L^p–ANALYSIS OF FINITE AND INFINITE DIMENSIONAL DIFFUSION OPERATORS

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Abstract. This paper consists of lectures given at the C.I.M.E. summer school on Kolmogorov equations held at Cetraro in 1998. The purpose of these lectures was to present an approach to Kolmogorov equations in infinite dimensions which is based on an $L^p(\mu)$–analysis of the corresponding diffusion operators w.r.t. suitably chosen measures. The main ideas and aims are explained, and an as complete as possible presentation is given of what has been achieved in this respect over the last few years.

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0. Introduction

The purpose of these lectures is to present an approach to Kolmogorov equations in infinite dimensions which is based on an $L^p(\mu)$–analysis of the corresponding diffusion operators w.r.t. suitably chosen measures μ. More precisely, we solve

$$\frac{du}{dt} = Lu \ , \ \ u(0,\cdot) = f \tag{0.1}$$

for functions f, $u(t,\cdot)$ on an infinite dimensional (vector) space E and diffusion operators L (cf. Definition 2.1 below) by a C^0–semigroup

$$u(t,\cdot) = T_t f$$

generated by some closed extension $\hat{L}$ of L on $L^p(E;\mu)$ for some appropriate measure μ. Here we think of L as a–priori *only* given explicitly on a set D of nice (smooth) functions. In principle there are at least two types of measures that appear appropriate in applications:

(1) A given reference (e.g. Gaussian) measure μ so that L can be written as a sum of an operator L_0, which is symmetric on $L^2(E;\mu)$, and a small perturbation in the sense of sectorial forms. (This case is refered to in the text below as the “sectorial case”).

(2) μ is an invariant measure for L (or more generally for $\alpha - L$, $\alpha \geq 0$) in the sense that
$$\int Lu \ d\mu = 0 \text{ for all } u \in D \ .$$
(This situation we refer to below as the “non–sectorial case”).

So, in case (2) the first problem is to find such a measure μ. The second problem is then to construct a closed extension $\hat{L}$ of L on $L^p(E;\mu)$ that generates a C^0–semigroup on $L^p(E;\mu)$ (“existence problem”), and the third is to prove that it is the only such extension (“uniqueness problem”).

In these notes we address all three problems presenting results obtained over the last few years in implementing this program. For the precise structure of the corresponding parts of the text, i.e. Sections 1 – 4 we refer to the list of contents. We would only like to point out here that we also include results on properties as “positivity preserving”, “(sub–)Markovian” of the C^0–semigroups solving (0.1), in regard to their probabilistic relevance as transition functions of Markov processes. The latter is made precise in Section 5. In Section 6 we present a case study where E is an infinite dimensional manifold rather than a vector space. In particular, we discuss ergodic properties of the above C^0–semigroups in this situation.

The main purpose of these lectures is to give an updated and an as complete as possible presentation of what has been achieved so far and what the aims are, rather than to repeat detailed proofs. We complement instead arguments needed to derive the statements given here from those recent results, giving precise reference for the latter.

The material presented in these lectures is taken from a number of joint papers with various co–authors as well as the recent work of Andreas Eberle and Wilhelm

Stannat. I would like to thank the latter two as well as all my co–authors, in particular Sergio Albeverio, Vladimir Bogachev, Yuri Kondratiev, Vitali Liskevich, Tatjana Tsikalenko and Tu–Sheng Zhang for permission to include very recent, so far unpublished joint results.

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1. Solution of Kolmogorov equations via sectorial forms

1.1 Preliminaries.

In this subsection we recall some well–known classical results. We assume the reader to be familiar with the notion of a C^0–semigroup $(T_t)_{t\geq 0}$ resp. its generator $(L, D(L))$ on a Banach space (cf. e.g. [Pa 85]). We only recall here that if $u \in D(L)$ then $T_t u \in D(L)$ for all $t \geq 0$ and

$$\frac{d}{dt}\, T_t u = L T_t u = T_t L u \text{ for all } t \geq 0\ . \tag{1.0}$$

Below let $(B, \|\cdot\|)$ be a real Banach space with dual B^* and corresponding dualization $\langle\ ,\ \rangle$. Let L with domain D be a densely defined linear operator on B.

Definition 1.1. (L, D) is called *dissipative*, if for all $u \in D$ there exists $u^* \in B^*$ such that

$$\langle u^*, u\rangle = \|u\|_B^2 = \|u^*\|_B^2 \text{ and } \langle u^*, Lu\rangle \leq 0 \tag{1.1}$$

or, equivalently, (cf. [Pa 85, Ch. I, Thm. 4.2])

$$\|\alpha u - Lu\| \geq \alpha\|u\| \text{ for all } \alpha > 0\,,\ u \in D\ .$$

Remark 1.2. If (L, D) is dissipative, then it is closable. Below we shall denote its closure by $(\overline{L}, \overline{D})$ or shortly $\overline{L}$.

Theorem 1.3. *(Lumer/Phillips) Suppose (L, D) is dissipative. Then $\overline{L}$ is the generator of a C^0–semigroup of contractions (i.e., the operator norms are bounded by 1) on B if and only if $(\alpha - L)(D)$ is dense in B for some (or equivalently all) $\alpha > 0$.*

Let $L^* : D(L^*) \subseteq B^* \to B^*$ denote the adjoint of (L, D). Then as a consequence of Theorem 1.3 and the Hahn–Banach theorem one immediately obtains.

Corollary 1.4. (1^{st} adjoint criterion) Let (L, D) be dissipative. Then $\overline{L}$ generates a C^0–semigroup of contractions on B if and only if $\ker(\alpha - L^*) = \{0\}$ for some (or equivalently all) $\alpha > 0$.

This in turn obviously implies:

Corollary 1.5. (2^{nd} adjoint criterion) Let both (L, D) and $(L^*, D(L^*))$ be dissipative. Then $\overline{L}$ generates a C^0–semigroup of contractions on B.

Later we shall also use the following "uniqueness result":

Theorem 1.6. *Suppose there exists a C^0–semigroup $(T_t)_{t\geq 0}$ on B such that its generator extends (L, D). Then the following assertions are equivalent:*

(i) *D is a core for the generator of $(T_t)_{t\geq 0}$ (i.e., D is dense in the domain of the generator w.r.t. the norm induced by its graph norm).*

(ii) *The closure $\overline{L}$ of (L, D) generates a C^0–semigroup.*

(iii) *$(T_t)_{t\geq 0}$ is the only C^0–semigroup on B whose generator extends (L, D).*

If $(T_t)_{t\geq 0}$ consists of contractions and if $\varepsilon > 0$, then (i) - (iii) is equivalent to

(iii') $(T_t)_{t\geq 0}$ is the only C^0–semigroup on B such that its generator extends (L, D) and $\|T_t\| \leq e^{\varepsilon t}$ for all $t \geq 0$.

Proof. See [Eb 98, Ch. I, Thm. 1.2] and [Ar 86, A - II, Thm. 1.33]. $\square$

A sufficient condition for uniqueness (which will, however, turn out to be typically *not applicable* to cases of interest in these lectures) is the following:

Proposition 1.7. *Let $(T_t)_{t\geq 0}$ be a C^0–semigroup on B whose generator extends (L, D). If there exists a dense subspace $B_0 \subset B$ such that $T_t(B_0) \subset D$ for all $t \geq 0$, then $(T_t)_{t\geq 0}$ is the only C^0–semigroup whose generator extends (L, D).*

Proof. See e.g. [EthKur 86, Ch. 1, Prop. 3.3]. $\square$

1.2 Sectorial forms.

The terminology w.r.t. "sectorial forms" used in these lectures is according to [MR 92]. Below we fix a Hilbert space $(\mathcal{H}, (\ ,\ ))$ and set $\|\cdot\| := (\ ,\ )^{1/2}$.

Definition 1.8. A bilinear form $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{R}$ with dense domain $D(\mathcal{E}) \subset \mathcal{H}$ is called a *(closed) coercive form* if it satisfies the following conditions:

(i) Its symmetric part $(\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))$ defined by $\tilde{\mathcal{E}}(u,v) := \frac{1}{2}(\mathcal{E}(u,v) + \mathcal{E}(v,u))$; $u, v \in D(\tilde{\mathcal{E}}) := D(\mathcal{E})$, is positive definite and *closed* (i.e., $D(\mathcal{E})$ is complete with respect to the norm $\tilde{\mathcal{E}}_1^{1/2}$ coming from the symmetric part of $\mathcal{E}_\alpha(u,v) := \mathcal{E}(u,v) + \alpha(u,v)$; $\alpha > 0$; $u, v \in D(\mathcal{E})$ for $\alpha := 1$).

(ii) $(\mathcal{E}, D(\mathcal{E}))$ satisfies the *weak sector condition*, i.e., for one (and hence all) $\alpha \in (0, \infty)$ there exists $K_\alpha \in (0, \infty)$ such that

$$|\mathcal{E}_\alpha(u,v)| \leq K_\alpha\ \mathcal{E}_\alpha(u,u)^{1/2}\ \mathcal{E}_\alpha(v,v)^{1/2} \text{ for all } u, v \in D(\mathcal{E})\ . \tag{1.2}$$

We shall use (1.2) below almost always with $\alpha := 1$ and set $K := K_1$.

Remark 1.9. (i) If $(\mathcal{E}, D(\mathcal{E}))$ is a coercive form on $\mathcal{H}$, then so is $(\mathcal{E}^*, D(\mathcal{E}^*))$ defined by

$$\mathcal{E}^*(u,v) := \mathcal{E}(v,u)\ ;\ \ u, v \in D(\mathcal{E}^*) := D(\mathcal{E})\ . \tag{1.3}$$

The same holds for $(\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))$.

(ii) The *antisymmetric part* $(\check{\mathcal{E}}, D\check{\mathcal{E}}))$ of a bilinear form $(\mathcal{E}, D(\mathcal{E}))$ on $\mathcal{H}$ is defined by

$$\check{\mathcal{E}}(u,v) := \tfrac{1}{2}(\mathcal{E}(u,v) - \mathcal{E}(v,u)) \ ; \ u,v \in D(\check{\mathcal{E}}) := D(\mathcal{E}) \ .$$

It is easy to see that if $(\mathcal{E}, D(\mathcal{E}))$ is positive definite, then $(\mathcal{E}, D(\mathcal{E}))$ satisfies the weak sector condition if and only if for some (hence all) $\alpha \in (0,\infty)$ there exists $C_\alpha \in (0,\infty)$ such that

$$\check{\mathcal{E}}(u,v) \le C_\alpha \ \mathcal{E}_\alpha(u,u)^{1/2} \ \mathcal{E}_\alpha(v,v)^{1/2} \text{ for all } u,v \in D(\mathcal{E}) \ .$$

A coercive form is always associated with a linear operator which generates a C^0–semigroup of contractions on $\mathcal{H}$. We restate and reprove this fact here (cf. e.g. [MR 92, Proposition 2.16]), which is an easy consequence of the Lax–Milgram Theorem.

Proposition 1.10. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a coercive form on $\mathcal{H}$. Define*

$$D(L) := \{u \in D(\mathcal{E}) | v \mapsto \mathcal{E}(u,v) \text{ is continuous w.r.t. } \|\cdot\| \text{ on } D(\mathcal{E})\} \ , \tag{1.4}$$

and for $u \in D(L)$ let Lu denote the unique element in $\mathcal{H}$ such that

$$(-Lu, v) = \mathcal{E}(u,v) \text{ for all } v \in D(\mathcal{E}) \ . \tag{1.5}$$

Then $(L, D(L))$ is a linear operator on $\mathcal{H}$ that generates a C^0–semigroup $(T_t)_{t\ge 0}$ of contractions on $\mathcal{H}$.

Proof. Clearly $(L, D(L))$ is a well–defined linear operator on $\mathcal{H}$ and applying the above definition to the coericve form $(\mathcal{E}^*, D(\mathcal{E}^*))$ we likewise obtain an operator $(L^*, D(L^*))$ which obviously is exactly the adjoint operator of $(L, D(L))$ on $\mathcal{H}$.

Claim 1. $(1-L)(D(L)) = \mathcal{H}$ and $D(L)$ is dense in $\mathcal{H}$.
Indeed, applying for $f \in \mathcal{H}$ the Lax–Milgram Theorem to the linear functional $v \mapsto (v,f)$ on $D(\mathcal{E})$ (which is continuous w.r.t. $\tilde{\mathcal{E}}_1^{1/2}$) we obtain that there are $G_1 f$, $G_1^* f \in D(\mathcal{E})$ such that for $v \in D(\mathcal{E})$

$$(f,v) = \mathcal{E}_1(G_1 f, v) = \mathcal{E}_1^*(G_1^* f, v) = \mathcal{E}_1(v, G_1^* f) \ .$$

This in turn implies that $G_1 f \in D(L)$, $G_1^* f \in D(L^*)$ and that $(1-L)G_1 f = f$ and $(1-L^*)G_1^* f = f$. Consequently, $(1-L)(D(L)) = \mathcal{H}$ and if $g \in \mathcal{H}$ such that $(g,u) = 0$ for all $u \in D(L)$, then for all $f \in \mathcal{H}$

$$0 = (g, G_1 f) = \mathcal{E}_1(G_1 f, G_1^* g) = (f, G_1^* g) \ .$$

Hence $G_1^* g = 0$, therefore, $g = (1-L^*)G_1^* g = 0$. So, $D(L)$ is dense in $\mathcal{H}$.

Claim 2. L is dissipative and closed
Clearly, for all $u \in D(L)$, $\alpha > 0$, we have

$$\alpha \|u\|^2 \le \mathcal{E}_\alpha(u,u) = ((\alpha - L)u, u) \le \|\alpha u - Lu\| \ \|u\| \ .$$

In particular, $1-L$ is invertible with bounded inverse $(1-L)^{-1}$ which by Claim 1 is defined on all of $\mathcal{H}$. Hence $(1-L)^{-1}$ and thus also L is closed.

Claims 1 and 2 prove the assertion. $\square$

It follows by (1.2) that the operator L in the previous proposition is *weakly sectorial*, i.e., for all $\alpha \in (0,\infty)$ there exists $K_\alpha \in (0,\infty)$ such that

$$|((\alpha - L)u, v)| \leq ((\alpha - L)u, u)^{1/2}((\alpha - L)v, v)^{1/2} \text{ for all } u, v \in D(L)\ . \tag{1.6}$$

In fact Proposition 1.10 establishes a one–to–one correspondence between sectorial forms and such operators resp. C^0–semigroups of a certain type.

Theorem 1.11. *The association of L and hence $(T_t)_{t\geq 0}$ to $(\mathcal{E}, D(\mathcal{E}))$ in Proposition 1.10 defines bijective maps between the following three sets of mathematical objects:*

(a) *Sectorial forms $(\mathcal{E}, D(\mathcal{E}))$ on $\mathcal{H}$.*

(b) *Weakly sectorial, closed, dissipative operators $(L, D(L))$ on $\mathcal{H}$ for which $(1-L)(D(L))$ is dense in $\mathcal{H}$.*

(c) *C^0–semigroups $(T_t)_{t\geq 0}$ of contractions on $\mathcal{H}$ having the property that for their natural extensions $(T_t^{\mathbb{C}})_{t\geq 0}$ to the complexification $\mathcal{H}_{\mathbb{C}}$ of $\mathcal{H}$ the semigroups $(e^t T_t^{\mathbb{C}})_{t\geq 0}$ are restrictions of holomorphic contraction semigroups $(e^z T_z^{\mathbb{C}})_{z\in K}$ where K is a sector in $\mathbb{C}$ containing $[0,\infty)$.*

For the proof of Theorem 1.11 we refer to [MR 92, Ch. I, Sect. 2] (in particular, cf. the diagram on p. 27). We only note here that $(T_t)_{t\geq 0}$ in Proposition 1.10 is, therefore, analytic. In particular,

$$T_t(\mathcal{H}) \subset D(L) \text{ for all } t > 0$$

(cf. e.g. [ReS 75, Corollary 2 of Theorem X.52]) and thus for all $f \in \mathcal{H}$

$$\begin{aligned} \frac{d}{dt}\, T_t f &= L T_t f \text{ for all } t > 0 \\ T_0 f &= f\ . \end{aligned}$$

1.3 Sectorial forms on $L^2(E;m)$.

Let $(E;\mathcal{B};m)$ be a measure space and $\mathcal{H} := L^2(E;m) := L^2(E;\mathcal{B};m)$ the corresponding (real) L^2–space with its usual inner product. In particular, $\mathcal{H}$ now has a vector lattice structure w.r.t. the usual order "$\leq$" (defined representativewise m–a.e.). As usual for $f \in L^2(E;m)$ we set $f^+ := \sup(f,0)$ and $f^- := -\inf(f,0)$.
In this subsection we recall a few results giving conditions on a generator (resp. the associated coercive form, if it exists) ensuring that the C^0–semigroup respects this order. These results (at least for symmetric semigroups) are known under the key–word "Beurling–Deny criteria". We first recall

Definition 1.12. A C^0–semigroup $(T_t)_{t\ge0}$ on $L^2(E;m)$ is called *positivity preserving*, if

$$f \in L^2(E;m)\ ,\ f \ge 0 \Rightarrow T_t f \ge 0 \text{ for all } t \ge 0\ ,$$

and *sub–Markovian* if

$$f \in L^2(E;m)\ ,\ 0 \le f \le 1 \Rightarrow 0 \le T_t f \le 1 \text{ for all } t \ge 0\ .$$

If $m(E) < \infty$, a positivity preserving C^0–semigroup is called *Markovian* if $T_t 1 = 1$ for all $t > 0$.

It is easy to see that "sub–Markovian" implies "positivity preserving".

Theorem 1.13. *Let $(T_t)_{t\ge0}$ be a C^0–semigroup of contractions on $L^2(E;m)$ with generator $(L, D(L))$. Then*

(i) *$(T_t)_{t\ge0}$ is positivity preserving if and only if*

$$(Lu, u^+) \le 0 \textit{ for all } u \in D(L)\ .$$

(ii) *$(T_t)_{t\ge0}$ is sub–Markovian if and only if*

$$(Lu, (u-1)^+) \le 0 \textit{ for all } u \in D(L)\ .$$

Proof. For (i) see e.g. [MR 95, Theorem 1.7] and for (ii) see e.g. [MR 92, Ch. I, Proposition 4.3]. □

Unfortunately, the conditions on L in the previous theorem are often hard to check. As we shall see below, this is different if $(T_t)_{t\ge0}$ is associated with a sectorial form, since then there are respective easily checkable conditions on the latter.

Theorem 1.14. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a sectorial form on $L^2(E;m)$ with corresponding (generator L and) C^0–semigroup $(T_t)_{t\ge0}$ of contractions. Then:*

(i) *$(T_t)_{t\ge0}$ is positivity preserving if and only if for all $u \in D(\mathcal{E})$ also $u^+ \in D(\mathcal{E})$ and*

$$\mathcal{E}(u, u^+) \ge 0\ .$$

(ii) *$(T_t)_{t\ge0}$ is sub–Markovian if and only if for all $u \in D(\mathcal{E})$ also $u^\# := \inf(u^+, 1) \in D(\mathcal{E})$ and*

$$\mathcal{E}(u, u - u^\#) \ge 0\ .$$

Proof. (i): [MR 95, Theorem 1.5 and Proposition 1.3].
(ii): [MR 92, Ch. I, Theorem 4.4] and its proof. □

Remark 1.15. (i) In case (i) of Theorem 1.14 the coercive form $(\mathcal{E}, D(\mathcal{E}))$ is also called *positivity preserving* while in case (ii) it is called a *Semi–Dirichlet form*. If also $(\mathcal{E}^*, D(\mathcal{E}^*))$ is a Semi–Dirichlet form, then $(\mathcal{E}, D(\mathcal{E}))$ is called a *Dirichlet form*.
(ii) We note that if $(\mathcal{E}, D(\mathcal{E}))$ is positivity preserving, then so is $(\mathcal{E}^*, D(\mathcal{E}^*))$ and therefore so is $(\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))$. (Indeed, setting $u_n := u^+ - \frac{1}{n}u^-$ we obtain for all $n \in \mathbb{N}$

$$\begin{aligned}\mathcal{E}(u^+, u) &= \mathcal{E}(u_n, u) + \tfrac{1}{n}\mathcal{E}(u^-, u)\\ &= \mathcal{E}(u_n, (u_n)^+) + n\mathcal{E}(-u_n, (-u_n)^+) + \tfrac{1}{n}\mathcal{E}(u^-, u) \geq \tfrac{1}{n}\mathcal{E}(u^-, u)\ .)\end{aligned}$$

But if $(\mathcal{E}, D(\mathcal{E}))$ is a Semi–Dirichlet form, $(\mathcal{E}^*, D(\mathcal{E}^*))$ might be not (cf. [MOR 95, Remark 2.2 (ii)]). So, not every Semi–Dirichlet form is a Dirichlet form.

