# ELEMENTS OF $K_1$ VIA THE TRIPLE $G$-CONSTRUCTION

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Abstract. We represent the elements of $K_1$ of an exact category by means of two- and three-edge loops in the triple $G$-construction. The corresponding algebraic data are double short exact sequences and non-commutative $3 \times 3$-diagrams of short exact sequences. We state conjectural relations for the elements of $K_1$ in terms of $3 \times 3$-diagrams.

## 1. Review of the $G$- and the triple $G$-construction

In [GG] Gillet and Grayson attached a simplicial set $G.\mathcal{A}$ to any exact category $\mathcal{A}$ and proved that $|G.\mathcal{A}|$ is homotopy equivalent to $\Omega|Q\,\mathcal{A}| \sim \Omega|S.\mathcal{A}|$, the equivalence being natural in $\mathcal{A}$. Thus one can take the formula

$$K_m(\mathcal{A}) = \pi_m(G.\mathcal{A})\,, \quad m \geq 0\,,$$

for a definition of the higher K-groups of $\mathcal{A}$.

An $n$-simplex in $G.\mathcal{A}$ is a pair of triangular diagrams in $\mathcal{A}$ of the form

$$\begin{array}{ccccccc}
 & & & & P_{n/n-1} \\
 & & & & \uparrow \\
 & & & & \cdots \\
 & & P_{2/1} & \to \dots \to & P_{n/1} \\
 & & \uparrow & & \uparrow \\
 & P_{1/0} \to & P_{2/0} & \to \dots \to & P_{n/0} \\
 & \uparrow & \uparrow & & \uparrow \\
P_0 \to & P_1 \to & P_2 & \to \dots \to & P_n
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & & P_{n/n-1} \\
 & & & & \uparrow \\
 & & & & \cdots \\
 & & P_{2/1} & \to \dots \to & P_{n/1} \\
 & & \uparrow & & \uparrow \\
 & P_{1/0} \to & P_{2/0} & \to \dots \to & P_{n/0} \\
 & \uparrow & \uparrow & & \uparrow \\
P'_0 \to & P'_1 \to & P'_2 & \to \dots \to & P'_n
\end{array}
\tag{1.1}$$

subject to the conditions:
(i) the quotient index subtriangles in both diagrams coincide;
(ii) all the squares commute;
(iii) all the sequences of the form $P_j \to P_k \to P_{k/j}$, $P'_j \to P'_k \to P_{k/j}$, and $P_{j/i} \to P_{k/i} \to P_{k/j}$ with $i \leq j \leq k$ are s.e.s's in $\mathcal{A}$. (For short, we will write *s.e.s.* for *short exact sequence*).

In particular, a vertex in $G.\mathcal{A}$ is a pair of objects $(P, P')$, and an edge connecting $(P_0, P'_0)$ to $(P_1, P'_1)$ is a pair of s.e.s's $(P_0 \to P_1 \to P_{1/0},\ P'_0 \to P'_1 \to P_{1/0})$ with equal cokernels.

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The $i$-th face of (1.1) amounts to deleting all the objects whose indices contain $i$. For instance, the faces of a generic 2-simplex

$$t = \left[\begin{array}{cccccc} & & P_{2/1} & & & P_{2/1} \\ & & \uparrow & & & \uparrow \\ & P_{1/0} \longrightarrow & P_{2/0} & & P_{1/0} \longrightarrow & P_{2/0} \\ & \uparrow & \uparrow & & \uparrow & \uparrow \\ P_0 \longrightarrow & P_1 \longrightarrow & P_2 & P'_0 \longrightarrow & P'_1 \longrightarrow & P'_2 \end{array}\right]$$

are given by

$$\begin{aligned} d_0 t &= (P_1 \to P_2 \to P_{2/1}, \quad P'_1 \to P'_2 \to P_{2/1}) \\ d_1 t &= (P_0 \to P_2 \to P_{2/0}, \quad P'_0 \to P'_2 \to P_{2/0}) \\ d_2 t &= (P_0 \to P_1 \to P_{1/0}, \quad P'_0 \to P'_1 \to P_{1/0}). \end{aligned}$$

