mathcal{O}'_\pi);\ \pi^{\underline{d}}x, \pi^{\nu-w(\underline{d})}\,\tilde{\mathrm{Fr}}^w(x)\in\mathfrak{gl}(n,\mathcal{O}')\}.$$

We obtain (note that $S_{\underline{d},\nu}=S_{\nu-w(\underline{d}),\nu}$)

$$\delta'_B(\pi^{\underline{d}}) \sum_{(x,\tilde{\mathrm{Fr}}^w(x))\in(S_{\underline{d},\nu,\pi}\times S_{\underline{d},\nu,\pi})(\hat{k})^{\tilde{\mathrm{Fr}}^w\circ\sigma}} \mathrm{tr}\{\tilde{\sigma}\circ\tilde{\mathrm{Fr}}^w;\ (\mathcal{A}_{\lambda,\nu,\pi}\boxtimes\mathcal{A}_{\lambda^\vee,\nu,\pi})_{(\pi^{\underline{d}}x,\mu;\tilde{\mathrm{Fr}}^w(\pi^{\underline{d}}x,\mu))}\}$$

$$=\delta'_B(\pi^{\underline{d}})\,\mathrm{tr}\{\tilde{\sigma}\circ\tilde{\mathrm{Fr}}^w;\ R\Gamma_c(\hat{S}_{\underline{d},\nu,\pi}\times\hat{S}_{\underline{d},\nu,\pi},\mathcal{A}_{\lambda,\nu,\pi}\boxtimes\mathcal{A}_{\lambda^\vee,\nu,\pi})\}.$$

This is again equal to $\delta'_B(\pi^{\underline{d}})\,\mathrm{tr}\{\mathrm{Fr}^2;\ R\Gamma_c(\hat{S}_{\underline{d},\nu,\pi},\mathcal{A}_{\rho,\nu,\pi})\}$, since given an involutive automorphism A ($=\{x\mapsto w^tx^{-1}w\}$) of V, embedding V in $V\otimes V$ as the span of $\{e_i\otimes Ae_i\}$, we can write $V\otimes V=V\oplus V'$, where $V'=\mathrm{Span}\{e_i\otimes Ae_j(i\neq j)\}$, and note that $\mathrm{tr}(\tilde{\sigma}\circ A;V\otimes V)$ is just $\mathrm{tr}\{1;V\}$. We conclude that $\tilde{f}_{\lambda,\nu}=f_{\lambda,\nu}=a_{\lambda,\nu}$.

The advantage of the spherical functions $\tilde{f}_{\lambda,\nu}$ is that their image under the base change homomorphism $b:\mathcal{H}'\to\mathcal{H}$ can be described in practical terms. In fact we claim that $b(\tilde{f}_{\lambda,\nu})$ is $\tilde{\phi}_{\lambda,\nu}$, where $\tilde{\phi}_{\lambda,\nu}(g,\mu)=\mathrm{tr}\{\tilde{\mathrm{Fr}}^w\circ\kappa;\ (\mathcal{A}_{\lambda,\nu,\pi}*\mathcal{A}_{\lambda^\vee,\nu,\pi})_{(g,\mu)}\}$, and (g,μ) is a fixed point of $\tilde{\mathrm{Fr}}^w$; recall: $\tilde{\mathrm{Fr}}^w(g,\mu)=(\mu\,\tilde{\mathrm{Fr}}^w(g),\mathrm{Fr}(\mu))$. Here κ is the commutativity isomorphism $\kappa:\mathcal{A}_{\lambda,\nu}*\mathcal{A}_{\lambda',\nu'}\to\mathcal{A}_{\lambda',\nu'}*\mathcal{A}_{\lambda,\nu}$ of the convolution product of [N2, 3.3]. To show that $b(\tilde{f}_{\lambda,\nu})=\tilde{\phi}_{\lambda,\nu}$ we compute the constant terms $\tilde{\phi}^B_{\lambda,\nu}(\pi^{\tilde{\underline{d}}},\mu)$ of the function $\tilde{\phi}_{\lambda,\nu}$ (times $\delta_B(\pi^{\tilde{\underline{d}}})^{-1}$):

$$\sum_{x\in N(F_\pi)/N(\mathcal{O}_\pi)} \tilde{\phi}_{\lambda,\nu}(\pi^{\tilde{\underline{d}}}x,\mu).$$

The non zero terms are indexed by x such that $(\pi^{\tilde{\underline{d}}}x,\mu)\in(\mathfrak{gl}(n,\mathcal{O}')\times\mathcal{O}')\cap\tilde{\mathrm{U}}$ and $\mu=\pi^\nu$, thus $\tilde{\underline{d}}=\nu-w(\tilde{\underline{d}})$, and so $|\tilde{\underline{d}}|=n\nu/2$. The set of these elements is the fixed point set of $\tilde{\mathrm{Fr}}^w$ acting on $S_{\tilde{\underline{d}},\nu,\pi}(\hat{k})$. By the Grothendieck fixed point formula, the sum over x in $S_{\tilde{\underline{d}},\nu,\pi}(\hat{k})^{\tilde{\mathrm{Fr}}^w}$ of

$$\tilde{\phi}_{\lambda,\nu}(\pi^{\tilde{\underline{d}}}x,\mu)=\mathrm{tr}\{\tilde{\mathrm{Fr}}^w\circ\kappa;\ (\mathcal{A}_{\lambda,\nu,\pi}*\mathcal{A}_{\lambda^\vee,\nu,\pi})_{(\pi^{\tilde{\underline{d}}}x,\mu)}\}$$

is equal to

$$\mathrm{tr}\{\tilde{\mathrm{Fr}}^w\circ\kappa;\ R\Gamma_c(\hat{S}_{\tilde{\underline{d}},\nu,\pi},\mathcal{A}_{\lambda,\nu,\pi}*\mathcal{A}_{\lambda^\vee,\nu,\pi})\}.$$

According to [N2, 5.4.2], there is a decomposition

$$R\Gamma_c(\hat{S}_{\tilde{\underline{d}},\nu,\pi},\mathcal{A}_{\lambda',\nu',\pi}*\mathcal{A}_{\lambda'',\nu'',\pi})=\oplus R\Gamma_c(\hat{S}_{\underline{d}',\nu',\pi},\mathcal{A}_{\lambda',\nu',\pi})\otimes R\Gamma_c(\hat{S}_{\underline{d}'',\nu'',\pi},\mathcal{A}_{\lambda'',\nu'',\pi}),$$

where the sum ranges over $\underline{d}' + \underline{d}'' = \tilde{\underline{d}}$, $\nu' + \nu'' = \nu$, $|\underline{d}'| = |\lambda'|$, $|\underline{d}''| = |\lambda''|$, and κ permutes the two factors of $\otimes$ in the direct sum. Thus

$$R\Gamma_c(\hat{S}_{\underline{d}',\nu',\pi}, \mathcal{A}_{\lambda',\nu',\pi}) \otimes R\Gamma_c(\hat{S}_{\underline{d}'',\nu'',\pi}, \mathcal{A}_{\lambda'',\nu'',\pi})$$

$$\xrightarrow{\kappa} R\Gamma_c(\hat{S}_{\underline{d}'',\nu'',\pi}, \mathcal{A}_{\lambda'',\nu'',\pi}) \otimes R\Gamma_c(\hat{S}_{\underline{d}',\nu',\pi}, \mathcal{A}_{\lambda',\nu',\pi})$$

$$\xrightarrow{\tilde{\mathrm{Fr}}^w} R\Gamma_c(\hat{S}_{\nu''-w(\underline{d}''),\nu'',\pi}, \mathcal{A}_{(\lambda'')^*,\nu'',\pi}) \otimes R\Gamma_c(\hat{S}_{\nu'-w(\underline{d}'),\nu',\pi}, \mathcal{A}_{(\lambda')^*,\nu',\pi}).$$

Consequently $\tilde{\mathrm{Fr}}^w \circ \kappa$ has fixed points only if $\underline{d}' = \nu'' - w(\underline{d}'')$, $\underline{d}'' = \nu' - w(\underline{d}')$, $\nu' = \nu''$, $2\nu' = \nu$, $\lambda = (\lambda')^*$, $|\underline{d}'| = |\lambda| = |\nu' - w(\underline{d}')|$, thus $|\underline{d}'| = n\nu'/2$ and $\tilde{\underline{d}} = \nu' + \underline{d}' - w(\underline{d}')$, thus $|\tilde{\underline{d}}| = n\nu/2$, and then – choosing λ' to be $\lambda^\vee$ – the constant term is

$$\delta_B(\pi^{\tilde{\underline{d}}}) \sum \mathrm{tr}\{\tilde{\mathrm{Fr}}^w \circ \kappa;\ R\Gamma_c(\hat{S}_{\underline{d}',\nu',\pi}, \mathcal{A}_{\lambda,\nu',\pi}) \otimes R\Gamma_c(\hat{S}_{\nu'-w(\underline{d}'),\nu',\pi}, \mathcal{A}_{\lambda^\vee,\nu',\pi})\},$$

where the sum ranges over $\nu' = \nu/2$, $\nu' + \underline{d}' - w(\underline{d}') = \tilde{\underline{d}}$, $|\underline{d}'| = |\lambda|$,

$$= \sum_{\{\underline{d}';\, \nu'+\underline{d}'-w(\underline{d}')=\tilde{\underline{d}}\}} \tilde{f}^B_{\lambda,\nu}(\pi^{\underline{d}'}, \mu'),$$

implying that $b(\tilde{f}_{\lambda,\nu}) = \tilde{\phi}_{\lambda,\nu}$, as asserted.