Since in many applications a coercive form $(\mathcal{E}, D(\mathcal{E}))$ is explicitly given only on a core $D \subset D(\mathcal{E})$, we recall the following useful result to check whether $(\mathcal{E}, D(\mathcal{E}))$ is positivity preserving. Analogous results hold for (Semi–)Dirichlet forms (see e.g. [MR 92, Prop. 4.7 and Ex. 4.8]).

Lemma 1.16. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a coercive form on $L^2(E; m)$. Let D be a subset of $D(\mathcal{E})$, dense w.r.t. $\tilde{\mathcal{E}}_1^{1/2}$ such that for all $u \in D$ the following holds: for all $\varepsilon > 0$ there exists a function $\varphi_\varepsilon : \mathbb{R} \to [-\varepsilon, \infty)$ such that $\varphi_\varepsilon(t) = t$ for all $t \in [0, \infty)$, $0 \leq \varphi_\varepsilon(t_2) - \varphi_\varepsilon(t_1) \leq t_2 - t_1$ if $t_1 \leq t_2$, $\varphi_\varepsilon \circ u \in D(\mathcal{E})$ and*

$$\sup_{\varepsilon>0}\ \mathcal{E}(\varphi_\varepsilon \circ u, \varphi_\varepsilon \circ u) \leq\ \ const.\ \mathcal{E}(u, u)$$

as well as

$$\limsup_{\varepsilon\to 0}\ \mathcal{E}(u, \varphi_\varepsilon \circ u) \geq 0\ .$$

Then $(\mathcal{E}, D(\mathcal{E}))$ is positivity preserving.

Proof. [MR 95, Proposition 1.2]. $\square$

1.4 Examples and Applications.

1.4.1 $E :=$ open subset of $\mathbb{R}^d$.

Let $E := U \subset \mathbb{R}^d$, U open, $\mathcal{B} := \mathcal{B}(U)$ the Borel σ–algebra on U, and let dx denote Lebesgue measure on U. For $1 \leq i, j \leq d$ let $a_{ij} \in L^1_{\text{loc}}\,(U; dx)$ such that $\overset{\vee}{a}_{ij} := \frac{1}{2}(a_{ij} - a_{ji}) \in L^\infty(U; dx)$, b_i, $\hat{b}_i \in L^\infty(U; dx) + L^d(U; dx)$, $c \in L^1_{\text{loc}}(U; dx)$ such that $c^- \in L^\infty(U; dx) + L^{d/2}(U; dx)$. Define $A := (a_{ij})_{1\leq i,j\leq d}$, $b := (b_1, \ldots, b_d)$, $\hat{b} = (\hat{b}_1, \ldots, \hat{b}_d)$ and let $\langle\ ,\ \rangle_{\mathbb{R}^d}$ denote the Euclidean inner product on $\mathbb{R}^d$. Assume that there exists $\delta > 0$ such that for all $\xi \in \mathbb{R}^d$

$$\langle A(x)\xi, \xi\rangle_{\mathbb{R}^d} \geq \delta\ \langle \xi, \xi\rangle_{\mathbb{R}^d} \text{ for } dx\text{–a.e. } x \in U\ .$$

Define for $u, v \in C_0^\infty(U)$ (: = the set of all infinitely differentiable real–valued functions on U with compact support).

$$\begin{aligned}\mathcal{E}(u, v) :&= \int_U \langle A(x)\nabla u(x), \nabla v(x)\rangle_{\mathbb{R}^d}\ dx + \int_U \langle b(x), \nabla u(x)\rangle_{\mathbb{R}^d}\ v(x)\ dx\\ &+ \int_U u(x)\langle \hat{b}(x), \nabla v(x)\rangle_{\mathbb{R}^d}\ dx + \int_U u(x)\ v(x)\ c(x)\ dx\ .\end{aligned}\tag{1.7}$$

Then there exists an $\alpha \in (0,\infty)$ such that $(\tilde{\mathcal{E}}_\alpha, C_0^\infty(U))$ is a closable, positive definite, symmetric bilinear form on $L^2(U;dx)$. Let $(\tilde{\mathcal{E}}_\alpha, D(\tilde{\mathcal{E}}_\alpha))$ denote its closure. $(\mathcal{E}_\alpha, C_0^\infty(U))$ satisfies the weak sector condition, and hence has a unique (*w.r.t.* $\tilde{\mathcal{E}}_\alpha^{1/2}$) continuous bilinear extension to the domain $D(\tilde{\mathcal{E}}_\alpha)$, and this extension, denoted by $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ (with $D(\mathcal{E}_\alpha) := D(\tilde{\mathcal{E}}_\alpha)$), is a coercive form on $L^2(U;dx)$. Furthermore, $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ is positivity preserving.

The proofs for all these statements are contained in [MR 95, Subsect. 2.1] (and they will be essentially repeated in the infinite dimensional case in the next subsection).

Let $L^{(\alpha)}$ denote the generator of $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ and $(T_t^{(\alpha)})_{t\geq 0}$ the corresponding positivity preserving C^0–semigroup of contractions on $L^2(U;dx)$. Define

$$L := L^{(\alpha)} + \alpha \ , \ D(L) := D(L^{(\alpha)}) \ ,$$
$$T_t := e^{\alpha t}\, T_t^{(\alpha)} \ , \ t \geq 0 \ .$$

Then for all $f \in L^2(U;dx)$

$$\frac{d}{dt}\, T_t f = L T_t f \text{ for all } t > 0 \text{ (and all } t \geq 0, \text{ if } f \in D(L))$$
$$T_0 f = f \ ,$$

where L (heuristically) is given by

$$Lu = \text{ div } (A\nabla u + \hat{b}u) - \langle b, \nabla u\rangle - c\ u \ . \tag{1.8}$$

$(T_t)_{t\geq 0}$ is positivity preserving, since so is $(T_t^{(\alpha)})_{t\geq 0}$.

1.4.2 $E :=$ topological vector space.

Let E be a locally convex topological real vector space which is a (topological) Souslin space, that is, the continuous image of a separable complete metric space (e.g. E is a separable real Banach space). Let $\mathcal{B} := \mathcal{B}(E)$ denote its Borel σ–algebra and E' its dual space. Let $(H, \langle\ ,\ \rangle_H)$ be a separable real Hilbert space, densely and continuously embedded into E. Identifying H with its dual H' we have that

$$E' \subset H \subset E \text{ densely and continuously} \tag{1.9}$$

and the dualization ${}_{E'}\langle\ ,\ \rangle_E$ between E' and E restricted to $E' \times H$ coincides with $\langle\ ,\ \rangle_H$. Here the dual E' of E is endowed with the strong topology. H should be thought of as a *tangent space* to E at each point.

For a dense linear subspace $K \subset E'$ we introduce the usual test function space $\mathcal{F}C_b^\infty(K)$ on E consisting of finitely based smooth bounded functions, i.e.,

$$\mathcal{F}C_b^\infty := \mathcal{F}C_b^\infty(K) := \{f(\ell_1,\ldots,\ell_m) \mid m \in \mathbb{N}, \ell_1,\ldots,\ell_m \in E', f \in C_b^\infty(\mathbb{R}^m)\}, \tag{1.10}$$

where $C_b^\infty(\mathbb{R}^m)$ denotes the set of all infinitely differentiable functions on $\mathbb{R}^m$ with bounded partial derivatives of all order. Note that since $\mathcal{B}(E) = \sigma(E')$, $\mathcal{F}C_b^\infty$ is dense in $L^2(E;m)$ for any finite positive measure m on $(E,\mathcal{B}(E))$.

Let $u \in \mathcal{F}C_b^\infty$. For $k \in E$ as usual we set

$$\begin{aligned}\frac{\partial u}{\partial k}(z) &:= \frac{d}{dt}\ u(z+sk)_{|s=0}\\ &= \sum_{i=1}^m \partial_i f(\ell_1(z),\dots,\ell_m(z))\ {}_{E'}\langle \ell_i, k\rangle_E,\ z \in E\ ,\end{aligned} \tag{1.11}$$

where ∂_i denotes derivative w.r.t. the i–th coordinate. The latter equality follows by the chain rule if u has the representation $u = f(\ell_1,\dots,\ell_m)$ as in (1.10). For $z \in E$ the H*–gradient* $\nabla_H u(z) \in H$ is defined as the unique element in H such that

$$\langle \nabla_H u(z), h\rangle_H = \frac{\partial u}{\partial h}(z) \tag{1.12}$$

(where we note that by (1.9), (1.11) the linear map $h \mapsto \frac{\partial u}{\partial h}(z)$ is continuous on $(H, \langle\ ,\ \rangle_H)$).

We now fix a finite positive measure μ on $(E, \mathcal{B}(E))$ such that $\mu(U) > 0$ for all non–empty open subsets $U \subset E$. Define

$$\mathcal{E}_\mu(u,v) := \int_E \langle \nabla_H u, \nabla_H v\rangle_H\ d\mu\ ;\ u,v \in \mathcal{F}C_b^\infty\ . \tag{1.13}$$

Then $(\mathcal{E}_\mu, \mathcal{F}C_b^\infty)$ is a positive definite symmetric bilinear form on $L^2(E;\mu)$ with dense domain. We assume

(C) $\qquad (\mathcal{E}_\mu, \mathcal{F}C_b^\infty)$ is closable on $L^2(E;\mu)$.

For sufficient (and essentially necessary) conditions implying (C) we refer to [AR 90, Sect. 3]. Next we are going to "build" coercive forms on $L^2(E;\mu)$ of type (1.7) starting with $(\mathcal{E}_\mu, \mathcal{F}C_b^\infty)$.

Let $\mathcal{L}_\infty(H)$ denote the set of all bounded linear operators on H with operator norm $\|\ \|$. Suppose $z \mapsto A(z)$, $z \in E$, is a map from E to $\mathcal{L}_\infty(H)$ such that $z \mapsto \langle A(z)h_1, h_2\rangle_H$ is $\mathcal{B}(E)$–measurable for all $h_1,\ h_2 \in H$ and $\|\tilde{A}\| \in L^1(E;\mu)$, $\|\check{A}\| \in L^\infty(E;\mu)$, where $\tilde{A} := \frac{1}{2}(A + A')$, $\check{A} := \frac{1}{2}(A - A')$ and $A'(z)$ denotes the adjoint of $A(z)$, $z \in E$, on H.

Furthermore, assume that the following "ellipticity" condition holds:

(E) There exists $\delta \in (0,\infty)$ such that

$$\langle \overset{(z)}{\tilde{A}}(z)h, h\rangle_H \geq \delta\ \langle h,h\rangle_H \text{ for all } h \in H\ .$$

Let $c \in L^1(E;\mu)$ and define

$$Q(u,v) := \int_E \langle \tilde{A}(x)\nabla_H u(x), \nabla_H v(x)\rangle_H\ \mu(dx) + \int_E u(x)v(x)c^+(x)\ \mu(dx)\ ;\ u,v \in \mathcal{F}C_b^\infty\ .$$

Then by (E) also $(Q, \mathcal{F}C_b^\infty)$ is closable on $L^2(E;\mu)$ (see [MR 92, Ch. II, Subsect. 3 e)]). Let $(Q, D(Q))$ denote its closure. Let $b, \hat{b} \in L^2(E \to H;\mu)$, $c \in L^1(E;\mu)$ such that they define a "small form perturbation" of $(Q, D(Q))$, i.e., they satisfy

(SFP) (i) there exist $\varepsilon \in (0,1)$ and $\alpha \in (0,\infty)$ such that

$$\int \left(\langle b+\hat{b}, \nabla_H u\rangle_H - c^- u\right) u \; d\mu \geq -\varepsilon \;\; Q(u,u) - \alpha \int u^2 \; d\mu$$

for all $u \in \mathcal{F}C_b^\infty$;

(ii) there exists $C \in (0,\infty)$ such that

$$\left|\int \langle b-\hat{b}, \nabla_H u\rangle_H \; v \; d\mu\right| \leq C \; Q_1(u,u)^{1/2} \; Q_1(v,v)^{1/2}$$

for all $u,v \in \mathcal{F}C_b^\infty$ (where for $\alpha > 0$ again $Q_\alpha(u,v) := Q(u,v) + \alpha \int u \; v \; d\mu$).

Then define

$$\begin{aligned}\mathcal{E}(u,v) := &\int_E \langle A(x)\nabla_H u(x)\,,\nabla_H v(x)\rangle_H \; \mu(dx) + \\ &\int_E \langle b(x), \nabla_H u(x)\rangle_H \; v(x) \; \mu(dx) + \int_E u(x) \; \langle \hat{b}(x), \nabla_H v(x)\rangle_H \; \mu(dx) \\ &+ \int_E u(x)v(x)c(x) \; \mu(dx)\,; \; u,v \in \mathcal{F}C_b^\infty \; . \end{aligned} \tag{1.14}$$

It follows by (SFP) (i) that for all $u \in \mathcal{F}C_b^\infty$

$$\begin{aligned}&\mathcal{E}_{\alpha+1}(u,u)(:= \mathcal{E}(u,u) + (\alpha+1)\int u^2 \; d\mu) \\ &\geq (1-\varepsilon) \; Q_1(u,u) + \varepsilon \; Q(u,u) + \int (\langle b+\hat{b}, \nabla_H u\rangle_H + (\alpha - c^-)u) \; u \; d\mu \\ &\geq (1-\varepsilon) \; Q_1(u,u)\end{aligned} \tag{1.15}$$

and by (SFP) (ii) that for all $u,v \in \mathcal{F}C_b^\infty$

$$|\check{\mathcal{E}}(u,v)| \leq (M+C) \; Q_1(u,u)^{1/2} \; Q_1(v,v)^{1/2}$$

where $M := \mu - \operatorname*{ess\,sup}_{x\in E} \|\check{A}(x)\|$. Hence by Remark 1.9 (ii) there exists $K_{\alpha+1} \in (0,\infty)$ such that for all $u,v \in \mathcal{F}C_b^\infty$

$$|\mathcal{E}_{\alpha+1}(u,v)| \leq K_{\alpha+1} \; Q_{\alpha+1}(u,u)^{1/2} \; Q_{\alpha+1}(v,v)^{1/2} \; . \tag{1.16}$$

(1.15) and (1.16) imply that $(\tilde{\mathcal{E}}_\alpha, \mathcal{F}C_b^\infty)$ is a closable, positive definite, symmetric bilinear form on $L^2(E;\mu)$, and for its closure $(\tilde{\mathcal{E}}_\alpha, D(\tilde{\mathcal{E}}_\alpha))$ we have that $D(\tilde{\mathcal{E}}_\alpha) = D(Q)$. By (1.16) $\mathcal{E}_\alpha$ uniquely extends to a (*w.r.t.* $Q_{\alpha+1}^{1/2}$) continuous bilinear form with domain $D(Q)$, and if $D(\mathcal{E}_\alpha) := D(Q)$, then (1.15), (1.16) imply that $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ is a coercive form on $L^2(E;\mu)$.

Now we want to show that $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ is positivity preserving. We shall use Lemma 1.16. So, for $\varepsilon > 0$ let $\varphi_\varepsilon \in C^\infty(\mathbb{R})$ such that $\varphi(\mathbb{R}) \subset [-\varepsilon,\infty)$, $\varphi_\varepsilon(t) = t$ for all $t \in [0,\infty)$, $0 < \varphi_\varepsilon' \leq 1$, and $\varphi_\varepsilon(t) = -\varepsilon$ for $t \in (-\infty,-2\varepsilon]$. Then the assumptions in Lemma 1.16 with $D := \mathcal{F}C_b^\infty$ are fulfilled. Indeed, obviously for all $u \in \mathcal{F}C_b^\infty$, $\varepsilon > 0$, $\varphi_\varepsilon \circ u \in \mathcal{F}C_b^\infty \subset D(\mathcal{E}_\alpha) = D(Q)$ and

$$Q_{\alpha+1}(\varphi_\varepsilon \circ u, \varphi_\varepsilon \circ u) \leq Q_{\alpha+1}(u,u) ,$$

hence by (1.15), (1.16)

$$\sup_{\varepsilon>0} \mathcal{E}_\alpha(\varphi_\varepsilon \circ u, \varphi_\varepsilon \circ u) \leq \text{ const. } \mathcal{E}_{\alpha+1}(u,u) .$$

Furthermore, since $|\varphi_\varepsilon(u)| \leq |u|$, $\varphi_\varepsilon(u) \underset{\varepsilon\searrow 0}{\longrightarrow} 1_{\{u\geq 0\}}u$, $|\varphi_\varepsilon'(u)| \leq 1$, $\varphi_\varepsilon'(u) \underset{\varepsilon\searrow 0}{\longrightarrow} 1_{\{u\geq 0\}}$, we obtain for $\varepsilon_n := \frac{1}{n}$, $n \in \mathbb{N}$, by Lebesgue's dominated convergence theorem

$$\begin{aligned}
&\lim_{n\to\infty} \mathcal{E}_\alpha(u, \varphi_{\varepsilon_n} \circ u) \\
&= \int 1_{\{u\geq 0\}} \Big(\langle A\nabla_H u, \nabla_H u\rangle_H + \langle b + \hat{b}, \nabla_H u\rangle_H \; u + u^2(c+\alpha)\Big) \; d\mu \\
&= \lim_{n\to\infty} \Big(\int \varphi_{\varepsilon_n}'(u)^2 \; \langle A\nabla_H u, \nabla_H u\rangle_H \; d\mu \\
&\quad + \int \varphi_{\varepsilon_n}'(u) \; \langle b + \hat{b}, \nabla_H u\rangle_H \; \varphi_{\varepsilon_n}(u) \; d\mu + \int \varphi_{\varepsilon_n}(u)^2 (c+\alpha) \; d\mu \Big) \\
&= \lim_{n\to\infty} \mathcal{E}_\alpha(\varphi_{\varepsilon_n} \circ u, \varphi_{\varepsilon_n} \circ u) \geq 0 .
\end{aligned}$$

So, Lemma 1.17 implies that $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ is positivity preserving.
Let $L^{(\alpha)}$ denote the generator of $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ and $(T_t^{(\alpha)})_{t\geq 0}$ the corresponding positivity preserving C^0–semigroup of contractions on $L^2(E;\mu)$. Define

$$\begin{aligned}
L &:= L^{(\alpha)} + \alpha \; , \; D(L) := D(L^{(\alpha)}) \; , \\
T_t &:= e^{\alpha t} \; T_t^{(\alpha)} \; , \; t \geq 0 \; .
\end{aligned}$$

Then for all $f \in L^2(E;\mu)$

$$\begin{aligned}
\frac{d}{dt} \; T_t f &= L T_t f \text{ for all } t > 0 \text{ (and all } t \geq 0\, , \text{ if } f \in D(L)) \\
T_0 f &= f, \qquad\qquad (1.17)
\end{aligned}$$

where L (heuristically) is given by

$$Lu = \operatorname{div}_\mu (A\nabla_H u + \hat{b}u) - \langle b, \nabla_H u\rangle_H - cu \; . \qquad (1.18)$$

Here div_μ denotes the divergence w.r.t. to μ, i.e., (-1) times the adjoint $\nabla_H^{*,\mu}$ of ∇_H on $L^2(\mu)$. $(T_t)_{t\geq 0}$ is positivity preserving, since so is $(T_t^{(\alpha)})_{t\geq 0}$.
To conclude this subsection we discuss conditions on $b, \hat{b}, c^-$ that ensure that (SFP) holds. This is e.g. in fact the case if $b, \hat{b}, c^-$ are bounded. This is easy to see and the details can be found in [MR 92, Ch. II, Subsect. 3 e)]. The unbounded case is more delicate. In finite dimensions (see Subsect. 1.4.1) (SFP) follows under the

"(L^d+L^∞)–condition" from Sobolev–embedding theorems, more precisely from the fact that

$$\left(\int |u|^{\frac{2d}{d-2}}\ dx\right)^{\frac{d-2}{2d}} \leq \text{ const. } \left(\int |\nabla u|^2_{\mathbb{R}^d}\ dx\right)^{1/2}$$

(cf. [MR 92, Ch. II, Subsect. 2 d)]). Such Sobolev embeddings do not hold in infinite dimensions even for very simple (e.g. Gaussian) measures μ. However, as a substitute in some cases the following, so–called *defective log–Sobolev inequality* holds for our triple (E, H, μ):

(DLS) There exists $\lambda_1,\ \lambda_2 \in [0,\infty)$ such that

$$\int_E u^2 \log|u|\ d\mu \leq \lambda_1 \int |\nabla_H u|^2_H\ d\mu + \|u\|^2_{L^2(\mu)} \log\|u\|_{L^2(\mu)} \\ +\ \lambda_2 \|u\|^2_{L^2(\mu)} \text{ for all } f \in \mathcal{F}C_b^\infty\ .$$

Here $\|u\|_{L^p(\mu)} := (\int |u|^p\ d\mu)^{1/p}$. (DLS) e.g. holds even with $\lambda_2 = 0$ if (E, H, μ) forms an abstract Wiener space (cf. [Gr 93]). We shall see other examples in the next subsection.