The reader is referred to [Ne1-3] for a better introduction to $G.\mathcal{A}$. In the present paper we will also need another simplicial set $G^{III}_{.}\mathcal{A}$ with the properties:
(i) a vertex in it is a triple of objects $(P', P, P'') \in \mathcal{A} \times \mathcal{A} \times \mathcal{A}$, the upper index III referring to this fact;
(ii) an edge from $(P'_0, P_0, P''_0)$ to $(P'_1, P_1, P''_1)$ is a diagram of the form

$$\begin{array}{ccccc} P'_0 & \longrightarrow & P'_1 & \longrightarrow & P'_{1/0} \\ & & & & \downarrow \\ P_0 & \longrightarrow & P_1 & \longrightarrow & P_{1/0} \\ & & & & \downarrow \\ P''_0 & \longrightarrow & P''_1 & \longrightarrow & P''_{1/0} \end{array} \tag{1.2}$$

consisting of four s.e.s's;
(iii) an arbitrary $n$-simplex is a triple of admissible filtrations

$$\begin{aligned} &P'_0 \to P'_1 \to \ldots \to P'_n \\ &P_0 \to P_1 \to \ldots \to P_n \\ &P''_0 \to P''_1 \to \ldots \to P''_n \end{aligned}$$

endowed with

a) a choice of quotients $P'_{j/i}$, $P_{j/i}$, $P''_{j/i}$ for all possible composed maps $P'_i \to P'_j$, $P_i \to P_j$, and $P''_i \to P''_j$;

b) short exact sequences

$$0 \to P'_{j/i} \to P_{j/i} \to P''_{j/i} \to 0 \tag{1.3}$$

for all $0 \leq i \leq j \leq n$ such that the arrows in (1.3) commute with the obvious projections and inclusions of the quotient objects.

One defines the face and degeneracy maps for $G^{III}$ in the obvious way, the same as for $G$. We refer to [G, sect. 8] for a more formal definition of $G^{III}$. In fact, an infinite series of "the long $G$'s" is introduced in [G], i.e., for any $k \geq 2$ there is a $G$-construction whose vertices are $k$-tuples of objects, and all these $G$-constructions are naturally homotopy equivalent. In particular, we have an obvious embedding $i : G \to G^{III}$ given on vertices by $(P, P') \to (P, P', 0)$ which is a homotopy equivalence, and we can therefore identify

$$K_1(\mathcal{A}) = \pi_1(G.\mathcal{A}) = \pi_1(G^{III}_{.}\mathcal{A}).$$

Given a s.e.s. $s = (A' \to A \to A'')$ we denote by $e(s)$ the edge from $(0, 0, 0)$ to $(A', A, A'')$ in $G^{III}_{.}\mathcal{A}$ given by

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "1"] & A' \arrow[d] \\
0 \arrow[r] & A \arrow[r, "1"] & A \arrow[d] \\
0 \arrow[r] & A'' \arrow[r, "1"] & A''
\end{tikzcd} \tag{1.4}$$

(cf. (1.2)). Note that a generic 2-simplex in $G^{III}_{.}\mathcal{A}$ is a diagram of the form

