Tightness of Sobolev capacities in infinite dimensions

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Abstract

We prove tightness of capacities generated by Sobolev classes in locally convex spaces with differentiable measures.

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1 Introduction

The problem of tightness of capacities generated by Sobolev classes of functions over smooth measures arises in measure theory and stochastic analysis. The question of tightness of the classical Sobolev capacities in $\mathbb{R}^n$ was answered positively (see, e.g., [5]), but in the infinite dimensional case specific difficulties appear. The problem under discussion has been considered in many works, in particular, in [2], [3], [1], [8]. The tightness of capacities generated by the classes $W^{p,r}$ is important, in particular, for the construction of the surface measure in infinite dimensional space, see [2], [6], [7], and [1].

In the present work, we consider the Sobolev classes generated by differentiable Radon measures on separable Fréchet spaces.

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2 Sobolev classes

Let (X, μ) be a locally convex space endowed with a Radon probability measure μ, and let H be a separable Hilbert space continuously and densely embedded in X. Here and further on we assume that the measure μ is Fomin differentiable along any vector $h \in H$ and that its logarithmic derivative β^μ (see [2, Definition 5.1.3]) satisfies the estimate

$$\|\beta_h^\mu\|_{L^2(\mu)} \leq const \cdot \|h\|_H \tag{1}$$

for each $h \in H$.

Let E be a separable Hilbert space. We shall denote by $\mathcal{H}(H, E)$ the class of all Hilbert–Schmidt operators from H to E; the Hilbert–Schmidt norm is specified by the formula

$$\|T\|^2_{\mathcal{H}(H,E)} = \sum_{m=1}^{\infty} \|Te_m\|^2_E,$$

where $\{e_m\}$ is an orthonormal basis of H.

The classes of polylinear Hilbert–Schmidt mappings are defined by induction: $\mathcal{H}_n(H, E) = \mathcal{H}(H, \mathcal{H}_{n-1}(H, E))$, $n = 2, 3, ...$, the corresponding norms are defined as follows:

$$\|T\|^2_{\mathcal{H}_n(H,E)} = \sum_{m_1=1}^{\infty} \dots \sum_{m_n=1}^{\infty} \|T(e_{m_1}, \dots e_{m_n})\|^2_E.$$

Definition 2.1. *(i) A μ-measurable mapping $F : X \longrightarrow E$ is said to be ray absolutely continuous (RAC) if for any $h \in H$, there exists a mapping $F_h : X \longrightarrow E$ such that $F = F_h$ μ-a.e. and for any $x \in X$ the mapping $t \longmapsto F_h(x + th)$ is absolutely continuous.*

(ii) A μ-measurable mapping $F : X \longrightarrow E$ is called stochastically Gâteaux differentiable (SGD) if there exists a μ-measurable mapping $D_H F : X \longrightarrow \mathcal{H}(H, E)$ such that for any $h \in H$ one has

$$\frac{F(x + th) - F(x)}{t} - D_H F(x)(h) \to 0$$

in measure μ as $t \to 0$.

The stochastic Gâteaux derivative of order $n > 1$ is defined inductively according to the formula: $D^n_H F = D_H(D^{n-1}_H F)$.

Definition 2.2. *Let $1 < p < \infty$. The Sobolev class $D^{p,1}(\mu, E)$ is defined as follows:*

$$D^{p,1}(\mu, E) = \{f \in L^p(\mu, E) : f \text{ is } RAC, SGD,\ D_H f \in L^p(\mu, \mathcal{H}(H, E))\}.$$

It is endowed with the norm

$$\|f\|_{p,1,E} = \|f\|_{L^p(\mu,E)} + \|D_H f\|_{L^p(\mu,\mathcal{H}(H,E))}$$

For $r = 2, 3, \dots$ we define $D^{p,r}(\mu, E)$ inductively as follows:

$$D^{p,r}(\mu, E) = \{f \in D^{p,r-1}(\mu, E) : \quad D_H f \in D^{p,r-1}(\mu, \mathcal{H}(H, E))\},$$

and the corresponding norm is

$$\|f\|_{p,r,E} = \|f\|_{L^p(\mu,E)} + \sum_{k=1}^{r} \|D_H^k f\|_{L^p(\mu,\mathcal{H}_k(H,E))} \tag{2}$$

Now we shall introduce another type of Sobolev classes. We shall call a function $f : X \to E$ smooth cylindrical if it has the form

$$f(x) = \sum_{k=1}^{n} u_k(l_{k1}(x), \dots l_{km}(x)) e_k,$$

where $n, m \in \mathbb{N}$, $u_k \in C_b^\infty(\mathbb{R}^m)$, $l_{kj} \in X^*$, and $\{e_k\}$ is an orthonormal basis of the space E. The class of such functions will be denoted by $\mathcal{FC}_b^\infty(X, E)$.