Lemma 1.17. *Suppose that in addition to (C), (E) above, (DLS) holds for (E, H, μ), that $\|\tilde{A}\| \in L^\infty(E;\mu)$, and that*

$$e^{|b-\hat{b}|^2_H} \in \bigcup_{p>0} L^p(E;\mu)\ ,$$
$$e^{|b+\hat{b}|^2_H} \in L^{p_1}(E;\mu)\ ,\ e^{c^-} \in L^{p_2}(E;\mu)$$

with $p_1, p_2 \in [1,\infty)$ satisfying $\frac{1}{2p_1} + \frac{1}{p_2} < \delta/4\lambda_1$ (where δ, λ_1 are as in (E) resp. (DLS)). Then (SFP) holds and all the above applies.

Proof. The proof is essentially a direct consequence of the following Young inequality, i.e.,

$$st \leq s\log s - s + e^t \text{ for all } t \in \mathbb{R},\ \ s \in [0,\infty)\ . \tag{1.19}$$

Let us first check (SFP) (i): Let $u \in \mathcal{F}C_b^\infty$. We may assume that $\|u\|_{L^2(\mu)} = 1$. Then by (1.19) and (DLS)

$$\begin{aligned}
&\int \langle b+\hat{b}, \nabla_H u\rangle_H\ u\ d\mu - \int u^2 c^-\ d\mu \\
\geq\ & -\left(\int |b+\hat{b}|^2_H\ u^2\ d\mu\right)^{1/2} \left(\int |\nabla_H u|^2_H\ d\mu\right)^{1/2} - \int u^2\ c^-\ d\mu \\
\geq\ & -\frac{1}{2p_1}\int u^2 \left(p_1|b+\hat{b}|^2_H\right)\ d\mu - \frac{1}{p_2}\int u^2\left(p_2 c^-\right)\ d\mu - \frac{1}{2}\int |\nabla_H u|^2_H\ d\mu \\
\underset{(1.18)}{\geq}\ & -\frac{1}{2p_1}\left(\int u^2\log u^2 - 1 + \int e^{p_1|b+\hat{b}|^2_H}\ d\mu\right) \\
& -\frac{1}{p_2}\left(\int u^2 \log u^2 - 1 + \int e^{p_2 c^-}\ d\mu\right) - \frac{1}{2}\int |\nabla_H u|^2_H\ d\mu \\
\underset{(DLS)}{\geq}\ & -\varepsilon\ Q(u,u) - \alpha
\end{aligned}$$

where $\varepsilon := \left(\frac{1}{2p_1}+\frac{1}{p_2}\right)\frac{2\lambda_1}{\delta}+\frac{1}{2}<1$ by assumption and

$$\alpha := \left(\frac{1}{2p_1}+\frac{1}{p_2}\right)2\lambda_2+\frac{1}{2p_1}\left(\|e^{|b+\hat{b}|_H^2}\|_{L^{p_1}(\mu)}^{p_1}-1\right) + \frac{1}{p_2}\left(\|e^{c^-}\|_{L^{p_2}(\mu)}^{p_2}-1\right) .$$

So, (SFP) (i) holds. To show (SFP) (ii) let $u,v\in\mathcal{F}C_b^\infty$. We may again assume that $\|u\|_{L^2(\mu)}=\|v\|_{L^2(\mu)}=1$. Then by (1.19) and (DLS) for any $p>0$

$$\begin{aligned}
&|\int\langle b-\hat{b},\nabla_H u\rangle_H\ v\ d\mu\\
\le\ & \frac{1}{\sqrt{p}}\left(\int v^2\left(p|b-\hat{b}|_H^2\right)d\mu\right)^{1/2}\left(\int|\nabla_H u|_H^2\ d\mu\right)^{1/2}\\
\underset{(1.18)}{\le}\ & \frac{1}{\sqrt{p}}\left(\int v^2\log v^2\ d\mu-1+\int e^{p|b-\hat{b}|_H^2}\ d\mu\right)^{1/2}\frac{1}{\sqrt{\delta}}\ Q(u,u)^{1/2}\\
\le\ & \frac{1}{\sqrt{p}}\left(\frac{2\lambda_1}{\delta}\ Q(v,v)+2\lambda_2+\|e^{|b-\hat{b}|_H^2}\|_{L^p(\mu)}^p-1\right)^{1/2}\frac{1}{\sqrt{\delta}}\ Q(u,u)^{1/2}\\
\le\ & K\ Q_1(u,u)^{1/2}\ Q_1(v,v)\ ,
\end{aligned}$$

where $K:=\frac{1}{\sqrt{p\delta}}\max\left(1,\frac{2\lambda_1}{\delta},2\lambda_2+\|e^{|b-\hat{b}|_H^2}\|_{L^p(\mu)}^p-1\right)$ and we used that $\|u\|_{L^2(\mu)}=\|v\|_{L^2(\mu)}=1$. $\square$

In order to apply all this to a *given* Kolmogorov equation

$$\frac{du}{dt}=Lu\ ,\ \ u(0,\cdot)=f$$

we have to find H and μ as above so that L is of type (1.18). Indeed, this is possible in a number of cases. We start investigating this in the next section by first looking at symmetric cases (i.e., where $\hat{b}=b\equiv 0$).

2. Symmetrizing measures

2.1 The classical finite dimensional case.

Consider the following operator on an open set $U\subset\mathbb{R}^d$

$$Lu:=\Delta u-\langle b,\nabla u\rangle_{\mathbb{R}^d}\ ,\ \ u\in D\,, \tag{2.1}$$

with domain $D:=C_0^\infty(U)$. Here Δ is the usual Laplacian on $\mathbb{R}^d$ and $b:U\to\mathbb{R}^d$ a vector field. So, we are in a special case of the situation considered in Subsection 1.4.1 (more precisely: $A\equiv Id_{\mathbb{R}^d}$, $\hat{b}\equiv 0$, $c\equiv 0$).

However, we do not assume that the components of b are in $L^\infty(U;dx)+L^d(U;dx)$, so cannot apply the results of Subsection 1.4.1. Instead we assume that

$$b = \nabla W$$

for some $W: U \to \mathbb{R}$, say C^1. Setting $\rho := e^{-W}$ we can rewrite (2.1) as

$$Lu = \Delta u + \langle \frac{\nabla \rho}{\rho}, \nabla u \rangle_{\mathbb{R}^d} \ ; \ u \in D \ ,$$

and see that for $\mu := \rho\ dx$ we have

$$-\int_U Lu\ v\ d\mu = \int_U \langle \nabla u, \nabla v \rangle_{\mathbb{R}^d}\ d\mu = -\int u\ Lv\ d\mu \ , \tag{2.2}$$

i.e., L is symmetric on $L^2(U;\mu)$. (This observation goes back to Kolmogorov, c.f. [Ko 37]). If we define for a vector field V on U

$$\begin{aligned} \mathrm{div}_\mu\ V :=&\ \mathrm{div}\ V + \langle \frac{\nabla \rho}{\rho}, V \rangle_{\mathbb{R}^d} \\ &(= -\nabla^{*,\mu} V \quad (\text{by } (2.2)))\ , \end{aligned} \tag{2.3}$$

(where $\nabla^{*,\mu}$ is the adjoint of ∇ on $L^2(\mu)$) we have

$$L = \mathrm{div}_\mu\ \nabla \text{ on } D = C_0^\infty(U)\ , \tag{2.4}$$

i.e, L is a so–called *μ–Laplacian*. So, indeed L fits in our framework, more precisely the one described in Subsection 1.4.2 (with $A \equiv Id_{\mathbb{R}}$, $b = \hat{b} \equiv 0$, $c = 0$, cf. (1.18)) at least if $U = \mathbb{R}^d$ and $\mu(\mathbb{R}^d) < \infty$. However, also the general case above with even less restrictions on ρ (resp. W) can be handled using coercive forms. We refer to [MR 92, Ch. II, Subsect. 2 a)].

The above case resp. its "manifold version" is, of course, quite easy since we are more or less given $H = \mathbb{R}^d$ and A in (1.18), i.e., we basically know in advance the underlying differential geometry and only have to find a measure μ that symmetrizes L, i.e., satisfies (2.2). So, this technique seems not suitable for infinite dimensions. It is, however, a remarkable fact, that for a large class of operators, so called *diffusion operators*, which includes all operators (also the infinite dimensional ones) which appear in these notes, this is sufficient. More precisely, if there exists a symmetrizing measure μ for such a diffusion operator, then there exists a "(co–)tangent bundle" (in a generalized sense) and a gradient ∇ so that L is a corresponding μ–Laplacian, $L = \mathrm{div}_\mu \nabla$ with $\mathrm{div} = -\nabla^{*,\mu}$. This result in its most general form (valid even in non–symmetric cases) is due to Andreas Eberle and we shall briefly explain this result in more detail in the next subsection. Subsequently, we shall see concrete infinite dimensional examples and applications.

2.2 Representation of symmetric diffusion operators

The material in this subsection is taken from [Eb 98, Ch. 1, Appendix B and Ch. 3 Appendix D, 7)].

Let $(E, \mathcal{B}, m)$ be a σ–finite measure space and fix $p \in [1, \infty)$.

Definition 2.1. A densely defined linear operator (L, D) on $L^p(E; m)$ is called *(abstract) diffusion operator* if

(i) $\varphi(u_1, \dots, u_k) \in D$ for all $k \in \mathbb{N}$; $u_1, \dots, u_k \in D$, and all $\varphi \in C^\infty(\mathbb{R}^k)$ such that $\varphi(0) = 0$, and, in this case

$$L\varphi(u_1, \dots, u_k) = \sum_{i=1}^{k} \partial_i \phi(u_1, \dots, u_k)\ Lu_i + \sum_{i,j=1}^{k} \partial_i \partial_j \phi(u_1, \dots, u_k)\ \Gamma(u_i, u_j)\ ,$$

where

$$\Gamma(u, v) := \frac{1}{2}\left(L(uv) - uLv - vLu\right)\ ;\ u, v \in D\ .$$

(ii) $\Gamma(u, u) \geq 0$ for all $u \in D$.

The bilinear operator $\Gamma : D \times D \to L(E; m)$ is called the *square field operator* of L, (where $L(E; m)$ denotes all m–equivalence classes of functions on E).

For the rest of this section we assume $L^2(E; m)$ to be separable and fix a diffusion operator (L, D) on $L^2(E; m)$. Note that by (i), D is automatically an algebra. We additionally assume that that $D \subset L^\infty(E; m)$, that D is dense in $L^2(E; m)$, and for simplicity that $1 \in D$ (in particular, $m(E) < \infty$). Note that (i) (applied to $k = 1$, $u_1 = 1$, $\varphi \equiv 1$) implies that $L1 = 0$.

Definition 2.2. Let $T'E = (T'_zE)_{z \in E}$ be a measurable field of Hilbert spaces over E (cf. e.g. [Eb 98, Ch. 3, Appendix D, 6)]). A map $d : D \to L^2(E \to T'E; m)$ (:= space of m–square integrable sections in $T'E$) is called L^2*–differential* (w.r.t. the co–tangent bundle $T'E$) if:

(i) The span of $\{u\ dv | u, v \in D\}$ is dense in $L^2(E \to T'E; m)$.

(ii) d is linear.

(iii) $d(u \cdot v) = u\ dv + v\ du$ for all $u, v \in D$.

Theorem 2.3. *(A. Eberle) Assume that m is a symmetrizing measure for L (i.e., $\int Lu\ v\ dm = \int\ u\ Lv\ dm$ for all $u, v \in D$). Then there exists a measurable field of Hilbert spaces $T'E = (T'_zE)_{z \in E}$, an L^2–differential $d : D \to L^2(E \to T'E; m)$ such that*

$$Lu = -d^* du\ ,\ u \in D\ , i.e.\ ,$$

$$-\int Lu\ v\ dm = \int_E (d_z u, d_z v)_{T'_zE}\ m\ (dz)\ \ \textit{for all } u, v \in D\ ,$$

where $d^* :\ dom\ (d^*) \subset L^2(E \to T'E; m) \to L^2(E; m)$ is the adjoint of d.

Proof. [Eb 98, Ch. 3, Theorem 3.11]. □

In [Eb 98] it is also proved that the above representation of L is unique "up to isomorphisms" and that there is a non–symmetric variant of Theorem 2.3 above.

In applications however, it is necessary to find natural and suitable representatives of $T'E$ and d. This will be done in quite simple "flat" cases in the next section. There, the choice of the tangent bundle and the gradient will be more or less obvious, so the only issue will be to find the symmetrizing measure μ. An important class of "non–flat" infinite dimensional examples for which the natural tangent bundle and gradient were only identified recently, will be discussed in Section 6.

2.3 Ornstein–Uhlenbeck type operators.

Let $(H, \langle \ , \ \rangle_H)$ be a Hilbert space and $(A, D(A))$ a self–adjoint operator on H such that for some $\varepsilon > 0$, $\langle Ah, h\rangle_H \geq \varepsilon \ \langle h, h\rangle_H$ for all $h \in H$. For $u \in \mathcal{F}C_b^\infty(D(A))$ (cf. (1.10) with $H = E$, $K := D(A) \subset H \equiv H'$)

$$Lu(x) := \ \mathrm{Trace}_H \ u''(x) - \langle x, A(\nabla_H u(x))\rangle_H \ , \ x \in H \ , \tag{2.4}$$

which makes sense, since by (1.11), $\nabla_H u(x) \in D(A)$. Here u'' denotes the second Fréchet derivative of u. Following the same idea as in Subsection 2.1 the correspon-ding symmetrizing measure should be given by the following heuristic formula

$$\mu(dx) = \text{“}\exp\left[-\tfrac{1}{2}\langle Ax, x\rangle\right] \ d^\infty x\text{”} \tag{2.5}$$

(where "$d^\infty x$" denotes "infinite dimensional Lebesgue measure"), i.e., μ should be a mean–zero Gaussian measure with covariance operator A^{-1}. Of course, μ will only exist on H if A^{-1} is trace class. But it always exists on a larger Hilbert (or Banach) space E as we shall see below. So, to find a symmetrizing measure in general we have to consider L on a larger state space E. In fact, E can be chosen in such a way that, as we shall see later, the solution of the corresponding Kolmogorov equation is given explicitly by a Mehler formula. More precisely, we have the following result:

Theorem 2.4. *There exists a separable real Hilbert space E and a C^0–semigroup $\left(e^{-tA_E}\right)_{t\geq 0}$ of linear operators on E with generator $(-A_E)$ such that*

(i) $H \subset D(A_E)$ with dense Hilbert–Schmidt embedding (where $D(A_E)$ is equipped with the graph norm given by A_E).

(ii) e^{-tA} is the restriction of e^{-tA_E} to H for all $t \geq 0$.

This result was first obtained in [BRS 97, Theorem 1.6] (even for the non–symmetric case, i.e., arbitrary generators A of C^0–semigroups on H). The simplest proof is given in [FR 97, Prop. 2.3]. Let A_E' be the adjoint of the generator A_E. Then A_E' is a linear operator on E' and if $K := D(A_E')$ (i.e., K denotes its domain), then

$$K \subset E' \subset D(A) \subset H \subset E \ , \tag{2.6}$$

and K is dense in E'. So, looking at test functions in the smaller space $\mathcal{F}C_b^\infty(K)$, we see that they have unique extensions to E, and L with domain $\mathcal{F}C_b^\infty(K)$ becomes an operator on the state space E, i.e., for $u \in \mathcal{F}C_b^\infty(K)$

$$Lu(x) := \Delta_H u(x) - \ {}_E\langle x, A_E'(\nabla_H u(x))\rangle_{E'} \ , \ x \in E \ , \tag{2.7}$$

where for $x \in E$

$$\begin{aligned} \Delta_H u(x) :&= \mathrm{Trace}_H \ u''(x) \ , \\ &= \sum_{i,j=1}^m \partial_i \partial_j f(\ell_1(x), \ldots, \ell_m(x)) \ \langle \ell_i, \ell_j\rangle_H \ , \end{aligned} \tag{2.8}$$

if $u = f(\ell_1, \ldots, \ell_m)$, $\ell_1, \ldots, \ell_m \in K$, $f \in C_b^\infty(\mathbb{R}^m)$, i.e., Δ_H is the *Gross–Laplacian*. Clearly (2.7) is exactly (2.4), if $x \in H$. u'' in (2.8) is the second Fréchet derivative of u on E and $\nabla_H u$ in (2.7) is defined as in (1.12).

Let now μ be the Gaussain probability measure on $(E, \mathcal{B}(E))$ with Fourier transform given by

$$\int_E e^{i_{E'}\langle \ell, x\rangle_E} \ \mu(dx) = e^{-\frac{1}{2}\|A^{-1/2}\ell\|_H^2} \ , \ \ell \in E' \ . \tag{2.9}$$

μ exists since the embedding $H \subset E$ is Hilbert–Schmidt and $A^{-1/2}$ is bounded (cf. e.g. [Ya 89, Theorem 3.1]). Then it is easy to check that for all $u, v \in \mathcal{F}C_b^\infty(K)$

$$\begin{aligned} \int_E Lu(x) \ v(x) \ \mu(dx) &= \int_E u(x) \ Lv(x) \ \mu(dx) \\ &= \int_E \langle \nabla_H u(x), \nabla_H v(x)\rangle_H \ \mu(dx) \\ &=: \mathcal{E}_\mu(u, v) \ . \end{aligned} \tag{2.10}$$

In particular, $(\mathcal{E}_\mu, \mathcal{F}C_b^\infty(K))$ is closable and its closure $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ is a symmetric, positive definite, densely defined bilinear form, hence in particular a coercive form on $L^2(E;\mu)$. $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ is even a Dirichlet form (cf. [MR 92, Ch. II, Prop. 3.8]). So, the results of Subsect. 1.4.2 (with $A \equiv Id_H$, $b = \hat{b} \equiv 0$, $c \equiv 0$, $\alpha = 0$) apply to give a solution to the corresponding Kolmogorov equation (1.17) via the sub–Markovian C^0–semigroup $(T_t)_{t\geq 0}$ on $L^2(E;\mu)$ associated with $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$.

