# Independent sets and 2-factors in edge-chromatic-critical graphs

Stefan Grünewald Eckhard Steffen
Universität Bielefeld
Fakultät für Mathematik
Postfach 100131
33501 Bielefeld, Germany

**Abstract**

In 1968, Vizing made the following two conjectures for graphs which are critical with respect to the chromatic index: (1) every critical graph has a 2-factor, and (2) every independent vertex set in a critical graph contains at most half of the vertices.

We prove both conjectures for critical graphs with many edges, and determine upper bounds for the size of independent vertex sets in those graphs.

## 1 Introduction

In this paper we consider simple graphs. Beside our definitions we use standard graph theoretical notation.

Vizing showed that the chromatic index $\chi'(G)$ of a graph $G$ equals its maximum vertex degree $\Delta(G)$ or $\Delta(G)+1$.

A graph $G$ is $\Delta$-*critical* if it has chromatic index $\Delta+1$ and each proper subgraph of $G$ has smaller chromatic index. If the value of $\Delta$ is unimportant we use *critical* graph instead of $\Delta$-critical.

A graph $G = (V, E)$ with more than $\Delta\lfloor \frac{|V|}{2} \rfloor$ edges cannot be colored with $\Delta$ colors, since it has too many edges. Such graphs are called *overfull*. Thus it is not the specific structure of the graph which causes its coloring properties. The same holds true for many critical graphs with an odd number of vertices and $\Delta\lfloor \frac{|V|}{2} \rfloor + 1$ edges, and it seems that for fixed $n$ the most critical graphs of order $n$ are overfull, see [1].

It is difficult to find critical graphs which are not overfull. For instance critical graphs of even order cannot be overfull and just recently the first infinite families of $\Delta$-critical graphs of even order, for each $\Delta \geq 3$, were constructed in [2].

There are only few results about the structure of critical graphs and some well known conjectures about their structure do not hold, see [2]. However the following two conjectures of Vizing are still open.

**Conjecture 1.1 ([3])** *Every critical graph has a 2-factor.*

Let $\alpha(G)$ be the *independence number* of a graph $G$, i.e. the maximum size of a set of pairwise non-adjacent vertices in $G$.

**Conjecture 1.2 ([3])** *For every critical graph $G = (V, E)$ holds $\alpha(G) \leq \frac{|V|}{2}$.*

Conjecture 1.1 implies Conjecture 1.2 since a 2-factor already does not allow an independent set with more than half of the vertices. From this also it follows that both conjectures are true for 2-critical graphs.

Let $K_0^\Delta$ be obtained from the complete bipartite graph $K_{\Delta,\Delta}$ by subdividing one edge. It is easy to see that $K_0^\Delta$ is $\Delta$-critical. We construct an infinite sequence of graphs $K_0^\Delta, K_1^\Delta \ldots$, where $K_{i+1}^\Delta$ $(i \geq 1)$ is obtained from $K_i^\Delta$ by replacing a vertex of maximum degree by a $K_{\Delta-1,\Delta}$. From results of [2] it follows that $K_k^\Delta$ is $\Delta$-critical for each $k \geq 1$. Further $|V(K_k^\Delta)| = 2(\Delta + k\Delta - k) + 1$ and $K_k^\Delta$ has an independent set of $\Delta + k\Delta - k$ vertices, and hence Conjecture 1.2 is best possible for every $\Delta$, if true.

The aim of this paper is to prove both conjectures for critical graphs with many edges, and to give upper bounds for the independence number of critical graphs.

## 2 Sufficient conditions for 2-factors and upper bounds for the independence number

Let $G = (V, E)$ be a graph, with $d_G(v)$ we denote the vertex-degree of vertex $v \in V$, and $s(G) = \sum_{v \in V} \Delta(G) - d_G(v)$ denotes the *deficiency* of $G$. We will make use of the well known Parity Lemma.

**Lemma 2.1 (Parity Lemma)** *Let $M$ be a multigraph whose edges are colored with colors $1, \ldots, k$, and let $a_i$ be the number of vertices $v$ in $M$ such that no edge incident to $v$ is colored $i$. Then for all $i = 1, \ldots, k : a_i \equiv |V(M)| \pmod 2$.*

**Theorem 2.2** *Let $\Delta \geq 3$ and $G = (V, E)$ be a $\Delta$-critical graph, and $k = \min\{d_G(u) + d_G(v) | uv \in E\}$.*

1. *If $|V|$ is odd, then*
   (a) *if $s(G) \leq 5\Delta - 2k + 1$ then $G$ has a 2-factor.*
   (b) *$\alpha(G) \leq \frac{1}{2}(|V| - 3) + \lfloor \frac{s(G)+2k-\Delta-2}{2\Delta} \rfloor$.*
2. *If $|V|$ is even, then*
   (a) *if $s(G) \leq 2\Delta - 6$ then $G$ has a 2-factor.*
   (b) *$\alpha(G) \leq \frac{1}{2}(|V| - 4) + \lfloor \frac{s(G)+2k-2}{2\Delta} \rfloor$.*

**Proof.** Let $e = vw$ be an edge with $d_G(v) + d_G(w) = k$. Since $G$ is critical, there is a $\Delta$-coloring $\phi$ of $G - e$. Note that no color can be missing at both $v$ and $w$, since otherwise we could color $G$ with $\Delta$ colors.

