When is a cellular algebra quasi–hereditary ?

by

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1 Introduction

To a large extent, algebraic representation theory of Lie algebras, algebraic groups and related finite groups deals with finite dimensional algebras which are cellular or quasi–hereditary. Group algebras of symmetric groups and their Hecke algebras are known to be cellular as well as various generalizations (e.g. Brauer algebras, cyclotomic Hecke algebras, Temperley–Lieb algebras, partition algebras). Several of these algebras also have been used in other contexts like topology (invariants of knots or manifolds) or statistical mechanics. Schur algebras associated with semisimple algebraic groups in any characteristic and blocks of the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ associated with semisimple complex Lie algebras are cellular as well, but they also satisfy the stronger condition to be quasi–hereditary. A quasi–hereditary structure comes both with desirable numerical properties (decomposition matrices are square matrices, the number of simple modules can be read off from a defining chain of ideals) and with homological structure (finite global dimension, vanishing results on certain cohomology groups, stratification of derived module categories, existence of 'tilting modules' and derived equivalences, possibility to define 'Kazhdan–Lusztig' theory), and also there is a categorical definition (which cannot exist for cellular algebras, see [8]). Many cellular algebras, in particular Brauer algebras and partition algebras, are known to be quasi–hereditary for some choice of parameters and not quasi–hereditary for some other choice (typically 'at zero').

The subject of this note are the following problems arising from this situation.

Problem 1. How to characterize among the cellular algebras the quasi–hereditary ones by a structural property?

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Problem 2. How to characterize among the cellular algebras the quasi–hereditary ones by a numerical property?

Problem 3. Given a cellular algebra with a cell chain of ideals, how to decide whether it is quasi–hereditary?

Problem 4. Determine precisely for which choice of parameters a Brauer algebra or a partition algebra is quasi–hereditary.

Our answer to these problems is contained in the following theorems. More precisely, problem 1 is answered by the equivalence of (a) and (c) in 1.1, problem 2 is answered by the equivalence of (a) and (d), problem 3 is answered by the equivalence of non–(a) and non–(b). Problem 4 is answered by theorems 1.3 and 1.4, which are applications of 1.1.

Theorem 1.1 *Let k be a field and A a cellular k–algebra (with respect to an involution i). Then the following are equivalent:*

(a) Some cell chain of A (with respect to some involution, possibly different from i) is a heredity chain as well, i.e. it makes A into a quasi–hereditary algebra.

(a') There is a cell chain of A (with respect to some involution, possibly different from i) whose length equals the number of isomorphism classes of simple A–modules.

(b) Any cell chain of A (with respect to any involution) is a heredity chain.

(c) A has finite global dimension.

(d) The Cartan matrix of A has determinant one.

Given a k–algebra A, it may be difficult to show that A is not cellular since the definition involves the choice of a basis. Our proof of theorem 1.1 provides us with an invariant of cellular algebras as follows:

Proposition 1.2 *Let k be a field and A a cellular k–algebra. Then the determinant of the Cartan matrix of A is a positive integer.*

We note that our result is in accordance with the socalled 'Cartan determinant conjecture' which states that a finite dimensional algebra of finite global dimension should have Cartan determinant one (not just plus or minus one, as Eilenberg had shown).

Applying the equivalence of (a) and (b) in the theorem, we can solve problem 4, namely we give a precise description for which parameters the Brauer algebras and partition algebras are quasi–hereditary.

Theorem 1.3 *Let k be any field, fix $\delta \in k$ and denote by $B(r,\delta)$ the Brauer algebra on $2r$ vertices and with parameter δ.*

Then $B(r,\delta)$ is quasi–hereditary if and only if

(1) δ is not zero or r is odd; and

(2) the characteristic of k is either zero or strictly bigger than r.

This extends previous results by Graham and Lehrer [6].

For partition algebras we have

Theorem 1.4 *Let k be any field, fix $\delta \in k$ and denote by $P(r,\delta)$ the partition algebra on $2r$ vertices and with parameter δ.*

Then $P(r,\delta)$ is quasi–hereditary if and only if δ is not zero and the characteristic of k is either zero or strictly bigger than r.