We note that if in this case for vector fields $V : E \to H$ of type

$$V(x) := \sum_{i=1}^n u_i \ \ell_i \tag{2.11}$$

$\ell_i \in K$, $u_i \in \mathcal{F}C_b^\infty(K)$, $n \in \mathbb{N}$, we define

$$\mathrm{div}_\mu \ V := \sum_{i=1}^n \left(\langle \nabla_H u_i, \ell_i\rangle_H - u_i \ {}_{E'}\langle A'_E \ell_i, \cdot\rangle_E\right) \ ,$$

then

$$L = \ \mathrm{div}_\mu \ \nabla_H \ .$$

By our special choice of E the semigroup $(T_t)_{t\geq 0}$ is given explicitly by a *Mehler formula.* In fact, it follows by [BR 95a, Sect. 5] that for any bounded $\mathcal{B}(E)$–measurable $f : E \to \mathbb{R}$, $t \geq 0$, and μ–a.e. $x \in E$

$$\begin{aligned} T_t f(x) &= \int_E f\left(e^{-tA_E} x + \sqrt{1 - e^{-2tA_E}} x'\right) \mu \ (dx') \\ &=: p_t f(x) \ . \end{aligned} \tag{2.12}$$

Remark 2.5. (i) Obviously, $(p_t)_{t\geq 0}$ from (2.12) is a *Feller semigroup*, i.e., $p_t(C_b(E)) \subset C_b(E)$ for all $t \geq 0$.
(ii) It can be checked that for all $u \in \mathcal{F}C_b^\infty(K)$

$$\frac{d}{dt}\ (p_t\, u\,(x))_{|t=0} = Lu(x) \text{ for all } x \in A_E \tag{2.13}$$

(cf. the proof of [BR 95a, Thm. 5.3]). So, restricting to H again we obtain that $p_t u$ is a solution to Kolmogorov's equations on our *original* Hilbert space H. So, if one is satisfied with initial conditions $u \in \mathcal{F}C_b^\infty(K)$ (or $u \in C_b^2(E)$), and "pointwise–differentiation" (i.e., for fixed x) in (2.13), one can avoid the machinery of sectorial forms in this special case by using Theorem 2.4 and the Mehler formula (2.12).
(iii) The above shows that really in general $H \subsetneqq E$ in this Gaussian case. Therefore, the framework presented in Subsection 1.4.2 was chosen in that way.
(iv) $(T_t)_{t\geq 0}$ above is just the first quantization of $\left(e^{-tA}\right)_{t\geq 0}$ and the generator of $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ is the second quantization of A (see e.g. [AR 91, Sect. 7 I]). Therefore, (DLS) holds for (E, H, μ) even with $\lambda_2 = 0$ (cf. [Re S 75, Thm. X.61] and [Gr 93, Thm. 3.12]). Therefore, the results of Subsect. 1.4.2 also apply, if we add small perturbations given by b, $\hat{b}$, c to L in (2.4).

2.4 Operators with non–linear drift.

Let $(E, \langle\ ,\ \rangle_E)$, $(H, \langle\ ,\ \rangle_H)$ be separable Hilbert spaces such that

$$H \subset E \text{ with dense Hilbert–Schmidt embedding }. \tag{2.14}$$

Again we identify H' with H so

$$E' \subset H \subset E$$

(cf. (1.9)). Let $\{\ell_n | n \in \mathbb{N}\} \subset E'$ be an orthonormal basis of H so that $\{\alpha_n \ell_n | n \in \mathbb{N}\}$, resp. $\{\gamma_n \ell_n | n \in \mathbb{N}\}$ are orthonormal basises for E' resp. E for some α_n, $\gamma_n \in (0, \infty)$, $n \in \mathbb{N}$. Define the projectors

$$P_n x := \sum_{m=1}^{n}\ {}_{E'}\langle \ell_m, x\rangle_E\ \ell_m\ ,\ x \in E\ . \tag{2.15}$$

The following result was obtained very recently (cf. [AKRT 98]).

Theorem 2.6. *Let E_1 be a Hilbert space containing E obtained by renormalizing the basis vectors ℓ_m, $m \in \mathbb{N}$, and let $\beta : E \to E_1$ be $\mathcal{B}(E)/\mathcal{B}(E_1)$ measurable such that the following conditions are satisfied:*
(i) There exists $N \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ there exists $c_m \in (0, \infty)$ such that

$$\left|{}_{E_1'}\langle \ell_m, \beta(x)\rangle_{E_1}\right| \leq c_m\,(1 + \|x\|_E)^N \text{ for all } x \in E\ .$$

(ii) There exist continuous $\tilde{\beta}_n = (\beta_{n1}, \ldots, \beta_{nn})$: $\mathbb{R}^n \to \mathbb{R}^n$, $n \in \mathbb{N}$, such that for all $n \in \mathbb{N}$

(a) *$\tilde{\beta}_n$ is the* vector logarithmic derivative *of some probability measure μ_n on $\mathbb{R}^n$, i.e.,*

$$\int_{\mathbb{R}^n} \partial_m u \; d\mu_n = -\int_{\mathbb{R}^n} u \; \beta_{nm} \; d\mu_n \text{ for all } 1 \le m \le n \text{ and all } u \in C_0^\infty(\mathbb{R}^n) \; ;$$

(b) *the maps $\beta_n : E \to E'$ defined by*

$$\beta_n(x) := \sum_{m=1}^{n} \beta_{nm}(P_n x) \; \ell_m \; , \; x \in E \; ,$$

satisfy the following coercivity condition (uniformly in n): there exist $M \in \left[\frac{N}{2}, \infty\right)$, $\delta > 0$, and $C(M, \delta) > 0$ such that for all $n \in \mathbb{N}$ and all $x \in E \setminus \{0\}$

$$-\langle \beta_n(x), x \rangle_E \ge 2M + \| P_n \, E' \subset P_n \, H \|_{H.S.}^2 \\ + \delta - C(M, \delta) \; \| P_n x \|_E^{-2M} \; ;$$

(c) *for all $m \in \mathbb{N}$, $\varepsilon > 0$ there exists $n(\varepsilon, m) \in \mathbb{N}$ such that for all $n > n(\varepsilon, m)$ and all $x \in E$*

$$\left| {}_{E_1'}\langle \ell_m, \beta(x) - \beta_n(x) \rangle_{E_1} \right| \le \varepsilon \, (1 + \|x\|_E)^N \; .$$

Then there exists a probability measure μ on $(E, \mathcal{B}(E))$ such that μ is a symmetrizing measure for the operator

$$L_0 u(x) := \Delta_H u(x) + \; {}_{E_1}\langle \beta(x), \nabla_H u(x) \rangle_{E_1'} \tag{2.16}$$

for all $u \in \mathcal{F}C_b^\infty(K)$, where $K(\subset E_1')$ is the linear span of $\{\ell_n | n \in \mathbb{N}\}$.

For the proof and further details, in particular, the relation and application to Gibbs measures in statistical mechanics we refer to [AKRT 98]. So as in the previous subsection all results in Subsection 1.4.2 with $b = \hat{b} \equiv 0$, $c \equiv 0$ apply to L_0.
We now give a concrete example to which Theorem 2.6 applies.

Example 2.7. Consider the lattice $\mathbb{Z}^d$ and the space $\ell^2(\mathbb{Z}^d)$ of square summable sequences indexed by $\mathbb{Z}^d$, so

$$\ell^2(\mathbb{Z}^d) \subset \mathbb{R}^{\mathbb{Z}^d} \; .$$

Define , in addition, for $p \in \mathbb{Z}$ the weighted sequence spaces

$$\ell_p^2(\mathbb{Z}^d) := \{ x \in \mathbb{R}^{\mathbb{Z}^d} \mid \|x\|_{\ell_p^2}^2 := \sum_{k \in \mathbb{Z}^d} (1 + |k|)^{2p} x_k^2 < \infty \} \; . \tag{2.17}$$

Let $\{\ell_k | k \in \mathbb{Z}^d\}$ be the standard orthonormal basis in $\ell^2(\mathbb{Z}^d)$ and $A := (a_{kj})_{k,j \in \mathbb{Z}^d}$ giving a linear operator on $\ell^2(\mathbb{Z}^d)$ such that

(i) there exists $\delta > 0$ such that $\langle Ax, x \rangle_{\ell^2} \ge \delta \; \|x\|_{\ell^2}^2$ for all $x \in \ell^2(\mathbb{Z}^d)$;

(ii) $a_{kj} = a(k-j)$ for all $k, j \in \mathbb{Z}^d$ ("translation invariance");
(iii) there exists $r > 0$ such that

$$a(k-j) = 0 \ \text{ if } |k-j|_{\mathbb{R}^d} > r \ \ (\text{"finite range"}).$$

Let P be a polynomial on $\mathbb{R}$ of even degree,

$$P(t) = c_N t^N + \cdots + c_1 t + c_0 \ , \ c_N > 0 \ , \ N \in \mathbb{N} \ , \ N \text{ even.} \tag{2.18}$$

Define $\beta := (\beta_k)_{k \in \mathbb{Z}^d} : \ell^2_{-p_1}(\mathbb{Z}^d) \to \ell^2_{-p_2}(\mathbb{Z}^d)$, where $p_2 > Np_1 + d/2$, $p_1 > d/2$, and

$$\beta_k(x) := -\sum_{j \in \mathbb{Z}^d} a(k-j) x_j - P'(x_k) \ , \ k \in \mathbb{Z}^d \ , \ x \in \ell^2_{-p_1}(\mathbb{Z}^d) \ . \tag{2.19}$$

Then Theorem 2.6 applies with $E := \ell^2_{-p_1}(\mathbb{Z}^d)$, $E_1 := \ell^2_{-p_2}(\mathbb{Z}^d)$, $H := \ell^2(\mathbb{Z}^d)$ to give a symmetrizing probability measure μ on E for the operator

$$L_0 u(x) = \Delta_H u(x) - {}_{\ell^2_{-p_2}}\langle Ax + (P'(x_k))_{k \in \mathbb{Z}^d} \ , \ \nabla_H u(x) \rangle_{\ell^2_{p_2}} \ ,$$

$x \in \ell^2_{-p_1}(\mathbb{Z}^d)$, $u \in \mathcal{F}C_b^\infty(K)$ with $K :=$ lin. span $\{\ell_k | k \in \mathbb{Z}^d\}$.

Remark 2.8. The measure μ in Example 2.7 is well–known. It is a Gibbs measure for the so–called $P(\Phi)_d$–lattice model (cf. [AKR 97, Subsections 5.1, 5.2]).

Furthermore, we have (DLS) in the case of Example 2.7 for small c_j, $1 \le j \le N$:

Theorem 2.9. *Let $c_0, \ldots, c_N$ in (2.18) be sufficiently small. Then (DLS) holds with $\lambda_2 = 0$ for (E, H, μ) as in Example 2.7. In particular, all results from Subsect. 1.4.2 apply.*

Proof. [AKR 95, Theorem 6]. □

The corresponding semigroup $(T_t)_{t \ge 0}$ constructed as in Subsect. 1.4.2 in the special case $A \equiv Id_H$, $\hat{b} \equiv 0$, then satisfies the Kolmogorov equation (1.17) with

$$\begin{aligned} Lu = \Delta_H u - {}_{\ell^2_{-p_2}}\langle Ax + (P'(x_k))_{k \in \mathbb{Z}^d} + b(x) \ , \ \nabla_H u(x) \rangle_{\ell^2_{p_2}} \\ - c(x) u(x) \ . \end{aligned} \tag{2.20}$$

In this case for vector fields $V : E \to H$ as in (2.11) we define

$$\mathrm{div}_\mu V(x) := \sum_{i=1}^n \langle \nabla_H u_i(x), \ell_i \rangle_H - u_i(x) \ {}_{\ell^2_{-p_2}}\langle Ax + (P'(x_k))_{k \in \mathbb{Z}^d} \ , \ \ell_i \rangle_{\ell^2_{p_2}} \ . \tag{2.21}$$

Then

$$L_0 = \ \mathrm{div}_\mu \nabla_H$$

and

$$L = \ \mathrm{div}_\mu \nabla_H - {}_{\ell^2_{-p_2}}\langle b, \nabla_H \cdot \rangle_{\ell^2_{p_2}} - c \ .$$

3. Non-sectorial cases: perturbations by divergence free vector fields

So, far we were only able to solve cases of Kolmogorov's equation on L^p (more precisely L^2) in symmetric situations or "small perturbations" (i.e., in the presence of a sector condition) thereof. In this section we shall look at a class of infinite dimensional operators which neither have a symmetrizing measure, nore are sectorial, but are perturbations of such by a divergence free vector field. This class is of special importance for applications (cf. Subsect. 3.4 and Sect. 4 below).

3.1 Diffusion operators on $L^p(E;m)$.

All the material in this subsection is taken from [Eb 98, Ch. 1, Appendix A and B]. Let $(E,\mathcal{B},m)$ be a σ–finite measure space and $p\in[1,\infty)$. Let (L,D) be a linear operator on $L^p(E;m)$ with dense domain D.

Lemma 3.1. *(i) (L,D) is dissipative if and only if*

$$\int sgn(u)\ |u|^{p-1}\ Lu\ dm\le 0 \text{ for all } u\in D\ ,$$

where $sign(x)=1_{(0,\infty)}-1_{(-\infty,0)}$.
(ii) Let $\lambda\ge 0$. Suppose that for every increasing $\psi\in C(\mathbb{R})$ such that $\psi(0)=0$, and $\psi(s)\le|s|^{p-1}$ for all $s\in\mathbb{R}$, we have

$$\int \psi\circ u\ Lu\ dm\ \le\lambda\int|u|^p\ dm \text{ for all } u\in D\ .$$

Then $(L-\lambda,D)$ is dissipative on $L^p(E;m)$.

Proof. (i) is obvious.
(ii) [Eb 98, Ch. 1, Lemma 1.7]. Choose smooth increasing functions $\psi_n:\mathbb{R}\to\mathbb{R}$, $n\in\mathbb{N}$, such that $\psi_n(s)=0$ if $|s|\le n^{-1}$, $\psi_n(s)=|s|^{p-1}\cdot$ sgn (s) if $|s|\ge 2n^{-1}$, and $|\psi_n(s)|\le|s|^{p-1}$ for all s. Then for any $u\in D$, $\psi_n\circ u$ is in $L^q(E;m)$, $\frac{1}{p}+\frac{1}{q}=1$, and

$$\int \text{sgn }(u)\ |u|^{p-1}(L-\lambda)\ u\ dm=\lim_{n\to\infty}\int\psi_n\circ u\ Lu\ dm-\lambda\int|u|^p\ dm\le 0$$

by dominated convergence. □

The following is important to check dissipativity.

Proposition 3.2. *Suppose (L,D) is a diffusion operator (in the sense of Definition 2.1) and that for some $\alpha\in[0,\infty)$ m is a* sub–invariant measure *for $(L-\alpha,D)$, i.e.,*

$$u\,,\ Lu\in L^1(E;m) \text{ and } \int Lu\ dm\le\alpha\int u\ dm \text{ for all } u\in D \text{ with } u\ge 0\ . \quad (3.1)$$

Then $\left(L - \frac{\alpha}{p}, D\right)$ *is dissipative on* $L^p(E;m)$. *In particular, the closure* $\overline{L}$ *of* (L, D) *generates a* C^0*–semigroup on* $L^p(E;m)$ *if* $(\lambda - L)(D)$ *is dense in* $L^p(E;m)$ *for some* $\lambda > \frac{\alpha}{p}$.

Proof. [Eb 98, Ch. 1, Lemma 1.8] Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth increasing function such that $|\psi(s)| \leq |s|^{p-1}$ for all s, and let $\Psi := \int_0^\bullet \psi(s)\,ds$. Then $0 \leq \Psi(t) \leq |t|^p/p$ for all $t \in \mathbb{R}$, and

$$L(\Psi \circ f) = \psi \circ f \ Lf + \psi' \circ f \ \Gamma(f, f) \geq \psi \circ f \ Lf \ m\text{–a.e.}$$

for all $f \in D$. Hence, by the sub–invariance,

$$\begin{aligned} \int \psi \circ f \ Lf \ dm \leq & \int L(\Psi \circ f) \ dm \\ \leq & \ \alpha \int \Psi \circ f \ dm \leq \frac{\alpha}{p} \int |f|^p \ dm \end{aligned}$$

for all $f \in D$. This implies the dissipativity by Lemma 3.1 (ii). The remaining part of the assertion now follows by Theorem 1.3 above. □

Condition (3.1) also implies that the C^0–semigroup in the last part of Proposition 3.2 is sub–Markovian since we have:

Theorem 3.3. *Let* (L, D) *be a diffusion operator. Suppose (3.1) holds and that* $(\lambda - L)(D)$ *is dense in* $L^p(E;m)$ *for some* $\lambda > \alpha/p$. *Then the semigroup* $(T_t)_{t \geq 0}$ *generated by the closure* $\overline{L}$ *of* (L, D) *is sub–Markovian.*

Proof. [Eb 98, Ch. 1, Lemma 1.9]. □

Below we shall mainly be working in the case $p = 1$. The corresponding results then have automatically consequences for $L^p(E;m)$ by the following:

Lemma 3.4. *Let* $(T_t^{(p)})_{t \geq 0}$ *be a sub–Markovian* C^0*–semigroup on* $L^p(E;m)$ *and let* $r \in [p, \infty)$. *Then the restrictions of* $T_t^{(p)}$, $t \geq 0$, *to* $L^r(E;m) \cap L^p(E;m)$ *are bounded w.r.t.* $\| \ \|_{L^r(m)}$. *The unique continuous extensions* $T_t^{(r)}$, $t \geq 0$, *of these operators to* $L^r(E;m)$ *form a sub–Markovian* C^0*–semigroup on* $L^r(E;m)$. *Furthermore, if* $(L^{(r)}, D(L^{(r)}))$ *denotes the generator of* $(T_t^{(r)})_{t \geq 0}$ *on* $L^r(E;m)$, *then*

$$D_1 := \{u \in D(L^{(p)}) \mid u, L^{(p)}u \in L^r(E;m)\} \subset D(L^{(r)})$$

and

$$L^{(p)}u = L^{(r)}u \text{ for all } u \in D_1 \ .$$

If $T_t^{(p)}$, $t \geq 0$, *are contractions on* $L^p(E;m)$, *then so are* $T_t^{(r)}$, $t \geq 0$, *on* $L^r(E;m)$.

Proof. [Eb 98, Ch. 1, Lemma 1.11]. □

Lemma 3.5. *Let $(T_t)_{t\geq 0}$ be a C^0–semigroup of* symmetric *operators on $L^2(E;m)$. Then each T_t, $t \geq 0$, is a contraction and the assertions of Lemma 3.4 hold for all $r \in [1,\infty)$.*

Proof. [Dav 89, Thm. 1.4.1]. □

3.2 Solution of Kolmogorov equations on $L^1(E;m)$.

The results of this subsection are due to W. Stannat (cf. [St 97]). We start with the finite dimensional case.

3.2.1 $E := \mathbb{R}^d$.

Let $E := \mathbb{R}^d$ and $\varphi \in H^{1,2}_{\text{loc}}(\mathbb{R}^d; dx)$, i.e., φ is locally in the Sobolev space (w.r.t. Lebesgue measure dx) of order 1 in $L^2(\mathbb{R}^d; dx)$. Assume $\varphi \neq 0$ dx–a.e. and define

$$\begin{aligned} \mu &:= \varphi^2 \, dx \ , \\ \beta^\mu &:= 2\frac{\nabla\varphi}{\varphi} \left(\in L^2_{\text{loc}}(\mathbb{R}^d;\mu)\right) \ . \end{aligned} \tag{3.2}$$

β^μ is hence the vector logarithmic derivative of μ. Define

$$\mathcal{E}_\mu(u,v) := \int_{\mathbb{R}^d} \langle \nabla u, \nabla v\rangle_{\mathbb{R}^d} \, d\mu \ ; \ u,v \in C_0^\infty(\mathbb{R}^d) \ . \tag{3.3}$$

Then $(\mathcal{E}_\mu, C_0^\infty(\mathbb{R}^d))$ is closable on $L^2(\mathbb{R}^d;\mu)$ and its closure $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ is a Dirichlet form (cf. Subsect. 1.4.1 and [MR 92, Ch. I, Subsect. 2 a)]). Let $\hat{L}_0$ denote the corresponding generator (cf. Proposition 1.10). Then $\widehat{L}_0$ is nothing but the Friedrichs' extension of

$$\begin{aligned} L_0 &= \Delta + \langle \beta^\mu, \nabla\cdot\rangle_{\mathbb{R}^d} \\ &= \ \text{div}_\mu \nabla \ , \ \ D(L_0) = C_0^\infty(\mathbb{R}^d) \end{aligned} \tag{3.4}$$

(cf. (2.4)). Let $(T_t^0)_{t\geq 0}$ be the sub–Markovian C^0–semigroup of contractions generated by $\hat{L}_0$ on $L^2(\mathbb{R}^d;\mu)$.