Definition 2.3. *A function $f \in L^p(\mu, E)$ belongs to the Sobolev class $W^{p,r}(\mu, E)$ if there exists a sequence of functions $f_n \in \mathcal{FC}_b^\infty(X, E)$ which converges to f in $L^p(\mu, E)$ and satisfies the Cauchy criterion with respect to the norm $\|\cdot\|_{p,r,E}$ defined by (2).*

The Sobolev derivative $D_H^k f$ of the function $f \in W^{p,r}(\mu, E)$ for $k = 1, \dots r$ is the $L^p(\mu, \mathcal{H}_k(H, E))$-limit of the functions $D_H^k f_n$. Such definition is correct provided the measure μ has full support.

It is known that for any Radon Gaussian measure γ the classes $W^{p,r}(\gamma, E)$ and $D^{p,r}(\gamma, E)$ coincide (see [2, §5.4]). In the general case we only have the inclusion $W^{p,r}(\mu, E) \subset D^{p,r}(\mu, E)$, herewith the Sobolev derivatives in $W^{p,r}$ coincide with the corresponding Gâteaux derivatives in $D^{p,r}$.

Everywhere below in the case of $E = \mathbb{R}$ we shall omit the index E in the notations introduced above.

Lemma 2.1. *If $p \le p'$, $r \le r'$, then $D^{p',r'}$ is continuously embedded into $D^{p,r}$ and $W^{p',r'}$ is continuously embedded in $W^{p,r}$; in both cases the norm of the embedding operator equals 1.*

Let us introduce an analogue of the Cameron–Martin space for an arbitrary differentiable measure on a locally convex space X. Let the space

$$H(\mu) = \big\{h \in X : \quad \beta_h^\mu \in L^2(\mu)\big\}.$$

be endowed with the norm $\|h\|_{H(\mu)} = \|\beta_h^\mu\|_{L^2(\mu)}$. Then it is a Hilbert space continuously embedded in X; if $L^2(\mu)$ is separable, then $H(\mu)$ is separable as well (see [3, Ch. 5], [4, §3]).

Now the condition (1) takes the form $H \subset H(\mu)$.

3 Capacities

Let $\mathcal{F}$ be some linear space of μ-measurable functions equipped with some norm $f \longmapsto \|f\|_0$. There exists a standard construction of a set function $C_{\mathcal{F}}$ associated with $\mathcal{F}$, which is defined on the class of all subsets of X and called the *capacity generated by* $\mathcal{F}$. We shall assume that functions in $\mathcal{F}$ coinciding μ-a.e. have equal norms.

Definition 3.1. *For any open set $U \subset X$, we put*

$$C_{\mathcal{F}}(U) = \inf\{\|f\|_0 : f \in \mathcal{F}, f \geq 0, \text{ and } f \geq 1 \text{ on } U \ \mu\text{-a.e.}\}.$$

For arbitrary subset A of the space X, we put

$$C_{\mathcal{F}}(A) = \inf\{C_{\mathcal{F}}(U) : A \subset U, \ U \text{ is open}\}.$$

The capacity $C_{\mathcal{F}}$ is subadditive, i.e., for any pair of sets A and B we have

$$C_{\mathcal{F}}(A \cup B) \leq C_{\mathcal{F}}(A) + C_{\mathcal{F}}(B).$$

This follows from the triangle inequality for the norm.