5. The Fourier orbital integrals of the spherical functions $\tilde{\phi}_{\lambda,\nu}$ on $\tilde{\mathrm{U}}$ has to be compared with an analogous integral of a function on $S^+_\pi = S(F_\pi)^+ = S(F_\pi) \cap (\mathfrak{gl}(n, \mathcal{O}') \times \mathcal{O}')$, where $S_\pi = S(F_\pi) = \{(s,\mu) \in G'_\pi \times F'^{\times}_\pi;\ (s,\mu) = {}^t\,\tilde{\mathrm{Fr}}^w(s,\mu)\}$ (thus $\mu \in F'_\pi$ and $sw\,\mathrm{Fr}(s)w = \mu$).

The group $G'_\pi \times F'^{\times}_\pi$ acts on S_π by

$$(g,\nu):\ (s,\mu) \mapsto (g,\nu)(s,\mu)^t\,\tilde{\mathrm{Fr}}^w(g,\nu) = (\nu gsw\,\mathrm{Fr}(g)^{-1}w,\ \mu\nu\,\mathrm{Fr}(\nu)).$$

Consider the space $\mathcal{H}^+_S = C_c(S^+_\pi/\tilde{\mathrm{K}}'_\pi, \hat{\mathbb{Q}}_\ell)$ of compactly supported $\hat{\mathbb{Q}}_\ell$-valued $\tilde{\mathrm{K}}'_\pi$-invariant functions on S^+_π, and the morphism

$$b':\ H'^{,+} = C_c(G'^{,+}_\pi//\tilde{\mathrm{K}}'_\pi, \hat{\mathbb{Q}}_\ell) \to H^+_S,$$

given by

$$b'(f)((g,\mu)\,\tilde{\mathrm{Fr}}^w({}^tg,\mu)) = \int_{G^w_\pi} f(gh,\mu)\,dh$$

where $G^w_\pi = \{h \in G'_\pi = \mathrm{GL}(n, F'_\pi);\ h = w\,\mathrm{Fr}(h)w\}$, and $b'(f)(s,\mu) = 0$ if $\mathrm{val}_\pi(\mu)$ is odd. Note that since f is spherical, namely K'-biinvariant, the dependence on μ is only through its valuation.

To express the function $\phi' = b'(f)$ which occurs in our comparison, we use the variety $\mathfrak{g}_{d',\nu,r}$, which is fibered over the affine scheme $Q_{d',r}$ over k' $(0 \le d' \le r)$ of pairs of polynomials (P,R) of degrees d', r over k', with $P|R$. The set of k'-points of the fiber above (P,R) is

$$\mathfrak{g}_{d',\nu,r}(P,R)(k') = \{(g,g',\mu) \in \mathfrak{gl}(n, \mathcal{O}'/(R))^2 \times \mathcal{O}'/(R);$$

$$\det(g) \in P'(\mathcal{O}'/(R))^\times,\ \deg\mu \equiv \nu(\mod r),\ \mathrm{Fr}(g)w^tg'w = \mu\}$$

(see [N2, 7.1]). We shall just write $\mu\,\tilde{\mathrm{Fr}}^w(g)$ for g' and refer to members of $\mathfrak{g}_{d',\nu,r}(P,R)(k')$ as pairs (g,μ) (for which g' exists).

The endomorphism $(\tau\times\tau)\circ\sigma\circ\tilde{\mathrm{Fr}}^w$ of $\mathfrak{g}^2_{d',\nu,r}$, where τ stands for transposition and σ for permutation of the two factors, maps $(g,\mu;g',\mu')$ to

$$(\mu'\,\mathrm{Fr}^w(g')^{-1},\mathrm{Fr}(\mu');\mu\,\mathrm{Fr}^w(g)^{-1},\mathrm{Fr}(\mu)).$$

Its fixed points are $(g,\mu;\mu\,\mathrm{Fr}^w(g)^{-1},\mathrm{Fr}(\mu))$.

There is a natural map $p = p_{d',r,\nu} : \mathfrak{g}_{d',r,\nu}(k') \to X_{d',\nu}(k')$, given by lifting $g \in \mathfrak{gl}(n,\mathcal{O}'/(R))$ with $\det(g) \equiv P(\mathcal{O}'/(R))^\times$ to $\tilde{g} \in \mathrm{GL}(n,\mathcal{O}')$ with $\det(\tilde{g}) = P$, lifting $\mu \in \mathcal{O}'/(R)$ to $\tilde{\mu} \in \mathcal{O}'$ of degree ν, such that $\tilde{\mu}\,\tilde{\mathrm{Fr}}^w(\tilde{g}) \in \mathfrak{gl}(n,\mathcal{O}')$. The group $G_{d',r} = G_r \times_{Q_r} Q_{d',r}$, whose stalk above (P,R) has k'-points $\mathrm{GL}(n,\mathcal{O}'/(R))$, acts on the right and on the left on $\mathfrak{g}_{d',\nu,r}$ by $(h',h)g = h'gh^{-1}$. The morphism p is smooth, the action is transitive, the morphism p is invariant under the right action of $G_{d',r}$ and equivariant relative to the left action of $G_{d',r}$ on $\mathfrak{g}_{d',\nu,r}$ and the action of $G_{d'}$ on $X_{d',\nu}$ ([N2, 7.1.1]).

For each ℓ-adic representation $\rho = \rho_\lambda$ of $S_{d'}$ put $\tilde{\mathcal{A}}_{\lambda,\nu} = \tilde{\mathcal{A}}^{(r)}_{\lambda,\nu} = p^*\mathcal{A}_{\lambda,\nu}$. Then the complex $\tilde{\mathcal{A}}^{(r)}_{\lambda,\nu}$ is a bi-$G_{d',r}$-equivariant perverse sheaf. The involution $\tau : \mathfrak{g}_{d',r,\nu} \to \mathfrak{g}_{d',r,\nu}$, $g \mapsto {}^tg$, extends to a morphism $\tilde{\tau}_{\lambda,\nu} : \tau^*\,\tilde{\mathcal{A}}^{(r)}_{\lambda,\nu} \to \tilde{\mathcal{A}}^{(r)}_{\lambda,\nu}$. Indeed, recall that the perverse sheaf $\mathcal{A}_{\rho,\nu}$ is the intermediate extension of $\mathcal{L}_\rho[\dim X_{d'}](\frac{1}{2}\dim X_{d'})$, where $\mathcal{L}_\rho$ is the local system $(\pi_{\mathrm{rss},*}\hat{\mathbb{Q}}_\ell \boxtimes V)^{S_d}$ on the open dense subset $X_{d',\nu,\mathrm{rss}}$ of $X_{d',\nu}$. Denoting by $Q_{d',r,\mathrm{rss}}$ the open subset of $Q_{d',r}$ whose set of k'-points consists of the polynomials (P,R) with separable P dividing R, then the restriction of $\tilde{\mathcal{A}}_{\rho,\nu}$ to $\mathfrak{g}_{d',r,\nu,\mathrm{rss}} = \mathfrak{g}_{d',r,\nu} \otimes_{Q_{d',r}} Q_{d',r,\mathrm{rss}}$ is the inverse image $p^*_{\mathrm{rss}}\mathcal{L}_\rho$ via the smooth morphism $p_{\mathrm{rss}} : \mathfrak{g}_{d',r,\nu,\mathrm{rss}} \to \mathfrak{g}_{d',r,\nu,\mathrm{rss}}$ (restriction of p). Since $p\circ\tau = p$, the isomorphism $\tau^*p^*_{\mathrm{rss}}\mathcal{L}_\rho \to p^*_{\mathrm{rss}}\mathcal{L}_\rho$ extends to the intermediate extension, giving $\tilde{\tau}_{\rho,\nu} : \tau^*\,\tilde{\mathcal{A}}_{\rho,\nu} \to \tilde{\mathcal{A}}_{\rho,\nu}$.