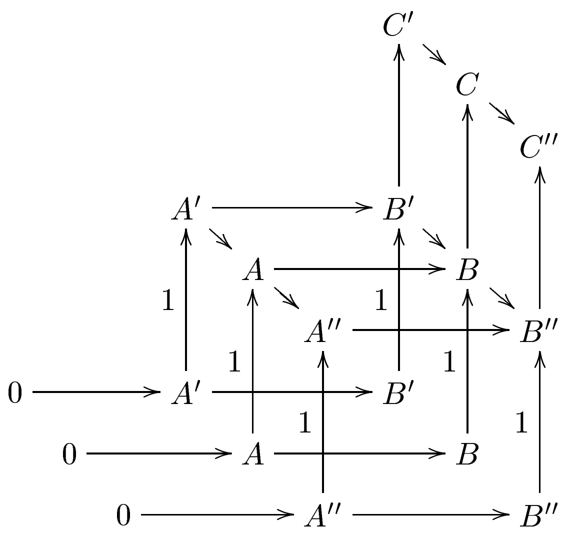
$$\begin{tikzcd}
 & & & & & & P'_{2/1} \arrow[dr] & & \\
 & & & & & & & P_{2/1} \arrow[dr] & \\
 & & & & & & & & P''_{2/1} \\
 & & & P'_{1/0} \arrow[rrr] \arrow[dr] & & & P'_{2/0} \arrow[uuu] \arrow[dr] & & \\
 & & & & P_{1/0} \arrow[rrr] \arrow[dr] & & & P_{2/0} \arrow[uuu] \arrow[dr] & \\
 & & & & & P''_{1/0} \arrow[rrr] & & & P''_{2/0} \arrow[uuu] \\
P'_0 \arrow[rrr] & & & P'_1 \arrow[uuu] \arrow[rrr] & & & P'_2 \arrow[uuu] & & \\
 & P_0 \arrow[rrr] & & & P_1 \arrow[uuu] \arrow[rrr] & & & P_2 \arrow[uuu] & \\
 & & P''_0 \arrow[rrr] & & & P''_1 \arrow[uuu] \arrow[rrr] & & & P''_2 \arrow[uuu]
\end{tikzcd}$$

In particular, given a commutative diagram of s.e.s's

$$d = \begin{array}{ccccc} A' & \longrightarrow & B' & \longrightarrow & C' \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ A'' & \longrightarrow & B'' & \longrightarrow & C'' \end{array} \tag{1.5}$$

we associate to it a 2-simplex $t(d)$ given by

(1.6)



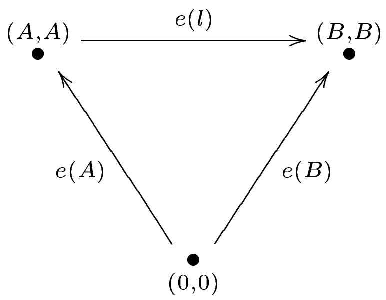

Note that the edges of $t(d)$ with the source $(0,0,0)$ are $e(s_A)$ and $e(s_B)$, where $s_A = (A' \to A \to A'')$ and $s_B = (B' \to B \to B'')$ are s.e.s's from (1.5).

## 2. Two-edge loops in $G^{III}$

Recall that a *double short exact sequence* (a d.s.e.s. for short)

$$l = (A \overset{f_1}{\underset{f_2}{\rightrightarrows}} B \overset{g_1}{\underset{g_2}{\rightrightarrows}} C)$$

is a pair of short exact sequences $s_1 = (A \xrightarrow{f_1} B \xrightarrow{g_1} C)$ and $s_2 = (A \xrightarrow{f_2} B \xrightarrow{g_2} C)$ on the same objects. In [Ne1] we proved that every element of $K_1(\mathcal{A})$ comes from a d.s.e.s. Namely, we associate to $l$ a loop $\mu(l)$ in $G.\mathcal{A}$ of the form