Remark 3.1. *It follows from Lemma 2.1 that for any set A the estimates $C_{D^{p,r}}(A) \leq C_{D^{p',r'}}(A)$ and $C_{W^{p,r}}(A) \leq C_{W^{p',r'}}(A)$ hold true if $p \leq p'$, $r \leq r'$.*

Definition 3.2. *Capacity $C_{\mathcal{F}}$ is said to be tight if for any $\varepsilon > 0$ there exists a compact $K_\varepsilon \subset X$ such that $C_{\mathcal{F}}(X \setminus K_\varepsilon) < \varepsilon$.*

4 Auxilliary results

Lemma 4.1. *Let $H \subset X$ be a separable Hilbert space such that $H \subset H(\lambda)$, where λ is a Radon probability measure. Let $f : X \longrightarrow E$ be a bounded mapping taking values in a separable Hilbert space E. Suppose that f is measurable with respect to any shift of the measure λ. Then the mapping*

$$F(x) = \int_X f(x+y)\lambda(dy) \tag{3}$$

is Gâteaux differentiable along H, and its derivative $D_H F(x)$ is a Hilbert–Schmidt operator with

$$\|D_H F(x)\|_{\mathcal{H}(H,E)} \leq c \cdot \|f(x+\cdot)\|_{L^2(\lambda,E)} \leq c \cdot \sup_y \|f(y)\|_E,$$

where c is the operator norm of the embedding $H \longrightarrow H(\lambda)$.

$\triangledown$ Proof. Let H_0 be the closure of H in $H(\lambda)$, endowed with the norm from $H(\lambda)$. The Hilbert space H_0 is separable, while $H(\lambda)$ may fail to be so. Since the embedding $H \longrightarrow H(\lambda)$ has finite norm c, we have

$$\|T\|_{\mathcal{H}(H,E)} \leq c\|T\|_{\mathcal{H}(H_0,E)}$$

for any Hilbert–Schmidt operator $T : H_0 \longrightarrow E$. Therefore, it suffices to prove our statement in the case of H and H_0 coinciding together with their norms. Existence of the Gâteaux derivative of F follows by [3, Example 2.1.15]. Let $\{e_n\}$ be an orthonormal basis in $H = H_0$. This implies that the sequence $\{\beta^\lambda_{e_n}\}$ is orthonormal in $L^2(\lambda)$. We have

$$\partial_{e_n} F(x) = -\int\limits_X f(x+y)\beta^\lambda_{e_n}(y)\lambda(dy).$$

Let $\{y_j\}$ be an orthonormal basis in E. Then $f(z) = \sum\limits_j f_j(z)y_j$. By the Bessel inequality we get

$$\sum_{n=1}^{\infty}\left(\int\limits_X f_j(x+y)\beta^\lambda_{e_n}(y)\lambda(dy)\right)^2 \leq \int\limits_X f_j(x+y)^2\lambda(dy).$$

Therefore

$$\sum_{n=1}^{\infty}\|\partial_{e_n}F(x)\|_E^2 = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\left(\int\limits_X f_j\left(x+y\right)\beta^\lambda_{e_n}\left(y\right)\lambda\left(dy\right)\right)^2 \leq$$

$$\leq \sum_{j=1}^{\infty}\int\limits_X f_j\left(x+y\right)^2\lambda(dy) = \int\limits_X \|f(x+y)\|_E^2\lambda(dy) =$$

$$= \|f(x+\cdot)\|^2_{L^2(\lambda,E)} \leq \sup_y \|f(y)\|_E^2.$$

The proof is complete. $\triangle$

Corollary 4.1. *Let μ be a probability measure on X and let the function $f : X \longrightarrow E$ be bounded and measurable with respect to the measure $\mu * \lambda$ and all shifts of the measure λ. Then for all $p > 1$ one has $F \in D^{p,1}(\mu, E)$. If $p \geq 2$, we have the estimates*

$$\|F\|_{L^p(\mu,E)} \leq \|f\|_{L^p(\mu*\lambda,E)},$$

$$\|D_H F\|_{L^p(\mu,\mathcal{H}(H,E))} \leq c\|f\|_{L^p(\mu*\lambda,E)}.$$

$\bigtriangledown$ Proof. Let us first show that the function F is μ-measurable. Let A be an arbitrary μ-measurable set. Put

$$\mu_F(A) = \int\limits_A \int f(x+y)\lambda(dy)\mu(dx) \equiv \int f(z)\big((\mu\,|_A) * \lambda\big)(dz).$$

This expression is well-defined since f is measurable with respect to the measure $(\mu\,|_A) * \lambda << \mu * \lambda$. The measure μ_F is absolutely continuous with respect to μ, and its Radon–Nikodym density coincides with F μ-a.e.

