Stochastic calculus of generalized Dirichlet forms and applications to stochastic differential equations in infinite dimensions

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1 Introduction

In this paper we systematically develop a stochastic calculus for *generalized* Dirichlet forms (cf. [St1]). In particular, we show *Fukushima's decomposition* of additive functionals and an *Itô-type formula* in this framework. The class of generalized Dirichlet forms is much larger than the well-studied class of symmetric and coercive Dirichlet forms as in [FOT 94] resp. [MR 92] and time dependent Dirichlet forms as in [O 92]. It contains examples of an entirely new kind (cf. Section 6, [St1]). Therefore, the results obtained in this paper lead to extensions of the corresponding results in the "classical" theories. In particular the proofs are "locally" completely different (cf. e.g. Theorem 2.3 and Theorem 2.5; though for the reader's convenience we tried to follow the line of argument in [FOT 94] as closely as possible). This difference has several reasons: First of all we *do not assume any sector condition*; in certain cases we have to handle $\mathcal{E}$-quasi-lower-semicontinuous functions instead of $\mathcal{E}$-quasi-continuous functions (cf. e.g. Remark 2.6 (iii)), and finally, the Dirichlet space of the classical situation generalizes to a space $\mathcal{F}$ which is not necessarily stable under composition with normal contractions. In contrast to the classical theory it is not known whether regularity or quasi-regularity alone imply the existence of an associated process. An additional structural assumption on $\mathcal{F}$ is made in $[\text{St1}, p.78, D3]$ (i.e. the existence of a nice intermediate space $\mathcal{Y}$ has to be assumed) in order to construct explicitly an associated m-tight special standard process $\mathbf{M}$.
In addition to the new theoretical results described above we also present new applications. In Section 6 we construct weak solutions to stochastic differential equations in infinite dimensions of the type

$$dX_t \;=\; dW_t + \frac{1}{2}\beta_H^\mu(X_t)dt + \overline{\beta}(X_t)dt \;,\; X_0 = z. \tag{1}$$

Here $(X_t)_{t\geq 0}$ takes values in some real separable Banach space E, $z \in E$, $(W_t)_{t\geq 0}$ is an E-valued Brownian motion, $\overline{\beta}$ is some square integrable vector field on E taking values in a real separable Hilbert space $H \subset E$ and $\beta_H^\mu : E \to E$ is the logarithmic derivative of μ associated with H (cf. Subsection 6.1). In the symmetric case, when $\overline{\beta} = 0$, equation (1)

has been studied intensively in [AR 91].
In Subsection 6.2 we give a first application of (1) for more explicit maps β_H^μ and $\overline{\beta}$ using existence results of [BR 95] on invariant probability measures for some given linear operator L, i.e. measures μ solving the equation $\int Lu\, d\mu = 0$ for all finitely based smooth functions u. We also use results of [St2] on the existence of diffusions associated to extensions of such operators. More precisely, in this case we assume E also to be a Hilbert space and that $H \subset E$ densely by a Hilbert-Schmidt map. We then apply (1) with $\overline{\beta} = \frac{1}{2}(B - \beta_H^\mu)$ where $B : E \to E$ is a Borel measurable vector field of the form $B = -id_E + v$, $v : E \to H$, satisfying *(B.1)-(B.3)* of Subsection 6.2. Under these assumptions on B there exists an invariant probability measure μ such that the stochastic differential equation

$$dX_t \;=\; dW_t - \frac{1}{2}X_t dt + \frac{1}{2}v(X_t)dt \;,\; X_0 = z \tag{2}$$

admits a weak solution $\mathbf{M} = (\Omega, (\mathcal{F})_{t\geq 0}, (X_t)_{t\geq 0}, (P_z)_{z\in E})$ for μ-a.e. (even (quasi-)every) $z \in E$. In particular μ is absolutely continuous w.r.t. the Gaussian measure γ on E with Radon-Nikodym derivative φ^2 where φ is in $H^{1,2}(E;\gamma)$, i.e., the Sobolev space over (E, H, γ). Moreover $\beta_H^\mu = -id_E + 2\frac{\nabla\varphi}{\varphi}$. It is known (see [BR 95, Theorem 3.10]) that the generator of $\mathbf{M}$ restricted to the finitely based smooth functions $L = \frac{1}{2}\Delta_H + \frac{1}{2}B \cdot \nabla$ is μ-symmetric if and only if $v = 2\frac{\nabla\varphi}{\varphi}$ or equivalently $B = \beta_H^\mu$. In our general, i.e., non-symmetric situation, $2\frac{\nabla\varphi}{\varphi}$ is the orthogonal projection of v on the closure of the set $\{\nabla u \mid u \in \mathcal{FC}_b^\infty\}$ in $L^2(E, H; \mu)$. The diffusion $\widehat{\mathbf{M}} = (\widehat{\Omega}, (\widehat{\mathcal{F}})_{t\geq 0}, (\widehat{X}_t)_{t\geq 0}, (\widehat{P}_z)_{z\in E})$ which is in duality to $\mathbf{M}$ w.r.t. μ weakly solves

$$d\widehat{X}_t \;=\; d\widehat{W}_t - \frac{1}{2}\widehat{X}_t dt + 2\frac{\nabla\varphi}{\varphi}(\widehat{X}_t)dt - \frac{1}{2}v(\widehat{X}_t)dt \;,\; \widehat{X}_0 = z \tag{3}$$

for μ-a.e. (even (quasi-)every) $z \in E$ where $(\widehat{W}_t)_{t\geq 0}$ is an E-valued $(\widehat{\mathcal{F}}_t)_{t\geq 0}$-Brownian motion starting at $0 \in E$ with covariance given by the inner product of H. Thus, adding the drifts of (2) and (3) we obtain β_H^μ as in the symmetric case (cf. e.g. [Fö 88]). In Section 6.3 we show that $\mathbf{M}$ also satisfies an Itô-type formula.
Let us now briefly summarize the contents of the remaining Sections 2-5. In Section 2 we introduce the framework and then establish an integral representation theorem for coexcessive functions. As a result we obtain a description of $\mathcal{E}$-exceptional sets in terms of an appropriate class of measures $\widehat{S}_{00}$. This is the key-point for the proof of Theorem 4.5 below. In the symmetric case our class of measures $\widehat{S}_{00}$ is smaller than the corresponding one in [FOT 94, *p.*78]. As a consequence the uniform convergence in Lemma 4.2 can be determined by a weaker semi-norm than in [FOT 94, *Lemma* 5.1.2.]. In Section 3 following [R 70] we associate to every additive functional of $\mathbf{M}$ its corresponding *Revuz measure*. Section 4 is devoted to the *Fukushima-decomposition*. Note that for the proof we *only* assume the coresolvent to be sub-Markovian and strongly continuous on $\mathcal{V}$. *No dual process* is needed. In Remark 4.6(iii) we give conditions for the extension of the decomposition which we use in Subsection 6.3 in an essential way. In Section 5 similarly to [FOT 94], [O 88] we derive an *Itô-type formula* for the transformation of the martingale part of the

decomposition. However, since $\mathcal{F}$ is not necessarily stable under composition with continuously differentiable Lipschitz functions we have to make some assumptions (cf. Theorem 5.6). But these assumptions are easy to check in applications (see Subsection 6.3).
For a large class of further examples and applications as well as for a more detailed presentation we refer to [Tr] and to forthcoming papers.

2 Framework and supplementary Potential Theory of generalized Dirichlet forms

Let E be a Hausdorff space such that its Borel σ-algebra $\mathcal{B}(E)$ is generated by the set $\mathcal{C}(E)$ of all continuous functions on E. Let m be a σ-finite measure on $(E, \mathcal{B}(E))$ such that $\mathcal{H} = L^2(E, m)$ is a separable (real) Hilbert space with inner product $(\cdot, \cdot)_{\mathcal{H}}$. Let $(\mathcal{A}, \mathcal{V})$ be a real valued coercive closed form on $\mathcal{H}$. Then $\mathcal{V}$ equipped with inner product $\widetilde{\mathcal{A}}_1(u, v) := \frac{1}{2}(\mathcal{A}(u, v) + \mathcal{A}(v, u)) + (u, v)_{\mathcal{H}}$ is again a separable real Hilbert space. Let $\|\cdot\|_{\mathcal{V}}$ be the corresponding norm. Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$ we have that $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ densely and continuously.
For a linear operator Λ defined on a linear subspace D of one of the Hilbert spaces $\mathcal{V}$, $\mathcal{H}$ or $\mathcal{V}'$ we will use from now on the notation (Λ, D). Let $(\Lambda, D(\Lambda, \mathcal{H}))$ be a linear operator on $\mathcal{H}$ satisfying the following conditions:

D1 (i) $(\Lambda, D(\Lambda, \mathcal{H}))$ generates a C_0-semigroup of contractions $(U_t)_{t \geq 0}$.
(ii) $(U_t)_{t \geq 0}$ can be restricted to a C_0-semigroup on $\mathcal{V}$.

Denote by $(\Lambda, D(\Lambda, \mathcal{V}))$ the generator corresponding to the restricted semigroup. From [St1, *Lemma* 2.3., *p.*12] we have that if $(\Lambda, D(\Lambda, \mathcal{H}))$ satisfies **D1** then $\Lambda : D(\Lambda, \mathcal{H}) \cap \mathcal{V} \to \mathcal{V}'$ is closable as an operator from $\mathcal{V}$ into $\mathcal{V}'$. Let $(\Lambda, \mathcal{F})$ denote its closure, then $\mathcal{F}$ is a real Hilbert space with corresponding norm

$$\|u\|^2_{\mathcal{F}} := \|u\|^2_{\mathcal{V}} + \|\Lambda u\|^2_{\mathcal{V}'}.$$

By [St1, *Lemma* 2.4., *p.*13] the adjoint semigroup $(\hat{U}_t)_{t \geq 0}$ of $(U_t)_{t \geq 0}$ can be extended to a C_0-semigroup on $\mathcal{V}'$ and the corresponding generator $(\hat{\Lambda}, D(\hat{\Lambda}, \mathcal{V}'))$ is the dual operator of $(\Lambda, D(\Lambda, \mathcal{V}))$. Let $\hat{\mathcal{F}} := D(\hat{\Lambda}, \mathcal{V}') \cap \mathcal{V}$. Then $\hat{\mathcal{F}}$ is a real Hilbert space with corresponding norm

$$\|u\|^2_{\hat{\mathcal{F}}} := \|u\|^2_{\mathcal{V}} + \|\hat{\Lambda} u\|^2_{\mathcal{V}'}.$$

It follows, from [St1, *Proposition* 3.4., *p.*19], that for all $\alpha > 0$ there exist continuous, linear bijections $W_\alpha : \mathcal{V}' \to \mathcal{F}$ and $\hat{W}_\alpha : \mathcal{V}' \to \hat{\mathcal{F}}$ such that $\mathcal{E}_\alpha(W_\alpha f, u) = \langle f, u \rangle = \mathcal{E}_\alpha(u, \hat{W}_\alpha f)$, $\forall f \in \mathcal{V}'$, $u \in \mathcal{V}$. $(W_\alpha)_{\alpha > 0}$ and $(\hat{W}_\alpha)_{\alpha > 0}$ satisfy the resolvent equation

$$W_\alpha - W_\beta = (\beta - \alpha) W_\alpha W_\beta \quad \textit{and} \quad \hat{W}_\alpha - \hat{W}_\beta = (\beta - \alpha) \hat{W}_\alpha \hat{W}_\beta$$

Restricting W_α to $\mathcal{H}$ we get a strongly continuous contraction resolvent $(G_\alpha)_{\alpha > 0}$ on $\mathcal{H}$ satisfying $\lim_{\alpha \to \infty} \alpha G_\alpha f = f$ in $\mathcal{V}$ for all $v \in \mathcal{V}$. The resolvent $(G_\alpha)_{\alpha > 0}$ is called the

resolvent associated with $\mathcal{E}$. Let $(\widehat{G}_\alpha)_{\alpha>0}$ be the adjoint of $(G_\alpha)_{\alpha>0}$ in $\mathcal{H}$. $(\widehat{G}_\alpha)_{\alpha>0}$ is called the *coresolvent associated with* $\mathcal{E}$.
Let the form $\mathcal{E}$ be given by

$$\mathcal{E}(u,v) := \begin{cases} \mathcal{A}(u,v) - \langle \Lambda u, v\rangle & \text{for } u \in \mathcal{F},\ v \in \mathcal{V} \\ \mathcal{A}(u,v) - \langle \hat{\Lambda} v, u\rangle & \text{for } u \in \mathcal{V},\ v \in \hat{\mathcal{F}} \end{cases}$$

and $\mathcal{E}_\alpha(u,v) := \mathcal{E}(u,v) + \alpha(u,v)_{\mathcal{H}}$ for $\alpha > 0$. $\mathcal{E}$ is called the *bilinear form associated with* $(\mathcal{A},\mathcal{V})$ and $(\Lambda, D(\Lambda,\mathcal{H}))$.
Here, $\langle\cdot,\cdot\rangle$ denotes the dualization between $\mathcal{V}'$ and $\mathcal{V}$. Note that $\langle\cdot,\cdot\rangle$ restricted to $\mathcal{H}\times\mathcal{V}$ coincides with $(\cdot,\cdot)_{\mathcal{H}}$ and that $\mathcal{E}$ is well-defined. By [St1, *Proposition* 4.6., *p.*24] we have that $(G_\alpha)_{\alpha>0}$ is sub-Markovian if and only if

$$\textbf{D2} \quad u \in \mathcal{F} \Rightarrow u^+ \wedge 1 \in \mathcal{V} \quad \textit{and} \quad \mathcal{E}(u, u - u^+ \wedge 1) \geq 0$$

is satisfied.

Definition 2.1 *The bilinear form* $\mathcal{E}$ *associated with* $(\mathcal{A},\mathcal{V})$ *and* $(\Lambda, D(\Lambda,\mathcal{H}))$ *is called a* **generalized Dirichlet form** *if* **D2** *holds.*

An element u of $\mathcal{H}$ is called 1-excessive (resp. 1-coexcessive) if $\beta G_{\beta+1} u \leq u$ (resp. $\beta\widehat{G}_{\beta+1}u \leq u$ for all $\beta \geq 0$. Let $\mathcal{P}$ (resp. $\hat{\mathcal{P}}$) denote the 1-excessive (resp. 1-coexcessive) elements of $\mathcal{V}$. An increasing sequence of closed subsets $(F_k)_{k\geq 1}$ is called an $\mathcal{E}$-nest, if for every function $u \in \mathcal{P}\cap\mathcal{F}$ it follows that $u_{F_k^c} \to 0$ in $\mathcal{H}$ and weakly in $\mathcal{V}$. A subset $N \subset E$ is called $\mathcal{E}$-exceptional if there is an $\mathcal{E}$-nest $(F_k)_{k\geq 1}$ such that $N \subset \cap_{k\geq 1} E \setminus F_k$. A property of points in E holds $\mathcal{E}$-quasi-everywhere ($\mathcal{E}$-q.e.) if the property holds outside some $\mathcal{E}$-exceptional set. A function f defined up to some $\mathcal{E}$-exceptional set $N \subset E$ is called $\mathcal{E}$-quasi-continuous ($\mathcal{E}$-q.c.)(resp. $\mathcal{E}$-quasi-lower-semicontinuous ($\mathcal{E}$-q.l.s.c.)) if there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$, such that $\bigcup_{k\geq 1} F_k \subset E\setminus N$ and $f_{|F_k}$ is continuous (resp. lower-semicontinuous) for all k.
For an $\mathcal{E}$-nest $(F_k)_{k\geq 1}$ let

$$C(\{F_k\}) = \{f : A \mapsto \mathbb{R} \mid \bigcup_{k\geq 1} F_k \subset A \subset E,\ f_{|F_k} \textit{ is continuous } \forall k\}$$

$$C_l(\{F_k\}) = \{f : A \mapsto \mathbb{R} \mid \bigcup_{k\geq 1} F_k \subset A \subset E,\ f_{|F_k} \textit{ is lower} - \textit{semicontinuous } \forall k\}$$

We denote by $\widetilde{f}$ an $\mathcal{E}$-q.c. m-version of f, conversely f denotes the m-class represented by an $\mathcal{E}$-q.c. m-version $\widetilde{f}$ of f.