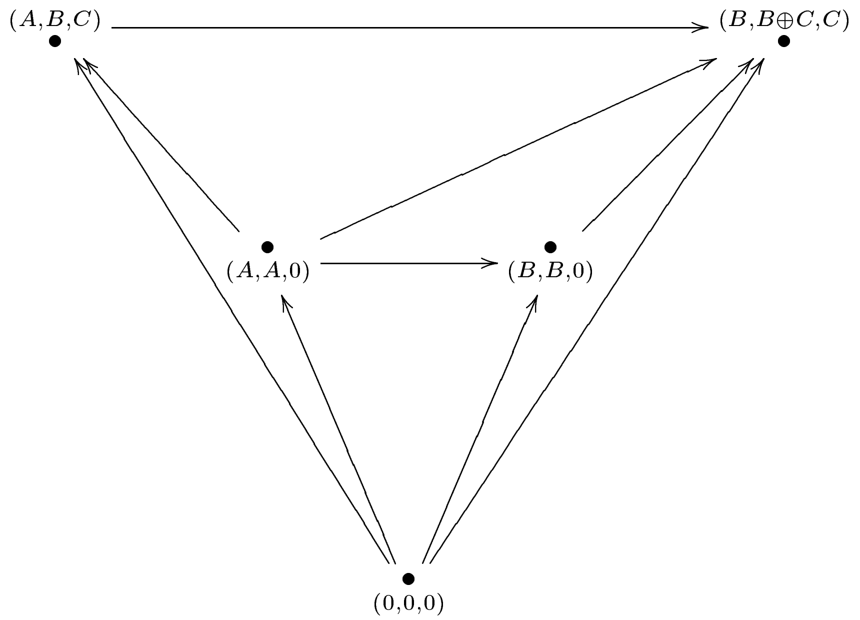
where $e(X)$ denotes the edge $(0 \to X \xrightarrow{1} X, \quad 0 \to X \xrightarrow{1} X)$ for any $X \in \mathcal{A}$ and $e(l)$ is the edge from $(A, A)$ to $(B, B)$ given by $l$, and prove that every loop in $G.\mathcal{A}$ is homotopic to $\mu(l)$ for some $l$. Since the map $i : G \to G^{III}$ is a homotopy equivalence, every loop in $G^{III}_{.}\mathcal{A}$ is homotopic to $i(\mu(l))$ for some $l$. On the other hand, we can associate to $l$ a simpler two-edge loop in $G^{III}_{.}\mathcal{A}$,

$$\kappa(l) := \begin{tikzcd} (0,0,0) \arrow[r, bend left, "e(s_1)"] \arrow[r, bend right, "e(s_2)"'] & (A,B,C) \end{tikzcd} \tag{2.1}$$

where for $i = 1, 2$, $e(s_i)$ denotes the edge given by (1.4), and we always consider depicted loops as oriented clockwise.

**Proposition 2.1.** *The loop $i(\mu(l))$ is homotopic to $\kappa(l)^{-1}$ for any d.s.e.s. $l$.*

*Proof.* We construct two configurations of 2-simplices in $G^{III}.\mathcal{A}$. The first configuration has the form