Let $p \geq 2$. Then we have

$$\int \|F(x)\|_E^p \mu(dx) \leq \int \|f(x+\cdot)\|^p_{L^1(\lambda,E)}\mu(dx) \leq$$

$$\leq \int \|f(x+\cdot)\|^p_{L^p(\lambda,E)}\mu(dx) = \|f\|^p_{L^p(\mu*\lambda,E)};$$

$$\int \|D_H F(x)\|^p_{\mathcal{H}(H,E)}\mu(dx) \leq c^p \cdot \int \|f(x+\cdot)\|^p_{L^2(\lambda,E)}\mu(dx) \leq$$

$$\leq c^p \cdot \int \|f(x+\cdot)\|^p_{L^p(\lambda,E)}\mu(dx) = \big(c\cdot \|f\|_{L^p(\mu*\lambda,E)}\big)^p.$$

In the case $p < 2$ we can apply Lemma 2.1. $\bigtriangleup$

5 Tightness of capacities generated by the Sobolev classes $D^{p,r}$

Tightness of Sobolev capacities over Gaussian measures is almost completely studied. Namely, there is the following result [2, Theorem 5.7.9]:

Theorem 5.1. *Let X be a locally convex space with a Radon Gaussian measure γ; the Hilbert space H be the Cameron–Martin space $H(\gamma)$ (or its subspace). Let $C_{p,r}$ be the capacity generated by the Sobolev class $D^{p,r}(\gamma) = W^{p,r}(\gamma)$, $r > 0$, $p > 1$. Then for any $\varepsilon > 0$ there exists a metrizable compact K_ε such that*

$$C_{p,r}(X \setminus K_\varepsilon) < \varepsilon.$$

Moreover,

$$C_{p,r}(B) = \sup\{C_{p,r}(K);\ K \subset B \textit{ is a metrizable compact }\}$$

for any Souslin set $B \subset X$.

Now let μ be a Radon probability measure on a complete locally convex space X. By μ_s we shall denote the probability measure specified by the formula

$$\mu_s(A) = \frac{1}{2}\big(\mu(A) + \mu(-A)\big).$$

If μ is Fomin differentiable along the vector h then so is μ_s, moreover, $|\beta_h^{\mu_s}(x)| \leq |\beta_h^{\mu}(x)| + |\beta_h^{\mu}(-x)|$. This implies that $H(\mu) \subset H(\mu_s)$.

Theorem 5.2. *Let X be a complete locally convex space with a Radon probability measure μ. Let H be a separable Hilbert space such that $H \subset H(\lambda)$, where λ is some symmetric Radon probability measure on X, e.g., μ_s or a centered Gaussian measure. Then for any $p \in [1; +\infty)$ the capacity $C_{D^{p,1}(\mu)}$ is tight.*

$\bigtriangledown$ Proof. Let $K_j,\ j \in \mathbb{N}$ be absolutely convex compact sets such that

$$(\mu + \lambda)(X \backslash K_j) < 10^{-j}.$$

Consider the function $\varphi_j(x) = \lambda(2K_j + x)$. For $x \in K_j$ we have $2K_j + x \supset K_j$, whence $\varphi_j(x) \geq \lambda(K_j) > \frac{9}{10}$. For $x \notin 2K_j$

$$\varphi_j(x) = \lambda(2K_j + x) = \lambda(-2K_j - x),$$

but the compact sets $2K_j + x$ and $-2K_j - x$ are disjoint: otherwise there exists y such that

$$y - x \in 2K_j, \quad -y - x \in 2K_j,$$

and by the convexity property we would get $-x \in 2K_j = -2K_j$. Therefore, $\varphi(x) \leq \frac{1}{2}$.

Now we shall estimate the norm of the gradient of the function φ_j. Note that $D_H\varphi_j(x) \equiv -D_H(1 - \varphi_j(x))$, and the function $1 - \varphi_j$ has the form (3) with $f(x) = I_{X\backslash 2K_j}(x)$. By Lemma 4.1 we get

$$\|D_H\varphi_j(x)\|_H \leq c \cdot \|I_{X\backslash(2K_j+x)}\|_{L^2(\lambda)},$$

where c is the operator norm of the embedding of H into $H(\lambda)$.