Definition 2.2 *The generalized Dirichlet form* $\mathcal{E}$ *associated with* $(\mathcal{A},\mathcal{V})$ *and* $(\Lambda, D(\Lambda,\mathcal{H}))$ *is called quasi-regular if:*
(i) There exists an $\mathcal{E}$*-nest* $(E_k)_{k\geq 1}$ *consisting of compact sets.*
(ii) There exists a dense subset of $\mathcal{F}$ *whose elements have* $\mathcal{E}$*-q.c. m-versions.*
(iii) There exist $u_n \in \mathcal{F}$, $n \in \mathbb{N}$, *having* $\mathcal{E}$*-q.c. m-versions* $\widetilde{u}_n$, $n \in \mathbb{N}$, *and an* $\mathcal{E}$*-exceptional set* $N \subset E$ *such that* $\{\widetilde{u}_n \mid n \in \mathbb{N}\}$ *separates the points of* $E \setminus N$.

From now on we assume that we have given a quasi-regular generalized Dirichlet form. We remark that by quasi-regularity every element in $\mathcal{F}$ admits an $\mathcal{E}$-q.c. m-version. Let $\mathcal{C}$,$\mathcal{D} \subset \mathcal{H}$. We define $\mathcal{D}_{\mathcal{C}} := \{u \in \mathcal{D} \mid \exists f \in \mathcal{C},\ u \leq f\}$. For a subset $\mathcal{G} \subset \mathcal{H}$ denote by $\widetilde{\mathcal{G}}$ all the $\mathcal{E}$-q.c. m-versions of elements in $\mathcal{G}$. In particular $\widetilde{\mathcal{P}}_{\mathcal{F}}$ denotes the set of all $\mathcal{E}$-q.c. m-versions of 1-excessive elements in $\mathcal{V}$ which are dominated by elements of $\mathcal{F}$. Note that $\widetilde{\mathcal{F} \cap \mathcal{P}} \subset \widetilde{\mathcal{P}}_{\mathcal{F}}$ and that $\widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$ is a linear lattice, that is $\widetilde{u} \wedge \alpha \in \widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$ for all $\alpha \geq 0$ and all $\widetilde{u} \in \widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$. We emphasize that an element in $\mathcal{P}_{\mathcal{F}}$ not necessarily admits an $\mathcal{E}$-q.c. m-version.
For an arbitrary Borel set $B \in \mathcal{B}(E)$ and an element $u \in \mathcal{H}$ such that $\{v \in \mathcal{H} \mid v \geq u \cdot 1_B\} \cap \mathcal{F} \neq \emptyset$ (resp. $\hat{u} \in \widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}$) let $u_B := e_{u \cdot 1_B}$ be the 1-*reduced function* (resp. $\hat{u}_B := \hat{e}_{\hat{u} \cdot 1_B}$ be the 1-*coreduced function*) of $u \cdot 1_B$ (resp. $\hat{u} \cdot 1_B$) as defined in [St1, *Definition* 1.8., *p.*65]. Let u_B^α (resp. $\hat{u}_B^\alpha$), $\alpha > 0$ denote the element $(u \cdot 1_B)_\alpha$ (resp. $(\widehat{\hat{u} \cdot 1_B})_\alpha$) of [St1, *Prop.* 1.6., *p.*62]. If $B = E$ we rather use the notation e_u instead of u_E.
We are now in the situation to state an integral representation theorem for elements in $\widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}$.

Theorem 2.3 *Let $\hat{u} \in \widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}$. Then there exists a unique σ-finite and positive measure $\mu_{\hat{u}}$ on $(E, \mathcal{B}(E))$ charging no $\mathcal{E}$-exceptional set, such that*

$$\int \widetilde{f}\, d\mu_{\hat{u}} = \lim_{\alpha \to \infty} \mathcal{E}_1(f, \alpha \widehat{G}_{\alpha+1} \hat{u}) \quad \forall \widetilde{f} \in \widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}\ .$$

Proof Set $I_{\hat{u}}(\widetilde{f}) = \lim_{\alpha \to \infty} \mathcal{E}_1(f, \alpha \widehat{G}_{\alpha+1} \hat{u})$, $\widetilde{f} \in \widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$. The limit exists since $\mathcal{E}_1(f, \alpha \widehat{G}_{\alpha+1} \hat{u})$ splits into two parts which are both increasing and bounded. Then $I_{\hat{u}}$ is a nonnegative linear functional on $\widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$. Let $(\widetilde{f}_n)_{n \in \mathbb{N}} \subset \widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$ such that $\widetilde{f}_n \downarrow 0$ pointwise on E for $n \to \infty$. Similar to the proof of Theorem 1 in [DoM 93] we will show that

$$I_{\hat{u}}(\widetilde{f}_n) \downarrow 0 \quad \text{as} \quad n \to \infty. \tag{4}$$

Fix $\varphi \in L^1(E; m) \cap \mathcal{B}(E)$ such that $0 < \varphi \leq 1$. By [St1, *Lemma* 3.10., *p.*73] there exists an $\mathcal{E}$-nest $(F_k)_{k \in \mathbb{N}}$, such that $\widetilde{G_1 \varphi} \geq \frac{1}{k}$ everywhere on F_k for all $k \in \mathbb{N}$. Since $\mathcal{E}$ is quasi-regular we may assume that F_k, $k \in \mathbb{N}$, is compact. We may further assume by [St1, *Lemma* 3.5., *p.*71] that $(\widetilde{f}_n)_{n \in \mathbb{N}} \subset C(\{F_k\})$. From Dini's Theorem we know that given $k_0 \in \mathbb{N}$ there exists $n(k_0) \in \mathbb{N}$, such that for all $n \geq n(k_0)$

$$f_n \leq \frac{1}{k_0} G_1 \varphi \quad \text{m–a.e. on } F_{k_0}.$$

Since $f_n \leq f_1 \in \mathcal{P}_{\mathcal{F}} - \mathcal{P}_{\mathcal{F}}$ there exists $f \in \mathcal{F}$ such that $f_n \leq f$ and therefore we have for all $n \in \mathbb{N}$

$$f_n \leq \frac{1}{k_0} G_1 \varphi + f_{F_{k_0}^c} \quad \text{m-a.e..}$$

Let $\hat{f} \in \widehat{\mathcal{F}}$ such that $\hat{u} \leq \hat{f}$. Then

$$I_{\hat{u}}(\widetilde{f}_n) \;=\; \lim_{\alpha \to \infty} \mathcal{E}_1(f_n, \alpha \widehat{G}_{\alpha+1} \hat{u})$$

$$\leq \limsup_{\alpha\to\infty} \mathcal{E}_1(\frac{1}{k_0} G_1\varphi + f_{F^c_{k_0}}\,,\, \alpha\widehat{G}_{\alpha+1}\hat{u})$$
$$\leq \mathcal{E}_1(\frac{1}{k_0} G_1\varphi + f_{F^c_{k_0}}\,,\, \hat{f}).$$

Since $\lim_{k\to\infty} \frac{1}{k} G_1\varphi + f_{F^c_k} = 0$ weakly in $\mathcal{V}$ we conclude that $\lim_{n\to\infty} I_{\hat{u}}(\widetilde{f}_n) = 0$ and (4) is shown. By the Theorem of Daniell-Stone there exists a unique measure say $\mu_{\hat{u}}$ on $\sigma(\widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}})$ such that $\widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}} \subset L^1(\mu_{\hat{u}})$. By [St1, *Proposition* 1.9., *p.*77] we know that $\widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$ separates the points of $E \setminus N$ where N is an $\mathcal{E}$-exceptional set and consequently $\sigma(\widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}) \supset \mathcal{B}(E \setminus N)$. Since $\mu_{\hat{u}}(\widetilde{N}) = \lim_{\alpha\to\infty} \mathcal{E}_1(1_{\widetilde{N}}, \alpha\widehat{G}_{\alpha+1}\hat{u}) = 0$ for every $\mathcal{E}$-exceptional set $\widetilde{N}$ we may assume that $\mu_{\hat{u}}$ is a Borel-measure. Finally $\int_E \widetilde{G_1\varphi}\, d\mu_{\hat{u}} \leq \mathcal{E}_1(G_1\varphi\,,\, \hat{f}) < \infty$ implies that $\mu_{\hat{u}}$ is σ-finite.

□

From now on we fix an m-tight special standard process $\mathbf{M} = (\Omega, (\mathcal{F}_t)_{t\geq 0}, (Y_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ with lifetime ζ such that the resolvent $R_\alpha f$ of $\mathbf{M}$ is an $\mathcal{E}$-q.c. m-version of $G_\alpha f$ for all $\alpha > 0$, $f \in \mathcal{H} \cap B_b(E)$. Note that in addition to quasi-regularity a structural assumption on $\mathcal{F}$ is made in [St1, *p.*78, *D*3] in order to construct explicitly an associated m-tight special standard process. Since we make no use of this technical assumption and since it may be subject to some further progress we instead prefer to assume the existence of $\mathbf{M}$. We remark that we use the resolvent of $\mathbf{M}$ in the proofs of Lemma 2.4 and Theorem 2.5 below but that the corresponding statements are independent of $\mathbf{M}$ and only depend on the generalized Dirichlet form.
We remark that whenever we fix a filtration in this paper, it will be the natural filtration. For $u \in \mathcal{P}$ we denote by $\overline{u}$ some $\mathcal{E}$-q.l.s.c. regularization, i.e. $\overline{u} := \sup_{n\geq 1} nR_{n+1}u$.

Lemma 2.4 *Let $F \subset E$ be closed. Then supp$(\mu_{(\widehat{G}_1\varphi)_F}) \subset F$.*

Proof Fix $\varphi \in L^1(E;m) \cap \mathcal{B}(E)$ such that $0 < \varphi \leq 1$. Define $\varphi^F := E.\Big[\int_{D_F}^\infty e^{-t}\varphi(Y_t)\, dt\Big]$ where $D_F = \inf\{t \geq 0 \mid Y_t \in F\}$. Using the strong Markov property of $\mathbf{M}$ it is easy to see that φ^F is a Borel measurable m-version of an 1-excessive element in $\mathcal{V}$ and that $\varphi^F \geq (G_1\varphi)_F$. In particular φ^F is 1-supermedian for $(R_\alpha)_{\alpha>0}$ and

$$\overline{(G_1\varphi)}_F = \sup_{n\geq 1} nR_{n+1}(G_1\varphi)_F \leq \sup_{n\geq 1} nR_{n+1}\varphi^F \leq \varphi^F \quad \mathcal{E}\text{-q.e.}, \tag{5}$$

furthermore

$$R_1\varphi - \varphi^F \text{ is } \begin{cases} > 0 & \mathcal{E}-q.e. \text{ on } F^c \\ 0 & \text{on } F. \end{cases}$$

Since $\mu_{(\widehat{G}_1\varphi)_F}$ does not charge $\mathcal{E}$-exceptional sets it follows from (5) that

$$\int R_1\varphi - \varphi^F\, d\mu_{(\widehat{G}_1\varphi)_F} \leq \int R_1\varphi - \overline{(G_1\varphi)}_F\, d\mu_{(\widehat{G}_1\varphi)_F}$$

but the expression on the right hand side is equal to zero since (for intermediate steps cf. [St1, p.60 ff])

$$\int R_1\varphi\, d\mu_{(\widehat{G}_1\varphi)_F} = \lim_{\alpha\to\infty} \alpha(G_1\varphi, ((\widehat{G}_1\varphi)^\alpha_F - \widehat{G}_1\varphi\cdot 1_F)^-)_{\mathcal{H}}$$

$$
\begin{aligned}
&= \lim_{\alpha\to\infty} \alpha((G_1\varphi)_F, ((\widehat{G}_1\varphi)^\alpha_F - \widehat{G}_1\varphi\cdot 1_F)^-)_{\mathcal{H}} \\
&= \lim_{\alpha\to\infty}\lim_{\beta\to\infty} \mathcal{E}_1(\beta\, R_{\beta+1}(G_1\varphi)_F, (\widehat{G}_1\varphi)^\alpha_F) \\
&\le \sup_{\beta\ge 1} \int \beta\, R_{\beta+1}(G_1\varphi)_F \, d\mu_{(\widehat{G}_1\varphi)_F} = \int \overline{(G_1\varphi)}_F \, d\mu_{(\hat{G}_1\varphi)_F}.
\end{aligned}
$$

Now $\mu_{(\widehat{G}_1\varphi)_F}(F^c) = 0$ follows by a standard argument, because $\mu_{(\widehat{G}_1\varphi)_F}$ is σ-finite.