Remark 3.6. In this and the next subsection the roles of $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ (resp. $\hat{L}_0, L_0$) could be replaced by a more general coercive form (resp. corresponding operators) as in Subsecton 1.4.1. Only for simplicity L_0 was chosen as in (3.4).

Now fix $b \in L^2_{\text{loc}}(\mathbb{R}^d \to \mathbb{R}^d;\mu)$ satisfying the following "divergence condition"

$$\text{(DC)} \qquad \int \langle b, \nabla u\rangle_{\mathbb{R}^d} \, d\mu = 0 \text{ for all } u \in C_0^\infty(\mathbb{R}^d) \, ,$$

or shortly written

$$\text{div}_\mu \ b = 0 \ . \tag{3.5}$$

Now consider the operator

$$\begin{aligned} L_b u &:= L_0 u - \langle b, \nabla u\rangle_{\mathbb{R}^d} \\ &= \Delta u + \langle \beta^\mu - b, \nabla u\rangle_{\mathbb{R}^d} \ , \ \ u \in C_0^\infty(\mathbb{R}^d) \ . \end{aligned} \tag{3.6}$$

Remark 3.7. (DC) is equivalent to

$$\int L_b u \; d\mu = 0 \text{ for all } u \in C_0^\infty(\mathbb{R}^d) \;, \tag{3.7}$$

i.e., μ is an invariant measure for L_b in the sense of Sect. 4 below. This holds since obviously, $\int L_0 u \; d\mu = 0$ for all $u \in C_0^\infty(\mathbb{R}^d)$. Hence by Proposition 3.2, $(L_b, C_0^\infty(\mathbb{R}^d))$ is, in particular dissipative, hence closable (cf. Remark 1.2) on $L^1(E;\mu)$.

Concerning the existence of solutions to Kolmogorov's equation for at least one extension $\hat{L}_b$ of $(L_b, C_0^\infty(\mathbb{R}^d))$, i.e. to

$$\frac{du}{dt} = \hat{L}_b u \;,\; u(0,\cdot) = f \tag{3.8}$$

which are positivity preserving w.r.t. the initial condition f, we have the following result by Wilhelm Stannat. Below for $W \subset L^1(\mathbb{R}^d;\mu)$ we set

$$W_b := \{f \in W | f \text{ bounded}\} \,,\; W_0 := \{f \in W | \text{ supp } |f|\cdot\mu \text{ compact}\} \,,\; W_{0,b} := W_0 \cap W_b \,.$$

Theorem 3.8. *Suppose condition (DC) holds. Then there exists a closed extension $\hat{L}_b$ of $(L_b, C_0^\infty(\mathbb{R}^d))$ on $L^1(\mathbb{R}^d;\mu)$ which generates a sub–Markovian C^0–semigroup $(T_t^b)_{t\geq 0}$ of contractions on $L^1(\mathbb{R}^d;\mu)$. $\hat{L}_b$ has the following properties:*
(i) $D(\hat{L}_b)_b \subset D(\mathcal{E}_\mu)$ and

$$\mathcal{E}_\mu(u,v) + \int \langle b, \nabla u\rangle v \; d\mu = -\int \hat{L}_b u \; v \; d\mu \qquad \text{for all } u \in D(\hat{L}_b)_b \,,\; v \in D(\mathcal{E}_\mu)_{0,b} \,. \tag{3.9}$$

In particular, $\hat{L}_b$ is uniquely determined by the (non–coercive) bilinear form on the left hand side of (3.9) (since $(D(\hat{L}_b)_b$ is dense in $D(\hat{L}_b)$ w.r.t. graph norm).

(ii)
$$\mathcal{E}_\mu(u,u) \leq -\int \hat{L}_b u \; u \; d\mu \text{ for } u \in D(\hat{L}_b)_b \,. \tag{3.10}$$

Proof. [St 97, Prop. 1.1]. □

As a consequence of Theorem 3.8 we get the desired solution to (3.8). However, it is not at all clear whether $(\hat{L}_b, D(\hat{L}_b))$ is unique. There could be many such extensions of the operator L_b explicitly given on $C_0^\infty(\mathbb{R}^d)$ in (3.6). The same drawback we also have in the case $b \equiv 0$, where by (3.9) $\hat{L}_0$ is just the Friedrichs' extension of $(L_0, C_0^\infty(\mathbb{R}^d))$ considered on $L^1(\mathbb{R}^d;\mu)$ (cf. Lemma 3.5). Also there, we could have many extensions that are generators. This problem (related to Theorem 1.6) will be addressed in Subsect. 3.3 below.

3.2.2 $E :=$ topological vector space.

Let E, H, μ and K be as in Subsect. 1.4.2, hence in particular $\mu(E) < \infty$. Define

$$\mathcal{E}_\mu(u,v) := \int_E \langle \nabla_H u(x), \nabla_H v(x)\rangle_H \ \mu(dx) \ ; \ u,v \in \mathcal{F}C_b^\infty(K) \ . \tag{3.11}$$

Suppose that

(C) $$(\mathcal{E}_\mu, \mathcal{F}C_b^\infty(K)) \ \text{ is closable on } L^2(E;\mu) \ .$$

Then the closure is a Dirichlet form (cf. Subsect. 1.4.2 and [MR 92, Ch. I, Proposition 3.8]). Let $\hat{L}_0$ be the associated generator and $(T_t^0)_{t\geq 0}$ the corresponding sub–Markovian C^0–semigroup of contractions on $L^2(E;\mu)$. Note that since obviously, $1 \in D(\hat{L}_0)$ and $\hat{L}_0 1 = 0$, it follows that $T_t^0 1 = 1$ for all $t \geq 0$. So, $(T_t^0)_{t\geq 0}$ is even Markovian.

Now let $b \in L^2(E \to H;\mu)$ again satisfying the "divergence condition"

$$\operatorname{div}_\mu \ b = 0 \ , \tag{3.12}$$

more precisely,

(DC) $$\int_E \langle b, \nabla_H u\rangle_H \ d\mu = 0 \text{ for all } u \in \mathcal{F}C_b^\infty(K) \ .$$

We assume for simplicity

$$\mathcal{F}C_b^\infty(K) \subset D(\hat{L}_0) \tag{3.13}$$

(see [St 97, Sect. 4] how to avoid this condition) and consider the operator

$$L_b u := \hat{L}_0 u - \langle b, \nabla_H u\rangle_H \ , \ u \in \mathcal{F}C_b^\infty(K) \ . \tag{3.14}$$

Remark 3.9. We note that also in this case (DC) is equivalent to

$$\int L_b u \ d\mu = 0 \text{ for all } u \in \mathcal{F}C_b^\infty(K) \ . \tag{3.15}$$

Consequently, $(L_b, \mathcal{F}C_b^\infty(K))$ is dissipative, hence closable on $L^1(E;\mu)$ (cf. Proposition 3.2, Remark 1.2 respectively).

Theorem 3.10. *Suppose conditions (C), (DC), and (3.13) hold. Then there exists a closed extension $\hat{L}_b$ of $(L_b, \mathcal{F}C_b^\infty(K))$ on $L^1(E;\mu)$ generating a Markovian C^0–semigroup $(T_t^b)_{t\geq 0}$ of contractions on $L^1(E;\mu)$. $\hat{L}_b$ has the following properties.*
(i) It is the closure of L_b, but considered with the larger domain $D(\hat{L}_0)_b$.
(ii) $D(\hat{L}_b)_b \subset D(\mathcal{E}_\mu)$ and

$$\mathcal{E}_\mu(u,v) + \int_E \langle b, \nabla_H u\rangle_H \ v \ d\mu = -\int_E \hat{L}_b u \ v \ d\mu \qquad \textit{for all } u \in D(\hat{L}_b)_b \ , \ v \in D(\mathcal{E}_\mu)_b \ . \tag{3.16}$$

In particular, $\hat{L}_b$ is uniquely determined by the (non–coercive) bilinear form on the left hand side of (3.16).

(iii)
$$\mathcal{E}_\mu(u,u) \leq -\int \hat{L}_b u\, u\, d\mu \text{ for all } u \in D(\hat{L}_b)_b\,. \tag{3.17}$$

Proof. [St 97, Prop. 4.1]. □

As in Subsect. 3.2.1 we obtain the desired solution of the corresponding Kolmogorov equation with the same drawback concerning uniqueness.

Remark 3.11. We emphasize that conditions (C) and (3.13) hold for μ defined as in the cases discussed in Subsect. 2.3 and Example 2.7.

3.3 Uniqueness problem.

Let $(E, \mathcal{B}, m)$ be a σ–finite measure space and $p \in [1, \infty)$. Let (L, D) be a linear operator on $L^p(E;m)$ with dense domain D. In the preceding subsection we presented results (for $p = 1$) ensuring that (L, D) has a closed extension $\hat{L}$ which generates a C^0–semigroup on $L^p(E;m)$. In this subsection we are concerned with the question whether $\hat{L}$ is the only extension generating a C^0–semigroup on $L^p(E;m)$. By Theorem 1.6 this is equivalent to the question whether the closure $\overline{L}$ of (L, D) on $L^p(E;m)$ generates a C^0–semigroup on $L^p(E;m)$.

Most of the results below are again taken from [St 97].

We recall that a C^0–semigroup $(T_t)_{t\geq 0}$ on $L^1(E;m)$ is called *conservative*, if for its dual semigroup $(T_t^*)_{t\geq 0}$ on $L^\infty(E;m)$ we have that $T_t^* 1 = 1$ for one (hence all) $t \geq 0$.

3.3.1 $E := \mathbb{R}^d$.

We consider the situation in Subsection 3.2.1 adopting all notations introduced there.

Remark 3.12. Obviously, $(T_t^0)_{t\geq 0}$ is conservative if $\mu(\mathbb{R}^d) < \infty$.

Theorem 3.13. *The closure $\overline{L}_0$ on $L^1(\mathbb{R}^d;\mu)$ of $(L_0, C_0^\infty(\mathbb{R}^d))$ generates a C^0–semigroup if and only if $(T_t^0)_{t\geq 0}$ (i.e., the C^0–semigroup generated by the Friedrichs' extension of $(L_0, C_0^\infty(\mathbb{R}^d))$ on $L^1(E,\mu)$) is conservative. This, in particular holds if $\mu(\mathbb{R}^d) < \infty$.*

Proof. [St 97, Thm. 2.1] (generalizing [Dav 85] to the case of non–smooth coefficients). □

As in Subsection 3.2.1 consider now

$$L_b = \Delta + \langle \beta^\mu - b, \nabla \cdot \,\rangle_{\mathbb{R}^d} \text{ on } C_0^\infty(\mathbb{R}^d)$$

with $b \in L^2_{\mathrm{loc}}(\mathbb{R}^d \to \mathbb{R}^d;\mu)$ satisfying (DC). Define $D(\mathcal{E}_\mu)_{\mathrm{loc}}$ to be the set of all $u : \mathbb{R}^d \to \mathbb{R}$ such that $\chi \cdot u \in D(\mathcal{E}_\mu)$ for all $\chi \in C_0^\infty(\mathbb{R}^d)$. Then the conservativity of $(T_t^b)_{t\geq 0}$ can be characterized as follows:

Proposition 3.14. *The following are equivalent*

(i) $(T_t^b)_{t\geq 0}$ *is conservative.*

(ii) There exist $\chi_n \in D(\mathcal{E}_\mu)_{\mathrm{loc}}$*,* $n \in \mathbb{N}$*, and* $\alpha \in (0,\infty)$ *such that* $(\chi_n - 1)^- \in D(\mathcal{E}_\mu)_{0,b}$*,* $\lim\limits_{n\to\infty} \chi_n = 0$ μ*–a.e. and*

$$\mathcal{E}_{\mu,\alpha}(\chi_n, v) \geq \int \langle b, \nabla\chi_n\rangle_{\mathbb{R}^d} v \, d\mu \text{ for all } v \in D(\mathcal{E}_\mu)_{0,b}\,,\ v \geq 0\ . \tag{3.18}$$

(iii) The closure on $L^1(\mathbb{R}^d;\mu)$ *of* L_b*, but considered on the* larger domain $D(L_0)_b$*, generates a* C^0*–semigroup on* $L^1(\mathbb{R}^d;\mu)$

Proof. [St 97, Proposition 1.9]. □

The following lemma provides simple conditions which ensure that the equivalent properties (i) – (iii) in Proposition 3.14 hold:

Lemma 3.15. *Proposition 3.14 (i) – (iii) hold if one of the following conditions are satisfied:*

(i) $b \in L^1(\mathbb{R}^d;\mu)$.

(ii) $\beta^\mu - b \in L^2(\mathbb{R}^d;\mu)$ *and* $\mu(\mathbb{R}^d) < \infty$.

(iii) There exists a Lyapunov–type function *for* L_b*, i.e., a* C^2*–function* $V : \mathbb{R}^d \to \mathbb{R}$ *such that*

$$\lim_{|x|_{\mathbb{R}^d}\to\infty} V(x) = \infty \text{ and } \lim_{|x|_{\mathbb{R}^d}\to\infty} L_b V(x) = -\infty\ . \tag{3.19}$$

This is particularly the case if there exists $M \in [0,\infty)$ *such that*

$$\langle (\beta^\mu - b)(x), x\rangle_{\mathbb{R}^d} \leq M\left(\ell n(|x|^2_{\mathbb{R}^d} + 1) + 1\right) \text{ for all } x \in \mathbb{R}^d\ . \tag{3.20}$$

Proof. For (i) see [St 97, Prop. 1.10 (a)].
(ii): By Theorem 4.8 (iii) below, $\beta^\mu - b \in L^2(\mathbb{R}^d;\mu)$ implies β^μ, $b \in L^2(\mathbb{R}^d;\mu)$. In particular, $b \in L^1(\mathbb{R}^d;\mu)$, so (i) implies the assertion.
(iii): See [St 97, Proofs of Prop. 1.10 (b) and Prop. 2.8]. □

Now we have the following generalization of Theorem 3.13 to the non–symmetric case.

Theorem 3.16. *The following are equivalent:*
(i) The closure of $(L_b, C_0^\infty(\mathbb{R}^d))$ *on* $L^1(\mathbb{R}^d;\mu)$ *generates a* C^0*–semigroup on* $L^1(\mathbb{R}^d;\mu)$.

(ii) One of the equivalent conditions in Prop. 3.14 holds.

Proof. By [St 97, Lemma 2.3] the proof is word by word the same as that of [St 97, Thm. 2.1]. □

Remark 3.17. The equivalent properties (i) and (ii) in Theorem 3.16 may not hold. Indeed, in [St 97, Example 1.12] it is proved that they do not hold in the following situation where μ is even Gaussian : $\varphi(x) := e^{-x^2/2}$, $b(x) := 6e^{x^2}$, $x \in \mathbb{R}^d$.

So far, we have discussed uniqueness results only in $L^1(\mathbb{R}^d;\mu)$, which hold under the weakest assumptions. Uniqueness in $L^p(\mathbb{R}^d;\mu)$ for $p \geq 1$ (in particular, for $p = 2$, i.e., if $b \equiv 0$, the question of *essential self–adjointness*) has also been studied by many authors. We refer to [Eb 98] for numerous new results and also for a survey, as well as [L 98] for a more recent special result which is not discussed in [Eb 98]. To give an idea of at least one type of results proved so far, we recall

Theorem 3.18. *Suppose $\mu(\mathbb{R}^d) < \infty$ and that $|\beta^\mu - b| \in L^p_{\text{loc}}(\mathbb{R}^d;dx)$ for some $p > d \geq 2$ such that*

$$|b| \in L^q(\mathbb{R}^d;\mu) \text{ for some } q \in [1,\infty] .$$

Let $r := 2 - \frac{2}{q+1}$(where $\frac{1}{\infty} := 0$). Then the closure of $(L_b, C_0^\infty(\mathbb{R}^d))$ on $L^r(\mathbb{R}^d;\mu)$ generates a C^0–semigroup on $L^r(\mathbb{R}^d;\mu)$.

Proof. [ABR 97, Theorem 1.4 (iii)]. □

3.3.2 $E :=$ topological vector space.

We consider the situation of Subsection 3.2.2 adopting all notations introduced there. In particular, we assume (3.13) to hold (for simplicity).

Remark 3.19. Obviously, $(T_t^0)_{t\geq 0}$ is always conservative.

Unfortunately, there is *no* analogue of Theorem 3.13 in this infinite dimensional case.

Theorem 3.20. *The closure of $(\hat{L}_0, \mathcal{F}C_b^\infty(K))$ does not always generate a C^0–semigroup on $L^1(E;\mu)$.*

Proof. See the counterexamples in [Eb 98, Subsect. 5 b)]. □

As in Subsection 3.2.2 consider now

$$L_b := \hat{L}_0 - \langle b, \nabla_H \cdot \rangle_H \text{ on } \mathcal{F}C_b^\infty(K)$$

with $b \in L^2(E \to H;\mu)$ satisfying (DC).
In this non–symmetric situation we have:

Theorem 3.21. *Suppose the closure of $(L_0, \mathcal{F}C_b^\infty(K))$ on $L^1(E;\mu)$ generates a C^0–semigroup on $L^1(E;\mu)$, then so does the closure of $(L_b, \mathcal{F}C_b^\infty(K))$ on $L^1(E;\mu)$.*

Proof. [St 97, Proposition 4.3]. □

Theorem 3.21 is the infinite dimensional generalization of Theorem 3.16 in part. Indeed, under our present assumptions it follows that if $E = H = \mathbb{R}^d$, and if $\mu = \varphi^2 dx$ for some $\varphi \in H^{1,2}_{\text{loc}}(\mathbb{R}^d;dx)$ (which in fact follows from (3.13)), then Theorem 3.16 (ii) holds by Lemma 3.15 (i), since in the present subsection b is assumed to be globally μ–square integrable and $\mu(E) < \infty$.

Remark 3.22. Also in the present infinite dimensional situation the uniqueness problem has been investigated for $p \in [1, \infty)$ rather than just the case $p = 1$. We again refer to [Eb 98, Ch. 5], and also [LR 97]. The latter paper proves L^p–uniqueness results in infinite dimensions for all $p \in [1, \infty)$ which are not covered by [Eb 98, Ch. 5]. They particularly apply to the operator generating the dynamics in the stochastic quantization of Euclidean field theory in finite volume (in contrast to Eberle's results also in case $p \geq 2$!).

Remark 3.23. We emphasize that the closure of $(L_0, \mathcal{F}C_b^\infty(K))$ even on $L^2(E;\mu)$ generates a C^0–semigroup on $L^2(E;\mu)$ for μ as in the cases discussed in Subsect. 2.3 and Example 2.7. For the latter this follows by [AKR 97, Theorem 5.13]. For the Gaussian situation in Subsect. 2.3 this follows by the explicit form of $(T_t^0)_{t\geq 0}$ given by the Mehler formula (2.12) and Proposition 1.7 applied to

$$B_0 := \text{ lin. span } \{\cos\left({}_{E'}\langle \ell, \cdot \rangle_E\right) \ , \ \sin\left({}_{E'}\langle \ell, \cdot \rangle_E\right) | \ell \in K\}$$

(cf. [BRS 96, Prop. 6.1]). So, Theorem 3.21 applies to all these cases.

3.4 Concluding remarks.

If we want to apply the results in Subsections 3.2, 3.3 above to an operator of type

$$L = \Delta + \langle B, \nabla \cdot \rangle_{\mathbb{R}^d} \text{ on } C_0^\infty(\mathbb{R}^d)$$

for some vector field $B : \mathbb{R}^d \to \mathbb{R}^d$ (or to its infinite dimensional analogue) in order to solve the corresponding Kolmogorov equation, we have to find a measure μ so that the decomposition

$$B = \beta^\mu - b$$

is such that all our assumptions are satisfied. By Remarks 3.7 resp. 3.9 we, therefore, have to look for invariant measures μ for L. If we can prove regularity for μ, i.e., that $\mu = \rho\, dx$ with ρ regular enough, then we can take

$$b := \frac{\nabla \rho}{\rho} - B\ .$$

This program will be pursued in the next section.