We can now introduce, for any n-partition λ of d', a new form of the spherical function $\tilde{f}_{\lambda,\nu}$, namely

$$\tilde{f}'_{\lambda,\nu}(g,\mu) = \mathrm{tr}\{(\tilde{\tau}_{\lambda,\nu}\times\tilde{\tau}_{\lambda^\vee,\nu})\circ\tilde{\sigma}\circ\tilde{\mathrm{Fr}}^w;\,(\tilde{\mathcal{A}}^{(r)}_{\lambda,\nu,\pi}\boxtimes\tilde{\mathcal{A}}^{(r)}_{\lambda^\vee,\nu,\pi})_{(g,\mu;\mu\,\mathrm{Fr}^w(g)^{-1},\mathrm{Fr}(\mu))}\}.$$

Here

$$(g,\mu) \in \mathfrak{g}_{d',r,\nu,\pi}(k') = \{(g,g',\mu) \in \mathfrak{gl}(n,\mathcal{O}'/\pi^r)^2 \times \mathcal{O}'/\pi^r;$$
$$\det(g) = \pi^{d'},\ \mu = \pi^\nu,\ \mathrm{Fr}(g)w^tg'w = \mu\}.$$

Since the complex $\tilde{\mathcal{A}}_{\lambda,\nu,\pi}$ is bi-$G^n_{r,\pi}\times G^1_{r,\pi}$-equivariant, where $G^n_{r,\pi}(k') = \mathrm{GL}(n,\mathcal{O}'/\pi^r)$, the function $\tilde{f}'_{\lambda,\nu}(g,\mu)$ is bi-$\tilde{\mathrm{K}}'_\pi = \mathrm{GL}(n,\mathcal{O}'_\pi)\times\mathcal{O}'^\times_\pi$-invariant.

To compare $\tilde{f}'_{\lambda,\nu}(g,\mu)$ with $\tilde{f}_{\lambda,\nu}(g,\mu) = \mathrm{tr}\{\tilde{\sigma}\circ\tilde{\mathrm{Fr}}^w;\,(\tilde{\mathcal{A}}^{(r)}_{\lambda,\nu,\pi}\boxtimes\tilde{\mathcal{A}}^{(r)}_{\lambda^\vee,\nu,\pi})_{(g,\mu;\tilde{\mathrm{Fr}}^w(g,\mu))}\}$, consider the commutative square

$$\begin{array}{ccc}
\tilde{\mathcal{A}}^{(r)}_{\lambda,\nu,(g,\mu)}\boxtimes\tilde{\mathcal{A}}^{(r)}_{\lambda^\vee,\nu,(\tilde{\mathrm{Fr}}^w(g,\mu))} & \xrightarrow{\tilde{\sigma}\circ\tilde{\mathrm{Fr}}^w} & \tilde{\mathcal{A}}^{(r)}_{\lambda,\nu,(g,\mu)}\boxtimes\tilde{\mathcal{A}}^{(r)}_{\lambda^\vee,\nu,(\tilde{\mathrm{Fr}}^w(g,\mu))} \\
1\boxtimes\tilde{\tau}_{\lambda^\vee,\nu}\downarrow & & \uparrow 1\boxtimes\tilde{\tau}^{-1}_{\lambda^\vee,\nu} \\
\tilde{\mathcal{A}}^{(r)}_{\lambda,\nu,(g,\mu)}\boxtimes\tilde{\mathcal{A}}^{(r)}_{\lambda^\vee,\nu,({}^t\,\tilde{\mathrm{Fr}}^w(g,\mu))} & \xrightarrow{(\tilde{\tau}_{\lambda,\nu}\times\tilde{\tau}_{\lambda^\vee,\nu})\circ\tilde{\sigma}\circ\tilde{\mathrm{Fr}}^w} & \tilde{\mathcal{A}}^{(r)}_{\lambda,\nu,(g,\mu)}\boxtimes\tilde{\mathcal{A}}^{(r)}_{\lambda^\vee,\nu,({}^t\,\tilde{\mathrm{Fr}}^w(g,\mu))}\,.
\end{array}$$

The equality $\tilde{f}'_{\lambda,\nu}(g,\mu)=\tilde{f}_{\lambda,\nu}(g,\mu)$ follows from the equality of the traces of the two horizontal maps.

6. The form of $\tilde{f}'_{\lambda,\nu}(g,\mu)$ is convenient in order to compute its image under the morphism b'.

Define the function $\tilde{\phi}'_{\lambda,\nu}(s,\mu)=\operatorname{tr}\{\kappa\circ\tilde{\tau}\circ\tilde{\operatorname{Fr}}^{w};(\tilde{\mathcal{A}}^{(r)}_{\lambda,\nu,\pi}*\tilde{\mathcal{A}}^{(r)}_{\lambda^\vee,\nu,\pi})_{(s,\mu)}\}$. It is defined on the set of fixed points (s,μ) of $\tau\circ\tilde{\operatorname{Fr}}^{w}$, thus $(s,\mu)=(\mu\tilde{\operatorname{Fr}}^{w}(s),\operatorname{Fr}(\mu))$. Here $(s,\mu)\in\mathfrak{g}_{d,r,\nu,\pi}(k')$. Since $\tilde{\mathcal{A}}_{\lambda,\nu}*\tilde{\mathcal{A}}_{\lambda^\vee,\nu}=\tilde{\mathcal{A}}_{\rho,\nu}$, $\rho=\operatorname{Ind}^{S_d}_{S_{d'}\times S_{d'}}(\rho_\lambda\otimes\rho_{\lambda^\vee})$, and $\tilde{\mathcal{A}}_{\rho,\nu,\pi}$ is a $G_{r,\pi}$-equivariant perverse sheaf, it follows that $\tilde{\phi}'_{\lambda,\nu}$ is a function in $\mathcal{H}^+_S$.

The virtue of the function $\tilde{\phi}'_{\lambda,\nu}$ is that it is $b'(\tilde{f}'_{\lambda,\nu})$. Indeed the integral defining $b'(\tilde{f}'_{\lambda,\nu})(\mu s,\mu\operatorname{Fr}(\mu))$ can also be described as the sum of $f'_{\lambda,\nu}(g,\mu)$ over $g\in\operatorname{GL}(n,F'_\pi)/\operatorname{GL}(n,F_\pi)^w$ with $g\operatorname{Fr}^w(g)^{-1}=s$, namely over the g in $\mathfrak{g}_{d',r,\pi}(k')/G^w_{r,\pi}(k)$, where $\mathfrak{g}_{d',r,\pi}(k')=\{g\in\operatorname{GL}(n,\mathcal{O}'_\pi/\pi^r);\ \det(g)=\pi^{d'}\}$ and

$$G^w_{r,\pi}(k)=\{g\in\operatorname{GL}(n,\mathcal{O}'_\pi/\pi^r);\ g=w\operatorname{Fr}(g)w\}.$$

The set of summation can be described alternatively using the map

$$m^\alpha:\mathfrak{g}_{d',d'',\nu',\nu'',r,\pi}/G^w_{r,\pi}\to\mathfrak{g}_{d,r,\nu,\pi}$$

as in [N2, 7.3], where $\nu=\nu'+\nu''$ and $d=d'+d''$. Here $\mathfrak{g}_{d',d'',\nu',\nu'',r}$ is a scheme over the affine scheme $Q_{d',d'',r}$ $(d'+d''=d<r)$ of triples of monic polynomials (P',P'',R) of degrees (d',d'',r) such that $P=P'P''$ divides R. One has $m_Q:Q_{d',d'',r}\to Q_{d,r}$ given by $(P',P'',R)\mapsto(P=P'P'',R)$. The fiber $\mathfrak{g}_{d',d'',\nu',\nu'',r}(P',P'',R)$ has k'-points

$$\{(g',g'_1,\mu';g'',g''_1,\mu'')\in(\mathfrak{gl}(n,\mathcal{O}'/(R))^2\times\mathcal{O}'/(R))^2;\det(g')\in P'(\mathcal{O}'/(R))^\times,$$

$$\det(g'')\in P''(\mathcal{O}'/(R))^\times,\ \operatorname{Fr}(g')w{}^tg'_1w=\mu',\ \operatorname{Fr}(g'')w{}^tg''_1w=\mu''\}.$$

We denote the members by $(g',\mu';g'',\mu'')$, and write $\mu'\tilde{\operatorname{Fr}}^{w}(g')$ for g'_1, and similarly for g''_1.