□

As a generalization of [FOT 94, *p.*78] we introduce the following class of measures

$$
\widehat{S}_{00} := \{\mu_{\hat{u}} \mid \ \hat{u} \in \widehat{\mathcal{P}}_{\widehat{G}_1\mathcal{H}^+_b} \text{ and } \mu_{\hat{u}}(E) < \infty\}
$$

where $\mathcal{H}^+_b$ are the positive and bounded elements in $\mathcal{H}$. Then we have

Theorem 2.5 *For $B \in B(E)$ the following conditions are equivalent:*
(i) B is $\mathcal{E}$-exceptional
(ii) $\mu(B) = 0\ \forall \mu \in \hat{S}_{00}$

Proof (i) $\Rightarrow$ (ii) is clear. Next we will show $\neg$(i) $\Rightarrow$ $\neg$(ii). Fix $\varphi \in L^1(E;m) \cap \mathcal{B}(E)$ such that $0 < \varphi \le 1$. If B is not $\mathcal{E}$-exceptional then, by [St1, *Proposition* 17, *p.*20], $\mathrm{cap}_\varphi(B) = \inf\{((\widehat{G}_1\varphi)_U, \varphi)_{\mathcal{H}} \mid U \supset B,\ U \text{ open}\} > 0$. Since cap_φ is regular there exists a compact $K \subset B$ with $\mathrm{cap}_\varphi(K) > 0$.
Let $D^+_0 - D^+_0 = \{f_n; n \in \mathbb{N}\}$ be a countable dense subset of bounded functions in $\mathcal{F}$ with $\mathcal{E}$-q.c. m-versions $\overline{D}^+_0 - \overline{D}^+_0 = \{\widetilde{f}_n; n \in \mathbb{N}\} \subset \widetilde{D}^+_0 - \widetilde{D}^+_0 \subset \widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$ which separate the points of $E \setminus N$ where N is an $\mathcal{E}$-exceptional set (cf. [St1, *Proposition* 1.9.(*ii*), *p.*77] for the existence). There exists further (cf. [St1, *Lemma* 1.10., *p.*77]) an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ consisting of compact metrizable sets such that $\{R_1\varphi, \widetilde{f}_n; n \in \mathbb{N}\} \subset C(\{F_k\})$ and such that $R_1\varphi \ge \frac{1}{k}$ $\mathcal{E}$-q.e. on F_k for all $k \ge 1$. We may further assume, that $N \subset \bigcap_{k\ge1} F^c_k$. Choose $n_0 \in \mathbb{N}$ such that $\mathrm{cap}_\varphi(K \cap F_k) > 0$ for all $k \ge n_0$. Since $\mathrm{cap}_\varphi(F^c_k) \underset{k\to\infty}{\longrightarrow} 0$, there exists $k_0 \ge n_0$ with

$$
\mathrm{cap}_\varphi(K \cap F_{k_0}) - \mathrm{cap}_\varphi(F^c_{k_0}) \ > \ 0. \tag{6}
$$

Let $\rho_{F_{k_0}}$ be a metric on F_{k_0} which is compatible with the relative topology on F_{k_0} inherited from E. Define for $n \in \mathbb{N}$

$$
B_n := \{z \in F_{k_0} \mid \rho_{F_{k_0}}(z, K \cap F_{k_0}) < \frac{1}{n}\}, \quad \overline{B}_n := \{z \in F_{k_0} \mid \rho_{F_{k_0}}(z, K \cap F_{k_0}) \le \frac{1}{n}\}.
$$

Then $B_n \cup F^c_{k_0} \subset E$ is open for all n and thus

$$
\begin{aligned}
\mathrm{cap}_\varphi(K \cap F_{k_0}) &= \inf_{\substack{K\cap F_{k_0}\subset U \\ U \text{ open}}} \mathrm{cap}_\varphi(U) \\
&\le \inf_{n\ge1} \mathrm{cap}_\varphi(B_n \cup F^c_{k_0}) \\
&= \inf_{n\ge1} \mathcal{E}_1(G_1\varphi, (\widehat{G}_1\varphi)_{B_n\cup F^c_{k_0}}) \\
&\le \inf_{n\ge1} \int R_1\varphi \, d\mu_{(\widehat{G}_1\varphi)_{\overline{B}_n}} + \mathrm{cap}_\varphi(F^c_{k_0})
\end{aligned}
$$

since $(\widehat{G}_1\varphi)_{B_n\cup F_{k_0}^c} \le (\widehat{G}_1\varphi)_{\overline{B}_n} + (\widehat{G}_1\varphi)_{F_{k_0}^c}$. It now follows from (6) that

$$0 \;<\; \int R_1\varphi \, d\mu_{(\widehat{G}_1\varphi)_\infty} \tag{7}$$

where $(\widehat{G}_1\varphi)_\infty$ is defined to be the weak limit of $((\widehat{G}_1\varphi)_{\overline{B}_n})_{n\in\mathbb{N}}$ in $\mathcal{V}$. Note that $(\widehat{G}_1\varphi)_\infty \in \widehat{\mathcal{P}}_{\widehat{G}_1\mathcal{H}_b^+}$ and thus there exists a unique $\mu_{(\widehat{G}_1\varphi)_\infty}$ by Theorem 2.3. For convenience we set $\hat{\mu}_n := \mu_{(\widehat{G}_1\varphi)_{\overline{B}_n}}$, $n \in \mathbb{N}$, and $\hat{\mu}_\infty := \mu_{(\widehat{G}_1\varphi)_\infty}$. We have to show $supp(\hat{\mu}_\infty) \subset K \cap F_{k_0}$, because then by (7)

$$0 < \int R_1\varphi \, d\hat{\mu}_\infty = \int R_1\varphi \, 1_{K\cap F_{k_0}} \, d\hat{\mu}_\infty \le \hat{\mu}_\infty(K \cap F_{k_0}) \le \hat{\mu}_\infty(B).$$

Clearly $supp(\hat{\mu}_\infty) \subset K \cap F_{k_0}$ implies $\hat{\mu}_\infty(E) < \infty$ and thus $\hat{\mu}_\infty \in \hat{S}_{00}$. We will proceed in several steps.

1. Step : There exists a subsequence such that $\hat{\mu}_{n_k}$ converges weakly to some μ:
By Lemma 2.4 we know that $supp(\hat{\mu}_n) \subset F_{k_0}$ for all $n \in \mathbb{N}$ and thus we have for all $f \in C_0(F_{k_0})$

$$\begin{aligned}
\sup_{n\in\mathbb{N}} \mid \int f \, d\hat{\mu}_n \mid \;&\le\; \|f\|_\infty \sup_{n\in\mathbb{N}} \hat{\mu}_n(F_{k_0}) \\
&\le\; \|f\|_\infty \sup_{n\in\mathbb{N}} \mathcal{E}_1(k_0\, G_1\varphi, (\widehat{G}_1\varphi)_{\overline{B}_n}) \\
&\le\; \|f\|_\infty \mathcal{E}_1(k_0\, G_1\varphi, \widehat{G}_1\varphi) \\
&<\; \infty.
\end{aligned}$$

It follows that $\{\hat{\mu}_n; n \in \mathbb{N}\}$ is relatively compact for the vague topology. Let us choose a subsequence $(\hat{\mu}_{n_k})_{k\in\mathbb{N}}$ which is convergent to some μ with respect to the vague topology. Since F_{k_0} is compact it follows that $(\hat{\mu}_{n_k})_{k\in\mathbb{N}}$ is weakly convergent to μ.

2. Step : μ is finite and $supp(\mu) \subset K \cap F_{k_0}$:
Since $1_{F_{k_0}} \in C_b(F_{k_0})$ it follows that

$$\mu(F_{k_0}) = \lim_{k\to\infty} \hat{\mu}_{n_k}(F_{k_0}) \le k_0\, \mathcal{E}_1(G_1\varphi, \widehat{G}_1\varphi) < \infty.$$

Further, since $\overline{B}_j{}^c \uparrow (K\cap F_{k_0})^c$ as $j \to \infty$ (the complements are taken in F_{k_0}) we conclude by the Porte-manteau-Theorem and Lemma 2.4 that

$$\mu((K \cap F_{k_0})^c) = \lim_{j\to\infty} \mu(\overline{B}_j{}^c) \le \lim_{j\to\infty} \liminf_{n_k\ge j} \hat{\mu}_{n_k}(\overline{B}_j{}^c) = 0.$$

3. Step : μ does not charge $\mathcal{E}$-exceptional sets:
Setting $\hat{\mu}(A) = \mu(A\cap F_{k_0})$ for $A \in \mathcal{B}(E)$ we may interpret μ as a Borel measure on E. We will make no distinction between μ and $\hat{\mu}$ in the following. Let $(E_k)_{k\in\mathbb{N}}$ be an arbitrary $\mathcal{E}$-nest. Then (with the complements in E)

$$\mu(\bigcap_{k\ge1} E_k^c) \;=\; \lim_{k\to\infty} \mu(E_k^c)$$

$$
\begin{aligned}
&\leq \lim_{k\to\infty}\liminf_{j\to\infty}\hat{\mu}_{n_j}(E_k^c)\\
&\leq \lim_{k\to\infty}\liminf_{j\to\infty}\int_{E_k^c} k_0\, R_1\varphi\, d\hat{\mu}_{n_j}\\
&\leq k_0\lim_{k\to\infty}\liminf_{j\to\infty}\int E_\cdot\Big[\int_{\sigma_{E_k^c}}^{\infty} e^{-t}\varphi(Y_t)\,dt\Big]d\hat{\mu}_{n_j}\\
&\leq k_0\lim_{k\to\infty}\mathrm{cap}_\varphi(E_k^c)\\
&= 0
\end{aligned}
$$

implies that $\mu(N)=0$ for every $\mathcal{E}$-exceptional set $N\in\mathcal{B}(E)$.
4. Step : $\mu=\hat{\mu}_\infty$:
Let $f\in\mathcal{F}$. There exists $(f_{m_k})_{k\in\mathbf{N}}\subset D_0^+-D_0^+$ such that $\lim_{k\to\infty}f_{m_k}=f$ in $\mathcal{F}$. By [St1, *Corollary* 3.8., *p.*73] we may assume that $(\widetilde{f}_{m_k})_{k\in\mathbf{N}}\subset\overline{D}_0^+-\overline{D}_0^+$ converges $\mathcal{E}$-q.e. to some $\mathcal{E}$-q.c m-version $\widetilde{f}$ of f. We will show that $(\widetilde{f}_{m_k})_{k\in\mathbf{N}}$ is $L^1(\mu)$-Cauchy. Since $|\,\widetilde{f}_{m_k}-\widetilde{f}_{m_l}\,|\in C_b(F_{k_0})$ and $|\,\widetilde{f}_{m_k}-\widetilde{f}_{m_l}\,|\leq\overline{e}_{f_{m_k}-f_{m_l}}+\overline{e}_{f_{m_l}-f_{m_k}}$ $\mathcal{E}$-q.e. where $\overline{e}_{f_{m_k}-f_{m_l}},\overline{e}_{f_{m_l}-f_{m_k}}$ are some $\mathcal{E}$-q.l.s.c. regularizations we have

$$
\begin{aligned}
\int|\,\widetilde{f}_{m_k}-\widetilde{f}_{m_l}\,|\,d\mu &\leq \lim_{j\to\infty}\int\overline{e}_{f_{m_k}-f_{m_l}}+\overline{e}_{f_{m_l}-f_{m_k}}\,d\hat{\mu}_{n_j}\\
&\leq \mathcal{E}_1(e_{f_{m_k}-f_{m_l}}+e_{f_{m_l}-f_{m_k}},\hat{G}_1\varphi)\\
&\leq \|e_{f_{m_k}-f_{m_l}}+e_{f_{m_l}-f_{m_k}}\|_{\mathcal{H}}\|\varphi\|_{\mathcal{H}}
\end{aligned}
$$

and we conclude by [St1, *Lemma* 2.2.(*i*), *p.*66]. Then for a new subsequence eventually

$$
\begin{aligned}
\int\widetilde{f}\,d\mu &= \lim_{k\to\infty}\lim_{j\to\infty}\int\widetilde{f}_{m_k}\,d\hat{\mu}_{n_j}\\
&= \lim_{k\to\infty}\lim_{j\to\infty}\mathcal{E}_1(f_{m_k},(\widehat{G}_1\varphi)_{\overline{B}_j})\\
&= \mathcal{E}_1(f,(\widehat{G}_1\varphi)_\infty),
\end{aligned}
$$

but since μ does not charge $\mathcal{E}$-exceptional sets by **3.Step** the equality holds for every $\mathcal{E}$-q.c. m-version $\widetilde{f}$ of f. From this it is easy to see that $\mu=\hat{\mu}_\infty$.

□

Remark 2.6 *(i) The assertion of Theorem* 2.5 *remains true if we replace* $\hat{S}_{00}$ *by* $\{\mu\in\hat{S}_{00}\mid\mu(E)=1\}$. *Note also that if* $(\widehat{G}_\alpha)_{\alpha>0}$ *is sub-Markovian we may replace* $\hat{S}_{00}$ *by the larger class* $\{\mu_{\widehat{u}}\mid\ \|\widehat{u}\|_\infty<\infty$ *and* $\mu_{\widehat{u}}(E)<\infty\}$ *and then our definition coincides with the one of* [*FOT 94*, *p.*78].
(ii) Similar to [FOT 94] it is possible to define the measures of finite (co-)energy integral and to show that these measures have properties similar to those in [FOT 94].We will also use the notation $\hat{U}_1\mu_{\widehat{u}}$ *for* $\widehat{u}$.
(iii) Note that if every element in $\mathcal{P}_{G_1\mathcal{H}_b^+}$ *admits an* $\mathcal{E}$*-q.c. m-version then before* **1.Step** : *in the proof of Theorem 2.5 one can show directly supp*$(\hat{\mu}_\infty)\subset K$. *Indeed we may assume that* $G_1\varphi$, $(G_1\varphi)_{\overline{B}_j}$, $\alpha\,R_{\alpha+1}(G_1\varphi)_{\overline{B}_j}$ *are continuous on* $\overline{B}_j$ *for every* j, $\alpha\in\mathbb{N}$. *We may*

also assume that $G_1\varphi \geq \frac{1}{k_0}$ *on each* $\overline{B}_j$. *We then have for each* $j \in \mathbb{N}$

$$\begin{aligned}\int G_1\varphi - (G_1\varphi)_{\overline{B}_j}\, d\hat{\mu}_\infty &= \lim_{\alpha\to\infty}\lim_{n\geq j}\lim_{\beta\to\infty}\int (G_1\varphi)_{\overline{B}_j} - \alpha\, R_{\alpha+1}(G_1\varphi)_{\overline{B}_j}\, d\mu_{(G_1\varphi)^\beta_{\overline{B}_n}} \\ &\leq \lim_{\alpha\to\infty} \|(G_1\varphi)_{\overline{B}_j} - \alpha\, R_{\alpha+1}(G_1\varphi)_{\overline{B}_j}\|_{\infty,\overline{B}_j} \int k_0 G_1\varphi\, d\hat{\mu}_\infty\end{aligned}$$

and the last expression is zero by Dini's Theorem (here $\|\cdot\|_{\infty,\overline{B}_j}$ *denotes the sup norm on the compact space* $\overline{B}_j$*). We then conclude as in the proof of Lemma 2.4.*

3 Additive functionals and Revuz measure

A family $(A_t)_{t\geq 0}$ of functions on Ω is called an *additive functional* (abbreviated AF) of $\mathbf{M} = (\Omega, (\mathcal{F}_t)_{t\geq 0}, (Y_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$, if:
(i) $A_t(\cdot)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$.
(ii) There exists a *defining* set $\Lambda \in \mathcal{F}_\infty$ and an $\mathcal{E}$-exceptional set $N \subset E$, such that $P_z[\Lambda] = 1$ for all $z \in E\backslash N$, $\theta_t(\Lambda) \subset \Lambda$ for all $t > 0$ and for each $\omega \in \Lambda$, $t \mapsto A_t(\omega)$ is right continuous on $[0,\infty[$ and has left limits on $]0,\zeta(\omega)[$, $A_0(\omega) = 0$, $|A_t(\omega)| < \infty$ for $t < \zeta(\omega)$, $A_t(\omega) = A_\zeta(\omega)$ for $t \geq \zeta(\omega)$ and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$ for $s, t \geq 0$. Two additive functionals $A := (A_t)_{t\geq 0}$, $B := (B_t)_{t\geq 0}$ are called equivalent (notation $A = B$), if there exists a common defining set Λ and a common $\mathcal{E}$-exceptional set N, such that $A_t(\omega) = B_t(\omega)$ for all $\omega \in \Lambda$ and $t \geq 0$.
An AF A is called a *continuous additive functional* (abbreviated CAF), if $t \mapsto A_t(\omega)$ is continuous on $[0,\infty)$, a *positive, continuous additive functional* (abbreviated PCAF) if $A_t(\omega) \geq 0$ and a finite AF, if $\mid A_t(\omega) \mid < \infty$ for all $t \geq 0, \omega \in \Lambda$. Two AF's A,B are said to be equivalent (in notation $A = B$) if for each $t > 0$ $P_z(A_t = B_t) = 1$ for $\mathcal{E}$-q.e. $z \in E$. For a Borel measure ν on E and $B \in \mathcal{B}(E)$ let $P_\nu(B) := \int P_z(B)\nu(dz)$ and E_ν be the expectation w.r.t. P_ν. The energy of an AF A of $\mathbf{M}$ is then defined by

$$e(A) = \lim_{\alpha\to\infty} \alpha^2 E_m\Big[\int_0^\infty e^{-\alpha t} A_t^2\, dt\Big], \tag{8}$$

whenever this limit exists in $[0,\infty]$. We will set $\overline{e}(A)$ for the same expression but with $\overline{\lim}$ instead of lim.
From now on let us assume that the coresolvent $(\widehat{G}_\alpha)_{\alpha>0}$ associated with $\mathcal{E}$ is sub-Markovian.