4. Invariant measures: regularity, existence and uniqueness

Let $(E, \mathcal{B})$ be a measurable space and let $L(E, \mathcal{B})$ denote the linear space of all $\mathcal{B}$–measurable real–valued functions on E. Let $D \subset L(E, \mathcal{B})$ be a linear subspace and $L : D \to L(E, \mathcal{B})$ a linear operator.

Definition 4.1. A probability measure μ on $(E, \mathcal{B})$ is called an *invariant measure for* (L, D) (or shortly for L if D is fixed) if

$$Lu \in L^1(E;\mu) \text{ and } \int Lu\, d\mu = 0 \text{ for all } u \in D\ . \tag{4.1}$$

We abbreviate (4.1) by $L^*\mu = 0$.

Remark 4.2. Suppose μ is an invariant measure for (L,D) and that (L,D) is a linear operator on $L^1(E;\mu)$ such that it has a closed extension $\hat{L}$ which generates a C^0–semigroup $(T_t)_{t\geq 0}$ on $L^1(E;\mu)$. Then obviously μ is $(T_t)_{t\geq 0}$–*invariant*, i.e.,

$$\int T_t f\ d\mu = \int f\ d\mu \text{ for all } f \in L^1(E;\mu)\ , \tag{4.2}$$

if and only if $(T_t)_{t\geq 0}$ is conservative (cf. Subsect. 3.3). So, by Remark 3.17, (4.1) in general does not imply (4.2). Clearly, however,

$$(4.2) \quad \Longleftrightarrow \quad \int \hat{L}u\ d\mu = 0 \text{ for all } u \in D(\hat{L})\ . \tag{4.3}$$

So, by Theorem 3.16 for $(L,D) := (L_b, C_0^\infty(\mathbb{R}^d))$, as considered there, (4.2) holds if and only if $\hat{L} =$ closure of (L,D) on $L^1(E;\mu)$.

4.1 Sectorial case.

We consider the situation of Subsect. 1.4.2 with $\hat{b} \equiv 0$, $c \equiv 0$. We assume that (C) and (SFP) hold. Let L_0 denote the generator of $(Q, D(Q))$ (defined as in Subsect. 1.4.2 with $c \equiv 0$). As in Subsect. 3.2 for simplicity we assume that

$$\mathcal{F}C_b^\infty(K) \subset D(L_0) \tag{4.4}$$

and set

$$L_b u := L_0 u - \langle b, \nabla_H u\rangle_H\ ,\ u \in \mathcal{F}C_b^\infty(K)\ . \tag{4.5}$$

So, L_b is nothing but the restriction of $L := L^{(\alpha)} + \alpha$ to $\mathcal{F}C_b^\infty(K)$ where $L^{(\alpha)}$ is the generator of the sectorial form $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ defined in Subsect. 1.4.2 with $A = Id_H$ $\hat{b} \equiv 0$, $c \equiv 0$. Let $(T_t^{(\alpha)})_{t\geq 0}$ be the corresponding C^0–semigroup and set $T_t := e^{\alpha t} T_t^{(\alpha)}$, $t \geq 0$.

Since $L_b u$ is only defined as a class of functions being μ–a.e. equal, invariant measures in the sense of Definition 4.1 are only well–defined if they are absolutely continuous w.r.t. μ. In the next subsection we shall present results about existence and uniqueness of such measures.

4.1.1 Results.

Theorem 4.3. *Suppose (C), (E), (SFP) and (4.4) hold. Assume furthermore:*
(i) (DLS) holds (cf. Subsect. 1.4.2).
(ii) There exist $l_n \in E'$, $n \in \mathbb{N}$, forming an orthonormal basis of H such for all $N \in \mathbb{N}$

$$\mu_N := \mu \circ (\ell_1, \ldots, \ell_N)^{-1} = \rho_N\ dx$$

for some $\mathcal{B}(\mathbb{R}^N)$–measurable $\rho_N : \mathbb{R}^N \to [0,\infty)$ such that

$$R(\rho_N) := \left\{ y \in \mathbb{R}^N \,|\, \int_{\{x | |x-y|_{\mathbb{R}^N} < \varepsilon\}} \frac{1}{\rho_N(x)}\ dx < \infty \text{ for some } \varepsilon > 0 \right\} \tag{4.6}$$

has full μ_N–measure.
Then there exists $\rho \in D(\mathcal{E})(= D(\mathcal{E}_\mu))$, $\rho \geq 0$, such that $\nu := \rho \cdot \mu$ is an invariant measure for L_b.

Proof. [BRZ 97, Theorem 3.6 and Remark 2.9] $\square$

4.1.2 Application to the uniqueness problem.

We still consider the situation of the previous subsection

Lemma 4.4. *In the situation of Theorem 4.3 we have that* $\left|\frac{\nabla_H \rho}{\rho}\right|_H \in L^2(E;\nu)$.

Proof. Let $k \in \mathbb{N}$. Then $ln(\rho + \frac{1}{k}) \in D(\mathcal{E})(= D(\mathcal{E}_\mu))$ and hence by [BRZ 97, Lemma 2.10] for all $k \in \mathbb{N}$

$$\begin{aligned}
&\int_E \langle \frac{\nabla_H \rho}{\rho + \frac{1}{k}} \ , \ \frac{\nabla_H \rho}{\rho + \frac{1}{k}} \rangle_H \left(\rho + \frac{1}{k}\right) \ d\mu \\
&= \mathcal{E}_\mu \left(\ell n \left(\rho + \frac{1}{k}\right) \ , \ \rho \right) \\
&= - \lim_{n\to\infty} \int_E \langle b, \nabla_H \left(\ell n \left(\rho + \frac{1}{k} \right)\right)\rangle_H \inf(\rho, n) \ d\mu \\
&\le \int_E |b|_H \ |\nabla_H \rho|_H \ d\mu \\
&\le \left(\int_E |b|_H^2 \ d\mu \right)^{1/2} \mathcal{E}_\mu(\rho,\rho)^{1/2} < \infty \ .
\end{aligned}$$

Now the assertion follows by Fatou's Lemma. □

Let us assume that μ has the following property:

(U) If for all $\rho \in D(\mathcal{E}_\mu)$, with $|\nabla_H \rho/\rho|_H \in L^2(E;\rho\cdot\mu)$ and $\rho \ge 0$, we define

$$L_\rho u := L_0 u + \langle \frac{\nabla_H \rho}{\rho} \ , \ \nabla_H u \rangle_H \ ; \ u \in \mathcal{F}C_b^\infty(K) \ ,$$

then $(L_\rho, \mathcal{F}C_b^\infty(K))$ is well–defined on $L^1(E;\rho\cdot\mu)$ and its closure generates a C^0–semigroup on $L^1(E;\rho\cdot\mu)$.

Remark 4.5. (i) There is a lot of examples of measures μ known to satisfy condition (U). For instance it follows by [Eb 98, Corollary 5.4] that (U) holds for many Gaussian measures as in Subsect. 2.3. At present we do not know, however, whether (U) holds for Example 2.7, though we expect it is the case.
(ii) In condition (U) we set $\frac{\nabla_H \rho}{\rho} := 0$ on $\{\rho = 0\}$, and to be precise we have to consider L_ρ on the $(\rho\cdot\mu)$–equivalence classes $\mathcal{F}C_b^\infty(K)^\sim$ of $\mathcal{F}C_b^\infty(K)$. Since possibly $\mu(\{\rho = 0\}) > 0$, there might be several different representatives in $\mathcal{F}C_b^\infty(K)$ for the same class in $\mathcal{F}C_b^\infty(K)^\sim$. Therefore, we have to assume that L_ρ is well–defined representativewise.

Theorem 4.6. *Assume μ satisfies (U) and consider the situation of Theorem 4.3. Let $\nu := \rho\cdot\mu$ be as defined there. Suppose $|b|_H$, $L_0 u \in L^4(E;\mu)$ for all $u \in \mathcal{F}C_b^\infty$. Then the closure of $(L_b, \mathcal{F}C_b^\infty(K))$ on $L^1(E;\nu)$ generates a C^0–semigroup $(T_t^\nu)_{t\ge 0}$ on $L^1(E;\nu)$. Furthermore, $T_t^\nu f = T_t f$ ν–a.e. for all $t \ge 0$, $f \in L^1(E;\mu)$.*

Proof. Applying Theorem 3.21 with $\mu := \nu$, $L_0 := L_\rho$, $b := \frac{\nabla_H \rho}{\rho} - b$ which is possible by Lemma 4.4, we obtain the first part of the assertion. The second part is obvious. □

Remark 4.7. (i) Theorem 4.6 assures us that even if we might loose uniqueness on $L^1(E;\mu)$ by passing from $(L_0, \mathcal{F}C_b^\infty(K))$ to $(L_b, \mathcal{F}C_b^\infty(K))$, we keep at least uniqueness on $L^1(E;\nu)$ where $\nu = \rho \cdot \mu$ is an invariant measure for L_b.
(ii) Again we note that Theorem 4.6 applies to all cases in Subsect. 2.3.
(iii) If (E, H, μ) is an abstract Wiener space and if $e^{|b|_H^2} \in \bigcup_{p>2} L^p(E;\mu)$, it has recently been proved in [Sh. 98, Sect. 3], that the closure of $(L_b, \mathcal{F}C_b^\infty(K))$ on $L^2(E;\mu)$ generates a C^0–semigroup on $L^2(E;\mu)$ (hence on $L^1(E,\mu)$). So, in this case one can analyze L_b on L^1 also w.r.t. the initial Gaussian measure μ, instead of the invariant measure.

4.2 Non–sectorial cases.

In this section for simplicity we shall assume that the second order part of our operator is the Laplacian resp. the Gross–Laplacian in the infinite dimensional case. For the general case we refer to the corresponding underlying literature [BR 95], [BKR 96], [BDPR 96], [ABR 97], [BKR 97], [BR 98].

4.2.1 Regularity and applications to Kolmogorov equations.

a) Let us start with the finite dimensional case: $E := \mathbb{R}^d$, $\mathcal{B} := \mathcal{B}(\mathbb{R}^d)$. Let $B : \mathbb{R}^d \to \mathbb{R}^d$ be Borel–measurable and define

$$L_B u := \Delta u + \langle B, \nabla u \rangle_{\mathbb{R}^d} \ , \ u \in C_0^\infty(\mathbb{R}^d) \ . \tag{4.7}$$

Note that if μ is a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $L_B^* \mu = 0$ in the sense of Definition 4.1, then, in particular, $L_B u \in L^1(\mathbb{R}^d;\mu)$ for all $u \in C_0^\infty(\mathbb{R}^d)$, hence necessarily

$$B \in L^1_{\text{loc}}(\mathbb{R}^d;\mu) \ . \tag{4.8}$$

We recall the following special case of the regularity result [BKR 97, Theorem 1].

Theorem 4.8. *Suppose μ is a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that*

$$L_B^* \mu = 0 \ . \tag{4.9}$$

Let for $p > d$, $|B|_{\mathbb{R}^d} \in L^p_{\text{loc}}(\mathbb{R}^d;dx)$ or $|B|_{\mathbb{R}^d} \in L^p_{\text{loc}}(\mathbb{R}^d;\mu)$. Then $\mu \ll dx$ and $\rho := \frac{d\mu}{dx}$ has the following properties:

(i) $\rho \in H^{1,p}_{\text{loc}}(\mathbb{R}^d;dx)$ (i.e., is locally in the Sobolev space (w.r.t. dx) of order 1 in $L^p(\mathbb{R}^d;dx)$). In particular, ρ is Hölder continuous of order $1 - d/p$.

(ii) $\inf_K \rho > 0$ for all compact $K \subset \mathbb{R}^d$.

(iii) If $|B|_{\mathbb{R}^d} \in L^2(\mathbb{R}^d;\mu)$, then $\frac{\nabla\rho}{\rho} \in L^2(\mathbb{R}^d;\mu)$.

Proof. (i): [BKR 97, Theorem 1].
(ii) : This is a consequence of the Harnack inequality for L_B proved in [Tr 73].
(iii) : [BR 95, Theorem 3.1]. $\square$

Consider the situation of Theorem 4.8 and decompose L_B as follows

$$L_B u := \Delta u + \langle \beta^\mu, \nabla u\rangle - \langle b, \nabla u\rangle \ ; \ u \in C_0^\infty(\mathbb{R}^d) \tag{4.10}$$

where $\beta^\mu := \frac{\nabla \rho}{\rho}$, $b := \beta^\mu - B$. Then by Remark 3.7 and Theorem 3.13 we know that Theorem 3.16 applies. In particular, in case $|B|_{\mathbb{R}^d} \in L^2(\mathbb{R}^d;\mu)$, we hence conclude by Proposition 3.15 (ii) that the closure of $(L_B, C_0^\infty(\mathbb{R}^d))$ on $L^1(\mathbb{R}^d;\mu)$ generates a C^0–semigroup on $L^1(\mathbb{R}^d;\mu)$.

b) Consider the situation in Subsect. 2.3 with $A := Id_H$, hence (E,H,μ) is an abstract Wiener space and $K = E'$. Define as in (2.7)

$$L_0 u(x) := \Delta_H u(x) - \ {}_E\langle x, \nabla_H u(x)\rangle_{E'} \ , \ u \in \mathcal{F}C_b^\infty(K) \ , \ x \in E \ . \tag{4.11}$$

Let $B : E \to H$ be $\mathcal{B}(E)/\mathcal{B}(H)$ measurable and define

$$L_B u := L_0 u + \langle B, \nabla_H u\rangle_H \ , \ u \in \mathcal{F}C_b^\infty(K) \ . \tag{4.12}$$

Then we have the following regularity result for invariant measures (see also [ABR 97, Theorem 4.3] for a generalization).

Theorem 4.9. *Suppose ν is a probability measure on $(E, \mathcal{B}(E))$ such that $L_B^*\nu = 0$. Assume in addition that ${}_{E'}\langle \ell, \cdot\rangle_E \in L^2(E;\nu)$ for all $\ell \in E'$ and that $|B|_H \in L^2(E;\nu)$. Then $\nu \ll \mu$ and $\rho := \frac{d\nu}{d\mu} \in D(\mathcal{E}_\mu)$.*

Proof. [BR 95, Theorem 3.5]. □

Consider the situation of Theorem 4.9 and decompose L_B as follows

$$L_B u := L_0 u + \langle \tfrac{\nabla_H \rho}{\rho}, \nabla_H u\rangle_H - \langle \tfrac{\nabla_H \rho}{\rho} - B, \nabla_H u\rangle_H \ , \ u \in \mathcal{F}C_b^\infty(K) \ . \tag{4.13}$$

since by Remark 4.5 condition (U) is satisfied we conclude by Remark 3.9 and Theorem 3.21 that the closure of $(L_B, \mathcal{F}C_b^\infty(K))$ on $L^1(E;\nu)$ generates a C^0–semigroup on $L^1(E;\mu)$.

4.2.2 Existence and uniqueness $E := \mathbb{R}^d$.

Consider the situation of part a) of Subsect. 4.2.1. We recall the following special case of the main result in [BR 98].

Theorem 4.10. *Suppose $|B| \in L^p_{\mathrm{loc}}(\mathbb{R}^d;dx)$ for $p > d$ and assume that there exists a Lyapunov–type function of L_B, i.e., there exists $V \in C^2(\mathbb{R}^d)$ such that*

$$\lim_{|x|\to\infty} V(x) = \infty \ , \ \lim_{|x|\to\infty} L_B V(x) = -\infty \ .$$

Then there exists a unique probability measure μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $L_B^*\mu = 0$.

Proof. [BR 98, Theorem 1.6 and Remark 1.10 (ii)]. □

There is a close connection between uniqueness of solutions to $L_B^*\mu = 0$ and whether the closure of $(L_B, C_0^\infty(\mathbb{R}^d))$ on $L^1(\mathbb{R}^d;\mu)$ generates a C^0–semigroup on $L^1(\mathbb{R}^d;\mu)$. More precisely, we have the following.

Theorem 4.11. *Suppose* $|B| \in L^p_{\mathrm{loc}}(\mathbb{R}^d; dx)$ *for* $p > d$. *Let* $\mathcal{K}$ *be a convex set of probability measures* μ *on* $(\mathbb{R}^d; \mathcal{B}(\mathbb{R}^d))$ *such that for every* $\mu \in \mathcal{K}$

(i) $L^*_B \mu = 0$;

(ii) the closure of $(L_B, C_0^\infty(\mathbb{R}^d))$ *on* $L^1(\mathbb{R}^d; \mu)$ *generates a* C^0*–semigroup on* $L^1(\mathbb{R}^d; \mu)$.

Then $\#\mathcal{K} \leq 1$. *In particular,*

$$\#\left\{\mu \middle|\ L^*_B\mu = 0 \text{ and } B - \frac{\nabla\rho}{\rho} \in L^1(\mathbb{R}^d; \mu) \text{ with } \rho := \frac{d\mu}{dx}\right\} \leq 1 \ .$$

Proof. [ABR 97, Theorem 1.2] and the discussion following it. □

Finally, we shall look at the special case where B is a gradient. More precisely, assume

$$\begin{aligned} &B \text{ is a (fixed) Borel–measurable } dx\text{–version of} \\ &2\frac{\nabla\varphi}{\varphi} \text{ for some } \varphi \in H^{1,2}_{\mathrm{loc}}(\mathbb{R}^d; dx)\ ,\ \int \varphi^2\ dx = 1\ . \end{aligned} \tag{4.14}$$

Here as usual we set $\frac{\nabla\varphi}{\varphi} := 0$ on $\{\varphi = 0\}$. Then obviously $\mu_0 := \varphi^2\ dx$ belongs to the following set of probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$

$$\begin{aligned} \mathcal{M}_1 := \{\mu|\ L^*_B\mu = 0 \text{ such that } &|B|_{\mathbb{R}^d} \in L^2_{\mathrm{loc}}(\mathbb{R}^d; \mu) \\ &\text{and } |B - \tfrac{\nabla\rho}{\rho}| \in L^2(\mathbb{R}^d; \mu)\}\ . \end{aligned} \tag{4.15}$$

Clearly, μ_0 is symmetrizing for $(L_B, C_0^\infty(\mathbb{R}^d))$, and, in fact, by the following result so are *all* other measures in $\mathcal{M}_1$. In some cases we even have uniqueness.

Theorem 4.12. *Let* B *be as in (4.14) and* $\mu_0 := \varphi^2\ dx$. *Then:*

(i) Every $\mu \in \mathcal{M}_1$ *is a symmetrizing measure for* $(L_B, C_0^\infty(\mathbb{R}^d))$.

(ii) If $B \in L^1_{\mathrm{loc}}(U; dx)$ *for some connected open set* $U \subset \mathbb{R}^d$ *such that* $dx(\mathbb{R}^d \setminus U) = 0$, *then*

$$\mathcal{M}_1 = \{\mu_0\}\ .$$

Remark 4.13. The condition in Theorem 4.12 (ii) cannot be dropped. Indeed, let $d = 1$ and $\varphi(x) := x\ (2\pi)^{-1/4}e^{-x^2/4}$, $x \in \mathbb{R}$. Then $B(x) = -x + \frac{2}{x}$, $x \in \mathbb{R}$, by (4.14), and $B \notin L^1_{\mathrm{loc}}(\mathbb{R}; dx)$. Let $\varphi_1 := (\int_0^\infty \varphi\, dx)^{-1} 1_{[0,\infty)}\varphi$ and $\varphi_2 := (\int_{-\infty}^0 \varphi\, dx)^{-1} 1_{(-\infty,0]}\varphi$. Then $\alpha\,\varphi_1\, dx + (1-\alpha)\varphi_2\, dx \in \mathcal{M}_1$ for all $\alpha \in [0, 1]$ (cf. [BR 95, Example 6.1]).

4.2.3 Existence and uniqueness for $E :=$ topological vector space.

Consider E, H, K as in Subsect. 1.4.2 and let $B : E \to E$ be $\mathcal{B}(E)/\mathcal{B}(E)$–measurable. Define

$$L_B u := \Delta_H u + {}_E\langle B, \nabla_H u\rangle_{E'}\ ,\ u \in \mathcal{F}C_b^\infty(K)\ . \tag{4.16}$$

Assume that also E is a Hilbert space with inner product $\langle\ ,\ \rangle_E$.