Martin [10] had shown this in case of characteristic zero and $\delta \neq 0$. In [14], the "if part" of 1.4 is proved.

In section two we recall the definitions of cellular and quasi–hereditary algebras and then we prove theorem 1.1.

In section three we deal with applications. In particular we recall the relevant definitions and prove theorems 1.3 and 1.4 and a few other results on related classes of algebras like Temperley–Lieb algebras and Jones' annular algebras.

2 The criterion

First we recall the two equivalent definitions of cellular and the definition of quasi–hereditary. For simplicity we stick to the ground ring being an (arbitrary) field k. By algebra we always mean a finite dimensional associative algebra with unit.

Definition 2.1 *(Graham and Lehrer, [6]) An associative k–algebra A is called a* **cellular algebra** *with cell datum (I,M,C,i) if the following conditions are satisfied:*

(C1) The finite set I is partially ordered. Associated with each $\lambda \in I$ there is a finite set $M(\lambda)$. The algebra A has a k–basis $C^\lambda_{S,T}$ where (S,T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in I$.

(C2) The map i is a k–linear anti–automorphism of A with $i^2 = id$ which sends $C^\lambda_{S,T}$ to $C^\lambda_{T,S}$.

(C3) For each $\lambda \in I$ and $S,T \in M(\lambda)$ and each $a \in A$ the product $aC^\lambda_{S,T}$ can be written as $(\sum_{U\in M(\lambda)} r_a(U,S)C^\lambda_{U,T}) + r'$ where r' is a linear combination of basis elements with upper index μ strictly smaller than λ, and where the coefficients $r_a(U,S) \in k$ do not depend on T.

In the following we shall call a k–linear anti–automorphism i of A with $i^2 = id$ an involution of A. In [7] it has been shown that this definition is equivalent to the following one.

Definition 2.2 *Let A be a k–algebra. Assume there is an antiautomorphism i on A with $i^2 = id$. A two–sided ideal J in A is called a* **cell ideal** *if and only if $i(J) = J$ and there exists a left ideal $\Delta \subset J$ such that Δ has finite k–dimension and that there is an isomorphism of A–bimodules $\alpha : J \simeq \Delta \otimes_k i(\Delta)$ (where $i(\Delta) \subset J$ is the i–image of Δ) making the following diagram commutative:*

$$\begin{tikzcd} J \arrow[r, "\alpha"] \arrow[d, "i"'] & \Delta \otimes_k i(\Delta) \arrow[d, "x \otimes y \mapsto i(y) \otimes i(x)"] \\ J \arrow[r, "\alpha"] & \Delta \otimes_k i(\Delta) \end{tikzcd}$$

The algebra A (with the involution i) is called **cellular** *if and only if there is a vector space decomposition $A = J_1' \oplus J_2' \oplus \ldots \oplus J_n'$ (for some n) with $i(J_j') = J_j'$ for each j and such that setting $J_j = \oplus_{l=1}^{j} J_l'$ gives a chain of two sided ideals of A: $0 = J_0 \subset J_1 \subset J_2 \subset \ldots \subset J_n = A$ (each of them fixed by i) and for each j ($j = 1, \ldots, n$) the quotient $J_j' = J_j / J_{j-1}$ is a cell ideal (with respect to the involution induced by i on the quotient) of A/J_{j-1}.*

The modules $\Delta(j), 1 \leq j \leq n$, are called standard modules of the cellular algebra A, and the above chain in A is called a cell chain. (Standard modules are called cell modules in [6]).

Let us also recall the definition of quasi-hereditary algebras introduced in [3].