Theorem 3.1 *Let* A *be a* PCAF *of* $\mathbf{M}$. *Then there exists a unique positive measure* μ_A *on* $(E, \mathcal{B}(E))$, *charging no* $\mathcal{E}$*-exceptional set and called the Revuz measure of* A, *such that*

$$\int_E f\, d\mu_A = \lim_{\alpha\to\infty} \alpha E_m\Big[\int_0^\infty e^{-\alpha t} f(Y_t)\, dA_t\Big] \quad \textit{for all } f \in \mathcal{B}(E)^+. \tag{9}$$

Furthermore, there exists an $\mathcal{E}$*-nest of compact sets* $(\widetilde{F}_n)_{n\geq 1}$ *such that* $\mu_A(\widetilde{F}_n) < \infty$ *for each* n *and such that* $U_A^1 1_{\widetilde{F}_n}$ *is an* $\mathcal{E}$*-q.l.s.c. m-version of some element in* $\mathcal{P}_{G_1\mathcal{H}_b^+}$ *for each* n.

Proof As usual we set $U_A^\alpha f(z) := E_z\big[\int_0^\infty e^{-\alpha t} f(Y_t)\, dA_t\big]$ for a PCAF A of $\mathbf{M}$ and $f \in \mathcal{B}(E)^+$. Then we have the following resolvent equations (cf. [Sh 88, 36.16]) for $0 \le \alpha < \beta$ and $f \in \mathcal{B}(E)^+$

$$U_A^\alpha f = U_A^\beta f + (\beta - \alpha) R_\alpha U_A^\beta f = U_A^\beta f + (\beta - \alpha) R_\beta U_A^\alpha f, \tag{10}$$

but one has to be careful not to substract because we make no finiteness assumptions on $U_A^\beta f$. Since $(\widehat{G}_\alpha)_{\alpha>0}$ is assumed to be sub-Markovian it follows for $0 \le \alpha < \beta$ and $f \in \mathcal{B}(E)^+$ that $\int \alpha U_A^\alpha f\, dm \le \int \beta U_A^\beta f\, dm$. This implies the existence of μ_A. Clearly μ_A does not charge $\mathcal{E}$-exceptional sets and it remains to show that μ_A is σ-finite. Fix $\varphi \in L^1(E;m) \cap \mathcal{B}(E)$ such that $0 < \varphi \le 1$. For $z \in E$ set

$$\Phi(z) := E_z\Big[\int_0^\zeta e^{-s}\varphi(Y_s) e^{-A_s}\, ds\Big].$$

Let $(p_t)_{t\ge 0}$ denote the transition semigroup of $\mathbf{M}$. We have for $\mathcal{E}$-q.e. $z \in E$

$$\begin{aligned} e^{-t} p_t(R_1\varphi - \Phi)(z) &= E_z\Big[\int_t^\zeta e^{-s}\varphi(Y_s)(1 - e^{-A_s} e^{A_t})\, ds\Big] \\ &\le E_z\Big[\int_t^\zeta e^{-s}\varphi(Y_s)(1 - e^{-A_s})\, ds\Big] \\ &\le (R_1\varphi - \Phi)(z) \end{aligned} \tag{11}$$

and

$$\lim_{t\searrow 0} e^{-t} p_t(R_1\varphi - \Phi)(z) = (R_1\varphi - \Phi)(z). \tag{12}$$

By (11) we know that $R_1\varphi - \Phi$ is 1-supermedian for $(R_\alpha)_{\alpha>0}$. Since $R_1\varphi - \Phi \le R_1\varphi \in L^2(E;m)$ it follows from [St1, *Lemma* 2.1., *p.*65] that $R_1\varphi - \Phi$ is an m-version of some 1-excessive element in $\mathcal{P}_{G_1\mathcal{H}_b^+}$. It is easy to see that (11) together with (12) imply

$$\sup_{\alpha\ge 1} \alpha R_{\alpha+1}(R_1\varphi - \Phi) = R_1\varphi - \Phi$$

$\mathcal{E}$-q.e. and therefore $R_1\varphi - \Phi$ is $\mathcal{E}$-q.l.s.c.. A simple calculation (cf. [R 70, *p.*509]) gives

$$U_A^1\Phi = R_1\varphi - \Phi \quad \mathcal{E}\text{-q.e.}$$

and then

$$\Phi = R_1\varphi - U_A^1\Phi$$

is $\mathcal{E}$-quasi-upper-semicontinuous. Hence

$$\widetilde{F}_n := \{\Phi \ge \frac{1}{n}\} \cap F_n,\ n \ge 1,$$

is closed in E, where $(F_n)_{n\ge 1}$ is an $\mathcal{E}$-nest of compact sets such that $-\Phi \in C_l(\{F_n\})$. We will show that $(\widetilde{F}_n)_{n\ge 1}$ is an $\mathcal{E}$-nest of compact sets. To see this set

$$B_n = \{z \in E \setminus N \mid \Phi < \frac{1}{n}\},\ n \ge 1,$$

where N is such that $U_A^1\Phi(z) = R_1\varphi(z) - \Phi(z)$ for all $z \in E \setminus N$. Let σ_B be the first hitting time of $B \in \mathcal{B}(E)$, i.e. $\sigma_B = inf\{t > 0 \mid Y_t \in B\}$. Then we have for $\mathcal{E}$-q.e. $z \in E$ and all $n \geq 1$

$$E_z\Big[\int_{\sigma_{B_n \cup F_n^c}}^{\zeta} e^{-s}\varphi(Y_s)e^{-A_s}\,ds\Big] \leq \frac{1}{n} + E_z\Big[\int_{\sigma_{F_n^c}}^{\zeta} e^{-s}\varphi(Y_s)e^{-A_s}\,ds\Big]. \tag{13}$$

In fact, let $\{t_i\}_{i\in\mathbb{N}} \subset [0,\infty)$ dense and define the stopping times $\sigma_k^n := \min\{t_i \mid 1 \leq i \leq k, Y_{t_i} \in B_n \cup F_n^c\}$, $k, n \geq 1$, with the convention $\min \emptyset = \infty$. Then $\lim_{k\to\infty} \sigma_k^n = \sigma_{B_n \cup F_n^c}$, $n \geq 1$, and (13) follows from Lebesgue's Theorem. Set $\tilde{\sigma} = lim_{n\to\infty}\sigma_{\widetilde{F}_n^c}$. Since $e^{-s}\varphi(Y_s)e^{-A_s} > 0$ on $s < \zeta$ $P_z - a.s$ for $\mathcal{E}$-q.e. $z \in E$ it follows again from Lebesgue's Theorem that

$$E_m\Big[\int_{\widetilde{\sigma}}^{\zeta} e^{-s}\varphi(Y_s)e^{-A_s}\,ds\Big] = 0$$

which further implies that $P_m(\tilde{\sigma} < \zeta) = 0$ and consequently $(\widetilde{F}_n)_{n\geq 1}$ is an $\mathcal{E}$-nest by [St1, *Remark* 3.6., *p.*91]. Finally

$$\begin{aligned} U_A^1 1_{\widetilde{F}_n}(z) &= E_z\Big[\int_0^{\zeta} e^{-t} 1_{\widetilde{F}_n}(Y_t)\,dA_t\Big] \\ &\leq n\,E_z\Big[\int_0^{\zeta} e^{-t}\Phi(Y_t)\,dA_t\Big] \\ &= n\,U_A^1\Phi(z) \;\leq\; n\,R_1\varphi(z) \end{aligned}$$

implies that $\mu_A(\widetilde{F}_n) < \infty$, $n \geq 1$. Indeed, by the resolvent equation (10) and the sub-Markovianity of $(\widehat{G}_\alpha)_{\alpha>0}$ we have for all $\beta \geq 1$, $n \geq 1$

$$\begin{aligned} \int \beta\,(R_\beta n\,\varphi - U_A^\beta 1_{\widetilde{F}_n})\,dm &= \int \beta\,(R_1 n\,\varphi - U_A^1 1_{\widetilde{F}_n}) - (\beta-1)\beta\,R_\beta(R_1 n\,\varphi - U_A^1 1_{\widetilde{F}_n})\,dm \\ &\geq \int R_1 n\,\varphi - U_A^1 1_{\widetilde{F}_n}\,dm \;\geq\; 0, \end{aligned}$$

hence $\int \beta\,R_\beta n\,\varphi\,dm \geq \int \beta\,U_A^\beta 1_{\widetilde{F}_n}\,dm$ for all $\beta \geq 1$, and therefore $\mu_A(\widetilde{F}_n) \leq \int n\,\varphi\,dm < \infty$, $n \geq 1$. By [St1, Lemma 2.1.(i),p.65] $U_A^1 1_{\widetilde{F}_n}$ is an m-version of some element in $\mathcal{P}_{G_1}\mathcal{H}_b^+$ for each n.

□

4 Fukushima's decomposition of AF's

For the proof of Theorem 4.5 below we follow the same strategy as in [FOT 94, Chapter 5] whereas the following three Lemmata including the proofs are somewhat different. The uniform convergence in Lemma 4.2 is determined by a weaker semi-norm than in [FOT 94, Lemma 5.1.2.]. After our preparations the proof of Theorem 4.5 is almost the same as in [FOT 94, p.203 ff].

Let $\widetilde{u} \in \widetilde{\mathcal{H}}$. Then $(\widetilde{u}(Y_t) - \widetilde{u}(Y_0))_{t\geq 0}$ is an AF of $\mathbf{M}$, and independent (up to equivalence) of the special choice $\widetilde{u}$. We then set

$$A_t^{[u]} \quad = \quad (\widetilde{u}(Y_t) - \widetilde{u}(Y_0))_{t\geq 0}. \tag{14}$$

It follows from the sub-Markovianity of $(\widehat{G}_\alpha)_{\alpha>0}$ that

$$\overline{e}(A^{[u]}) \leq \overline{\lim_{\alpha\to\infty}}\, \alpha(u - \alpha G_\alpha u, u)_{\mathcal{H}}. \tag{15}$$

Note that in the case where $(\widehat{G}_\alpha)_{\alpha>0}$ is strongly continuous on $\mathcal{V}$ the right hand side above is equal to $\mathcal{E}(u,u)$ for all $u \in \mathcal{F}$. Define

$$\begin{aligned} \mathcal{M} \quad := \quad & \{M | M \text{ is a finite AF}, E_z\Big[\, M_t^2 \,\Big] < \infty, E_z\Big[\, M_t \,\Big] = 0 \\ & \text{for } \mathcal{E}\text{-q.e } z \in E \text{ and all } t \geq 0\}. \end{aligned}$$

$M \in \mathcal{M}$ is called a *martingale additive functional* (MAF). Furthermore define

$$\overset{\circ}{\mathcal{M}} \quad = \quad \{M \in \mathcal{M} |\, e(M) < \infty\}. \tag{16}$$

The elements of $\overset{\circ}{\mathcal{M}}$ are called MAF's of finte energy.
Let $M \in \mathcal{M}$. There exists an $\mathcal{E}$-exceptional set N_M, such that $(M_t, \mathcal{F}_t, P_z)_{t\geq 0}$ is a square integrable martingale for all $z \in E \setminus N_M$. Analogous to [Me 67, III.Théorème 3] or see also [FOT 94, A.3] there exists after a method of perfection a unique (up to equivalence) PCAF $< M >$, called the quadratic variation or sharp bracket of M, such that $(M_t^2 - < M >_t, \mathcal{F}_t, P_z)_{t\geq 0}$ is a martingale for all $z \in E \setminus N_M$.
It then follows that one half of the total mass of the Revuz measure $\mu_{<M>}$ associated to the quadratic variation of $M \in \mathcal{M}$ is equal to the energy of M, i.e.

$$e(M) = \frac{1}{2}\int_E d\mu_{<M>}. \tag{17}$$

For $M,\, L \in \overset{\circ}{\mathcal{M}}$ let

$$< M, L >_t = \frac{1}{2}\,(< M + L >_t - < M >_t - < L >_t).$$

Then $(< M, L >_t)_{t\geq 0}$ is a CAF of bounded variation on each finite interval of t and satisfies

$$E_z(M_t\, L_t) = E_z(< M, L >_t) \quad \forall t \geq 0,\ \mathcal{E}\text{-q.e. } z \in E.$$

Furthermore the finite signed measure $\mu_{<M,L>}$ defined by $\mu_{<M,L>} = \frac{1}{2}(\mu_{<M+L>} - \mu_{<M>} - \mu_{<L>})$ is related to $< M, L >$ in the sense of Theorem 3.1. If $f \in \mathcal{B}_b(E)^+$, then $f \bullet \mu_{<\cdot,\cdot>}$ is symmetric, bilinear and positive on $\overset{\circ}{\mathcal{M}} \times \overset{\circ}{\mathcal{M}}$, where $f \bullet \mu_{<M,L>}(A) := \int_A f\, d\mu_{<M,L>}$ for every $A \in \mathcal{B}(E)$ and every pair $(M, L) \in \overset{\circ}{\mathcal{M}} \times \overset{\circ}{\mathcal{M}}$. Define

$$\begin{aligned} \mathcal{N}_c \quad = \quad & \{N | N \text{ is a finite CAF}, e(N) = 0, E_z\Big[\, |N_t| \,\Big] < \infty \\ & \text{for } \mathcal{E}\text{-q.e.} z \in E \text{ and all } t \geq 0\}. \end{aligned}$$

The "isometry" (17) and the continuity statement (15) are fundamental for the stochastic calculus related to $\mathcal{E}$.
We set $\mathcal{A} = \overset{\circ}{\mathcal{M}} \oplus \mathcal{N}_c$. Namely $\mathcal{A}$ consists of AF's such that

$$A_t = M_t + N_t \quad M \in \overset{\circ}{\mathcal{M}},\ N \in \mathcal{N}_c.$$

$\mathcal{A}$ is a linear space of AF's of finite energy. Furthermore this decomposition is unique as will be seen in the proof of Theorem 4.5. We define the mutual energy of $A, B \in \mathcal{A}$ by

$$e(A,B) = \lim_{\alpha\to\infty} \alpha^2 E_m\Big[\int_0^\infty e^{-\alpha\, t} A_t\, B_t\, dt\Big].$$

By the Cauchy-Schwarz inequality we know that $e(A,B) = 0$ when either A or B is in $\mathcal{N}_c$. Therefore

$$e(A) = e(M) \ \text{ if } \ A = M + N,\ M \in \overset{\circ}{\mathcal{M}},\ N \in \mathcal{N}_c. \tag{18}$$

Lemma 4.1 *Let $\widetilde{u} \in \widetilde{\mathcal{H}}_{\mathcal{F}^+_-}$ where $\mathcal{H}_{\mathcal{F}^+_-} := \{u \in \mathcal{H} \mid u, -u \in \mathcal{H}_{\mathcal{F}}\}$. Then we have for any $\mu \in \widehat{S}_{00}$ and $T > 0$*

$$P_\mu(\sup_{0\le t\le T} \mid \widetilde{u}(Y_t) \mid > \varepsilon) \ \le \ \frac{e^T}{\varepsilon}\|h\|_{\mathcal{H}}\|e_u + e_{-u}\|_{\mathcal{H}},$$

where h is in $\mathcal{H}_b^+$ such that $\widehat{U}_1\,\mu \le \widehat{G}_1\, h$.