There is a morphism $m:\mathfrak{g}_{d',d'',\nu',\nu'',r}(P',P'',R)\to\mathfrak{g}_{d,\nu,r}(P,R)$ given by

$$m(g',\mu';g'',\mu'')=(g'g'',\nu'\mu'').$$

The group G^w_r acts on $\mathfrak{g}_{d',d'',\nu',\nu'',r}$ by $\alpha(h,(g',g''))=(g'h^{-1},hg'')$, and m^α denotes the morphism induced by m on $\mathfrak{g}_{d',d'',\nu',\nu'',r}/G^w_r$. As usual, a subscript π means taking completion at π, or what amounts to the same: taking P of degree d to be π^d. Thus m^α maps $\mathfrak{g}_{d',d'',\nu',\nu'',r,\pi}(k')/G^w_{r,\pi}(k')\to\mathfrak{g}_{d,\nu,r,\pi}(k')$, taking classes

$$(g',\mu';g'',\mu'')\equiv(g'h^{-1},\mu';hg'',\mu'')\overset{\tau\times\tau\circ\sigma\circ\tilde{\operatorname{Fr}}^{w}}{\longmapsto}(\mu''\operatorname{Fr}^w(g'')^{-1},\operatorname{Fr}(\mu'');\mu'\operatorname{Fr}^w(g')^{-1},\operatorname{Fr}(\mu'))$$

to

$$(g=g'g'',\mu=\mu'\mu'')\overset{\tau\circ\tilde{\operatorname{Fr}}^{w}}{\longmapsto}(\mu\operatorname{Fr}^w(g)^{-1},\operatorname{Fr}(\mu)).$$

The fixed points of $(\tau\times\tau)\circ\sigma\circ\tilde{\mathrm{Fr}}^w$ are $(g,\mu;\mu\,\mathrm{Fr}^w(g)^{-1},\mathrm{Fr}(\mu))$, and there are fixed points only when $d''=n\nu'-d'$ and $\nu'=\nu''$. It follows that the summation ranges over the fixed point set of $(\tau\times\tau)\circ\sigma\circ\tilde{\mathrm{Fr}}^w$ acting on the fiber $(m^\alpha)^{-1}(\mu s,\mu\,\mathrm{Fr}(\mu))$.

The sum consists of the terms

$$\mathrm{tr}\{(\tilde\tau_{\lambda,\nu}\times\tilde\tau_{\lambda^\vee,\nu})\circ\tilde\sigma\circ\tilde{\mathrm{Fr}}^w;(\tilde{\mathcal{A}}_{\lambda,\nu,\pi}\boxtimes\tilde{\mathcal{A}}_{\lambda^\vee,\nu,\pi})_{(g,\mu;\tau\circ\tilde{\mathrm{Fr}}(g,\mu))}\}.$$

To compute this trace recall that we have a morphism

$$m:\mathfrak{g}_{d',d'',\nu',\nu'',r}(P',P'',R)\to\mathfrak{g}_{d,\nu,r}(P,R)$$

which factorizes through m^α, which fits in the following commutative square:

$$\begin{array}{cccccc}\tilde{\mathcal{A}}_{\rho',\nu'}\boxtimes^\alpha\tilde{\mathcal{A}}_{\rho'',\nu''} & \mathfrak{g}^\alpha_{d',d'',\nu',\nu'',r}(P',P'',R) & \overset{m^\alpha}{\to} & \mathfrak{g}_{d,\nu,r}(P,R) & Rm^\alpha_*(\tilde{\mathcal{A}}_{\rho',\nu'}\boxtimes^\alpha\tilde{\mathcal{A}}_{\rho'',\nu''})\simeq\tilde{\mathcal{A}}_{\rho,\nu}\\ & p^\alpha\downarrow & & p\downarrow & \\ \mathcal{A}_{\rho',\nu'}\tilde\boxtimes\mathcal{A}_{\rho'',\nu''} & X_{d'}\tilde\times X_{d''} & \overset{\mu}{\to} & X_d & \mathcal{A}_{\rho,\nu}.\end{array}$$

Here we used the notations m^α, $\boxtimes^\alpha$, $\tilde\boxtimes$, etc., of [N2, 7.3].

The complexes of perverse sheaves are related by the corresponding morphisms

$$\begin{array}{ccc}((\tau_{d',\nu}\times\tau_{d',\nu})\circ\sigma\circ\tilde{\mathrm{Fr}}^w)^*(\tilde{\mathcal{A}}_{\lambda,\nu,\pi}\boxtimes^\alpha\tilde{\mathcal{A}}_{\lambda^\vee,\nu,\pi}) & \overset{Rm^\alpha_*}{\to} & (\tau\circ\tilde{\mathrm{Fr}}^w)^*(\tilde{\mathcal{A}}_{\rho,\nu,\pi})\\ (\tilde\tau_{\lambda,\nu}\times\tau_{\lambda^\vee,\nu})\circ\tilde\sigma\circ\tilde{\mathrm{Fr}}^w\downarrow & & \downarrow\tilde\tau_{\rho,\nu}\circ\kappa\circ\tilde{\mathrm{Fr}}^w\\ \tilde{\mathcal{A}}_{\lambda,\nu,\pi}\boxtimes^\alpha\tilde{\mathcal{A}}_{\lambda^\vee,\nu,\pi} & \overset{Rm^\alpha_*}{\to} & \tilde{\mathcal{A}}_{\rho,\nu,\pi}\end{array}$$

where $\rho=\mathrm{Ind}^{S_d}_{S_{d'}\times S_{d'}}(\rho_\lambda\otimes\rho_{\lambda^\vee})$. The left vertical morphism is mapped to the right one by Rm^α_*. It follows that our trace is

$$=\mathrm{tr}\{\tilde\tau_{\rho,\nu}\circ\kappa\circ\tilde{\mathrm{Fr}}^w;(\tilde{\mathcal{A}}_{\rho,\nu,\pi})_{(\mu s,\mu\,\mathrm{Fr}(\mu))}\}=\tilde\phi'_{\lambda,\nu}(\mu s,\mu\,\mathrm{Fr}(\mu)).$$

7. In summary, we introduced the spherical functions $\tilde f_{\lambda,\nu}$ and $\tilde f'_{\lambda,\nu}$ which are equal to each other and to $f_{\lambda,\nu}=a_{\lambda,\nu}$ of [N2], and the spherical functions $\tilde\phi_{\lambda,\nu}$ and $\tilde\phi'_{\lambda,\nu}$, for which we have $b(\tilde f_{\lambda,\nu})=\tilde\phi_{\lambda,\nu}$ and $b'(\tilde f'_{\lambda,\nu})=\tilde\phi'_{\lambda,\nu}$. We can now give a geometric description of the Fourier orbital integrals which we wish to compare.

The first such integral is

$$I_\pi(a,\alpha,\tilde\phi_{\lambda,\nu})=\sum\tilde\phi_{\lambda,\nu}(x^{-1}wax')\psi_\alpha(x^{-1}x')=\sum\tilde\phi_{\lambda,\nu}({}^txax')\psi_0(h_\alpha(x,x')),$$

where the sum ranges over (x,x') in

$$X_\pi(a)(k)=\{(x,x')\in[N(F_\pi)/N(\mathcal{O}_\pi)]^2;\ {}^txax'\in\mathfrak{gl}(n,\mathcal{O}')\}.$$

The last displayed equality follows from $\psi_\alpha(w^t x w) = \psi_\alpha(x)$. Here $\psi_0 : k \hookrightarrow \hat{\mathbb{Q}}_\ell^\times$ is an embedding. For each $\alpha \in (k^\times)^{n-1}$, $\alpha = (\alpha_1, \dots, \alpha_{n-1})$ with $\alpha_i = \alpha_{n-i}$ $(1 \le i < n)$, $h_\alpha(x, x') = \mathrm{res}_\pi[\iota \sum_{1 \le i < n} \alpha_i(x_i + x_i')d\pi]$ where $x = (x_{ij})$ and $x_i = x_{i,i+1}$. The element $\iota \in k' - k$ satisfies $\mathrm{Fr}(\iota) = -\iota$, and $x = \tilde{\mathrm{Fr}}^w(x)$ implies that $x_i = -\mathrm{Fr}(x_{n-i})$, hence $h_\alpha(x, x') \in k$ for any $(x, x') \in X_\pi(a)(k)$.