Take a decreasing function $\psi \in C^\infty(\mathbb{R})$ such that $\psi\,|_{[0;\frac{1}{2}]} = 1,\ \psi\,|_{[\frac{9}{10};1]} = 0$. Denote $g_j = \psi \circ \varphi_j$. Then $g_j \geq 0,\ g_j\,|_{X\backslash 2K_j} = 1$. ¿From $\sup|g_j| = 1,\ g_j\,|_{K_j} = 0$ we get

$$\|g_j\|_{L^p(\mu)} \leq \big(\mu(X\backslash K_j)\big)^{1/p} \to 0$$

as $j \to \infty$. In addition,

$$\int \|D_H g_j(x)\|_H^p \mu(dx) \leq (c \cdot \sup_{[0;1]} |\psi'|)^p \int \|I_{X\backslash(2K_j+x)}\|_{L^2(\lambda)}^p \mu(dx) \leq$$

$$\leq c_1(p)\Big[\int\limits_{X\backslash K_j}\mu(dx)+\int\limits_{K_j}\big(\lambda(X\backslash K_j)\big)^{p/2}\mu(dx)\Big]\leq$$

$$\leq c_1(p)\big(10^{-j}+10^{-jp/2}\big)\to 0.$$

Therefore, $C_{D^{p,1}}(X\backslash 2K_j)\leq \|g_j\|_{p,1}<\varepsilon$ for sufficiently large j. $\triangle$

Now we obtain certain result for the capacities $C_{D^{p,r}}$ with arbitrary $r\in\mathbb{N}$.

Theorem 5.3. *Let X be a complete locally convex space with a Radon probability measure μ. Let H be a separable Hilbert space, and let λ be a symmetric Radon probability measure on X such that $H\subset H(\lambda)$. Then for all $p,r\geq 1$ the capacities $C_{D^{p,r}(\mu)}$ are tight. In particular, this is true if $H\subset H(\mu)$.*

$\bigtriangledown$ Proof. Let c be the norm of the operator of embedding $H\longrightarrow H(\lambda)$. Let ψ be an increasing function from $C_b^\infty(\mathbb{R})$, such that $\psi\mid_{(-\infty;\frac{1}{10}]}=0,\ \ \psi\mid_{[\frac{1}{2};+\infty)}=1$. Fix $p,r\in\mathbb{N}$.

Let K be an absolutely convex compact set such that $\mu(X\backslash K)\leq\varepsilon,\ \lambda(X\backslash K)\leq\varepsilon$, where $\varepsilon\in(0;\frac{1}{10})$. Put $f^{*0}=I_{X\backslash(r+1)K}$. We shall construct the functions φ^{*k}, f^{*k} via the following recurrent formulas:

$$\varphi^{*k}(x)=\int f^{*(k-1)}(x+y)\lambda(dy),$$

$$f^{*k}(x)=\psi(\varphi^{*k}(x)),\quad k=1,\dots,r. \tag{4}$$

Then the functions f^{*k} possess the following properties:

a) f^{*k} are universally measurable on X;

b) f^{*k} has the Hilbert–Schmidt valued Gâteaux derivatives of any order $l\leq k$;

c) $0\leq f^{*k}\leq 1;\ \ f^{*k}\mid_{X\backslash(r+1)K}=1;\ \ f^{*k}$ and all its derivatives of any order $l\leq k$ vanish on $(r+1-k)K$;

d) $\sup\limits_x\|D_H^l f^{*k}(x)\|_{\mathcal{H}_l(H,\mathbb{R})}\leq const(l)$ for $l=0,1,\dots k$, where $const(l)$ depends only on c, ψ, and l.

For $k=0$ these properties are obviously true. Suppose that they hold true for $k-1$. Then:

a) Let ν be an arbitrary probability measure on X. Then the $\nu*\lambda$-measurability of $f^{*(k-1)}$ implies the ν-measurability of the functions φ^{*k} and f^{*k}.

b) Applying Lemma 4.1 to $D_H^{k-1}f^{*(k-1)}$ we get the existence of $D_H^k\varphi^{*k}$ and $D_H^k f^{*k}$.

c) For $x\notin(r+1)K$ by the same reasoning as in Theorem 5.2 we get that $\lambda((r+1)K-x)\leq\frac{1}{2}$, whence $\varphi^{*k}(x)\geq\frac{1}{2}\Rightarrow f^{*k}(x)=1$. For $x\in(r+1-k)K$ we have $x+K\subset(r+1-(k-1))K$, whence $\varphi^{*k}(x)\leq\varepsilon$, therefore, $f^{*k}(x)=0$. ¿From $\psi^{(j)}(\varphi^{*k}(x))=0,\ \ j=1,\dots k$, it follows that $D_H^j f^{*k}(x)=0$ for $j=1,\dots k$.