Proof Set $U = \{\mid \widetilde{u} \mid > \varepsilon\}$. Since

$$\begin{aligned}\{\sup_{0\le t\le T} \mid \widetilde{u}(Y_t) \mid > \varepsilon\} &= \{\exists t \in [0,T] \mid Y_t \in U; t < \zeta\}\\ &\subset \{\sigma_U \le T; \sigma_U < \zeta\}\end{aligned}$$

we have P_z-a.s. for $\mathcal{E}$-q.e. $z \in E$

$$\frac{e^{T-\sigma_U}}{\varepsilon} \mid \widetilde{u}(Y_{\sigma_U}) \mid \ \text{ is } \ \begin{cases} \ge 1 & \text{on} \quad \{\sigma_U \le T; \sigma_U < \zeta\} \\ \ge 0 & \text{elsewhere}\end{cases}$$

because $t \mapsto \widetilde{u}(Y_t)$ is right continuous on $[0,\zeta)$ P_z-a.s. for $\mathcal{E}$-q.e. $z \in E$ and $\widetilde{u}(Y_t) = 0$ for $t \ge \zeta$. Let $\overline{e}_u$, $\overline{e}_{-u}$ be $\mathcal{E}$-q.l.s.c. regularizations of e_u, e_{-u}. Since $\mid \widetilde{u} \mid \le \overline{e}_u + \overline{e}_{-u}$ $\mathcal{E}$-q.e. it follows that

$$\begin{aligned}P_\mu(\sup_{0\le t\le T} \mid \widetilde{u}(Y_t) \mid > \varepsilon) &\le \frac{e^T}{\varepsilon} E_\mu\Big[e^{-\sigma_U} \mid \widetilde{u}(Y_{\sigma_U}) \mid \Big]\\ &\le \frac{e^T}{\varepsilon}\int \overline{e}_u + \overline{e}_{-u}\, d\mu\\ &\le \frac{e^T}{\varepsilon}\mathcal{E}_1(e_u + e_{-u}, \widehat{G}_1\, h)\\ &\le \frac{e^T}{\varepsilon}\|h\|_{\mathcal{H}}\|e_u + e_{-u}\|_{\mathcal{H}}.\end{aligned}$$

□

Lemma 4.2 *Let $(\widetilde{u}_n)_{n\in\mathbb{N}} \subset \widetilde{\mathcal{H}}_{\mathcal{F}^+_-}$ be a sequence of $\mathcal{E}$-q.c. functions, such that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence w.r.t. $\|u\|_e := \|e_u + e_{-u}\|_{\mathcal{H}}$. Then there exists a subsequence $(\widetilde{u}_{n_k})_{k\in\mathbb{N}}$, such that for $\mathcal{E}$-q.e. $z \in E$*

$$P_z(\widetilde{u}_{n_k}(Y_t) \text{ converges uniformly in } t \text{ on each compact interval of } [0,\infty)) \;=\; 1.$$

Proof Choose a subsequence such that $\|u_{n_k} - u_{n_{k+1}}\|_e \le 2^{-2k}$ for all $k \in \mathbb{N}$ and set $\Lambda_k = \{\sup_{0\le t\le T} \mid \widetilde{u}_{n_k} - \widetilde{u}_{n_{k+1}} \mid (Y_t) > 2^{-k}\}$. Then using Lemma 4.1 we have that $P_\mu(\limsup_{k\to\infty}\Lambda_k) = 0$ for every $\mu \in \widehat{S}_{00}$ by the Lemma of Borel-Cantelli. Clearly this implies $P_z(\limsup_{k\to\infty}\Lambda_k) = 0$ for $\mathcal{E}$-q.e. $z \in E$ in view of Theorem 2.5.

□

Lemma 4.3 *Let A be a* PCAF *and let μ_A be the associated Revuz measure. Then we have for any $\nu \in \widehat{S}_{00}$ and $t > 0$*

$$E_\nu\Big[\, A_t\Big] \;\le\; e^t \|\widehat{U}_1\,\nu\|_\infty \mu_A(E).$$

Proof From Theorem 3.1 we know that there exists an $\mathcal{E}$-nest of compact sets $(\widetilde{F}_n)_{n\in\mathbb{N}}$ such that $\mu_A(\widetilde{F}_n) < \infty$ and such that $U^1_A 1_{\widetilde{F}_n}$ is an $\mathcal{E}$-q.l.s.c. m-version of some element in $\mathcal{P}_{G_1\mathcal{H}^+_b}$. Then

$$\begin{aligned}
E_\nu\Big[\, A_t\Big] &= \lim_{n\to\infty} \int E_\cdot\Big[\int_0^t 1_{\widetilde{F}_n}(Y_s)\, dA_s\Big]\, d\nu \\
&\le e^t \lim_{n\to\infty} \int E_\cdot\Big[\int_0^\infty e^{-s} 1_{\widetilde{F}_n}(Y_s)\, dA_s\Big]\, d\nu \\
&= e^t \lim_{n\to\infty}\lim_{\alpha\to\infty} \mathcal{E}_1(\alpha R_{\alpha+1}\, U^1_A 1_{\widetilde{F}_n}, \widehat{U}_1\nu) \\
&= e^t \lim_{n\to\infty}\lim_{\alpha\to\infty} \mathcal{E}_1(R_1(\alpha U_A^{\alpha+1} 1_{\widetilde{F}_n}), \widehat{U}_1\nu) \\
&\le e^t \|\widehat{U}_1\,\nu\|_\infty \lim_{n\to\infty} \mu_A(\widetilde{F}_n) \\
&= e^t \|\widehat{U}_1\,\nu\|_\infty \mu_A(E).
\end{aligned}$$

□

Using Theorem 2.5 and Lemma 4.3 the proof of the following Theorem is similar to [FOT 94, p.203].

Theorem 4.4 *Let $(M^n)_{n\in\mathbb{N}} \subset \overset{\circ}{\mathcal{M}}$ be e-Cauchy. Then there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ and a unique $M \in \overset{\circ}{\mathcal{M}}$, such that $\lim_{n\to\infty} e(M^n - M) = 0$ and for $\mathcal{E}$-q.e. $z \in E$*

$$P_z(\lim_{k\to\infty} M^{n_k}_t = M_t \text{ uniformly in } t \text{ on each compact interval of } [0,\infty)) \;=\; 1.$$

Let us assume from now on that $(\widehat{G}_\alpha)_{\alpha>0}$ is strongly continuous on $\mathcal{V}$.

Theorem 4.5 *Let $u \in \mathcal{F}$. There exists a unique $M^{[u]} \in \overset{\circ}{\mathcal{M}}$ and a unique $N^{[u]} \in \mathcal{N}_c$ such that*

$$A^{[u]} \;=\; M^{[u]} + N^{[u]}. \tag{19}$$

Proof Uniqueness: If $M \in \overset{\circ}{\mathcal{M}}$ is of zero energy then by Lemma 4.3

$$E_\mu\Big[\, M_t^2 \,\Big] \;=\; 0 \quad \forall \mu \in \widehat{S}_{00}$$

and consequently M is equivalent to the zero AF by Theorem 2.5.
Existence: Let $f \in L^2(E;m) \cap \mathcal{B}(E)$. For $t \geq 0$ set

$$N_t^{[R_1 f]} = \int_0^t (R_1 f - f)(Y_s)\, ds \quad \text{and} \quad M_t^{[R_1 f]} = R_1 f(Y_t) - R_1 f(Y_0) - N_t^{[R_1 f]}.$$

1) We show $N_t^{[R_1 f]} \in \mathcal{N}_c$:
Since $R_1 g$ is finite $\mathcal{E}$-q.e for all $g \in L^2(E;m) \cap \mathcal{B}(E)$ and

$$\begin{aligned} E_z\Big[\, \mid N_t^{[R_1 f]} \mid \,\Big] &\leq E_z\Big[\int_0^t \mid R_1 f - f \mid (Y_s)\, ds \Big] \\ &\leq e^t R_1 \mid R_1 f - f \mid (z) \end{aligned}$$

the same is true for $E_\cdot\Big[\, \mid N_t^{[R_1 f]} \mid \,\Big]$. Surely $N^{[R_1 f]}$ is then a CAF and it remains only to show that $e(N^{[R_1 f]}) = 0$. Let $g := R_1 f - f$, then

$$\begin{aligned} & \alpha^2 E_m\Big[\int_0^\infty e^{-\alpha t} (N_t^{[R_1 f]})^2\, dt \Big] \\ = \;& \alpha^2 E_m\Big[\int_0^\infty e^{-\alpha t} 2 \int_0^t \int_0^s g(Y_s) g(Y_v)\, dv ds\, dt \Big] \\ = \;& 2\alpha E_m\Big[\int_0^\infty \alpha e^{-\alpha t} \int_0^t (g(Y_s) \int_0^s g(Y_v)\, dv) ds\, dt \Big] \\ = \;& 2\alpha E_m\Big[\int_0^\infty e^{-\alpha t} g(Y_t) \int_0^t g(Y_v)\, dv\, dt \Big] \\ = \;& 2\alpha E_m\Big[\int_0^\infty g(Y_v) \int_v^\infty e^{-\alpha t} g(Y_t)\, dt\, dv \Big] \\ = \;& 2\alpha E_m\Big[\int_0^\infty e^{-\alpha v} g(Y_v) \int_0^\infty e^{-\alpha t} g(Y_{t+v})\, dt\, dv \Big] \\ = \;& 2\alpha E_m\Big[\int_0^\infty e^{-\alpha v} g(Y_v) R_\alpha g(Y_v)\, dv \Big] \\ = \;& 2\alpha \int R_\alpha(g\, R_\alpha g)\, dm \;\leq\; 2 \int g\, R_\alpha g\, dm \underset{\alpha \to \infty}{\longrightarrow} 0. \end{aligned}$$

2) $M_t^{[R_1 f]} \in \overset{\circ}{\mathcal{M}}$:
Let $g \in L^2(E;m) \cap \mathcal{B}(E)$. It follows by Fatou's Lemma that $\int R_1(g^2)\, d\mu \leq \|\widehat{U}_1 \mu\|_\infty (g,g)$ for any $\mu \in \widehat{S}_{00}$. Theorem 2.5 then implies that $R_1(g^2)$ is finite $\mathcal{E}$-q.e.. Now, since

$$\begin{aligned} E_z\Big[(M_t^{[R_1 f]})^2 \Big] &\leq 4E_z\Big[R_1(f^2)(Y_t) \Big] + 4R_1(f^2)(z) + 4E_z\Big[t \int_0^t ((R_1 f)^2 + f^2)(Y_s)\, ds \Big] \\ &\leq 8e^t R_1(f^2)(z) + 4te^t R_1((R_1 f)^2 + f^2)(z) \end{aligned}$$

we conclude that $E.\Big[(M_t^{[R_1 f]})^2\Big]$ is finite $\mathcal{E}$-q.e.. It is easy to see that $E.\Big[M_t^{[R_1 f]}\Big] = 0$ $\mathcal{E}$-q.e. and $M^{[R_1 f]}$ has finite energy since by (15) we have

$$\begin{aligned} e(M^{[R_1 f]}) &\leq 2\,\overline{e}(A^{[R_1 f]}) + 2\,e(N^{[R_1 f]}) \\ &\leq 2\,\overline{\lim_{\alpha\to\infty}}\,\alpha(R_1 f - \alpha G_\alpha R_1 f, R_1 f)_{\mathcal{H}} \;=\; 2\,\mathcal{E}(R_1 f, R_1 f) \;<\; \infty. \end{aligned}$$

3) We have shown the statement of the Theorem for $u = R_1 f$, $f \in L^2(E;m) \cap \mathcal{B}(E)$. For general $u \in \mathcal{F}$ there exists $(f_n)_{n\in\mathbb{N}} \subset L^2(E;m)$ such that $R_1 f_n \to u$ in $\mathcal{F}$ as $n \to \infty$. Let $u_n := R_1 f_n$, $n \in \mathbb{N}$. By the uniqueness of the decomposition we know that $M^{[u_n]} - M^{[u_m]} = M^{[u_n - u_m]}$. With (15) and (18) we have

$$e(M^{[u_n-u_m]}) \;=\; e(A^{[u_n-u_m]}) \;\leq\; \mathcal{E}(u_n - u_m, u_n - u_m)$$

and since $u_n \to u$ in $\mathcal{F}$ it follows that $(M^{[u_n]})_{n\in\mathbb{N}} \subset \overset{\circ}{\mathcal{M}}$ is e-Cauchy and then by Theorem 4.4 it makes sense to set

$$\begin{aligned} M^{[u]} &= \lim_{n\to\infty} M^{[u_n]} \;\text{ in }\; (\overset{\circ}{\mathcal{M}}, e) \\ N^{[u]} &= A^{[u]} - M^{[u]}. \end{aligned}$$

It only remains to show $N^{[u]} \in \mathcal{N}_c$. Note that there exists a subsequence n_k such that for $\mathcal{E}$-q.e. $z \in E$

$$P_z(N_t^{[u_{n_k}]} \text{ converges uniformly in } t \text{ on each compact interval of } [0,\infty)) \;=\; 1$$

since the same is true for $A^{[u]}$ and $M^{[u]}$ by Lemma 4.2 and Theorem 4.4. Therefore $N^{[u]}$ is a CAF. Finally

$$\begin{aligned} \overline{e}(N^{[u]}) &= \overline{e}(A^{[u-u_n]} - (M^{[u]} - M^{[u_n]}) + N^{[u_n]}) \\ &\leq 3\,\overline{e}(A^{[u-u_n]}) + 3\,e(M^{[u]} - M^{[u_n]}) \\ &\leq 6\,\mathcal{E}(u - u_n, u - u_n). \end{aligned}$$

□

Remark 4.6 *(i) In general it is not possible to find a decomposition of the additive functional $(\overline{u}(Y_t) - \overline{u}(Y_0))_{t\geq 0}$ for all $u \in \mathcal{P}$ of type (19) where $N_t^{[u]}$ is of zero energy. Here $\overline{u}$ denotes a 1-excessive regularization of u. As an example consider the uniform motion (to the right) on the real line and $\overline{u}(x) = e^x 1_{(-\infty,0)}(x)$, $x \in \mathbb{R}$. (In respect to this there appears to be a mistake in the proof of [O 92], Theorem 6.4., where the contrary is claimed).*
(ii) If we do not require the strong continuity of $(\widehat{G}_\alpha)_{\alpha>0}$ on $\mathcal{V}$ Theorem 4.5 applies for all $u \in \mathcal{F}$, such that $\alpha\widehat{G}_\alpha u \to u$ in $\mathcal{V}$ as $\alpha \to \infty$.
(iii) Similar to 3) of the preceding proof we can see that the decomposition (19) extends to all $\widetilde{u} \in \widetilde{\mathcal{H}}_{\mathcal{F}_-^+}$ for which there exist $\widetilde{u}_n \in \widetilde{\mathcal{H}}_{\mathcal{F}_-^+}$, $n \in \mathbb{N}$, such that we have (19) for $A^{[u_n]}$, $n \in \mathbb{N}$ and such that $e_{u_n-u} + e_{u-u_n} \underset{n\to\infty}{\longrightarrow} 0$ in $\mathcal{H}$ and $\overline{e}(A^{[u_n-u]}) \underset{n\to\infty}{\longrightarrow} 0$.