To express geometrically our sum, denote by $\mathcal{L}_{\psi_0}$ the Artin-Schreier sheaf on $\mathbb{G}_a$ associated with the character $\psi_0 : k \hookrightarrow \hat{\mathbb{Q}}_\ell^\times$, and note that ([La, 1.1.3.3]) $\psi_0(x) = \mathrm{tr}\{\mathrm{Fr}; (\mathcal{L}_{\psi_0})_x\}$. The trace of the Frobenius is a function on the set of fixed points $x = \mathrm{Fr}(x)$ of the Frobenius acting on $\mathbb{G}_a(\hat{k}) = \hat{k}$, namely on k. Moreover, for a morphism $h_\alpha : X_\pi(a) \to \mathbb{G}_a$, with $h_\alpha(\tilde{\mathrm{Fr}}^w(x, x')) = \mathrm{Fr}(h_\alpha(x, x'))$, by [La, 1.1.1.4] we have that

$$\mathrm{tr}\{\mathrm{Fr};\, (\mathcal{L}_{\psi_0})_{h_\alpha(x,x')}\} = \mathrm{tr}\{\tilde{\mathrm{Fr}}^w;\, (h_\alpha^* \mathcal{L}_{\psi_0})_{(x,x')}\}.$$

We need a similar expression for the values of the function

$$\tilde{\phi}_{\lambda,\nu}(g, \mu) = \mathrm{tr}\{\tilde{\mathrm{Fr}}^w \circ \kappa; (\mathcal{A}_{\lambda,\nu,\pi} * \mathcal{A}_{\lambda^\vee,\nu,\pi})_{(g,\mu)}\}$$

on the set of fixed points (g, μ) of $\tilde{\mathrm{Fr}}^w(g, \mu) = (\mu \tilde{\mathrm{Fr}}^w(g), \mathrm{Fr}(\mu))$ in $X_{d,\nu,\pi}(k')$, $\nu = 2d$. To describe the values of this function, we use the following morphisms (where $d_n < r$):

$$\tilde{p}_r : \mathfrak{g}_{d_n,\nu,r,\pi}(k) \to X_{d_n,\nu,\pi}(k).$$

Here

$$X_{d_n,\nu,\pi}(k) = \{(gK, \mu) \in (\mathfrak{gl}(n, \mathcal{O}') \times \mathcal{O}') \cap \tilde{\mathrm{U}}(n, F_\pi)/\,\tilde{\mathrm{U}}(n, \mathcal{O}_\pi);$$

$$\det(g) = \pi^{d_n}, g\,\mathrm{Fr}^w(g) = \mu,\ \mathrm{val}_\pi(\mu) = \nu\}.$$

Note that $g\,\mathrm{Fr}^w(g) = \mu$ implies $2d_n = n\nu$. Also

$$\mathfrak{g}_{d_n,\nu,r,\pi}(k) = \{(g, \mu) \in \mathfrak{gl}(n, \mathcal{O}'_\pi/\pi^r\mathcal{O}'_\pi) \times ((\mathcal{O}'_\pi/\pi^r) - \{0\});$$

$$gw\,\mathrm{Fr}(g)w = \mu \in \mathcal{O}_\pi/\pi^r,\ \det(g) \equiv \pi^{d_n} (\mod \pi^r)\}.$$

The morphism $\tilde{p}_r$ consists of lifting $g \in \mathfrak{gl}(n, \mathcal{O}'_\pi/\pi^r)$ to $\mathfrak{gl}(n, \mathcal{O}'_\pi)$.

Denote by g_i the main minor of size i – the $i \times i$-matrix on the top left – of g. Then

$$X_{\pi,\nu,r}(a, \mu)(k) = \{(g, \mu) \in \mathfrak{gl}(n, \mathcal{O}'_\pi/\pi^r\mathcal{O}'_\pi) \times ((\mathcal{O}'_\pi/\pi^r) - \{0\});$$

$$gw\,\mathrm{Fr}(g)w = \mu \in \mathcal{O}_\pi/\pi^r,\ \mathrm{val}_\pi(\mu) = \nu (\mod r),\ \det(g_i) \equiv a_i (\mod \pi^r),\ 1 \le i \le n\},$$

where $a = \mathrm{diag}(a_1, a_2/a_1, \dots, a_n/a_{n-1}) \in \tilde{\mathrm{U}}(n, F_\pi)$, $a_i \in \mathcal{O}' - \{0\}$, embeds naturally in $\mathfrak{g}_{d_n,\nu,r,\pi}(k)$. Denote by $\iota : X_{\pi,\nu,r}(a, \mu) \hookrightarrow \mathfrak{g}_{d_n,\nu,r,\pi}$ this embedding. Note that if $X_{\pi,\nu,r}(a, \mu)(k)$ is non empty, then $a\,\tilde{\mathrm{Fr}}^w(a) \equiv \mu (\mod \pi^r)$.

Each $g \in \mathfrak{gl}(n, \mathcal{O}'_\pi)$, $g\,\tilde{\mathrm{Fr}}^w(g) = \mu$, can be expressed in a unique way in the form $g = {}^t xax'$, with x, x' in $N(F_\pi)/N(\mathcal{O}_\pi)$, $a\,\tilde{\mathrm{Fr}}^w(a) = \mu$. We then obtain a morphism $p_r : X_{\pi,\nu,r}(a,\mu) \to X_\pi(a,\mu)$,

$$X_\pi(a,\mu)(k) = \{(x,x') \in [N(F_\pi)/N(\mathcal{O}_\pi)]^2;\ {}^t xax' \in \mathfrak{gl}(n,\mathcal{O}'_\pi),\ a\,\tilde{\mathrm{Fr}}^w(a) = \mu\},$$

on lifting g from $\mathfrak{gl}(n, \mathcal{O}'_\pi/\pi^r)$ to $\mathfrak{gl}(n, \mathcal{O}'_\pi)$.

On these schemes we have the following complexes of perverse sheaves. The sheaves $\mathcal{A}_{\lambda,\nu}$ and $\mathcal{A}_{\lambda,\nu,\pi}$ on $X_{d,\nu}$ and $X_{d,\nu,\pi}$ are associated to representations $\rho :\ S_d \to \mathrm{GL}(V_\rho)$ parametrized by n-partitions λ of d. Further we have $\tilde{\mathcal{A}}^{(r)}_{\rho,\nu} = \tilde{p}_r^* \mathcal{A}_{\rho,\nu}$ and $\tilde{\mathcal{A}}^{(r)}_{\rho,\nu,\pi} = \tilde{\mathcal{A}}^{(r)}_{\rho,\nu}[-d_n](-\frac{1}{2}d_n)$, and we write $\tilde{\mathcal{A}}^{[r]}_{\rho,\nu,\pi} = \iota^*\,\tilde{\mathcal{A}}^{(r)}_{\rho,\nu,\pi}$ for the restriction to $X_{\pi,\nu,r}(a,\mu)$. As in [N2, 9.2.2], put $\dot{\mathcal{A}}_{\rho,\nu,\pi} = Rp_{r,!}\,\tilde{\mathcal{A}}^{[r]}_{\rho,\nu,\pi}[2d_r](d_r)$, where d_r is the relative dimension of p_r, and note that $\tilde{\mathcal{A}}^{[r]}_{\rho,\nu,\pi} = p_r^* \dot{\mathcal{A}}_{\rho,\nu,\pi}$.