d) For $l=0$ the statement is obvious, $const(0)=1$. Let $1 \le l \le k$. According to Lemma 4.1

$$\|D_H^l \varphi^{*k}(x)\|_{\mathcal{H}_l(H,\mathbb{R})} \le c \cdot \sup_y \|D_H^{l-1} f^{*(k-1)}(y)\|_{\mathcal{H}_{l-1}(H,\mathbb{R})} \le c \cdot const(l-1).$$

Applying the chain rule for the l-th derivative (see [3, Lemma 7.1.13]) and the identity

$$\|A \otimes B\|_{\mathcal{H}_{m+n}} = \|A\|_{\mathcal{H}_m} \cdot \|B\|_{\mathcal{H}_n},$$

we obtain the estimate

$$\|D_H^l f^{*k}(x)\|_{\mathcal{H}_l(H,\mathbb{R})} \le$$

$$\le \|\psi\|_{C_b^l} \cdot l! \cdot \sum_{m=1}^{l} \left[\sum_{i_1,\dots i_m \in \mathbb{N}, \Sigma i_j = l} \left\| \bigotimes_{j=1}^{m} D_H^{i_j} \varphi^{*k}(x) \right\|_{\mathcal{H}_l(H,\mathbb{R})} \right] =$$

$$= \|\psi\|_{C_b^l} \cdot l! \cdot \sum_{m=1}^{l} \left[\sum_{i_1,\dots i_m \in \mathbb{N}, \Sigma i_j = l} \left(\prod_{j=1}^{m} \|D_H^{i_j} \varphi^{*k}(x)\|_{\mathcal{H}_{i_j}(H,\mathbb{R})} \right) \right] \le$$

$$\le P_l(const(1), \dots const(l-1)) =: const(l).$$

Here P_l is a polynomial of the l-th degree, whose coefficients depend only on c, ψ and l. So we can express the values $const(l)$ via some recurrent formula.

The properties a)–d) are proved. They imply that

$$C_{D^{p,r}}(X \setminus (r+1)K) \le \|f^{*r}\|_{p,r} \le$$

$$\le \sum_{l=0}^{r} \left(\int\limits_{X \setminus K} const(l)^p \lambda(dx) \right)^{1/p} \le C(r) \cdot \varepsilon^{1/p},$$

which tends to zero as $\varepsilon \to 0$.

In the case $H \subset H(\mu)$ we can take the measure μ_s for the measure λ, since $H(\mu) \subset H(\mu_s)$. $\quad \triangle$

6 Tightness of capacities generated by the Sobolev classes $W^{p,r}$

Tightness of the $W^{p,1}$-generated capacities over arbitrary full-supported probability measures on separable Banach spaces was proved in [8]. We will consider a more general case, but we impose a certain restriction on the Hilbert subspace H, so the result in [8] is not only the restriction of Theorem 6.1 to the Banach case with $r=1$.

Lemma 6.1. *Let K be an absolutely convex compact set in a separable Fréchet space X with a probability measure μ having full support. Let a separable Hilbert space H be continuously and densely embedded into X. Let λ be a probability measure such that $H \subset H(\lambda)$. Then the function $f(x) = \lambda(K + x)$ belongs to the class $W^{p,1}(\mu)$ for all $p \geq 1$.*

$\bigtriangledown$ Proof. Due to separability and completeness of metrizable space X there exists a decreasing sequence of convex closed cylinders B_n, such that $\bigcap_{n=1}^{\infty} B_n = K$. Consider the functions

$$f_n(x) = \lambda(B_n + x).$$

Since H is dense in X, for each B_n we can choose a finite dimensional subspace $H_n \subset H$ such that B_n has the form

$$B_n = B_n' + Y_n,$$

where B_n' is a convex closed subset of H_n and Y_n is the linear topological complement of H_n. Then it can be easily verified that the function f_n is cylindrical and H-Lipschitzian, therefore, it belongs to $\bigcap\limits_{q>1} W^{q,1}(\mu)$. For all x one has $f_n(x) \to f(x)$, whence due to the uniform boundedness of the functions f_n it follows by the Lebesgue theorem that $\|f_n - f\|_{L^p(\mu)} \to 0$. Let $n > m$. By Corollary 4.1 we obtain

$$\|D_H f_n(x) - D_H f_m(x)\|_{L^p(\mu,H)} \leq c \cdot \|I_{B_n \setminus B_m}\|_{L^{\max\{p;2\}}(\mu*\lambda)} \longrightarrow 0$$

uniformly in $n > m$ as $m \to \infty$. So, $\{f_n\}$ is the desired approximating sequence for f. $\bigtriangleup$