5 An Itô-type formula

Lemma 5.1 *Let $f \in \mathcal{B}_b(E)$ and $M \in \overset{\circ}{\mathcal{M}}$. Then there exists a unique element denoted by $f \bullet M \in \overset{\circ}{\mathcal{M}}$, such that*

$$\frac{1}{2}\int_E f\, d\mu_{<M,L>} \;=\; e(f \bullet M, L) \quad \textit{for all } L \in \overset{\circ}{\mathcal{M}}\,. \tag{20}$$

Proof Let first $f \in \mathcal{B}_b(E)^+$ and $M, L \in \overset{\circ}{\mathcal{M}}$. Since

$$\begin{aligned} |\,\frac{1}{2}\int_E f\, d\mu_{<M,L>}\,| &\le (\frac{1}{2}\int_E f\, d\mu_{<M>})^{\frac{1}{2}}(\frac{1}{2}\int_E f\, d\mu_{<L>})^{\frac{1}{2}} \\ &\le \|f\|_\infty e(M)^{\frac{1}{2}} e(L)^{\frac{1}{2}} \end{aligned}$$

there exists a unique bounded linear map $T_f : \overset{\circ}{\mathcal{M}} \longrightarrow \overset{\circ}{\mathcal{M}}$, such that $\frac{1}{2}\int_E f\, d\mu_{<M,L>} = e(T_f(M), L)$. Finally we set $f \bullet M := T_f(M)$ and $T_f(M) := T_{f^+}(M) - T_{f^-}(M)$ for $f = f^+ - f^- \in \mathcal{B}_b(E)$.

□

Let us assume from now on that in **D1** (ii) the adjoint semigroup $(\hat{U}_t)_{t\ge 0}$ of $(U_t)_{t\ge 0}$ can also be restricted to a C_0-semigroup on $\mathcal{V}$. Let $(\hat{\Lambda}, D(\hat{\Lambda}, \mathcal{H}))$ denote the generator of $(\hat{U}_t)_{t\ge 0}$ on $\mathcal{H}$, $\hat{\mathcal{A}}(u,v) := \mathcal{A}(v,u)$, u,$v \in \mathcal{V}$ and let the coform $\widehat{\mathcal{E}}$ be defined as the bilinear form associated with $(\hat{\mathcal{A}}, \mathcal{V})$ and $(\hat{\Lambda}, D(\hat{\Lambda}, \mathcal{H}))$. Note that since $(\widehat{G}_\alpha)_{\alpha>0}$ was assumed to be sub-Markovian the corresponding statement of **D2** holds for the coform. Let us further assume from now on that the coform $\widehat{\mathcal{E}}$ is quasi-regular too.
We fix an m-tight special standard process $\widehat{\mathbf{M}} = (\widehat{\Omega}, (\widehat{\mathcal{F}}_t)_{t\ge 0}, (\widehat{Y}_t)_{t\ge 0}, (\widehat{P}_z)_{z\in E_\Delta})$ such that $\widehat{R}_\alpha f = \hat{E}.[\int_0^\infty e^{-\alpha t} f(\widehat{Y}_t)\, dt]$ is an $\widehat{\mathcal{E}}$-q.c. ($=\mathcal{E}$-q.c.) m-version of $\widehat{G}_\alpha f$ for all $f \in \mathcal{H} \cap \mathcal{B}_b(E)$. Necessary and sufficient conditions for the existence of such a process are given in [St1]. $\widehat{\mathbf{M}}$ is then in duality to $\mathbf{M}$ w.r.t. m.

Remark 5.2 *(i) Let $(\widehat{\mathcal{P}}_{\widehat{\mathcal{F}}})_b$ denote the set of bounded elements of $\widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}$. By quasi-regularity of $\widehat{\mathcal{E}}$ we know that $(\widetilde{\widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}})_b - (\widetilde{\widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}})_b$ separates the points of $E \setminus N$, where N is an $\widehat{\mathcal{E}}$-exceptional set. However, note that an element in $(\widehat{\mathcal{P}}_{\widehat{\mathcal{F}}})_b$ not necessarily admits an $\widehat{\mathcal{E}}$-q.c. m-version. Let μ, ν be finite measures on $(E, \mathcal{B}(E))$ charging no $\widehat{\mathcal{E}}$-exceptional set. Then by [Sh 88, Appendices A.0.8] it follows that $\mu = \nu$ if $\int \widehat{R}_\alpha \tilde{f}\, d\mu = \int \widehat{R}_\alpha \tilde{f}\, d\nu$ for all $\tilde{f} \in (\widetilde{\widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}})_b$.*
(ii) Let A, B be two PCAF*'s. If $\mu_A = \mu_B$ then $A = B$. Indeed if $\mu_A = \mu_B$ then by (10) and (21) for every $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ of compact sets $U_A^\beta 1_{F_k} = U_B^\beta 1_{F_k}$ $\mathcal{E}$-q.e. for each $k \in \mathbb{N}$ and $\beta > 0$. We may even choose $(F_k)_{k\in\mathbb{N}}$ (cf. the proof of Theorem 3.1) such that $U_B^1 1_{F_k} \le kR_1\varphi$ for every $k \in \mathbb{N}$. Then with $A^{(1)} := \int_0^\cdot 1_{F_k}(X_s) dA_s$, $A^{(2)} := \int_0^\cdot 1_{F_k}(X_s) dB_s$ for fixed k remarking that $\int v_{ij}\, d\nu \le \int 2kR_1\varphi\, d\nu < \infty$ for any $\nu \in \hat{S}_{00}$ we conclude as in the proof of [FOT 94, Theorem 5.1.2.] and with the same notations. Since $k \in \mathbb{N}$ was arbitrary we get $A = B$.*

Lemma 5.3 *For $f \in \mathcal{B}_b(E)$ and $M, L \in \overset{\circ}{\mathcal{M}}$. Then we have*

$$\mu_{<f\bullet M,L>} \;=\; f \bullet \mu_{<M,L>}.$$

Proof Let $M \in \overset{\circ}{\mathcal{M}}$ and $f \in \mathcal{B}_b(E)$. Analogous to [Me 67, Théorème 4, p. 127] or [FOT 94, Theorem A.3.19.] there exists (after perfection) a unique $\widetilde{M} \in \overset{\circ}{\mathcal{M}}$, such that for all $L \in \overset{\circ}{\mathcal{M}}$, $t > 0$ and $\mathcal{E}$-q.e. $z \in E$

$$E_z\Big[< \widetilde{M}, L >_t \Big] \;=\; E_z\Big[\int_0^t f(Y_s) d < M, L >_s \Big].$$

For all $g \in L^2(E;m) \bigcap \mathcal{B}_b(E), \alpha > 0$, we then have

$$\int_E \widehat{R}_\alpha g \, d\mu_{<\widetilde{M},L>} \;=\; \int_E (\widehat{R}_\alpha g) f \, d\mu_{<M,L>},$$

and consequently

$$\mu_{<f\bullet M,L>} \;=\; f \bullet \mu_{<M,L>},$$

since $e(\widetilde{M} - f \bullet M, L) = 0$, for all $L \in \overset{\circ}{\mathcal{M}}$ by the preceding Lemma.

□

Let μ_A be the measure defined in Theorem 3.1. Then as in [O 88, *Lemma* 4.1.7., *p.*91] one shows that for every $f \in \mathcal{B}(E)^+$, $g \in L^2(E;m) \cap \mathcal{B}_b(E)^+$, $\beta > 0$

$$\int f \widehat{R}_\beta g \, d\mu_A \;=\; \lim_{\alpha\to\infty} \alpha (U_A^{\alpha+\beta} f, \widehat{R}_\beta g)_{\mathcal{H}}. \tag{21}$$

The proof of the next Theorem is based on the proof of Theorem 5.3.2. in [O 88, p.160]. For convenience we write $\mu_{<u,v>}$ instead of $\mu_{<M^{[u]},M^{[v]}>}$ and $\mu_{<u>}$ instead of $\mu_{<M^{[u]}>}$.

Theorem 5.4 (Product rule) *Let $\widetilde{f} = (\widetilde{f}_1, ..., \widetilde{f}_n)$ be an n-tuple of $\mathcal{E}$-q.c. m-versions of bounded elements in $\mathcal{H}$ such that $A^{[\Phi(f)]}$ admits the decomposition of Theorem* 4.5 *for all $\Phi \in C^1(\mathbb{R}^n)$ with $\Phi(0) = 0$ and let $\overline{e}(A^{[f_i - kR_k f_i]}) \to 0$, as $k \to \infty$, $1 \le i \le n$. Let further the martingale part $M^{[u]}$ of the decomposition of Theorem* 4.5 *be continuous for all u in $G_1\mathcal{H}_b$. Let Φ, $\Psi \in C^1(\mathbb{R}^n)$, $\Phi(0) = \Psi(0) = 0$ and $\widetilde{w} \in \widetilde{\mathcal{H}}$, w bounded such that we have the decomposition of Theorem* 4.5 *for $A^{[w]}$, then:*

$$\mu_{<\Phi(f)\cdot\Psi(f),w>} \;=\; \Phi(\widetilde{f}_1, ..., \widetilde{f}_n) \bullet \mu_{<\Psi(f),w>} + \Psi(\widetilde{f}_1, ..., \widetilde{f}_n) \bullet \mu_{<\Phi(f),w>}. \tag{22}$$

Proof It is enough to show

$$\int_E h \, d\mu_{<\Phi^2(f),w>} \;=\; 2 \int_E h \, \Phi(\widetilde{f}_1, ..., \widetilde{f}_n) \, d\mu_{<\Phi(f),w>}$$

for $h = \widehat{R}_\beta g$, $g \in \mathcal{B}_b(E)^+ \bigcap L^1(E;m)$, $\beta > 0$, because then we will consider $\mu_{<(\Phi(f)+\Psi(f))^2,w>}$. We may furthermore assume that $\int h\,dm = 1$. Then by (21)

$$\begin{aligned}
&\int_E h\,d\mu_{<\Phi^2(f),w>}\\
=\;& \lim_{\alpha\to\infty} \alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t} M_t^{[\Phi^2(f)]} M_t^{[w]}\,dt\Big]\\
=\;& \lim_{\alpha\to\infty} \alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t}(\Phi^2(\widetilde{f})(Y_t)-\Phi^2(\widetilde{f})(Y_0))(\widetilde{w}(Y_t)-\widetilde{w}(Y_0))\,dt\Big]\\
=\;& 2\lim_{\alpha\to\infty} \alpha\,(\alpha+\beta)E_{h\Phi(\widetilde{f})\,m}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t}(\Phi(\widetilde{f})(Y_t)-\Phi(\widetilde{f})(Y_0))(\widetilde{w}(Y_t)-\widetilde{w}(Y_0))\,dt\Big]\\
&+\lim_{\alpha\to\infty} \alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t}(\Phi(\widetilde{f})(Y_t)-\Phi(\widetilde{f})(Y_0))^2(\widetilde{w}(Y_t)-\widetilde{w}(Y_0))\,dt\Big]\\
=\;& 2\lim_{\alpha\to\infty} I_\alpha + \lim_{\alpha\to\infty} II_\alpha.
\end{aligned}$$

By (21) and Lebesgue's Theorem we have

$$\begin{aligned}
\lim_{\alpha\to\infty} I_\alpha &= \lim_{\alpha\to\infty} \alpha(U^{\alpha+\beta}_{<\Phi(f),w>}1, h\Phi(\widetilde{f}))\\
&= \lim_{\alpha\to\infty} \alpha \lim_{\gamma\to\infty}(\gamma\, R_{\gamma+\alpha+\beta}\, U^{\alpha+\beta}_{<\Phi(f),w>}1, h\Phi(\widetilde{f}))\\
&= \lim_{\alpha\to\infty} \alpha \lim_{\gamma\to\infty}(\gamma\, U^{\gamma+\alpha+\beta}_{<\Phi(f),w>}1, \widehat{R}_{\alpha+\beta}(h\Phi(\widetilde{f})))\\
&= \lim_{\alpha\to\infty} \alpha \int_E \widehat{R}_{\alpha+\beta}(h\Phi(\widetilde{f}))\,d\mu_{<\Phi(f),w>}\\
&= \int_E h\Phi(\widetilde{f})\,d\mu_{<\Phi(f),w>}
\end{aligned}$$

and for some constant $L > 0$ $\lim_{\alpha\to\infty} II_\alpha$ is dominated by

$$nL\sum_{i=1}^n \overline{\lim_{\alpha\to\infty}}\ \alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t}(\widetilde{f}_i(Y_t)-\widetilde{f}_i(Y_0))^2 \mid \widetilde{w}(Y_t)-\widetilde{w}(Y_0)\mid dt\Big].$$

For $1 \le i \le n$, $k \in \mathbb{N}$, we set $f_{ki} = kR_k\widetilde{f}_i$. Since we assumed that $\overline{e}(A^{[f_i-f_{ki}]}) \to 0$, as $k \to \infty$, $1 \le i \le n$, it is enough to show that for $1 \le i \le n$

$$\overline{\lim_{\alpha\to\infty}}\ \alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t}(f_{ki}(Y_t)-f_{ki}(Y_0))^2 \mid \widetilde{w}(Y_t)-\widetilde{w}(Y_0)\mid dt\Big]$$

tends to zero as $k \to \infty$. By our assumption $(M_t^{[f_{ki}]}, \mathcal{F}_t, P_{hm})_{t\ge 0}$ is a continuous square integrable martingale and consequently by the Burkholder-Davis-Gundy inequality

$$E_{hm}\Big[(M_t^{[f_{ki}]})^4\Big] \;\le\; CE_{hm}\Big[< M^{[f_{ki}]} >_t^2\Big]$$

where $C > 0$ is a constant independent of f_{ki}. Then

$$\begin{aligned}
&\overline{\lim_{\alpha\to\infty}}\ \alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t}(f_{ki}(Y_t)-f_{ki}(Y_0))^2 \mid \widetilde{w}(Y_t)-\widetilde{w}(Y_0)\mid dt\Big]\\
\le\;& 2\,\overline{\lim_{\alpha\to\infty}}\ (\alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t}(M_t^{[f_{ki}]})^4\,dt\Big])^{\frac12}(\int h\,d\mu_{<w>})^{\frac12}\\
\le\;& 2\,\overline{\lim_{\alpha\to\infty}}\ (\alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t} < M^{[f_{ki}]} >_t^2\,dt\Big])^{\frac12}(\int h\,d\mu_{<w>})^{\frac12}.
\end{aligned}$$

Then

$$\begin{aligned}
&\overline{\lim_{\alpha\to\infty}}\ \alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t}(< M^{[f_{ki}]} >_t)^2\,dt\Big]\\
=\ &\overline{\lim_{\alpha\to\infty}}\ \alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t}\,2\int_0^t < M^{[f_{ki}]} >_t - < M^{[f_{ki}]} >_s\ d < M^{[f_{ki}]} >_s\ dt\Big]\\
=\ &2\ \overline{\lim_{\alpha\to\infty}}\ \alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty\int_s^\infty e^{-(\alpha+\beta)\,t} < M^{[f_{ki}]} >_{t-s}\circ\vartheta_s\,dtd < M^{[f_{ki}]} >_s\ \Big]\\
=\ &2\ \overline{\lim_{\alpha\to\infty}}\ \alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,s}E_{Y_s}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t} < M^{[f_{ki}]} >_t\ dt\Big]\,d < M^{[f_{ki}]} >_s\ \Big]\\
=\ &2\ \overline{\lim_{\alpha\to\infty}}\ \alpha\int h\,U^{\alpha+\beta}_{<M^{[f_{ki}]}>}U^{\alpha+\beta}_{<M^{[f_{ki}]}>}1\,dm\\
\leq\ &2\int h\,U^{\gamma}_{<M^{[f_{ki}]}>}1\,d\mu_{<M^{[f_{ki}]}>}
\end{aligned}$$

for every $\gamma>0$. Now since $U^1_{<M^{[f_{ki}]}>}1(z)$ is bounded by the same constant for $\mathcal{E}$-q.e. z in E and since

$$U^{\gamma+1}_{<M^{[f_{ki}]}>}1(z)\ =\ U^1_{<M^{[f_{ki}]}>}1(z)-\gamma R_{\gamma+1}U^1_{<M^{[f_{ki}]}>}1(z)\ \downarrow\ 0$$

for $\mathcal{E}$-q.e. z in E as $\gamma\to\infty$ we conclude by Lebesgue's Theorem.