Definition 2.3 *(Cline, Parshall and Scott [3]) Let A be a k-algebra. An ideal J in A is called a* **heredity ideal** *if J is idempotent, $J(rad(A))J = 0$ and J is a projective left (or, right) A-module. The algebra A is called* **quasi-hereditary** *provided there is a finite chain $0 = J_0 \subset J_1 \subset J_2 \subset \ldots \subset J_n = A$ of ideals in A such that J_j/J_{j-1} is a heredity ideal in A/J_{j-1} for all j. Such a chain is then called a heredity chain of the quasi-hereditary algebra A.*

We also need the notion of a **Cartan matrix** in the following abstract sense (which coincides with the one used in group theory if A is the group algebra of a finite group over a splitting field). Denote the simple A–modules by $L(1), \ldots, L(m)$ and their projective covers by $P(1), \ldots, P(m)$. The entries $c_{j,h}$ of the Cartan matrix $C(A)$ are the composition multiplicities $[P(j) : L(h)]$. The determinant of $C(A)$ is called the Cartan determinant. In general this can be any integer.

Proof of theorem 1.1.

First we collect a number of auxiliary statements for later use.

For the first three assertions we fix a cellular algebra A with involution i and cell chain $0 = J_0 \subset J_1 \subset \ldots \subset J_n = A$. We note that these assertions are already implicit in [6].

(1) There is a natural bijection between isomorphism classes of simple A–modules and indices $l \in \{1, \ldots, n\}$ such that $J_l^2 \not\subset J_{l-1}$. The inverse of this bijection is given by sending such an l to $\Delta(l)/rad(\Delta(l))$ (which in this case is simple). This is proposition 4.1 in [7]. In the following we index the simple modules in this way by a subset of the set $\{1, \ldots, n\}$.

(2) If l is the index of a simple module $L(l)$ as in (1), then the composition factors $L(j)$ of the standard module $\Delta(l)$ satisfy $j \geq l$ and $j = l$ occurs with multiplicity one (and this factor is the unique simple quotient $\Delta(l)/rad(\Delta(l))$. This follows again from proposition 4.1 of [7].

(3) The given cell chain of A is a heredity chain (making A into a quasi–hereditary algebra) if and only if all J_l satisfy $J_l^2 \not\subset J_{l-1}$ if and only if n equals the number of isomorphism classes of simple modules. This is corollary 4.2 in [7]. Note that we will have to prove the stronger statement that if one given cell chain is not a heredity chain then there is no heredity chain at all.

We note that this implies the equivalence between (a) and (a') in the theorem.

(4) A quasi–hereditary algebra has finite global dimension (Parshall and Scott [12]), which actually is bounded by $2n - 2$ where n is the length of the heredity chain (Dlab and Ringel [4]).

(5) A quasi–hereditary algebra has Cartan determinant one. This has been shown by Burgess and Fuller [2]. It also follows from (a slight modification of) the computation of the Cartan determinant of a cellular algebra which we give below.

(6) If an algebra A has finite global dimension, then its Cartan determinant has absolute value one. This has been observed by Eilenberg [5]. It also follows directly from the observation that the Grothendieck group of A–mod is free abelian of finite rank, both the simple modules and the projective modules form a basis, and the Cartan matrix transforms one of these bases into the other.

We shall also need the following property of symmetric matrices with real entries. The identity matrix is denoted by I. For a matrix X, we denote by X^{tr} the transposed matrix of X.

(7) Let X be a positive definite matrix, Y a positive semidefinite matrix and Z a 'square root' of X, that is, $Z^2 = X$ and $Z = Z^{tr}$ and Z is positive definite (this always exists, an easy exercise in linear algebra). Then the matrix $U = Z^{-1}YZ^{-1}$ is positive

semidefinite and has the same eigenvalues as the matrix $V = X^{-1}Y$. In fact, $\lambda X - Y$ equals (for any $\lambda \in k$) the product $Z(\lambda I - Z^{-1}YZ^{-1})Z$. Hence a vector x satisfies $(\lambda X - Y)x = 0$ if and only if the vector $u = Zx$ satisfies $(\lambda I - U)u = 0$. Moreover, an eigenvalue λ of V to the eigenvector v can be written as $\lambda = \frac{v^{tr}Yv}{v^{tr}Xv}$, hence it is a non–negative real number.