Corollary 6.1. *Let X be a separable Fréchet space. Choose any $p > 1$. Let $f^{*0}, \dots f^{*r}$ be the functions constructed according to formulas (4) starting with $f^{*0} = I_{X\setminus(r+1)K}$. Let us denote by $f_n^{*0}, \dots f_n^{*r}$ the functions constructed according to the same formulas starting with $f_n^{*0} = I_{X\setminus B_n}$, where B_n are decreasing convex closed cylinders whose intersection is $(r+1)K$. Then for any $k = 1, \dots r$, the functions f_n^{*k} approximate the function f^{*k} in the norm $\|\cdot\|_{p,k}$ as $n \to \infty$, and f^{*r} belongs to $W^{p,r}(\mu)$.*

$\bigtriangledown$ Proof. It suffices to consider the case $p \geq 2$. We first verify the approximation by using induction in k. The base of induction is the fact that $f_n^{*0} - f^{*0} \to 0$ as $n \to \infty$ in each of the spaces L^p over the measures

$$\mu,\ \mu * \lambda,\ \mu * \lambda * \lambda,\ \text{ etc.} \tag{5}$$

The step of induction uses Corollary 4.1 and the fact that $\|\psi \circ f - \psi \circ g\|_{p,l}$ can be estimated via $\|f - g\|_{p,l}$. It yields estimates of the form

$$\|D_H^l f_n^{*(k+1)} - D_H^l f_m^{*(k+1)}\|_{L^p(\nu,\mathcal{H}_l(H,\mathbb{R}))} \leq$$

$$\le C_{\psi,l} \cdot \|D^l_H f^{*k}_n - D^l_H f^{*k}_m\|_{L^p(\nu*\lambda,\mathcal{H}_l(H,\mathbb{R}))}$$

$$\text{and} \quad \|D^{l+1}_H f^{*(k+1)}_n - D^{l+1}_H f^{*(k+1)}_m\|_{L^p(\nu,\mathcal{H}_{l+1}(H,\mathbb{R}))} \le$$

$$\le c_{\psi,l} \cdot \|D^l_H f^{*k}_n - D^l_H f^{*k}_m\|_{L^p(\nu*\lambda,\mathcal{H}_l(H,\mathbb{R}))},$$

where $0 \le l \le k$, and ν is an arbitrary measure of the form (5). Moreover, from Corollary 4.1 we obtain $\|f^{*r}_n - f^{*r}\|_{L^p(\mu)} \to 0$. Since f^{*r}_n is a cylindrical function having bounded Sobolev derivatives of order up to r, it belongs to $W^{p,r}(\mu)$, whence it follows that $f^{*r} \in W^{p,r}(\mu)$. $\triangle$

Theorem 6.1. *If X is a separable Fréchet space, then Theorem 5.2 and Theorem 5.3 remain valid when replacing the classes $D^{p,1}$ and $D^{p,r}$ by the classes $W^{p,1}$ and $W^{p,r}$ respectively.*

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References

[1] H. Airault, P. Malliavin, Intégration géométrique sur l'espace de Wiener, Bull. Sci. Math., 2e serie **112** (1988), p. 3–52.

[2] V. I. Bogachev, Gaussian measures, Amer. Math. Soc., Providence, 1998.

[3] V. I. Bogachev, Differentiable measures and the Malliavin calculus, J. Math. Sci. **87** (1997), n 5, p. 3577–3731.

[4] V. I. Bogachev, O. G. Smolyanov, Analytic properties of infinite dimensional distributions, Russian Math. surveys **45** (1990), n 3, p. 1–104.

[5] V. M. Goldstein, Yu. G. Reshetnyak, Introduction to the theory of functions with generalized derivatives and quasiconform mappings, Moscow, "Nauka", 1983.

[6] O. V. Pugachev, Construction of non-Gaussian surface measures by the Malliavin method, Mat. Zametki, 1998 (to appear).

[7] O. V. Pugachev, The Gauss–Ostrogradskii formula in infinite-dimensional space, Sbornik Math. **189** (1998), n 5, p.757–770.

[8] M. Röckner, B. Schmuland, Tightness of general $C_{1,p}$ capacities on Banach space, Journal of Functonal Analysis, **108** (1992), n 1, p. 1–12.