□

Theorem 5.5 (Chain rule) *Let $\widetilde{f}=(\widetilde{f}_1,...,\widetilde{f}_n)$ be an n-tuple of $\mathcal{E}$-q.c. m-versions of bounded elements in $\mathcal{H}$ such that $A^{[\Phi(f)]}$ admits the decomposition of Theorem 4.5 for all $\Phi\in C^1(\mathbb{R}^n)$ with $\Phi(0)=0$ and let $\overline{e}(A^{[f_i-kR_kf_i]})\to 0$, as $k\to\infty$, $1\leq i\leq n$. Let further the martingale part $M^{[u]}$ of the decomposition of Theorem 4.5 be continuous for all u in $G_1\mathcal{H}_b$. Let $\Phi\in C^1(\mathbb{R}^n)$, $\Phi(0)=0$ and $\widetilde{w}\in\widetilde{\mathcal{H}}$, w bounded such that we have the decomposition of Theorem 4.5 for $A^{[w]}$, then*

$$\mu_{<\Phi(f),w>}\ =\ \sum_{i=1}^n\frac{\partial\Phi}{\partial x_i}(\widetilde{f})\bullet\mu_{<f_i,w>}. \tag{23}$$

Proof We first observe that any powers of the coordinate functions satisfy (23) by the product rule and then by the product rule again all polynomials of n variables vanishing at the origin.
Let $K\subset\mathbb{R}^n$ be a compact set such that $\widetilde{f}(z)\in K$ for $\mathcal{E}$-q.e. $z\in E$. Let $\chi\in C_0^\infty(\mathbb{R}^n)$, $\chi=1$ on $K\subset K'=supp(\chi)$.
There exists (cf. [CH 68, II.4.2, 4.3, p.57]) a sequence of polynomials vanishing at the origin $(p_j)_{j\in\mathbb{N}}$, such that $p_j\underset{j\to\infty}{\longrightarrow}\Phi$, $\frac{\partial p_j}{\partial x_i}\underset{j\to\infty}{\longrightarrow}\frac{\partial\Phi}{\partial x_i}$, $1\leq i\leq n$, uniformly on K'. Note that $[(\Phi-p_j)\chi](\widetilde{f})(z)=(\Phi-p_j)(\widetilde{f})(z)$ for $\mathcal{E}$-q.e. $z\in E$. Then we have for all $g\in L^2(E;m)\cap\mathcal{B}_b(E)$, $\alpha>0$

$$\begin{aligned}
\sum_{i=1}^n\int\widehat{R}_\alpha\,g\,\frac{\partial\Phi}{\partial x_i}(\widetilde{f})\,d\mu_{<f_i,w>}\ &=\ \sum_{i=1}^n\lim_{j\to\infty}\int\widehat{R}_\alpha\,g\,\frac{\partial p_j}{\partial x_i}(\widetilde{f})\,d\mu_{<f_i,w>}\\
&=\ \lim_{j\to\infty}\int\widehat{R}_\alpha\,g\,d\mu_{<p_j(f),w>}\\
&=\ \int\widehat{R}_\alpha\,g\,d\mu_{<\Phi(f),w>}
\end{aligned}$$

where the second identity followed from the product rule. Because of (17) the third identity follows since we assumed to have the decomposition of Theorem 4.5 for all $\Phi(f)$ like above and therefore

$$\begin{aligned} e(M^{[\Phi(f)]} - M^{[p_j(f)]}) &= e(A^{[((\Phi - p_j)\chi)(f)]}) \\ &\leq n \sum_{i=1}^{n} \left\| \frac{\partial((\Phi - p_j)\chi)}{\partial x_i} \right\|_\infty^2 \overline{e}(A^{[f_i]}). \end{aligned}$$

The last expression tends to zero as $j \to \infty$.

□

Summarizing we get the following

Theorem 5.6 *Let $\widetilde{f} = (\widetilde{f}_1, ..., \widetilde{f}_n)$ be an n-tuple of $\mathcal{E}$-q.c. m-versions of bounded elements in $\mathcal{H}$ such that $A^{[\Phi(f)]}$ admits the decomposition of Theorem 4.5 for all $\Phi \in C^1(\mathbb{R}^n)$ with $\Phi(0) = 0$ and let $\overline{e}(A^{[f_i - kR_k f_i]}) \to 0$, as $k \to \infty$, $1 \leq i \leq n$. Let further the martingale part $M^{[u]}$ of the decomposition of Theorem 4.5 be continuous for all u in $G_1\mathcal{H}_b$. Then we have*

$$\Phi(\widetilde{f}_1, \ldots, \widetilde{f}_n)(Y_t) - \Phi(\widetilde{f}_1, \ldots, \widetilde{f}_n)(Y_0) = \sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i}(\widetilde{f}_1, \ldots, \widetilde{f}_n) \bullet M_t^{[f_i]} + N_t^{[\Phi(f_1, \ldots, f_n)]}$$

for all Φ like above and this decomposition is orthogonal w.r.t. $e(\cdot, \cdot)$.

Proof The assertion follows by Lemma 5.1, Lemma 5.3 and Theorem 5.5 because

$$e(M^{[\Phi(f)]} - \sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i}(\widetilde{f}) \bullet M^{[f_i]}) = \frac{1}{2} \int d\mu_{<M^{[\Phi(f)]} - \sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i}(\widetilde{f}) \bullet M^{[f_i]}>} = 0.$$

□

6 An example

6.1 Weak solutions of SDE's in infinite dimensions

Let E be a separable real Banach space and $(H, \langle \cdot, \cdot \rangle_H)$ a separable real Hilbert space such that $H \subset E$ densely and continuously. Identifying H with its topological dual H' we obtain that $E' \subset H \subset E$ densely and continuously. Define the linear space of finitely based smooth functions on E by

$$\mathcal{FC}_b^\infty := \{f(l_1, ..., l_m) \mid m \in \mathbb{N}, f \in \mathcal{C}_b^\infty(\mathbb{R}^m), l_1, ..., l_m \in E'\}.$$

Here $\mathcal{C}_b^\infty(\mathbb{R}^m)$ denotes the set of all infinitely differentiable (real-valued) functions on $\mathbb{R}^m$ with all partial derivatives bounded. For $u \in \mathcal{FC}_b^\infty$, $k \in E$ let

$$\frac{\partial u}{\partial k}(z) := \frac{d}{ds} u(z + sk) \mid_{s=0}, \; z \in E.$$

It follows that if $u = f(l_1, ..., l_m)$ and $k \in H$ we have that

$$\frac{\partial u}{\partial k}(z) = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(l_1(z), ..., l_m(z))\langle l_i, k\rangle_H,\ z \in E.$$

Consequently, $k \mapsto \frac{\partial u}{\partial k}(z)$ is continuous on H and we can define $\nabla u(z) \in H$ by

$$\langle \nabla u(z), k\rangle_H = \frac{\partial u}{\partial k}(z).$$

Let μ be finite positive measure on $(E, \mathcal{B}(E))$. Assume for simplicity $supp(\mu) \equiv E$. An element k in E is called *well-μ-admissible* if there exist $\beta_k^\mu \in L^2(E;\mu)$ such that for all u, v in $\mathcal{FC}_b^\infty$

$$\int \frac{\partial u}{\partial k}\, d\mu = -\int u\beta_k^\mu\, d\mu.$$

Let us assume

(A.1) There exists a dense linear subspace K of E' consisting of well-μ-admissible elements.

Then it is well known that the densely defined positive definite symmetric bilinear form

$$\mathcal{E}^0(u, v) := \frac{1}{2}\int \langle \nabla u, \nabla v\rangle_H\, d\mu \quad u, v \in \mathcal{FC}_b^\infty$$

is closable on $L^2(E;\mu)$ and that the closure $(\mathcal{E}^0, D(\mathcal{E}^0))$ is a symmetric quasi-regular Dirichlet form. Let $(L^0, D(L^0))$ be the associated generator. Let $\overline{\beta} \in L^2(E, H;\mu)$ (i.e. $\overline{\beta} : E \to E$ is $\mathcal{B}(E)/\mathcal{B}(E)$-measurable, $\overline{\beta}(E) \subset H$ and $\|\overline{\beta}\|_H \in L^2(E;\mu)$) be such that

$$\int \langle \overline{\beta}, \nabla u\rangle_H\, d\mu = 0 \quad \text{for all } u \in \mathcal{FC}_b^\infty. \tag{24}$$

Since $\mathcal{FC}_b^\infty$ is dense in $D(\mathcal{E}^0)$ (24) implies that $\int \langle \overline{\beta}, \nabla u\rangle_H\, d\mu = 0$ for all $u \in D(\mathcal{E}^0)$ and thus $\int \langle \overline{\beta}, \nabla u\rangle_H\, v\, d\mu = -\int \langle \overline{\beta}, \nabla v\rangle_H\, u\, d\mu$ for all u, $v \in D(\mathcal{E}^0)_b$. Let

$$Lu := L^0 u + \langle \overline{\beta}, \nabla u\rangle_H\ , \quad u \in D(L^0)_b.$$

It then follows from [St2, *Proposition* 4.1.] that $(L, D(L^0)_b)$ is closable on $L^1(E;\mu)$ and that the closure $(\overline{L}, D(\overline{L}))$ generates a Markovian C_0-semigroup of contractions. Furthermore $D(\overline{L})_b \subset D(\mathcal{E}^0)$ and for $u \in D(\overline{L})_b$, $v \in D(\mathcal{E}^0)_b$

$$\mathcal{E}^0(u, v) - \int \langle \overline{\beta}, \nabla u\rangle_H\, v\, d\mu = -\int \overline{L}u\, v\, d\mu. \tag{25}$$

Let $(L, D(L))$ with associated resolvent $(G_\alpha)_{\alpha>0}$ be the part of $(\overline{L}, D(\overline{L}))$ on $L^2(E;\mu)$, $(L', D(L'))$ with associated resolvent $(G'_\alpha)_{\alpha>0}$ be the adjoint of $(L, D(L))$ in $L^2(E;\mu)$. According to [St1, $I.4.9.(ii)$] $(L, D(L))$ is associated with the *generalized Dirichlet form*

$$\mathcal{E}(u,v) := \begin{cases} (-Lu, v) & \text{for } u \in D(L),\ v \in L^2(E;\mu) \\ (u, -L'v) & \text{for } u \in L^2(E;\mu),\ v \in D(L') \end{cases}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(E;\mu)$. There exists (cf. [St2, $Th.$4.6., $Prop.$4.7.]) a μ-tight special standard process $\mathbf{M} = (\Omega, \mathcal{F}_\infty, (X_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ with life time ζ that is associated with $(L, D(L))$ in the sense that $R_\alpha f(\cdot) := E_\cdot[\int_0^\infty e^{-\alpha t} f(X_t)dt]$ is an $\mathcal{E}^0$-q.c. m-version of $G_\alpha f$ for all $f \in \mathcal{B}_b(E) \cap L^2(E;\mu)$, $\alpha > 0$. Furthermore $P_z[\zeta = +\infty] = 1$, $P_z[t \mapsto X_t$ is continuous on $[0,\infty)] = 1$ for $\mathcal{E}^0$-q.e $z \in E$.
We emphasize that by [St2, $Lemma$ 4.5] L-nests and $\mathcal{E}^0$-nests coincide. Therefore $\mathcal{E}$-exceptional and $\mathcal{E}^0$-exceptional sets coincide.
Since $-\overline{\beta}$ satisfies the same assumptions as $\overline{\beta}$ the closure $(\overline{L}', D(\overline{L}'))$ of $L'u := L^0 u - \langle\overline{\beta}, \nabla u\rangle_H$, $u \in D(L^0)_b$ on $L^1(E;\mu)$ generates a Markovian C_0-semigroup of contractions too, $D(\overline{L}')_b \subset D(\mathcal{E}^0)$ and for $u \in D(\overline{L}')_b$, $v \in D(\mathcal{E}^0)_b$ $\mathcal{E}^0(u,v) + \int\langle\overline{\beta}, \nabla u\rangle_H\, v\, d\mu = -\int \overline{L}' u\, v\, d\mu$. It is easy to see that the part of $(\overline{L}', D(\overline{L}'))$ on $L^2(E;\mu)$ is $(L', D(L'))$. Let $(R'_\alpha)_{\alpha>0}$ denote the resolvent of the associated coprocess. Since $(G'_\alpha)_{\alpha>0}$ is sub-Markovian and strongly continuous on $\mathcal{V} = L^2(E;\mu)$, Theorem 4.5 applies for $u \in D(L)$ with $N_t^{[u]} = \int_0^t Lu(X_s)ds$. Let $v \in D(L)_b$, $g \in L^2(E;\mu) \cap \mathcal{B}_b(E)^+$, $\gamma > 0$, then by (21)

$$\begin{aligned} \int R'_\gamma g\, d\mu_{<v>} &= 2\mathcal{E}^0(v, vR'_\gamma g) - \mathcal{E}^0(v^2, R'_\gamma g) \\ &= \int R'_\gamma g\, \langle\nabla v, \nabla v\rangle_H\, d\mu. \end{aligned}$$

Now let $u_n \in D(L)_b$ such that $u_n \underset{n\to\infty}{\longrightarrow} u$ in $D(L)$. Since by (25) $u_n \underset{n\to\infty}{\longrightarrow} u$ in $D(\mathcal{E}^0)$ we have $\int R'_\gamma g\, d\mu_{<u>} = \int R'_\gamma g\, \langle\nabla u, \nabla u\rangle_H\, d\mu$. In particular $\int R'_\gamma g\, d\mu_{<u>} = \int R'_\gamma g\, d\mu_A$ where $A = \int_0^\cdot \langle\nabla u(X_s), \nabla u(X_s)\rangle_H\, ds$. Therefore by Remark 5.2 (i) and (ii) it follows that $\langle M^{[u]}\rangle_t = \int_0^t \langle\nabla u(X_s), \nabla u(X_s)\rangle_H\, ds$. Note that $\langle M^{[u]}\rangle$ is finite since $\langle\nabla u, \nabla u\rangle_H \in L^1(E;\mu)$. Assume

(A.2) $\quad u_k(\cdot) :=_{E'} \langle k, \cdot\rangle_E \in L^2(E;\mu)$ for all $k \in K$.