Now we are going to look at the Cartan matrix of a cellular algebra. We fix a base field k and a cellular algebra A with an involution i and a cell chain $0 = J_0 \subset J_1 \subset \ldots \subset J_n = A$. Each subquotient J_l/J_{l-1} in the chain has the form $\Delta(l) \otimes_k i(\Delta(l))$. Denote the number of isomorphism classes of simple A–modules by m. As in [6] we form the 'decomposition matrix' D of A (with respect to the given cell chain). It is an $n \times m$–matrix with integer entries $d_{a,b} = [\Delta(a) : L(b)]$, the composition multiplicity of $L(b)$ in $\Delta(a)$. By assertion (1) above, the indices of simple modules can be naturally identified with the indices of those ideals J_l satisfying $J_l^2 \not\subset J_{l-1}$. Using this we can rearrange the rows of D in such a way that it gets the form $\begin{pmatrix} D_1 \\ D_2 \end{pmatrix}$, where both D_1 and D_2 are integer matrices and D_2 (which corresponds to the subset of indices associated with simple modules) is a square matrix. Here we put $D_1 = 0$ (of some size, which does not matter) in case $n = m$, and this will not change the result of our computation. We note that D equals D_2 if and only if n equals m. By assertion (3) above, this happens if and only if the given cell chain is a heredity chain.

By assertion (2) above, D_2 actually is lower triangular with all diagonal entries equal to one.

The Cartan matrix C of A satisfies $C = D^{tr}D$ (see [6]). In fact, the composition multiplicity $[\Delta(j) : L(h)]$ equals $dim_k(e(h)\Delta(j))$ if $e(h)$ is a primitive idempotent generating a projective cover $Ae(h)$ of $L(h)$. This projective module is filtered by $J_0e(h) \subset J_1e(h) \subset \ldots \subset J_ne(h) = Ae(h)$. The subquotients in this filtration are of the form $\Delta(l) \otimes_k i(\Delta(l))e(h)$. Hence $\Delta(j)$ occurs in this filtration with multplicity $dim_k(i(\Delta(j))e(h))$ which equals $dim_k(e(h)\Delta(j))$, which is the desired claim.

Next we are going to compute the determinant of C, which by definition of C must be an integer. We have $C = (D_1^{tr}, D_2^{tr})\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = (D_1^{tr}D_1 + D_2^{tr}D_2)$. Denote by C_1 the product $D_2^{tr}D_2$. The unitriangularity of D_2 gives $det(C_1) = 1$. Hence $det(C) = det(C_1^{-1}C) = det(I + C_1^{-1}D_1^{tr}D_1)$ where I denotes the identity matrix. Clearly C_1 is positive definite and $D_1^{tr}D_1$ is positive semidefinite. Of course, $D_1^{tr}D_1$ is zero if and only if $n = m$.

The positive definite symmetric matrix C_1 can be written as C_2^2 for some symmetric C_2. The above statement (7) implies that the matrix $C_3 = C_2^{-1}D_1^{tr}D_1C_2^{-1}$ has the same eigenvalues as the matrix $C_1^{-1}D_1^{tr}D_1$. By construction, C_3 is symmetric and its eigenvalues are non–negative real numbers. Moreover, all eigenvalues are zero precisely in the case $n = m$. Going back to the matrix $I + C_1^{-1}D_1^{tr}D_1$ we conclude that all its eigenvalues are of the form $1+\lambda$, where λ are eigenvalues of the matrix $C_1^{-1}D_1^{tr}D_1$ and therefore non–negative real numbers, and all λ equal zero if and only $n = m$. We note that this implies statement (5) above (under the assumption that the algebra is cellular – for quasi–hereditary algebras without an involution i one has to slightly modify the argument).

Altogether we have shown: $det(C)$ is a positive integer and it is equal to one if and only if for the given cell chain we have $n = m$. But of course, $det(C)$ does not depend on the choice of the cell chain. Hence we have shown: if the length n of some cell chain satisfies $n = m$, then the same must be true for any cell chain. This proves the equivalence of (a'), (b) and (d).

Assertion (4) shows now that (b) implies (c).