Consequently the value of the spherical function $\tilde{\phi}_{\lambda,\nu}$ at $({}^t xax', \mu)$ can be expressed as follows:

$$\begin{aligned}\tilde{\phi}_{\lambda,\nu}({}^t xax',\mu) &= \mathrm{tr}\{\tilde{\mathrm{Fr}}^w \circ \kappa;\ (\mathcal{A}_{\lambda,\nu,\pi} * \mathcal{A}_{\lambda^\vee,\nu,\pi})_{\tilde{p}_r(\iota({}^t xax',\mu))}\}\\ &= \mathrm{tr}\{\tilde{\mathrm{Fr}}^w \circ \kappa;\ \iota^* \tilde{p}_r^* (\mathcal{A}_{\lambda,\nu,\pi} * \mathcal{A}_{\lambda^\vee,\nu,\pi})_{({}^t xax',\mu)}\}.\end{aligned}$$

The last complex of sheaves is equal to

$$\iota^*(\tilde{\mathcal{A}}^{(r)}_{\lambda,\nu,\pi} * \tilde{\mathcal{A}}^{(r)}_{\lambda^\vee,\nu,\pi}) = \tilde{\mathcal{A}}^{[r]}_{\lambda,\nu,\pi} * \tilde{\mathcal{A}}^{[r]}_{\lambda^\vee,\nu,\pi} = \tilde{\mathcal{A}}^{[r]}_{\rho,\nu,\pi} = p_r^* \dot{\mathcal{A}}_{\rho,\nu,\pi}.$$

Here $\rho = \mathrm{Ind}^{S_d}_{S_{d'} \times S_{d'}} (\rho_\lambda \otimes \rho_\lambda^\vee)$. Hence

$$\tilde{\phi}_{\lambda,\nu}({}^t xax',\mu) = \mathrm{tr}\{\tilde{\mathrm{Fr}}^w \circ \kappa;\ (p_r^* \dot{\mathcal{A}}_{\rho,\nu,\pi})_{({}^t xax',\mu)}\} = \mathrm{tr}\{\tilde{\mathrm{Fr}}^w \circ \kappa;\ (\mathcal{A}_{\rho,\nu,\pi})_{(x,x')}\}.$$

Note also that κ acts trivially on $h_\alpha^* \mathcal{L}_{\psi_0}$ over $X_\pi(a)$. It follows that our Fourier orbital integral takes the form

$$\begin{aligned}I_\pi((a,\mu),\alpha,\tilde{\phi}_{\lambda,\nu}) &= \sum_{(x,x') \in X_\pi(a,\mu)(k)} \mathrm{tr}\{\tilde{\mathrm{Fr}}^w \circ \kappa;\ (\dot{\mathcal{A}}_{\rho,\nu,\pi})_{(x,x')}\} \cdot \mathrm{tr}\{\tilde{\mathrm{Fr}}^w \circ \kappa;\ (h_\alpha^* \mathcal{L}_{\psi_0})_{(x,x')}\}\\ &= \sum_{(x,x') \in X_\pi(a,\mu)(k)} \mathrm{tr}\{\tilde{\mathrm{Fr}}^w \circ \kappa;\ (\dot{\mathcal{A}}_{\rho,\nu,\pi} \otimes h_\alpha^* \mathcal{L}_{\psi_0})_{(x,x')}\}\\ &= \mathrm{tr}\{\tilde{\mathrm{Fr}}^w \circ \kappa;\ R\Gamma_c(\hat{X}_\pi(a,\mu), \dot{\mathcal{A}}_{\rho,\nu,\pi} \otimes h_\alpha^* \mathcal{L}_{\psi_0})\}.\end{aligned}$$

The last equality follows from the Grothendieck fixed point formula.

This expression will be compared with the Fourier orbital integral

$$J_\pi((a,\mu),\alpha,\tilde{\phi}'_\lambda) = \sum_{x \in N(F'_\pi)/N(\mathcal{O}'_\pi)} \tilde{\phi}'_\lambda({}^t\,\tilde{\mathrm{Fr}}^w(x)ax,\mu)\psi'_\alpha(\tilde{\mathrm{Fr}}^w(x)x),$$

where $\alpha = (\alpha_1, \dots, \alpha_{n-1}) \in k^{\times n-1}$ with $\alpha_{n-i} = \alpha_i$, and $\psi'_\alpha(\tilde{\mathrm{Fr}}^w(x)x) = \psi_0(h_\alpha(\tilde{\mathrm{Fr}}^w(x), x))$. If $\tau(x,y) = (y,x)$, and $\tilde{\tau} : \tau^*\mathcal{A} \to \mathcal{A}$ is the induced action on a sheaf $\mathcal{A}$ on $X \times X$, then this last value can be expressed as

$$\mathrm{tr}\{\mathrm{Fr};\ (\mathcal{L}_{\psi_0})_{h_\alpha(\tilde{\mathrm{Fr}}^w(x),x)}\} = \mathrm{tr}\{\tilde{\mathrm{Fr}}^w \circ \tilde{\tau};\ (h_\alpha^*\mathcal{L}_{\psi_0})_{(\tilde{\mathrm{Fr}}^w(x),x)}\}.$$

The value of the spherical function occurring in the sum is

$$\tilde{\phi}'_{\lambda,\nu}({}^t\,\tilde{\mathrm{Fr}}^w(x)ax, \mu) = \mathrm{tr}\{\tilde{\tau} \circ \tilde{\mathrm{Fr}}^w \circ \kappa;\ (\tilde{\mathcal{A}}_{\lambda,\nu,\pi} * \tilde{\mathcal{A}}_{\lambda^\vee,\nu,\pi})_s\},$$

where $s = \iota({}^t\,\tilde{\mathrm{Fr}}^w(x)ax, \mu)$ is a fixed point of ${}^t\,\tilde{\mathrm{Fr}}^w$. But $\iota^*(\tilde{\mathcal{A}}_{\lambda,\nu,\pi} * \tilde{\mathcal{A}}_{\lambda^\vee,\nu,\pi}) = \iota^*(\tilde{\mathcal{A}}_{\rho,\nu,\pi}) = p_r^*\dot{\mathcal{A}}_{\rho,\nu,\pi}$, hence we get

$$\mathrm{tr}\{\tilde{\tau} \circ \tilde{\mathrm{Fr}}^w \circ \kappa;\ (\dot{\mathcal{A}}_{\rho,\nu,\pi})_{p_r({}^t\,\tilde{\mathrm{Fr}}^w(x)ax,\mu)}\} = \mathrm{tr}\{\dot{\tau} \circ \tilde{\mathrm{Fr}}^w \circ \kappa;\ (\dot{\mathcal{A}}_{\rho,\nu,\pi})_{(\tilde{\mathrm{Fr}}^w(x),x)}\},$$

where $(\tilde{\mathrm{Fr}}^w(x), x)$ is a fixed point of $\sigma \circ \tilde{\mathrm{Fr}}^w$ on $X_\pi(a,\mu)(\hat{k})$. The sum then ranges over the set $X_\pi(a,\mu)(\hat{k})^{\tau\circ\tilde{\mathrm{Fr}}^w}$ of fixed points $(x, \tilde{\mathrm{Fr}}^w(x))$ of $\tau \circ \tilde{\mathrm{Fr}}^w$ in $X_\pi(a,\mu)(\hat{k})$, of the terms

$$\begin{aligned}
&\mathrm{tr}\{\dot{\tau} \circ \tilde{\mathrm{Fr}}^w \circ \kappa;\ (\dot{\mathcal{A}}_{\rho,\nu,\pi})_{(x,\tilde{\mathrm{Fr}}^w(x))}\} \cdot \mathrm{tr}\{\tilde{\mathrm{Fr}}^w \circ \tau;\ (h_\alpha^*\mathcal{L}_{\psi_0})_{(x,\tilde{\mathrm{Fr}}^w(x))}\} \\
&= \mathrm{tr}\{\dot{\tau} \circ \tilde{\mathrm{Fr}}^w \circ \kappa;\ (\dot{\mathcal{A}}_{\rho,\nu,\pi} \otimes h_\alpha^*\mathcal{L}_{\psi_0})_{(x,\tilde{\mathrm{Fr}}^w(x))}\}.
\end{aligned}$$

Here κ is taken to act trivially on $h_\alpha^*\mathcal{L}_{\psi_0}$. By the Grothendieck fixed point formula we conclude that

$$J_\pi((a,\mu), \alpha, \tilde{\phi}'_\lambda) = \mathrm{tr}\{\dot{\tau} \circ \tilde{\mathrm{Fr}}^w \circ \kappa;\ R\Gamma_c(\hat{X}_\pi(a,\mu), \dot{\mathcal{A}}_{\rho,\nu,\pi} \otimes h_\alpha^*\mathcal{L}_{\psi_0})\}.$$

We can now use [N2, Theorem 4A], which asserts that the involution $\dot{\tau}$ acts on

$$R\Gamma_c(\hat{X}_\pi(a,\mu), \dot{\mathcal{A}}_{\rho,\nu,\pi} \otimes h_\alpha^*\mathcal{L}_{\psi_0})$$

as multiplication by $(-1)^{\mathrm{val}_\pi(a_1 \dots a_{n-1})}$ for any representation ρ of S_d and for any $\alpha \in (k^\times)^{n-1}$, from which our assertion follows.