Let $C_1$ be the set of colors present at only one of $v, w$ and $C_2$ be the set of colors present at both. We may assume that $C_1 = \{1, 2, \ldots, 2\Delta - k + 2\}$ and $C_2 = \{2\Delta - k + 3, \ldots, \Delta\}$, where $C_2 = \emptyset$ if $k = \Delta + 2$.

Let $V' = V \setminus \{v, w\}$, and for $1 \leq i \leq \Delta$, let $a_i$ be the number of vertices in $V'$ where color $i$ is missing. Let $a_l = \min\{a_i | i = 1, \ldots, \Delta\}$.

If $l \in C_1$ then, since the elements of $C_1$ and $C_2$ have different parity, it follows that

$$\begin{aligned} s(G) - 2\Delta + k &= \sum_{i=1}^{\Delta} a_i \\ &\geq a_l|C_1| + (a_l + 1)|C_2| \\ &= a_l\Delta + k - \Delta - 2, \end{aligned}$$

and hence

$$a_l \leq \lfloor \frac{s(G) + 2}{\Delta} \rfloor - 1. \tag{1}$$

**Remark C1:** We may assume that $l$ is missing at $v$ and let $m$ be a color missing at $w$. Since $G$ is critical there is a path $P_{m,l}(v, w)$ with terminal ends $v$ and $w$, whose edges are colored $l$ and $m$ alternately. Path $P_{m,l}(v, w)$ together with $e = vw$ forms an odd circuit $C_{m,l}(e)$ in $G$.

If $l \in C_2$ then, since the elements of $C_1$ and $C_2$ have different parity, it follows that

$$\begin{aligned} s(G) - 2\Delta + k &= \sum_{i=1}^{\Delta} a_i \\ &\geq (a_l + 1)|C_1| + a_l|C_2| \\ &= a_l\Delta + 2\Delta - k + 2. \end{aligned}$$

and hence

$$a_l \leq \lfloor \frac{s(G) + 2k - 2}{\Delta} \rfloor - 4. \tag{2}$$

1) Let $|V|$ be odd. In this case it follows from Lemma 2.1 that for $1 \leq i \leq \Delta$:

$$a_i \equiv 1 \pmod 2 \text{ if } i \in C_2 \text{ and } a_i \equiv 0 \pmod 2 \text{ if } i \in C_1.$$

We have to consider two cases according to whether $l$ is in $C_1$ or $C_2$.
Let $l \in C_1$. From Remark C1 it follows that $G$ contains an odd circuit $C_{m,l}(e)$ which allows only an independent set of $\frac{|V(C_{m,l}(e))|-1}{2}$ vertices. Considering the subgraph whose edges are colored $m$ or $l$ shows that $\alpha(G) \leq \frac{1}{2}(|V| + a_l - 1)$.
Further $a_l$ is even and therefore if $\lfloor \frac{s(G)+2}{\Delta} \rfloor$ is even, Equation (1) gives a bound which is at least $a_l + 1$. To make the bound best possible for all cases we replace (1) by

$$a_l \leq 2\lfloor \frac{s(G) + 2 - \Delta}{2\Delta} \rfloor,$$

and hence

$$\alpha(G) \leq \frac{1}{2}(|V| + 2\lfloor \frac{s(G) + 2 - \Delta}{2\Delta} \rfloor - 1) \tag{3}$$

in this case.
Let $l \in C_2$. Then $a_l$ is odd and hence if $\lfloor \frac{s(G)+2k-2}{\Delta} \rfloor$ is even, Equation (2) gives a bound which is at least $a_l + 1$. To make the bound best possible for all cases we replace (2) by

$$a_l \leq 2\lfloor \frac{s(G) + 2k - \Delta - 2}{2\Delta} \rfloor - 3.$$

In this case we have $\alpha(G) \leq \frac{1}{2}(|V| + a_l)$ and hence

$$\alpha(G) \leq \frac{1}{2}(|V| + 2\lfloor \frac{s(G) + 2k - \Delta - 2}{2\Delta} \rfloor - 3). \tag{4}$$

Since $G$ is critical it follows that $k \geq \Delta + 2$ and therefore

$$\begin{aligned} 2\lfloor \frac{s(G) + 2k - \Delta - 2}{2\Delta} \rfloor - 3 \quad &\geq \quad 2\lfloor \frac{s(G) + \Delta + 2}{2\Delta} \rfloor - 3 \\ &= \quad 2\lfloor \frac{s(G) - \Delta + 2}{2\Delta} \rfloor - 1. \end{aligned} \tag{5}$$

Now statement 1.$b$ directly follows from (3), (4) and (5).