Assertion (6) finally tells us that we can conclude (d) from (c). ■

We remark that the computation of $det(C)$ goes through in a more general situation:

Proposition 2.1 *Let A be a finite dimensional k–algebra with a chain $0 = J_0 \subset J_1 \subset \ldots \subset J_n = A$ of two–sided ideals whose subquotients are of the form $\Delta(l)\otimes\nabla(l)$ such that for all primitive idempotents $e \in A$ there is an isomorphism of vector spaces $e\Delta(l) \simeq \nabla(l)e$. Then the Cartan determinant of A is a positive integer.*

3 Applications

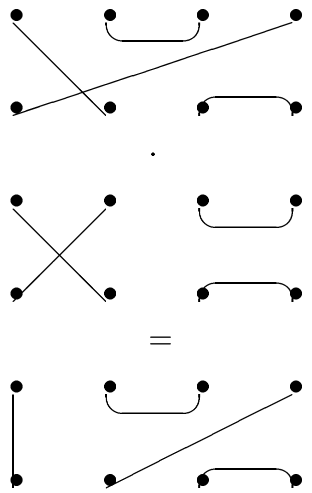
First we discuss **Brauer algebras**. They have been introduced by Brauer [1] in order to extend classical Schur–Weyl duality to semisimple algebraic groups of types B and C. More precisely, Schur–Weyl duality means the following double centraliser property: Fix a field k and two natural numbers n and r. Then the general linear group $GL_n(k)$ acts (diagonally) from the left on the vector space $(k^n)^{\otimes r}$ whereas the symmetric group Σ_r acts from the right by permuting places of tensors. These two actions centralise each other. Replacing in this setup the algebraic group $GL_n(k)$ by a subgroup of type B or C, one has to replace the symmetric group by the Brauer algebra if one wants to keep the double centraliser property.

For the definition of the Brauer algebra, we fix a field k, an indeterminate x, and a natural number r. Then $B(r, x)$ has a basis consisting of all diagrams, which consist of $2r$ vertices, divided into 2 ordered sets, the r top vertices and the r bottom vertices, and r edges such that each edge belongs to exactly 2 vertices and each vertex belongs to exactly one edge. Multiplication of basis elements is defined by concatenating diagrams: Assume we are given two basis elements, say a and b. First, draw an edge from bottom vertex i of a to top vertex i of b (for each $i = 1, \ldots, r$). This produces a diagram which is almost of the desired form except that there may be cycles not attached to any of the (new) top and bottom vertices. Denote the number of these cycles by d. Then delete all cycles; the result is a basis element, say c. Now the product ab is by definition $x^d c$.

Of course, for a field element, say δ, the Brauer algebra $B(r, \delta)$ is defined by using δ instead of x, that is, by forming the quotient of $B(r, x)$ modulo $x - \delta$. (For Brauer's application to orthogonal or symplectic groups one has to choose δ to be an integer.)

Let us illustrate this definition by an example. We multiply two elements in $B(4, \delta)$:

$\delta^1 \cdot$



Proof of theorem 1.3.

It has been shown in [6] (and later on in a different way also in [9]) that Brauer algebras are cellular for any choice of r and δ. In the following we are going to use the notation of [8], where cell structures are written as 'inflations', that is, by data of linear algebra. The cell chain exhibited both in [6] and in [9] shows that as a free k–module, A is equal to

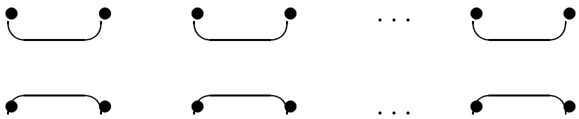
$$k\Sigma_r \oplus (V_{r-2} \otimes V_{r-2} \otimes k\Sigma_{r-2}) \oplus (V_{r-4} \otimes V_{r-4} \otimes k\Sigma_{r-4}) \oplus \dots,$$

and the iterated inflation starts with $k\Sigma_r$, inflates it along $V_{r-2} \otimes V_{r-2} \otimes k\Sigma_{r-2}$ and so on, ending with an inflation of $k = k\Sigma_1$ or $k = k\Sigma_0$ as bottom layer (depending on whether r is odd or even).

Now we have to distinguish several cases.