Here ${}_{E'}\langle\cdot,\cdot\rangle_E$ denotes the dualization between E and E'. Then clearly $u_k \in D(L)$,

$$Lu_k = \frac{1}{2}\beta_k^\mu + \langle\overline{\beta}, k\rangle_H, \quad k \in K, \tag{26}$$

and

$$\langle M^{[u_k]}, M^{[u_{k'}]}\rangle_t = t\langle k, k'\rangle_H, \quad k, k' \in K. \tag{27}$$

Choosing an ONB $K_0 \subset K$ of H which separates the points of E by Theorem 4.5 applied to u_k, $k \in K_0$ we get a countable system of 1-dimensional SDE's with independent 1-dimensional Brownian motions according to (27) and drifts given according to (26). If we assume

(A.3) For one (and hence all) $t > 0$ there exists a probability measure μ_t on $(E, \mathcal{B}(E))$, such that

$$\int e^{i_{E'}\langle k,z\rangle_E}\mu_t(dz) = e^{\frac{1}{2}t\|k\|_H^2} \text{ for all } k \in E'$$

similar to [AR 91, Theorem 6.6] it is then possible to lift the countable system of 1-dimensional equations to a single equation on E, namely we have

Theorem 6.1 *There exist maps $W, N^0 : \Omega \longrightarrow C([0,\infty), E)$ with the following properties:*
(i) $\omega \mapsto W_t(\omega) := W(\omega)(t)$ and $\omega \mapsto N_t^0(\omega) := N^0(\omega)(t)$ are both $\mathcal{F}_t/\mathcal{B}(E)$ measurable for $t \geq 0$.
(ii) There exists an $\mathcal{E}^0$-exeptional set $S \subset E$ such that under each P_z, $z \in E\backslash S$, $W = (W_t)_{t\geq 0}$ is an E-valued $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion starting at $0 \in E$ with covariance $\langle\cdot,\cdot\rangle_H$ (i.e. under each P_z, $z \in E\backslash S$, for all $0 \leq s < t$ $W_t - W_s$ is independent of $\mathcal{F}_s$ and $\langle k, W_t - W_s\rangle_E$ is mean zero Gaussian with variance $(t-s)\|k\|_H^2$).
(iii) For each $k \in K$, $t \geq 0$ and $\mathcal{E}^0$-q.e. $z \in E$ we have P_z-a.s.

$$_{E'}\langle k, W_t\rangle_E = M_t^{[u_k]} \text{ and } _{E'}\langle k, N_t^0\rangle_E = \frac{1}{2}\int_0^t \beta_k^\mu(X_s)ds.$$

(iv) For $\mathcal{E}^0$-q.e. $z \in E$ we have P_z-a.s.

$$X_t = z + W_t + N_t^0 + \int_0^t \overline{\beta}(X_s)ds \tag{28}$$

where the last integral is in the sense of Bochner (cf. the following Remark 6.2) and where for $k \in K$ $\langle k, \int_0^t \overline{\beta}(X_s)ds\rangle_H = \int_0^t \langle\overline{\beta}(X_s), k\rangle_H\, ds$.

Remark 6.2 *(i) The assumption that the Gaussian measures satisfying (A.3) exist is of course, necessary. It just means that there exists (cf. [Ku 75, p.74]) a Brownian semigroup on E with covariance $\langle\cdot,\cdot\rangle_H$, i.e., there is a Brownian motion on E over H. Hence (A.3) is the best result one could hope for.*
(ii) In the above general situation there is no garantee that $k \mapsto \beta_k^\mu(z)$, $k \in K$, is represented by an element in E for μ-a.e. $z \in E$. But if we assume

(A.4) *There exists a $\mathcal{B}(E)/\mathcal{B}(E)$-measurable map $\beta_H^\mu : E \to E$ such that*
(i) $_{E'}\langle k, \beta_H^\mu\rangle_E = \beta_k^\mu$ μ-a.s. for each $k \in K$,
(ii) $\|\beta_H^\mu\|_E \in L^1(E;\mu)$

then we may define the process N^0 in Theorem 6.1 as a Bochner integral. In fact, by 2) of the proof of Theorem 4.5 we see that $\|\beta_H^\mu\|_E \in L^1(E;\mu) \cap \mathcal{B}(E)$ implies that the AF $\int_0^\cdot \|\beta_H^\mu\|_E(X_s)\, ds$ *is finite. We then see by [Yo 78, Theorem 1, p.133, Corollary 2, p.134] that $N_t^0 := \frac{1}{2}\int_0^t \beta_H^\mu(X_s)\, ds$, $t \geq 0$ (where the integral is in the sense of Bochner P_z-a.s for $\mathcal{E}^0$-q.e. $z \in E$) has the desired properties.*
(iii) It is easy to see that (A.4) is equivalent to the following assumption:

(A.4') *There exists a $\mathcal{B}(E)/\mathcal{B}(E)$-measurable map $B : E \to E$ such that*
(i) $\frac{1}{2}{}_{E'}\langle k, B\rangle_E = Lu_k$ *μ-a.s. for each $k \in K$,*
(ii) $\|B\|_E \in L^1(E;\mu)$.

Analogous to (ii) we may then replace $N_t^0 + \int_0^t \overline{\beta}(X_s)ds$ *in* (28) *by the Bochner integral* $\frac{1}{2}\int_0^t B(X_s)ds$.

6.2 Applications

In this subsection we assume that E is a separable real Hilbert space with inner product $\|\cdot\|_E := \langle\cdot,\cdot\rangle_E^{\frac{1}{2}}$ and that $H \subset E$ densely by a Hilbert-Schmidt map. Then there exists a nonnegative definite injective self-adjoint Hilbert-Schmidt operator T on E such that $H = T(E)$ and $\|\cdot\|_H = \|T^{-1}\cdot\|_E$. Analogous to [Ku 75, *Theorem* 4.4 *Step* 3.] we see that $\|\cdot\|_E$ is measurable over H, hence *(A.3)* holds. Let $B : E \to E$ be a Borel measurable vector field satisfying the following conditions:

(B.1) $\lim_{\|z\|_E\to\infty}\langle B(z), z\rangle_E = -\infty$,
(B.2) ${}_{E'}\langle l, B\rangle_E : E \to \mathbb{R}$ is weakly continuous for all $l \in E'$.
(B.3) There exist C_1, C_2, $d \in (0,\infty)$, such that $\|B(z)\|_E \le C_1 + C_2\|z\|_E^d$ for all $z \in E$.

Then by [BR 95, *Theorem* 5.2.] there exists a probability measure μ on $(E, \mathcal{B}(E))$ such that ${}_{E'}\langle l, B\rangle_E \in L^2(E;\mu)$ for all $l \in E'$ and such that

$$\int \frac{1}{2}\Delta_H u + \frac{1}{2}{}_{E'}\langle \nabla u, B\rangle_E \, d\mu \;=\; 0 \;\text{ for all }\; u \in \mathcal{FC}_b^\infty \tag{29}$$

where Δ_H is the Gross-Laplacian, i.e., $\Delta_H u = \sum_{i,j=1}^m \frac{\partial f}{\partial x_i \partial x_j}(l_1(z),...,l_m(z))\langle l_i, l_j\rangle_H$ if $u = f(l_1,...,l_m) \in \mathcal{FC}_b^\infty$. Assume that $B(z) = -z + v(z)$, $v : E \to H$. Because of *(B.1)*, *(B.3)* it follows by [BR 95, *Lemma* 5.1.] that $v \in L^2(E,H;\mu)$. In particular we have $\|z\|_E \in L^p(E,\mu)$ for all $p \ge 1$. Let γ be a Gaussian measure on E with covariance $\langle\cdot,\cdot\rangle_H$. By [BR 95, *Theorem* 3.5.] $d\mu = \varphi^2\, d\gamma$ where φ is in the Sobolev space $H^{1,2}(E;\gamma)$. Furthermore the logarithmic derivative β_H^μ of μ associated with H exists and admits the representation $\beta_H^\mu(z) = -z + \frac{2\nabla\varphi}{\varphi}(z)$. Note that possibly $supp(\mu) \not\equiv E$. Nevertheless, since every $k \in E'$ is well-μ-admissible (thus in particular *(A.1)* holds)

$$\mathcal{E}^0(u,v) := \frac{1}{2}\int \langle \nabla u, \nabla v\rangle_H \, d\mu, \quad u, v \in \mathcal{FC}_b^\infty,$$

is well-defined and closable on $L^2(E;\mu)$ and the closure is a symmetric quasi-regular Dirichlet form. Let $(L^0, D(L^0))$ be the associated generator. It is easy to see that $\mathcal{FC}_b^\infty \subset D(L^0)$ and

$$L^0 u \;=\; \frac{1}{2}\Delta_H u + \frac{1}{2}{}_{E'}\langle \nabla u, \beta_H^\mu\rangle_E, u \in \mathcal{FC}_b^\infty.$$

Set $\overline{\beta} := \frac{1}{2}(B - \beta_H^\mu)$. Clearly $\overline{\beta} \in L^2(E,H;\mu)$ and by (29) since $\int L^0 u \, d\mu = 0$, $u \in \mathcal{FC}_b^\infty$

$$\int \langle \overline{\beta}, \nabla u\rangle_H \, d\mu \;=\; 0 \text{ for all } \; u \in \mathcal{FC}_b^\infty. \tag{30}$$

As in 6.1 we then construct a conservative diffusion $\mathbf{M} = (\Omega, \mathcal{F}_\infty, (X_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ associated to the part on $L^2(E;\mu)$ (which we denote by $(L, D(L))$) of the closure on $L^1(E,\mu)$ of $L^0u + \langle\overline{\beta}, \nabla u\rangle_H$, $u \in D(L^0)_b$. Note that $Lu = \frac{1}{2}\Delta_H u + \frac{1}{2}{}_{E'}\langle \nabla u, B\rangle_E$, $u \in \mathcal{FC}_b^\infty$. Surely *(A.2)* is satisfied and clearly $Lu_k = \frac{1}{2}{}_{E'}\langle k, B\rangle_E$ hence *(A.4')* holds. By Theorem 6.1 and Remark 6.2(ii) we then have P_z-a.s. for $\mathcal{E}^o$-q.e. $z \in E$ (thus in particular P_μ-a.s.)

$$X_t \;=\; z + W_t - \frac{1}{2}\int_0^t X_s ds + \frac{1}{2}\int_0^t v(X_s)ds \tag{31}$$

where $(W_t)_{t\geq 0}$ is an E-valued $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion starting at $0 \in E$ with covariance $\langle\cdot,\cdot\rangle_H$ and where $\frac{1}{2}\int_0^t X_s ds$, $\frac{1}{2}\int_0^t v(X_s)ds$ are in the sense of Bochner P_z-a.s. for $\mathcal{E}^o$-q.e. $z \in E$. Note that $\int\langle v - 2\frac{\nabla\varphi}{\varphi}, \nabla u\rangle_H \, d\mu = 0$ for all u in $\mathcal{FC}_b^\infty$.
Let $(L', D(L'))$ denote the adjoint operator of $(L, D(L))$ on $L^2(E;\mu)$. Clearly, since $L'u_k = {}_{E'}\langle k, \frac{1}{2}id_E + \frac{2\nabla\varphi}{\varphi} - \frac{1}{2}v\rangle_E$ and $\|id_E\|_E$, $\|\frac{2\nabla\varphi}{\varphi}\|_E$, $\|v\|_E \in L^2(E;\mu)$ the coprocess $\widehat{\mathbf{M}} = (\widehat{\Omega}, \widehat{\mathcal{F}}_\infty, (\widehat{X}_t)_{t\geq 0}, (\widehat{P}_z)_{z\in E_\Delta})$ associated to $(L', D(L'))$ weakly solves

$$\widehat{X}_t \;=\; z + \widehat{W}_t - \frac{1}{2}\int_0^t \widehat{X}_s ds + \int_0^t 2\frac{\nabla\varphi}{\varphi}(\widehat{X}_s)ds - \frac{1}{2}\int_0^t v(\widehat{X}_s)ds$$

for $\mathcal{E}^o$-q.e. $z \in E$ where $(\widehat{W}_t)_{t\geq 0}$ is an E-valued $(\widehat{\mathcal{F}}_t)_{t\geq 0}$-Brownian motion starting at $0 \in E$ with covariance $\langle\cdot,\cdot\rangle_H$.

6.3 An Itô-type formula

Let $(\overline{G}_\alpha)_{\alpha>0}$ be the resolvent associated to $(\overline{L}, D(\overline{L}))$. Since $\overline{G}_{\alpha|L^2} = G_\alpha$, $\alpha > 0$, it follows that $D(L)_b \subset D(\overline{L})_b$ is dense w.r.t. the L^1-graph norm. Let $u_n := n\overline{G}_n u$, $u \in D(\overline{L})_b$. Then by [St1, *Lemma* 4.4.(*iii*)] we have that $e_{u-u_n} + e_{u_n-u} \underset{n\to\infty}{\longrightarrow} 0$ in $D(\mathcal{E}^0)$. Furthermore $\overline{e}(A^{[u-u_n]}) = (-\overline{L}(u-u_n), u-u_n) \underset{n\to\infty}{\longrightarrow} 0$. Now by Remark 4.6(iii) Theorem 4.5 extends to $u \in D(\overline{L})_b$. Similar to the finite dimensional case [St1, *Remark* 1.7.(*iii*)] one can show that $D(\overline{L})_b$ is an algebra. Therefore with the same notations as in the proof of Theorem 5.5 and Lemma 4.1 we conclude that

$$P_\mu(\sup_{0\leq t\leq T} |\,(\Phi - p_j)(\tilde{f})(X_t)\,| > \varepsilon) \leq \frac{e^T}{\varepsilon}\|h\|_{\mathcal{H}} n \sum_{i=1}^n \left\| \frac{\partial((\Phi - p_j)\chi)}{\partial x_i} \right\|_\infty \|e_{f_i} + e_{-f_i}\|_{\mathcal{H}}$$

and that

$$\overline{e}(A^{[(\Phi-p_j)(f)]}) \leq n\sum_{i=1}^n \left\| \frac{\partial((\Phi - p_j)\chi)}{\partial x_i} \right\|_\infty^2 (-\overline{L}f_i, f_i)$$

where $f_i \in D(\overline{L})_b$, $1 \leq i \leq n$, and then similar to Remark 4.6(iii) Theorem 4.5 applies. As a consequence Theorem 5.6 also applies.

ACKNOWLEDGEMENT. I would like to thank M.Röckner for suggesting me to further develop the theory of generalized Dirichlet forms in the way implemented in this paper. I would also like to thank W.Stannat for valuable discussions.

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