In this situation we have the following existence result:

Proposition 4.14. *Assume that*

$$\lim_{|x|_E\to\infty} \langle B(x), x\rangle_E = -\infty \ , \tag{4.17}$$

that $x \mapsto {}_{E'}\langle \ell, B(x)\rangle_E$ *is weakly continuous on* E *for all* $\ell \in E'$, *and that there exist* C_1, C_2, $\alpha \in (0,\infty)$ *such that*

$$|B(x)|_E \le C_1 + C_2\ |x|_E^\alpha \text{ for all } x \in E \ . \tag{4.18}$$

Then there exists a probability measure μ *on* $(E, \mathcal{B}(E))$ *such that* $L_B^*\mu = 0$.

Proof. [BR 95, Theorem 5.2]. □

The above result is not optimal. There is work in progress towards generalizations. The question, which conditions imply uniqueness for invariant measures of $(L_B, \mathcal{F}C_b^\infty(K))$, are in this general case largely open.
Concerning the question under what conditions invariant measures are symmetrizing in this infinite dimensional case, we have the following result.

Proposition 4.15. *Let* (E, H, μ) *be an abstract Wiener space and let* $(L_B, \mathcal{F}C_b^\infty(K))$ *be as defined in (4.12) with* $K := E'$. *Suppose* $B = -Id_E + b$ *such that*

$$\begin{gathered} b \ \text{ is a } \mu\text{–version of } \ 2\frac{\nabla_H\varphi_0}{\varphi_0} \text{ for some } \varphi_0 \in D(\mathcal{E}_\mu) \text{ with} \\ \int \varphi_0^2\, d\mu = 1 \quad (\text{where } \nabla_H\varphi_0/\varphi_0 := 0 \text{ on } \{\varphi_0 = 0\}) \ . \end{gathered} \tag{4.19}$$

(i) Assume ν *is a probability measure on* $(E, \mathcal{B}(E))$ *such that* ${}_{E'}\langle \ell, \cdot\rangle_E \in L^2(E;\nu)$ *for all* $\ell \in E'$, $|b|_H \in L^2(E;\nu)$ *and* $L_B^*\nu = 0$. *Then* $\nu = \varphi^2\cdot\mu$ *for some* $\varphi \in D(\mathcal{E}_\mu)$ *and*

$$\frac{\nabla_H\varphi}{\varphi} = \frac{\nabla_H\varphi_0}{\varphi_0} \ \mu\text{–a.e.}$$

(Obviously, $\nu_0 := \varphi_0^2\cdot\mu$ *is such a measure.) In particular,* ν *is symmetrizing for* $(L_B, \mathcal{F}C_b^\infty(K))$.

(ii) If $|\frac{\nabla_H\varphi_0}{\varphi_0}|_H \in L^2(E;\mu)$, *then* $\nu_0 := \varphi_0\cdot\mu$ *is the only probability measure satisfying the assumptions in (i).*

Proof. [ABR 97, Theorem 4.5]. □

5. Corresponding diffusions and relation to Martingale problems

In this section we show that the positivity preserving C^0–semigroups constructed above, in most cases are really transition semigroups of Markov processes. So, to call the equation

$$\frac{d}{dt}\, T_t u = LT_t u \tag{5.1}$$

a “Kolmogorov equation” is indeed justified. Though we do not use these processes at all to construct solutions to our Kolmogorov equations, this section is enclosed to complete the picture (however, in a slightly sketchy style). In addition, the process can be used subsequently to get more specific information about $(T_t)_{t\ge0}$ and L.

5.1 Existence of associated diffusions.

5.1.1 Sectorial case.

Let E be a topological space and $\mathcal{B}(E)$ its Borel σ–algebra. Let m be a σ–finite positive measure on $(E, \mathcal{B}(E))$. Let $(\mathcal{E}, D(\mathcal{E}))$ be a sectorial form on $L^2(E;m)$ with associated generator $(L, D(L))$ and C^0–semigroup $(T_t)_{t\geq 0}$.

Definition 5.1. (i) An increasing sequence $(E_k)_{k\in\mathbb{N}}$ of closed subsets of E is called an $\mathcal{E}$*–nest* if

$$\{u \in D(\mathcal{E}) \mid u = 0 \ m\text{–a.e. on } E \setminus E_k \text{ for some } k \in \mathbb{N}\}$$

is dense in $D(\mathcal{E})$ w.r.t. $\tilde{\mathcal{E}}_1^{1/2}$.
(ii) $N \subset E$ is called $\mathcal{E}$*–exceptional*, if $N \subset E \setminus \bigcup_{k=1}^{\infty} E_k$ for some $\mathcal{E}$–nest $(E_k)_{k\in\mathbb{N}}$. A property of points $x \in E$ is said to hold $\mathcal{E}$*–quasi–everwhere* (abbreviated $\mathcal{E}$*–q.e.*) if it holds outside an $\mathcal{E}$–exceptional set.
(iii) A function $f : A \to \mathbb{R}$, $A \subset E$, is called $\mathcal{E}$*–quasi–continuous* if $f_{|E_k}$ is continuous for all $k \in \mathbb{N}$ for some $\mathcal{E}$–nest $(E_k)_{k\in\mathbb{N}}$ such that $\bigcup_{k=1}^{\infty} E_k \subset A$.

Definition 5.2. $(\mathcal{E}, D(\mathcal{E}))$ is said to be *quasi–regular* if the following conditions hold:
(i) There exists an $\mathcal{E}$–nest $(E_k)_{k\in\mathbb{N}}$ consisting of compact sets.
(ii) There exists an $\tilde{\mathcal{E}}_1^{1/2}$–dense subset of $D(\mathcal{E})$ whose elements have $\mathcal{E}$–quasi–continuous m–versions.
(iii) There exist $u_n \in D(\mathcal{E})$, $n \in \mathbb{N}$, having $\mathcal{E}$–quasi–continuous m–versions $\tilde{u}_n$, $n \in \mathbb{N}$, and an $\mathcal{E}$–exceptional set $N \subset E$ such that $\{\tilde{u}_n | n \in \mathbb{N}\}$ separates the points of $E \setminus N$.

Definition 5.3. Let $(\mathcal{E}, D(\mathcal{E}))$ be a Semi–Dirichlet form. A *diffusion process* $\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t), (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ on E with lifetime ζ (i.e., a strong Markov process with continuous sample paths on $[0,\zeta)$) is called *associated to* $(\mathcal{E}, D(\mathcal{E}))$ if for every $f \in L^2(E;m) \cap L^\infty(E;m)$ and any m–version $\hat{f}$ of f and all $\alpha > 0$

$$x \mapsto E_x\left[\int_0^\infty e^{-\alpha t} \hat{f}(X_t)\ dt\right] \tag{5.2}$$

is an $\mathcal{E}$–quasi–continuous m–version of $G_\alpha f := \int_0^\infty e^{-\alpha t}\ T_t f\ dt$.

Remark 5.4. $\mathbb{M}$ is associated to $(\mathcal{E}, D(\mathcal{E}))$ if and only if for every $f \in L^2(E;m) \cap L^\infty(E;m)$ and any m–version $\hat{f}$ or f and all $t \geq 0$

$$x \mapsto E_x\left[\hat{f}(X_t)\right]\ ,\ x \in E\ , \tag{5.3}$$

is an $\mathcal{E}$–quasi–continuous m–version of $T_t f$.

Proof. [MR 92, Ch. IV, Prop. 2.8]. □

Theorem 5.5. *Let $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ be as in Subsections 1.4.1 or 1.4.2 with $\alpha > 0$ as defined there. Assume that E in Subsect. 1.4.2 is a separable Banach space. Then:*
(i) $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ has the local property, i.e.,

$$\mathcal{E}_\alpha(u,v) = 0 \text{ whenever } u, v \in D(\mathcal{E}) \text{ such that } \quad supp\ (|u| \cdot m) \cap \ supp\ (|v| \cdot m) = \emptyset\ ,$$

(ii) $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ is quasi–regular.
(iii) Suppose $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ is a Semi–Dirichlet form (which is e.g. the case if $\hat{b} \equiv 0$). Then there exists a diffusion process $\mathbb{M}$ on E associated with $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$.

Proof. (i): [MR 92, Ch. V, Examples 1.12 (i)].
(ii): [MR 92, Ch. IV, Sect. 4].
(iii): By the main result in [MR 92, Ch. IV] more precisely its “Semi– Dirichlet form –version” in [MOR 95] we know that quasi–regularity is equivalent to the existence of a so–called m–special standard process $\mathbb{M}$ on E associated with $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$. Because of (i) it follows that $\mathbb{M}$ is indeed a diffusion (cf. [MR 92, Ch. V, Sect. 1]). A proof that $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ is a Semi–Dirichlet form if $\hat{b} \equiv 0$, can be found in [MR 95, Remark 2.6]. □

Remark 5.6. Even if Theorem 5.5 only gives a process $\mathbb{M}$ with transition semigroup determined by $(T_t^{(\alpha)})_{t\geq 0}$ (cf. Subsect. 1.4), “unkilling” $\mathbb{M}$ by using the multiplicative functional $e^{\alpha t}$, $t \geq 0$, we obtain a process with transition semigroup determined by $(T_t)_{t\geq 0}$.

5.1.2 Non–sectorial cases.

Now we consider the situations of Subsections 3.2.1 and 3.2.2. Let $(T_t^b)_{t\geq 0}$ be the sub–Markovian C^0–semigroups of contractions on $L^1(\mathbb{R}^d;\mu)$ resp. $L^1(E;\mu)$ appearing in Theorems 3.8 and 3.9 respectively.

In [St 96] a theory has been developed, called the *“Theory of Generalized Dirichlet forms”*, which includes the theory of sectorial forms, time dependent versions of it, as well as the cases studied in Subsections 3.2.1 and 3.2.2 above. Also the notion of quasi–regularity extends to this more general framework, yielding (as in the sectorial case in [MR 92]) a complete analytic characterization of all generalized Semi–Dirichlet forms associated with (minimally regular) strong Markov processes (in the sense of Definition 5.3). So, there is a complete analogue of Theorem 5.5 for the semigroups $(T_t^b)_{t>0}$ above (cf. [St 97, Thm. 3.5, Prop. 3.6 resp. Thm. 4.6, Prop. 4.7]). We only mention here that in the situation of Subsect. 3.2.2 we even have that $\zeta = \infty$. The only draw–back is that Remark 5.4 above does not hold in these cases. The function

$$x \mapsto E_x[\hat{f}(X_t)]\ ,\ x \in E\ ,$$

is still an m–version of $T_t f$, but in general no longer $\mathcal{E}$–quasi–continuous.

5.2 Solution of the martingale problem.

The following is a special case of a general result in [T 97, 98].

Theorem 5.7. *(i) The diffusion process* $\mathbb{M}$ *in Theorem 5.5 solves the martingale problem for* (L, D) *with* $D := C_0^\infty(\mathbb{R}^d)$ *resp.* $D := \mathcal{F}C_b^\infty(K)$ *in the situations of Subsections 1.4.1 resp. 1.4.2, i.e., for all* $u \in D$
(a) $\int_0^t Lu(X_s)\, ds$, $t \geq 0$, *is* P_x*–a.s. independent of the* μ*–version for* Lu *for* μ*–a.e.* $x \in E$.
(b) $u(X_t) - u(X_0) - \int_0^t Lu(X_s)\, ds$, $t \geq 0$, *is an* $(\mathcal{F}_t)$*–martingale under* P_x *for* μ*–a.e.* $x \in E$.
(ii) An analogous statement holds for the cases discussed in Subsect. 5.1.2 with L_b *replacing* L.

5.2 Uniqueness.

We recall that if $\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0},\ (X_t)_{t\geq 0},\ (P_x)_{x\in E_\Delta})$ is a diffusion process on E, a probability measure is called *subinvariant* for $\mathbb{M}$ if for all $f : E \to \mathbb{R}_+$, $\mathcal{B}(E)$–measurable

$$\int E_x[f(X_t)]\ \mu(dx) \leq \int f(x)\ \mu(dx) \text{ for all } t \geq 0\ . \tag{5.4}$$

The entire Sect. 3 of these notes was devoted to obtain criteria to ensure that the closure of a diffusion operator (L, D) on $L^1(E; \mu)$ generates a C^0–semigroup on $L^1(E; \mu)$. One application of this is to obtain uniqueness for Markov processes solving the corresponding martingale problem:

Theorem 5.8. *(i) Consider the situation of Theorem 5.7 (i). Assume that the closure of* (L, D) *on* $L^1(E; \mu)$ *generates a* C^0*–semigroup on* $L^1(E; \mu)$*. Let* $\mathbb{M}' = (\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0},\ (X'_t)_{t\geq 0}.\ (P'_x)_{x\in E_\Delta})$ *be a diffusion (or even only a right) process on* E *solving the martingale problem for* (L, D) *such that* μ *is subinvariant for* $\mathbb{M}'$ *(*$\mathbb{M}$ *from Theorem 5.5 is such a process). Then* $x \mapsto E'_x[\hat{f}(X'_t)]$ *is a* μ*–version of* $T_t f$ *for all* $f \in L^1(\mathbb{R}^d; \mu) \cap L^\infty(\mathbb{R}^d; \mu)$ *and all* μ*–versions* $\hat{f}$ *of* f*, i.e.,* $\mathbb{M}' = \mathbb{M}$ *up to* μ*–equivalence.*
(ii) An analogous statement holds for the cases discussed in Subsect 5.1.2 with L_b *replacing* L.

Proof. [AR 95] and [St 97, Prop. 2.6]. □

6. Appendix

6.1 Kolmogorov equations in $L^2(E; \mu)$ for infinite dimensional manifolds E: a case study from continuum statistical mechanics.

The purpose of this section is to give an important *"non–flat"* example where the measurable field of Hilbert spaces, to represent an m–symmetric diffusion operator L as in Theorem 2.3, can be constructed explicitly in a natural way giving rise to numerous applications. Let us first describe our framework.

6.1.1 Framework and relevant operators L.

Let X be a connected, oriented C^∞ Riemannian manifold such that $m(X) = \infty$ where m is the volume element. Let $\langle\ ,\ \rangle_{TX}$ denote the Riemannian metric and ∇^X, Δ^X the corresponding gradient resp. Laplacian. There is a natural and simple infinite dimensional structure associated with X, the so–called *configuration space* Γ_X over X defined as follows:

$$\Gamma_X := \{\gamma \subset X \mid \gamma \cap K \text{ is a finite set for each compact } K \subset X\} \ . \tag{6.1}$$

$\gamma \in \Gamma_X$ is identified with the $\mathbb{Z}_+ \cup \{+\infty\}$–valued Radon measure

$$\gamma := \sum_{x\in\gamma} \varepsilon_x \ , \tag{6.2}$$

where ε_x denotes Dirac measure at x. Γ_X can therefore be endowed with the vague topology. Let $\mathcal{B}(\Gamma_X)$ denote the corresponding Borel σ–algebra.

Functions $f \in C_0^\infty(X)$ on X are *lifted to* Γ_X as follows:

$$\langle f, \gamma\rangle := \sum_{x\in\gamma} f(x) = \int_X f(x)\ \gamma(dx) \ . \tag{6.3}$$

This gives rise to the following algebra of *test functions* on Γ_X:

$$\mathcal{F}C_b^\infty := \{g(\langle f_1, \cdot\rangle, \ldots, \langle f_N, \cdot\rangle) \mid N \in \mathbb{N},\ f_1, \ldots, f_N \in C_0^\infty(X)\ ,\ g \in C_b^\infty(\mathbb{R}^N)\} \ .$$

Typical differential operators of interest on Γ_X are (cf. [AKR 98 a, b], [R 98] and in particular the references there in):

$$\begin{aligned}
&LF(\gamma)\\
&= \sum_{i,j=1}^N \partial_i\partial_j g(\langle f_1,\gamma\rangle, \ldots, \langle f_N,\gamma\rangle) \int_X \langle \nabla^X f_i(x), \nabla^X f_j(x)\rangle_{T_xX}\ \gamma(dx)\\
&+ \sum_{i=1}^N \partial_i g(\langle f_1,\gamma\rangle, \ldots, \langle f_N,\gamma\rangle) \int_X \Delta^X f_i(x)\ \gamma(dx)\\
&- \sum_{i=1}^N \partial_i g(\langle f_1,\gamma\rangle, \ldots, \langle f_N,\gamma\rangle) \sum_{\{x,y\}\subset\gamma} \langle \nabla_x^X \phi(x,y)\ ,\ \nabla^X f_i(x)\rangle_{T_xX}
\end{aligned} \tag{6.4}$$

where $F = g(\langle f_1, \cdot\rangle, \ldots, \langle f_N, \cdot\rangle) \in \mathcal{F}C_b^\infty$ and $\phi : X \times X \to \mathbb{R}$ is a given function satisfying suitable conditions. Below for simplicity we shall only handle the case $\phi \equiv 0$ and refer for the case with non–trivial *interaction potential* ϕ to [AKR 98 b] and [R 98].

One immediately checks that L in (6.4) is a diffusion operator in the sense of Definition 2.1. There are various ways to find a corresponding natural "differential geometric" structure on Γ_X giving a concrete model for the representation in Theorem 2.3. We shall obtain such a structure by lifting the geometry on X to Γ_X.

6.1.2 A Riemannian–type structure on configuration space.

a) Lifting of flows.

Let $v \in V_0(X)$ (:= all smooth vector fields on X with compact support). Let ϕ_t^v, $t \in \mathbb{R}$, be the corresponding flow on X, i.e., the unique solution to

$$\frac{d}{dt}\ \phi_t^v(x) = v(\phi_t^v(x))\ ,\ \phi_0^v(x) = x \in X\ . \tag{6.5}$$

We define a corresponding *flow on* Γ_X by

$$\phi_t^v(\gamma) := \{\phi_t^v(x) \mid x \in \gamma\} = \sum_{x\in\gamma} \varepsilon_{\phi_t^v(x)}\ . \tag{6.6}$$

We immediately get from this the following:

b) Lifting of directional derivatives.

On X for $v \in V_0(X)$, $f \in C_0^\infty(X)$ we have

$$\nabla_v^X f(x) := \frac{d}{dt}\ f(\phi_t^v(x))_{|t=0} = \langle \nabla^X f(x), v(x)\rangle_{T_xX}\ . \tag{6.7}$$

The corresponding *directional derivative on* Γ_X is hence for $F = g(\langle f_1, \cdot\rangle, \ldots, \langle f_N, \cdot\rangle) \in \mathcal{F}C_b^\infty$ and $\gamma \in \Gamma_X$

$$\begin{aligned}\nabla_v^\Gamma F(\gamma) :&= \frac{d}{dt}\ F\left(\phi_t^v(\gamma)\right)_{|t=0} \\ &= \sum_{i=1}^N\ \partial_i g\left(\langle f_1, \gamma\rangle, \ldots, \langle f_N, \gamma\rangle\right) \int_X \langle \nabla_v^X f(x), v(x)\rangle_{T_xX}\ \gamma(dx)\ . \end{aligned} \tag{6.8}$$

Comparing (6.7) and (6.8) we obtain:

c) Lifting of gradients and tangent bundle.

Define for $F = g(\langle f_1, \cdot\rangle, \ldots, \langle f_N, \cdot\rangle) \in \mathcal{F}C_b^\infty$, $\gamma \in \Gamma_X$,

$$\nabla^\Gamma F(\gamma) := \sum_{i=1}^N\ \partial_i g(\langle f_1, \gamma\rangle, \ldots, \langle f_N, \gamma\rangle)\nabla^X f_i\ \ (\in V_0(X))\ . \tag{6.9}$$

Then by (6.7), (6.8) for $v \in V_0(X)$

$$\begin{aligned}\nabla_v^\Gamma F(\gamma) &= \int \langle \nabla^\Gamma F(\gamma)(x), v(x)\rangle_{T_xX}\ \gamma(dx) \\ &= \langle \nabla^\Gamma F(\gamma), v\rangle_{L^2(X\to TX;\gamma)}\ , \end{aligned} \tag{6.10}$$

where $L^2(X \to TX;\gamma)$ denotes the space of (γ–classes of) γ–square integrable sections in the tangent bundle $TX = (T_xX)_{x\in X}$. Hence the appropriate "tangent bundle" on Γ_X is given by

$$T_\gamma\Gamma_X := L^2(X \to TX;\gamma)\ ,\ \gamma \in \Gamma_X\ , \tag{6.12}$$

and the metric $\langle\ ,\ \rangle_{T_\gamma\Gamma_X}$ is just given by the usual inner product in this vector–valued L^2–space.