Appendix. Matching Fourier orbital integrals on a unitary group and perverse sheaves: unit element.

The purpose of this appendix is to show that the proof of the fundamental lemma by Ngô [N], [N1] in the context of Fourier orbital integrals on the general linear group and the unit element of the Hecke algebra, can be adapted – on making only minor changes – to prove the analogous fundamental lemma in the context of such integrals on the unitary group, as conjectured in [F], for local fields of positive characteristic and for regular elements. That such a simple adaptation is possible is yet another manifestation of the power of the intersection cohomological techniques used by Ngô [N], [N1].

Let us recall (part of) the conjecture of [F]. Let F be a local non archimedean field with ring $\mathcal{O}$ of integers and $\operatorname{char} F \neq 2$. Let F'/F be a quadratic unramified field extension and $\mathcal{O}'$ the ring of integers in F'. Let $x \mapsto \overline{x}$ denote the action of $\mathrm{Gal}(F'/F)$. Fix a character $\psi_0 : F \to \mathbb{C}^\times$ with conductor $\mathcal{O}$, and $\iota \in \mathcal{O}'^\times$ with $\overline{\iota} = -\iota$. Write $G = GL(r, F)$ and $G' = GL(r, F')$, and signify by w the matrix $(\delta_{i,r+1-j})$ in G, whose non zero entries are ones placed on the anti-diagonal. The unitary group U is $\{g \in G';\ gw{}^t\overline{g} = w\}$, where t signifies transpose and $\overline{g} = (\overline{g}_{i,j})$ for $g = (g_{i,j})$. Write N for the unipotent upper triangular subgroup of U, and N' for that of G'. Write A for the diagonal subgroup of U, and A' for that of G'.

Put $\psi(n) = \psi_0(\iota^{-1} \sum_{1 \leq i < r} n_{i,i+1})$. It is a $\mathbb{C}^\times$-valued character on N, and $\psi'(n) = \psi(n\overline{n}^{-1})$ is on N'. Let $K' = GL(r, \mathcal{O}')$ be the standard maximal compact subgroup of G', and $K = K' \cap U$ that of U. Denote by 1_K the characteristic function of K in U, by $1_{K'}$ that of K' in G'. Signify by dn the Haar measure on N normalized to assign $K \cap N$ the volume one, as well as the Haar measure on N' which assigns $K' \cap N'$ the volume one.

The local conjecture relates (in particular) the Fourier orbital integral

$$\Psi(a) = \int_N \int_N 1_K(n^{-1}wan')\psi(n^{-1}n')\, dn\, dn' = \int_N \int_N 1_K({}^tnan')\psi(nn')\, dn\, dn'$$

at the diagonal matrix a in $A \subset U$ (for the last equality note that $\sum_{1 \leq i < r} n_{i,i+1}$ is the same for $w{}^tnw$ and for n, and that $w \in K$), with the twisted Fourier orbital integral

$$\Psi'(a) = \int_{N'} 1_{K'}(\overline{n}^{-1}wan)\psi'(n)\, dn.$$

Theorem. *Suppose that* $\operatorname{char} F > 2$. *Denote by* $|a|$ *the product of the first* $[r/2]$ *entries down the diagonal of* a. *Then* $\Psi'(a) = (-1)^{(r-1)\mathrm{val}_{F'}(|a|)}\Psi(a)$.

The conjecture of [F] takes $\operatorname{char} F$ to include zero, and deals with general corresponding spherical functions, not only the unit element 1_K. Moreover we deal here only with the open set of elements of the form ${}^tnan'$ in U and $\overline{n}^{-1}wan$ of G', while [F] considers also singular elements (or other elements of order two in the Weyl group). To compare with [F] note that we define here the unitary group using the form w, not J of [F] whose anti-diagonal entries alternate from 1 to -1, as this simplifies the notations. Then for the entries of n we have $n_{i,i+1} = -\overline{n}_{r-i,r-i+1}$. For applications of the conjectures of [F] to liftings of automorphic forms from the unitary group $U(r, \mathbb{A}'/\mathbb{A})$ to $GL(r, \mathbb{A}')$, and for local applications, see [F] and [F1].

Soon after the conjecture of [F] was announced, an analogous conjecture – where U is replaced by G – was formulated by Jacquet and Ye [JY]. This analogous conjecture is relevant to base change of automorphic forms from G to G'. Jacquet and Ye [JY1] proved their conjecture (for the unit element of the Hecke algebra and regular elements) for local F of any characteristic for $GL(3)$ by brute force computations of the Mellin transforms of both sides of the equality. For F of positive characteristic and for the regular elements, the conjecture of [JY] has been proven by Ngô [N], [N1] for $GL(r)$ on translating it to a question on Kloosterman sums, and reducing it to a result of Deligne [D] by means of perverse sheaves.

We shall follow Ngô in translating our conjecture to a geometric statement, where the equality amounts to a description of the action of a certain involution. Our observation is that this involution is already described in Ngô [N], [N1], and the conjecture of [F] follows. In contrast, the technique of brute force computations would require working out the Mellin transforms in the situation of [F] anew, and no simple reduction to the computations of [JY1] is known.

We hope to deal with general corresponding spherical functions – not only the unit element 1_K – in the near future. Dealing with non regular orbits (not of the form ${}^tnan'$, or $\overline{n}^{-1}wan$), is a question concerning asymptotic expansion of orbital integrals; perhaps it can also be dealt with directly. It will be very interesting to examine whether the results in positive characteristic could be used to imply the conjecture in characteristic zero (cf. Kazhdan [K]).

The local field F has the form $F = k((\pi))$, where k is a finite field, and π is a generator of the maximal ideal in the ring $\mathcal{O} = k[[\pi]]$ of integers in F. Further $F' = k'((\pi))$ and $\mathcal{O}' = k'[[\pi]]$, where k' is the quadratic extension of k in its algebraic closure $\hat{k}$. Fix $\iota \in k'^{\times}$ with $\overline{\iota} = -\iota$; it is used in the definition of ψ. Our integrals are simply sums:

$$\Psi(a) = \sum_{X(a)(k)} \psi(nn'), \qquad \Psi'(a) = \sum_{X'(a)(k)} \psi'(n),$$

ranging over the finite sets

$$X(a)(k) = \{(n, n') \in [N/N \cap K]^2;\ {}^tnan' \in K' = GL(r, \mathcal{O}')\}$$

and

$$X'(a)(k) = \{n \in N'/N' \cap K';\ \overline{n}^{-1}wan \in K'\}.$$

These sums can be expressed by means of Deligne's cohomological interpretation of exponential sums. Let ℓ be a prime other than $p = \mathrm{char}(k)$, and let $\psi_1 : k \hookrightarrow \hat{\mathbb{Q}}_\ell^\times$ be a $\hat{\mathbb{Q}}_\ell$-valued character of k. If X is an algebraic variety of finite type over k with a k-morphism $h : X \to \mathbb{G}_a$, then the exponential sum $\sum_{x \in X(k)} \psi_1(h(x))$ takes the form $\mathrm{tr}\{\mathrm{Fr}; R\Gamma_c(X \times_k \hat{k}, h^*\mathcal{L}_{\psi_1})\}$ by virtue of the Grothendieck fixed point formula. Here $\mathcal{L}_{\psi_1}$ is the Artin-Schreier $\hat{\mathbb{Q}}_\ell$-sheaf on $\mathbb{G}_a$ associated to ψ_1, $h^*\mathcal{L}_{\psi_1}$ is its pullback via h to a $\hat{\mathbb{Q}}_\ell$-sheaf on the $\hat{k}$-variety $X \times_k \hat{k}$, Fr is the Frobenius endomorphism (generator of $\mathrm{Gal}(\hat{k}/k)$) acting on the complex $R\Gamma_c(X \times_k \hat{k}, h^*\mathcal{L}_{\psi_1})$.