Let $s(G) \leq 5\Delta - 2k + 1$. If $l \in C_1$ then if there is a second color $m \neq l$ in $C_1$ and $a_m = 0$, then it can be arranged by possibly recoloring that $l$ is missing at $v$ and $m$ at $w$. Then $C_{m,l}(e)$ together with the $(m,l)$-circuits of even length form a 2-factor of $G$.
Let us assume to the contrary that such a color $m \in C_1$ does not exist, i.e. $a_i \geq 2$ for each $i \in C_1$ and $i \neq l$. Then it follows that

$$\begin{aligned} s(G) - 2\Delta + k &= \sum_{i=1}^{\Delta} a_i \\ &\geq 2(|C_1| - 1) + |C_2| \\ &= 3\Delta - k + 2 \end{aligned}$$

and hence $s(G) \geq 5\Delta + 2k - 2$, contradiction.
If $l \in C_2$ then it follows from (2) that $a_l = 0$, contradicting the fact that $a_l$ is odd in this case. This proves statement 1.$a$.

2) Let $|V|$ be even. In this case it follows from Lemma 2.1 that for $1 \leq i \leq \Delta$:

$$a_i \equiv 0 \pmod 2 \text{ if } i \in C_2 \text{ and } a_i \equiv 1 \pmod 2 \text{ if } i \in C_1.$$

Again we have to consider two cases according to whether $l$ is in $C_1$ or $C_2$.
Let $l \in C_1$. From Remark C1 it follows that $G$ contains an odd circuit $C_{m,l}(e)$ which allows only an independent set of $\frac{|V(C_{m,l}(e))|-1}{2}$ vertices. Considering the subgraph whose edges are colored $m$ or $l$ shows that $\alpha(G) \leq \frac{1}{2}(|V| + a_l - 1)$.
Further $a_l$ is odd and therefore if $\lfloor \frac{s(G)+2}{\Delta} \rfloor$ is odd, Equation (1) gives a bound which is at least $a_l + 1$. To make the bound best possible for all cases we replace (1) by

$$a_l \leq 2\lfloor \frac{s(G)+2}{2\Delta} \rfloor - 1.$$

Thus

$$\alpha(G) \leq \frac{1}{2}(|V| + 2\lfloor\frac{s(G)+2}{2\Delta}\rfloor - 2) \tag{6}$$

in this case.

Let $l \in C_2$. Then $a_l$ is even and hence if $\lfloor\frac{s(G)+2k-2}{\Delta}\rfloor$ is odd, Equation (2) gives a bound which is least $a_l + 1$. To make the bound best possible for all cases we replace (2) by

$$a_l \leq 2\lfloor\frac{s(G)+2k-2}{2\Delta}\rfloor - 4.$$

In this case we have $\alpha(G) \leq \frac{1}{2}(|V| + a_l)$ and hence

$$\alpha(G) \leq \frac{1}{2}(|V| + 2\lfloor\frac{s(G)+2k-2}{2\Delta}\rfloor - 4). \tag{7}$$

Since $G$ is critical it follows that $k \geq \Delta + 2$ and therefore

$$\begin{aligned} 2\lfloor\frac{s(G)+2k-2}{2\Delta}\rfloor - 4 &\geq 2\lfloor\frac{s(G)+2\Delta+2}{2\Delta}\rfloor - 4 \\ &= 2\lfloor\frac{s(G)+2}{2\Delta}\rfloor - 2. \end{aligned} \tag{8}$$

Now statement $2.b$ directly follows from (6), (7) and (8).

It remains to prove statement $2.a$. Since $G-e$ is $\Delta$-colorable and of even order, Lemma 2.1 implies that each color is missing on an even number of vertices. Since $s(G-e) \leq 2\Delta - 4$ there are at least two colors, say 1 and 2, which are present at every vertex. Thus the subgraph with the edges colored 1 or 2 forms a 2-factor in $G$.

□

**Corollary 2.3** *Every overfull critical graph $G$ has a 2-factor.*

**Proof.** Since $|V(G)|$ is odd, $s(G) = \Delta - 2$, and $d_G(v) + d_G(w) \leq 2\Delta$ for every edge $e = vw$, the result is implied by Theorem 2.2 $1.a$.

□

**Corollary 2.4** *Let $G$ be a critical graph of even order, and $k = \min\{d_G(u) + d_G(v) | uv \in E\}$. If $s(G) \leq 6\Delta - 2k + 1$ then $\alpha(G) \leq \frac{1}{2}|V|$.*

**Proof.** This directly follows from Theorem 2.2 2.$b$.

□

# References


[1] G. Brinkmann, E. Steffen, Chromatic index critical graphs of orders 11 and 12, to appear in *European Journal of Combinatorics*

[2] S. Grünewald, E. Steffen, Chromatic Index Critical graphs of even order, to appear in *Journal of Graph Theory*

[3] V. G. Vizing, Some unsolved problems in graph theory, *Russian Mathematical Surveys* **23 (6)** (1968) 125-142