6.1.3 L as a μ–Laplacian.

Let $\mathcal{VF}C_b^\infty$ denote the set of all vector fields V on Γ_X (i.e., all sections in $T\Gamma_X := (T_\gamma\Gamma_X)_{\gamma\in\Gamma_X}$) of the form

$$\gamma \mapsto V(\gamma) = \sum_{i=1}^N F_i(\gamma)\ v_i \quad (\in T_\gamma\Gamma_X)\,,\ F_i \in \mathcal{F}C_b^\infty\ , \\ v_i \in V_0(X)\,,\ N \in \mathbb{N}\ . \tag{6.13}$$

For such V define

$$\gamma \mapsto\ \mathrm{div}^\Gamma V(\gamma) := \sum_{i=1}^N \left(\nabla_{v_i}^\Gamma F_i(\gamma) + F_i(\gamma)\langle\ \mathrm{div}^X v_i, \gamma\rangle\right)\ . \tag{6.14}$$

Note that this is the exact lifting of the divergence on X, since by (6.3) we have necessarily

$$\mathrm{div}^\Gamma v(\gamma) := \langle\ \mathrm{div}^X v, \gamma\rangle\ ,\ \gamma \in \Gamma_X\ ,$$

hence requiring the usual product rule for div^Γ we arrive at (6.14) for vector fields V as in (6.13).
Obviously, for the operator $(L, \mathcal{F}C_b^\infty)$ defined in (6.4) with $\phi \equiv 0$ we have

$$L =\ \mathrm{div}^\Gamma\nabla^\Gamma =: \Delta^\Gamma \text{ on } \mathcal{F}C_b^\infty\ . \tag{6.15}$$

But so far we have no measure μ so that L is really a μ–Laplacian, i.e., $-\ \mathrm{div}^\Gamma$ is the adjoint of $\left(\nabla^\Gamma, \mathcal{F}C_b^\infty\right)$ on $L^2(\Gamma_X;\mu)$, or shortly so that

$$-\left(\nabla^\Gamma\right)^{*,\mu} =\ \mathrm{div}^\Gamma \text{ on } \mathcal{VF}C_b^\infty\ . \tag{6.16}$$

In fact, one can characterize all such measures.
Recall there exists a unique measure π_m on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ with Laplace transform

$$\int_{\Gamma_X} e^{<f,\gamma>}\ \pi_m(d\gamma) = e^{\int_X (e^f-1)\,dm} \text{ for all } f \in C_0^\infty(X) \tag{6.17}$$

called *(pure) Poisson measure* with intensity (or mean) measure m (cf. e.g. [R 98, Subsect. 2.3]).

Theorem 6.1. *Let μ be a probability measure on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ such that*

$$\int_{\Gamma_X} |\langle f,\gamma\rangle|\ \mu(d\gamma) < \infty \textit{ for all } f \in C_0^\infty(X) \tag{6.18}$$

(i.e., the mean measure *of μ is Radon). Then the following are equivalent:*
(i) $(\nabla^\Gamma)^{*,\mu} = -\ \mathit{div}^\Gamma$.

(ii) μ is a mixed Poisson measure*, i.e., there exists a probability measure λ on $([0,\infty),\mathcal{B}([0,\infty)))$ such that*

$$\mu = \int_{[0,\infty)} \pi_{z\cdot m}\ \lambda(dz)\ . \tag{6.19}$$

In particular, in this case for all $F,G \in \mathcal{F}C_b^\infty$

$$-\int_{\Gamma_X} LF(\gamma)\ G(\gamma)\ \mu(d\gamma) = \int_{\Gamma_X} \langle \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma)\rangle_{T_\gamma\Gamma}\ \mu(d\gamma)\ , \tag{6.20}$$

i.e., $L = \ div^\Gamma \nabla^\Gamma$ is a μ–Laplacian.

Proof. [R 98, Theorem 2.2] and [AKR 98 b, Theorem 4.1]. □

Remark 6.2. (i) Note that since we a–priori assume (6.18), λ in (6.19) must satisfy

$$\int_{[0,\infty)} z\ \lambda(dz) < \infty\ . \tag{6.21}$$

(ii) Tracing through the proof of [AKR 98 b, Theorem 4.1, (i) $\Rightarrow$ (ii)] one sees that the set of all measures, which have Radon mean and *symmetrize* L, is indeed exactly the set of all mixed Poisson measures with λ satisfying (6.21).

There is a well–known theorem in differential geometry characterizing the volume element m on a Riemannian manifold X as the up to a constant unique Radon measure σ on X so that div^X is the adjoint of ∇^Γ on $L^2(X;\sigma)$. In this sense Theorem 6.1 can be interpreted as follows: it characterizes the mixed Poisson measures with λ satisfying (6.21) as exactly the *volume elements* on our "manifold" Γ_X (with tangent bundel $(T_\gamma\Gamma_X)_{\gamma\in\Gamma}$).

Theorem 6.1 has its analogue for non–trivial interaction potentials ϕ (cf. [AKR 98 b]) with so–called *Ruelle measures* as symmetrizing measures. It also has a number of applications. One concerns ergodicity and will be presented in the next subsection.

6.2 Ergodicity.

The results of this subsection are of a very general nature and hold for arbitrary diffusion operators. We shall explain them here only in the ("non–flat"!) situation of the previous subsection. We refer to [Eb 98, Ch. 3, Theorem 3.8] for the general case and to [AKR 97 a, Sect. 5], [AKR 98, Sect. 4], [AKR 98 b, Sect. 6] for cases of various types of Gibbs measures on the lattice or in continuum including *infinite volume* Euclidean quantum fields and also including the situation of Example 2.7 above.

We introduce the following notation

$$\mu_\lambda := \int_{[0,\infty)} \pi_{z\cdot m}\ \lambda(dz)\ , \tag{6.22}$$

where $\pi_{z\cdot m}$ is as defined in (6.17) with $z\cdot m$ replacing m for $z\in[0,\infty)$ and λ is a probability measure on $([0,\infty),\ \mathcal{B}([0,\infty))$ satisfying (6.21). Let $\mathcal{M}$ denote the set of all such μ_λ. By Theorem 6.1, $\mathcal{M}$ is exactly the set of all volume elements on Γ_X.

6.2.1 Sobolev spaces on Γ_X.

Fix $\mu_\lambda \in \mathcal{M}$ and consider

$$\mathcal{E}^\Gamma_{\mu_\lambda}(F,G) := \int_{\Gamma_X} \langle \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma)\rangle_{T_\gamma \Gamma_X} \ \mu_\lambda(d\gamma) \ , \ F, \ G \in \mathcal{F}C_b^\infty \ . \tag{6.23}$$

By Theorem 6.1 (ii) $\Rightarrow$ (i), $\left(\mathcal{E}^\Gamma_{\mu_\lambda}, \mathcal{F}C_b^\infty\right)$ is closable on $L^2(\Gamma_X;\mu_\lambda)$. Let $\left(\mathcal{E}^\Gamma_{\mu_\lambda}, D(\mathcal{E}^\Gamma_{\mu_\lambda})\right)$ denote its closure. Then we set

$$H_0^{1,2}(\Gamma_X;\mu_\lambda) := D\left(\mathcal{E}^\Gamma_{\mu_\lambda}\right) \ , \tag{6.24}$$

i.e., we consider $D(\mathcal{E}^\Gamma_{\mu_\lambda})$ as the $(1,2)$–*Sobolev* space on Γ_X, equipped with the inner product

$$\begin{aligned}\langle F,G\rangle_{H_0^{1,2}(\Gamma_X;\mu_\lambda)} :&= \int_{\Gamma_X} \langle \nabla^\Gamma F, \nabla^\Gamma G\rangle_{T\Gamma_X} \ d\mu_\lambda + \int FG \ d\mu_\lambda \\ &\left(= \mathcal{E}^\Gamma_{\mu_\lambda,1}(F,G)\right) \ ; \ F,G \in H_0^{1,2}(\Gamma_X;\mu_\lambda)\end{aligned} \tag{6.25}$$

and corresponding norm $\|\cdot\|_{H_0^{1,2}(\Gamma_X;\mu_\lambda)}$. We are now going to construct an extension of $H_0^{1,2}(\Gamma_X;\mu_\lambda)$ which is the exact analogue on Γ_X of a weak Sobolev space on a finite dimensional manifold. This is more or less a special case of a construction by A. Eberle in [Eb 98, Ch. 3 b)].

We first note that, as is easy to check, the μ_λ–equivalence classes determined by $\mathcal{V}\mathcal{F}C_b^\infty$ (cf. (6.13)) are dense in $L^2(\Gamma_X \to T\Gamma_X;\mu_\lambda)$, i.e., the Hilbert space of (μ–classes of) μ–square integrable sections in $T\Gamma_X$. Let $((\mathrm{div}^\Gamma)^{*,\mu_\lambda}, D((\mathrm{div}^\Gamma)^{*,\mu_\lambda}))$ be the adjoint of $(\mathrm{div}^\Gamma, \mathcal{V}\mathcal{F}C_b^\infty)$ as an operator from $L^2(\Gamma_X \to T\Gamma_X;\mu_\lambda)$ to $L^2(\Gamma_X;\mu_\lambda)$.

Remark 6.3. As an adjoint the operator $((\mathrm{div}^\Gamma)^{*,\mu_\lambda}, \ D((\mathrm{div}^\Gamma)^{*,\mu_\lambda}))$ is automatically closed, and by definition it is an operator from $L^2(\Gamma_X;\mu_\lambda)$ to $L^2(\Gamma_X \to T\Gamma_X;\mu_\lambda)$. Furthermore, again by definition, $G \in L^2(\Gamma_X;\mu_\lambda)$ belongs to $D((\mathrm{div}^\Gamma)^{*,\mu_\lambda})$ if and only if there exists $V_G \in L^2(\Gamma_X \to T\Gamma_X;\mu_\lambda)$ such that

$$\int G \ \mathrm{div}^\Gamma V \ d\mu_\lambda = -\int \langle V_G, V\rangle_{T\Gamma_X} \ d\mu \ \text{ for all } V \in \mathcal{V}\mathcal{F}C_b^\infty \ . \tag{6.26}$$

In this case $(\mathrm{div}^\Gamma)^{*,\mu_\lambda} G = V_G$.

Because of Remark 6.3 we set

$$W^{1,2}(\Gamma_X;\mu_\lambda) := D((\mathrm{div}^\Gamma)^{*,\mu_\lambda}) \ , \ d^{\mu_\lambda} := (\mathrm{div}^\Gamma)^{*,\mu_\lambda} \tag{6.27}$$

and think of $W^{1,2}(\Gamma_X;\mu_\lambda)$ as a *weak* $(1,2)$–*Sobolev space on* Γ_X with inner product

$$\begin{aligned}\langle F,G\rangle_{W^{1,2}(\Gamma_X;\mu_\lambda)} := \int_{\Gamma_X} \langle d^{\mu_\lambda} F, \ d^{\mu_\lambda} G\rangle_{T\Gamma_X} \ d\mu_\lambda + \int_{\Gamma_X} FG \ d\mu_\lambda \ , \\ F, \ G \in W^{1,2}(\Gamma_X;\mu_\lambda)\end{aligned} \tag{6.28}$$

and corresponding norm $\|\cdot\|_{W^{1,2}(\Gamma_X;\mu_\lambda)}$. Remark 6.3 immediately implies that

$$\mathcal{F}C_b^\infty \subset W^{1,2}(\Gamma_X;\mu_\lambda) \text{ and } d^{\mu_\lambda} = \nabla^\Gamma \text{ on } \mathcal{F}C_b^\infty , \tag{6.29}$$

hence

$$H_0^{1,2}(\Gamma_X;\mu_\lambda) \subset W^{1,2}(\Gamma_X;\mu_\lambda) \text{ and } d^\mu = \nabla^\Gamma \text{ on } H_0^{1,2}(\Gamma_X;\mu_\lambda) . \tag{6.30}$$

The above construction can be carried out for a large class of probability measures μ on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ (cf. [R 98, Subsect. 4.1]) and in general we *do not* have equality in (6.30) with such μ replacing μ_λ. It is a remarkable fact that we do have equality for mixed Poisson measures μ_λ in many cases.

Proposition 6.4. *Suppose X is complete, and $\lambda = \varepsilon_z$ for some $z \in [0,\infty)$ or Brownian motion on X is conservative. Then*

$$H_0^{1,2}(\Gamma_X;\mu_\lambda) = W^{1,2}(\Gamma_X;\mu_\lambda) . \tag{6.31}$$

Proof. [R 98, Proposition 4.6]. □

6.2.2 Ergodicity and extremality.

Obviously, the set $\mathcal{M}$ introduced at the beginning of Subsect. 6.2 is convex. Let $\mathcal{M}_{ex}$ denote the set of its extreme points.

Remark 6.5. It is not hard to check that

$$\mathcal{M}_{ex} = \{\pi_{z\cdot m} \mid z \in [0,\infty)\} . \tag{6.32}$$

Now we can formulate the main result of Subsect. 6.2.

Theorem 6.6. *Suppose the conditions in Proposition 6.4 hold and let $\mu \in \mathcal{M}$. Then the following are equivalent:*

(i) $\mu \in \mathcal{M}_{ex}$ *(i.e., by Remark 6.5, $\mu = \pi_{z\cdot m}$ for some $z \in [0,\infty)$).*

(ii) $\{\nu \in \mathcal{M} \mid \nu = \rho\cdot\mu$ *for some* $\rho : \Gamma_X \to [0,\infty)$, *bounded,* $\mathcal{B}(\Gamma_X)$*–measurable*$\}$ $= \{\mu\}$.

(iii) $(\mathcal{E}_\mu^\Gamma; D(\mathcal{E}_\mu^\Gamma))$ *is irreducible (i.e., $F \in D(\mathcal{E}_\mu^\Gamma)$, $\mathcal{E}_\mu^\Gamma(F,F) = 0$ implies $F = const.$).*

(iv) If $(L_\mu, D(L_\mu))$ denotes the generator of the Dirichlet form $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ (i.e., $(L_\mu, D(L_\mu))$ is the Friedrichs' extension of $(\Delta^\Gamma, \mathcal{F}C_b^\infty)$, cf. (6.15)), then: $F \in D(L_\mu)$, $L_\mu F = 0$ implies $F = const.$ ("uniqueness of ground state").

(v) $(T_t^\mu)_{t\geq 0} := (e^{tL_\mu})_{t\geq 0}$ is irreducible, (i.e., if $F \in L^2(\Gamma_X;\mu)$ such that $T_t^\mu(FG) = FT_t^\mu G$ for all $G \in L^\infty(\Gamma_X;\mu)$, $t \geq 0$, then $F = const.$).

(vi) $F \in L^2(\Gamma_X;\mu)$, $T_t^\mu F = F$, $t \geq 0$, implies $F = const.$

(vii) $(T_t^\mu)_{t\geq 0}$ is L^2–ergodic (i.e., for all $F \in L^2(\Gamma_X;\mu)$

$$\int \left(T_t^\mu F - \int F\, d\mu\right)^2 d\mu \to 0 \textit{ as } t \to \infty) .$$

Remark 6.7. It can be shown that $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ as in Theorem 6.6 has a diffusion process associated to it in the sense of Definition 5.3 and that (i) - (vii) above is equivalent to the time–ergodicity of that diffusion if it is started with distribution μ (cf. [R 98, Prop. 3.17]).

Proof of Theorem 6.6. (cf. [R 98 , Theorem 9.4]).
The equivalence of (iii) – (vii) is standard and is proved e.g. entirely analogously to [AKR 97a, Proposition 2.3]. So, we only prove (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).
(i) $\Rightarrow$ (ii): Assume (i) holds. Let $\rho : \Gamma_X \to [0,\infty)$ be bounded and $\mathcal{B}(\Gamma_X)$–measurable such that $\nu := \rho \cdot \mu \in \mathcal{M}$, and let $M := \sup_{\gamma \in \Gamma_X} \rho(\gamma)$. Define

$$\mu_1 := \frac{M-\rho}{M-1}\ \mu\ .$$

Then $\mu_1 \in \mathcal{M}$ and $\mu = \frac{M-1}{M}\mu_1 + \frac{1}{M}\nu$. By assumption (i) it follows that $\rho = 1$, and (ii) is proved.
(ii) $\Rightarrow$ (i): Assume (ii) holds. Let μ_1, $\mu_2 \in \mathcal{M}$ and $t \in (0,1)$ such that $\mu = t\mu_1 + (1-t)\mu_2$. Then they are both absolutely continuous w.r.t. μ with bounded densities. By assumption (ii) it follows that $\mu_1 = \mu = \mu_2$. Consequently $\mu \in \mathcal{M}$.
(ii) $\Rightarrow$ (iii): Assume (ii) holds. Let $G \in H_0^{1,2}(\Gamma_X;\mu)(= D(\mathcal{E}_\mu))$ such that $\mathcal{E}_\mu(G,G) = 0$. Let $G_n := \sup(\inf(G,n),-n)$. Then

$$\lim_{n\to\infty} \|G - G_n\|_{H_0^{1,2}(\Gamma_X;\mu)} = 0$$

(cf. e.g. [MR 92, Ch. I, Prop. 4.17]) and

$$\mathcal{E}_\mu(G_n,G_n) \le \mathcal{E}_\mu(G,G) = 0$$

(cf. e.g. [MR 92, Ch. I, Theorem 4.12]). Since $G = \text{const.}$ if all $G_n = \text{const.}$, we may assume that G is bounded and, since $1 \in D(\mathcal{E}_\mu)$ and $\mathcal{E}_\mu(1,1) = 0$, also that $G \ge 0$ (otherwise we add a large constant) and that $\int G\, d\mu = 1$. Define $\nu := G \cdot \mu$. Then, since $\nabla^\Gamma G = 0$ and $\mu \in \mathcal{M}$, we have

$$\begin{aligned}\int_{\Gamma_X} \langle \nabla^\Gamma F, V\rangle_{T\Gamma_X}\ d\nu &= \int_{\Gamma_X} \langle \nabla^\Gamma (FG), V\rangle_{T\Gamma_X}\ d\mu \\ &= -\int F\ G\ \operatorname{div}^\Gamma V\ d\mu \\ &= -\int F\ \operatorname{div}^\Gamma V\ d\nu\end{aligned}$$

for all $F \in \mathcal{F}C_b^\infty$, $V \in \mathcal{VF}C_b^\infty$. Hence $\nu \in \mathcal{M}$ and thus by assumption (ii), $G = 1$.
(iii) $\Rightarrow$ (ii): Assume (iii) holds. Let $\rho : \Gamma_X \to [0,\infty)$, $\mathcal{B}(\Gamma_X)$–measurable and bounded so that $\nu := \rho \cdot \mu \in \mathcal{M}$. Then there exist $F_n \in \mathcal{F}C_b^\infty$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \|\rho - F_n\|_{H_0^{1,2}(\Gamma_X;\mu)} = 0\ .$$

Hence for all $V \in \mathcal{VFC}_b^\infty$, since $\nu \in \mathcal{M}$,

$$\int \operatorname{div}^\Gamma V \ \rho \, d\mu = 0 \ .$$

It follows by Remark 6.3, (6.26) and (6.27) that

$$\rho \in W^{1,2}(\Gamma_X;\mu) \text{ and } d^\mu \rho = 0 \ .$$

Proposition 6.4 and (6.30) now imply that $\rho \in H_0^{1,2}(\Gamma_X;\mu)$ and that $\nabla^\Gamma \rho = d^\mu \rho = 0$, i.e., $\mathcal{E}_\mu(\rho,\rho) = 0$. By assumption (iii) we conclude that $\rho \equiv 1$. □

Remark 6.8. If we are dealing with measures μ where $H_0^{1,2}(\Gamma_X;\mu) \subsetneqq W^{1,2}(\Gamma_X;\mu)$, there is an analogue of Theorem 6.6 where $H_0^{1,2}(\Gamma_X;\mu)$ is replaced by $W^{1,2}(\Gamma_X;\mu)$. We refer to [R 98, Sect. 9] (or [AKR 98 b, Sect. 6]) for details and examples.

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