To define the affine k-varieties $X(a)$ and $X'(a)$, if R is a k-algebra, put $R' = R \otimes_k k'$. The automorphism $\Phi_0 \neq 1$ of k' over k acts on R' (fixing R), on $F \otimes_k R'$ (recall: $F = k((\pi))$), and on $M(r, F \otimes_k R')$ – $r \times r$ matrices – componentwise. Write $\mathbf{G}$ for $GL(r)$, $\mathbf{A}$ for its diagonal subgroup, and $\mathbf{N}$ for its subgroup of unipotent upper triangular matrices. Let Φ be the involution $\Phi(g) = w^t\Phi_0(g)^{-1}w$ of $\mathbf{G}(R')$, and define the unitary group $\mathbf{U}$ by $\mathbf{U}(R) = \mathbf{G}(R')^\Phi$, and its subgroups $\mathbf{A}_U(R) = \mathbf{A}(R')^\Phi$, $\mathbf{N}_U(R) = \mathbf{N}(R')^\Phi$. It is an affine k-group which splits over k': if R is a k'-algebra then $R' = R \times R$ and $\Phi_0(a,b) = (b,a)$, and $\mathbf{U}(R) = \{(g, w^tg^{-1}w);\ g \in \mathbf{G}(R)\}$ is $\mathbf{G}(R)$. An element $a = \operatorname{diag}(a_1, \dots, a_r) \in \mathbf{A}(F')$ lies in $\mathbf{A}_U(F)$ when $a_{r+1-i} = \overline{a}_i^{-1}$ for all i. We write $\overline{a}_i$ for $\Phi_0(a_i)$ when $a_i \in F' = k'((\pi))$.

For such an $a \in \mathbf{A}_U(F)$ define the functor $X(a)$ to assign to any k-algebra R the set

$$X(a)(R) = \{(n,n') \in [\mathbf{N}(F' \otimes_{k'} R')^\Phi / \mathbf{N}(\mathcal{O}' \otimes_{k'} R')^\Phi]^2;\ {}^tnan' \in \mathbf{G}(\mathcal{O}' \otimes_{k'} R')^\Phi\}.$$

If $s_i(x)$ signifies the top-left $i \times i$-minor of an $r \times r$-matrix x, then $\det s_i({}^tnan')$ is $\prod_{1 \le j \le i} a_j$, and $X(a)(R)$ is empty unless $\prod_{1 \le j \le i} a_j \in \mathcal{O}' = k'[[\pi]]$ for each $i \le r$.

That $X(a)$ is represented by an affine k-variety of finite type can be seen on writing $n' = (I_r + y_2') \dots (I_r + y_r')$ and ${}^tn = (I_r + {}^ty_r) \dots (I_r + {}^ty_2)$. Inductively,

$$(I_i + {}^ty_i) \cdot \operatorname{diag}(x_{i-1}, a_i) \cdot (I_i + y_i') = \begin{pmatrix} x_{i-1} & x_{i-1}y_i' \\ {}^ty_ix_{i-1} & {}^ty_ix_{i-1}y_i' + a_i \end{pmatrix}.$$

This has entries in $\mathcal{O}' \otimes_{k'} R'$ only if y_i and y_i' have entries in $(\det x_{i-1})^{-1}(\mathcal{O}' \otimes_{k'} R')$. But

$$(\prod_{1 \le j < i} a_j)^{-1}\mathcal{O}' \otimes_{k'} R' / \mathcal{O}' \otimes_{k'} R' = ((\prod_{1 \le j < i} a_j^{-1})\mathcal{O}'/\mathcal{O}') \otimes_{k'} R'$$

and $(\prod_{1 \le j < i} a_j^{-1})\mathcal{O}'/\mathcal{O}'$ is a finite dimensional k'-space. Consequently

$$X''(a)(R) = \{(n,n') \in [\mathbf{N}(F' \otimes_{k'} R')/\mathbf{N}(\mathcal{O}' \otimes_{k'} R')]^2;\ {}^tnan' \in \mathbf{G}(\mathcal{O}' \otimes_{k'} R')\}$$

is represented by an affine k'-variety of finite type. The functor $X(a)$ is represented by this variety and the $\operatorname{Gal}(k'/k)$-action $\Phi(n,n') = (\Phi(n), \Phi(n'))$, thus it is an affine k-variety of finite type.

When R is a k'-algebra, the set $X(a)(R)$ is isomorphic to $\{(n,n') \in [\mathbf{N}(F \otimes_k R)/\mathbf{N}(\mathcal{O} \otimes_k R)]^2;\ {}^tnan' \in \mathbf{G}(\mathcal{O} \otimes_k R)\}$.

For each $\alpha \in (k^\times)^{r-1}$, $\alpha_i = \alpha_{r-i}$, the k-variety $X(a)$ has a k-morphism $h_\alpha : X(a) \to \mathbb{G}_a$, given by $h_\alpha(n,n') = \sum_{1 \le i < r} \alpha_i \operatorname{res}[\iota^{-1}(n_{i,i+1} + n'_{i,i+1})d\pi]$, and an involution $\sigma(n,n') = (n',n)$.

We also consider the functor $X'(a)$ which associates to a k-algebra R the set

$$X'(a)(R) = \{n \in \mathbf{N}(F' \otimes_{k'} R')/\mathbf{N}(\mathcal{O}' \otimes_{k'} R');\ \Phi_0(n)^{-1}wan \in \mathbf{G}(\mathcal{O}' \otimes_{k'} R')\}.$$

This set is the set of fixed points of the involution $\Phi \circ \sigma$ on the set

$$X'(a)(R') = \{(n,n') \in [\mathbf{N}(F' \otimes_{k'} R')/\mathbf{N}(\mathcal{O}' \otimes_{k'} R')]^2;\ {}^tnan' \in \mathbf{G}(\mathcal{O}' \otimes_{k'} R')\}.$$

Indeed, (n, n') equals $\Phi \circ \sigma(n, n') = (w^t\Phi_0(n')^{-1}w, w^t\Phi_0(n)^{-1}w)$ means $n = w^t\Phi_0(n')^{-1}w$, and then ${}^tnan'$ is $\Phi_0(n')^{-1}wan'$. Again $X'(a)$ is represented by an affine k-variety of finite type, it has an involution $n \mapsto \Phi_0(n)$ and a k-morphism $h_\alpha : X'(a) \to \mathbb{G}_a$ for any $\alpha \in (k^\times)^{r-1}$, given by the restriction of $h_\alpha(n, n') = \iota^{-1} \sum_{1 \le i < r} \alpha_i(n_{i,i+1} + n'_{i,i+1})$ to the set of $\Phi \circ \sigma$-fixed points $n = (w^t\Phi_0(n)^{-1}w, n)$, namely by $h_\alpha(n) = \iota^{-1} \sum_{1 \le i < r} \alpha_i(n_{i,i+1} - \Phi_0(n_{i,i+1}))$.

The varieties $X(a)$ and $X'(a)$ are isomorphic over k', thus $X(a) \times_k k' = X'(a) \times_k k'$ and in particular $X(a) \times_k \hat{k} = X'(a) \times_k \hat{k}$; write $\hat{X}(a)$ for this last $\hat{k}$-variety. But the k-structures on $\hat{X}(a)$ determined by $X(a)$ and $X'(a)$ are different. If Fr and Fr$'$ are the corresponding Frobenius morphisms, then they are related by $\mathrm{Fr}' = \mathrm{Fr} \circ \sigma$. We conclude that for $\alpha = 1$,

$$\Psi(a) = \mathrm{tr}\{\mathrm{Fr};\, R\Gamma_c(\hat{X}(a), h_\alpha^*\mathcal{L}_{\psi_0})\} \qquad \text{and} \qquad \Psi'(a) = \mathrm{tr}\{\mathrm{Fr} \circ \sigma;\, R\Gamma_c(\hat{X}(a), h_\alpha^*\mathcal{L}_{\psi_0})\}.$$

Since the $\hat{k}$-variety $\hat{X}(a)$, the k-morphism h_α, and involution σ (but not Fr) are the same as in Ngô [N], [N1], we are now in a position to use Ngô's result [N], [N1]. It asserts that σ acts on the complex $R\Gamma_c(\hat{X}(a), h_\alpha^*\mathcal{L}_{\psi_0})$ as multiplication by $(-1)^{\mathrm{val}(\|a\|)}$, where $\|a\| = \prod_{1 \le i < r} a_i^{r-i}$. In our case $a_{r+1-i} = \overline{a}_i^{-1}$. Up to an element of an even valuation, $\|a\|$ is $a_1^{r-1} \prod_{1 < i \le r/2} a_i^{r-i} a_{r+1-i}^{r-i} = \prod_{1 \le i \le r/2} a_i^{r-1}$, as asserted.

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