If r is even and δ equals zero, then the above cell chain contains a nilpotent ideal, namely the layer $V_0 \otimes V_0 \otimes k$, whose elements are the scalar multiples of the diagram d

$d :=$



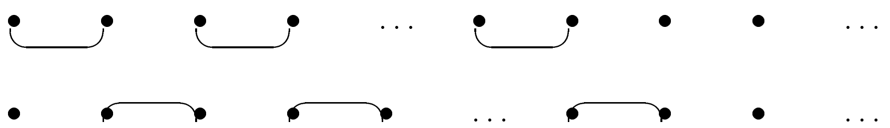
whose square is zero. Hence this chain is not a heredity chain. By theorem 1.1, the algebra cannot be quasi–hereditary.

For any choice of parameters, the Brauer algebra has a quotient $k\Sigma_r$, the group algebra of the symmetric group on r letters, and this quotient actually arises as $B(r,\delta)/J$ for some ideal in the cell chain. In particular, factoring out J of the cell chain of $B(r,\delta)$ yields a cell chain of $k\Sigma_r$. If the characteristic of k divides the order of Σ_r, that is, if $2 \leq char(k) \leq r$, the group algebra $k\Sigma_r$ has infinite global dimension. By theorem 1.1, the length of the cell chain of $k\Sigma_r$ and hence also that of $B(r,\delta)$ must be strictly bigger than the number of simple modules. Therefore, again by theorem 1.1, the algebra $B(r,\delta)$ cannot be quasi–hereditary.

If the characteristic of k is zero or bigger than r, then all the above group algebras $k\Sigma_{r-2l}$ are semisimple and the subquotients in the cell chain of $B(r,\delta)$ are inflations of the simple components of these algebras. The multiplication in such a subquotient is non–zero if and only if there is an idempotent in this layer (see e.g. proposition 4.1 of [7]). Hence the given cell chain is already a heredity chain (and $B(r,\delta)$ is quasi–hereditary), provided we can show that the multiplication in each layer is non–zero. To do this, we fix a layer in the filtration, say coming from $V_{r-2l} \otimes V_{r-2l} \otimes k\Sigma_{r-2l}$ for some l. Denote V_{r-2l} by V. By the construction in [6] or [9], the chosen layer has the form $V \otimes V \otimes J$, where the k–space V has a basis consisting of configurations formed by l connecting $2l$ out of $2r$ vertices. And J is some subquotient in a cell chain of $k\Sigma_{r-2l}$, hence by our present assumption, J is just a full matrix algebra over k. Multiplication in this layer (i.e. neglecting terms in lower layers of the cell chain) is of the form $(a \otimes b \otimes x)(c \otimes d \otimes y) = a \otimes d \otimes x\varphi(b,c)y$, where a, b, c, d are in V, x, y are in J and the bilinear form $\varphi : V \times V \to J$ assigns to a pair b, c of configurations a scalar multiple of a some element in J. If r is strictly bigger than $2l$, than we choose:

$b :=$

$c :=$



Then (see e.g. [9], proof of lemma 5.3) $\varphi(b,c)$ is a non–zero element in J and we can choose elements x and y in the simple algebra J such that the whole product $(a \otimes b \otimes x)(c \otimes d \otimes y) = a \otimes d \otimes x\varphi(b,c)y$ becomes non–zero. This gives us a non–zero structure constant in case $r > 2l$. In the remaining case $r = 2l$ we have to use the assumption that δ is not zero. Then choosing the above diagram d for both b and c we still get a non–zero structure constant. This finishes the proof. ∎

Now we turn to the partition algebras defined in [10]. These algebras are of interest in statistical mechanics [11]. Let us recall some definitions from [10].

Let M be a finite set. We denote by E_M the set of all equivalent relations on, or equivalently all partitions of the set M:

$$E_M := \{\rho = ((M_1)(M_2)\cdots(M_i)\cdots) \mid \emptyset \neq M_i \subset M, \cup_i M_i = M, \\ M_i \cap M_j = \emptyset\,(i \neq j)\}$$

For example, we take $M = \{1, 2, 3\}$, then

$$E_M = \{(123), (1)(23), (12)(3), (13)(2), (1)(2)(3)\}.$$

if $\mu \in E_M$ and $\nu \in E_N$, then we define $\mu \cdot \nu \in E_{M \cup N}$ to be the smallest ρ in $E_{M \cup N}$ such that ρ contains both μ and ν.

To definine the partition algebras, we put

$$M = \{1, 2, \cdots, n, 1', 2', \ldots, n'\}, M' = \{1', 2', \cdots, n', 1'', 2'', \ldots, n''\}.$$

Let $f : E_M \times E_M \longrightarrow \mathbf{Z}$ be such that $f(\mu, \nu)$ is the number of parts of $\mu \cdot \nu \in E_{M \cup M'}$ (note that $|M \cup M'| = 3n$) containing exclusively elements j' with a single prime.

For example, in case $n = 3$, $((123)(1'2')(3')) \cdot ((1')(2'3')(1'')(2'')(3'')) = ((123)(1'2'3')(1'')(2'')(3''))$ and $f(\mu, \nu) = 1$.

Let $C : E_M \times E_M \longrightarrow E_M$ be such that $C(\mu, \nu)$ is obtained by deleting all single primed elements of $\mu \cdot \nu$ (discarding the $f(\mu, \nu)$ empty brackets so produced), and replacing all double primed elements with single primed ones.

The partition algebra $P(n, q)$ is defined as follows.

Let k be a field, $q \in k$, and let n be a natural number. We define a product on E_M:

$$E_M \times E_M \longrightarrow E_M \qquad (\mu, \nu) \longmapsto \mu\nu = q^{f(\mu,\nu)} C(\mu, \nu)$$

This product is associative. Let $P(n, q)$ denote the vector space over k with the basis E_M. Then, by linearly extending the product on E_M, the vector space $P(n, q)$ becomes a finite dimensional algebra over k with the above product. $P(n, q)$ is called the **partition algebra**.

If we take $B_M = \{\rho \in E_M \mid \text{ each part of } \rho \text{ has exactly two elements of } M\}$ and define the product of two elements in B_M in the same way as in $P(n, q)$, then the subspace $B(n, q)$ of $P(n, q)$ with the basis B_M becomes a finite dimensional algebra. This is just the Brauer algebra $B(n, q)$. Similarly, if we take $P_M = \{\rho \in B_M \mid \rho \text{ is planar}\}$, then we get the Temperley–Lieb algebra $A_n(q)$ with the basis P_M.

The **proof of theorem 1.4** is quite similar to that of theorem 1.3. The partition algebras have been shown to be cellular in [14], and the proof given there contains all the ingredients for adapting the above proof.

Along similar lines (using again [6] or adapting [9]) one also can show the following extensions of results of Westbury [13] and of Graham and Lehrer [6].

Proposition 3.1 *Let $A_n(\delta)$ be a Temperley–Lieb algebra of type A. Then $A_n(\delta)$ is quasi–hereditary if and only if $\delta \neq 0$ or n odd.*

Proposition 3.2 *Let $J_n(\delta)$ be the Jones' annular algebra. Then $J_n(\delta)$ is quasi–hereditary if and only if $\delta \neq 0$.*

Finally we give an example of an application of proposition 1.2.

Proposition 5.3 of [7] states that a Brauer tree algebra is cellular if and only if the Brauer tree is a straight line. For example, associated with a straight line consisting of three edges there is an algebra A which as left module over itself looks as follows:

$$
{}_AA = \begin{array}{c} 1 \\ 2 \\ 1 \end{array} \oplus \begin{array}{ccc} & 2 & \\ 1 & & 3 \\ & 2 & \end{array} \oplus \begin{array}{c} 3 \\ 2 \\ 3 \end{array}
$$

Slightly changing the structure constants we get another algebra, say B, which looks as follows:

$$
{}_BB = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \oplus \begin{array}{ccc} & 2 & \\ 1 & & 3 \\ & 2 & \end{array} \oplus \begin{array}{c} 3 \\ 2 \\ 1 \end{array}
$$

Both algebras A and B are self–injective. But A is cellular as mentioned before, whereas B has Cartan determinant zero, hence by 1.2 it cannot be cellular.

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