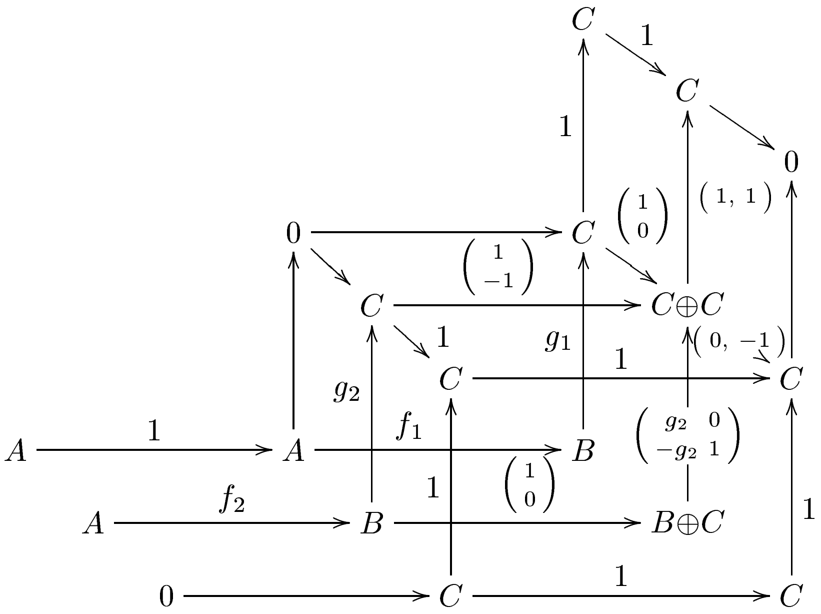


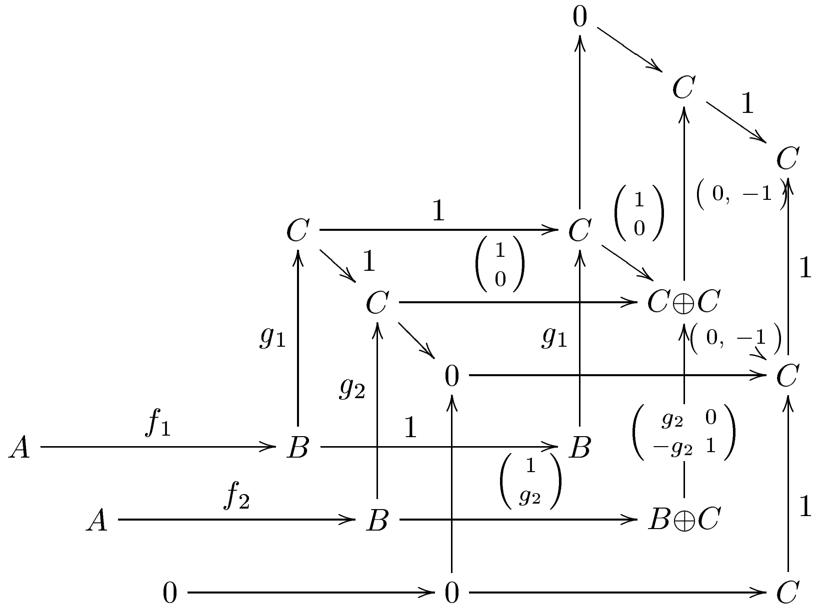
(2.2)

It has $i(\mu(l))$ as its inner loop and consists of four 2-simplices. The side 2-simplices are the simplices associated to the diagrams

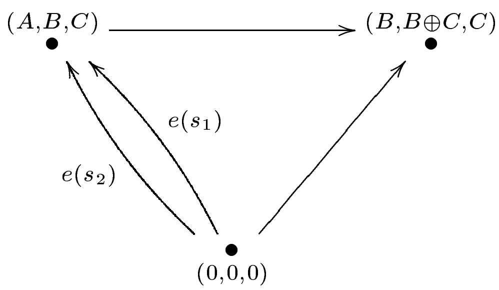
$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "1"'] & A \arrow[r] \arrow[d, "f_2"] & 0 \arrow[d] \\ A \arrow[r, "f_2"] \arrow[d] & B \arrow[r, "g_2"] \arrow[d, "g_2"] & C \arrow[d, "1"'] \\ 0 \arrow[r] & C \arrow[r, "1"] & C \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B \arrow[r, "1"] \arrow[d, "1"'] & B \arrow[r] \arrow[d, "\binom{1}{g_2}"] & 0 \arrow[d] \\ B \arrow[r, "\binom{1}{g_2}"] \arrow[d] & B \oplus C \arrow[r, "{(g_2,-1)}"] \arrow[d, "{(g_2,-1)}"] & C \arrow[d, "1"] \\ 0 \arrow[r] & C \arrow[r, "1"] & C \end{tikzcd}$$

as in (1.6). The upper simplices are given by





The second configuration




(2.3)

consists of $\kappa(l)^{-1}$ and a single 2-simplex

$$t\begin{pmatrix}
\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "f_1"] & B \arrow[r, "g_1"] \arrow[d, "\binom{1}{g_1}"] & C \arrow[d, "1"'] \\
B \arrow[r, "\binom{1}{0}"] \arrow[d, "g_1"] & B \oplus C \arrow[r, "(0,\ 1)"] \arrow[d, "(g_1,\ -1)"] & C \arrow[d] \\
C \arrow[r, "1"] & C \arrow[r] & 0
\end{tikzcd}
\end{pmatrix}.$$

The outer loops in (2.2) and (2.3) differ only in the edge from $(0,0,0)$ to $(B, B \oplus C, C)$, hence the difference between $i(\mu(l))$ and $\kappa(l)^{-1}$ is homotopic to the loop

$$\begin{tikzcd}
(0,0,0) \arrow[r, bend left, "e(r_1)"] \arrow[r, bend right, "e(r_2)"'] & (B, B \oplus C, C)
\end{tikzcd} \tag{2.4}$$

where

$$r_1 = (B \xrightarrow{\binom{1}{g_1}} B \oplus C \xrightarrow{(g_1,-1)} C), \quad r_2 = (B \xrightarrow{\binom{1}{g_2}} B \oplus C \xrightarrow{(g_2,-1)} C).$$

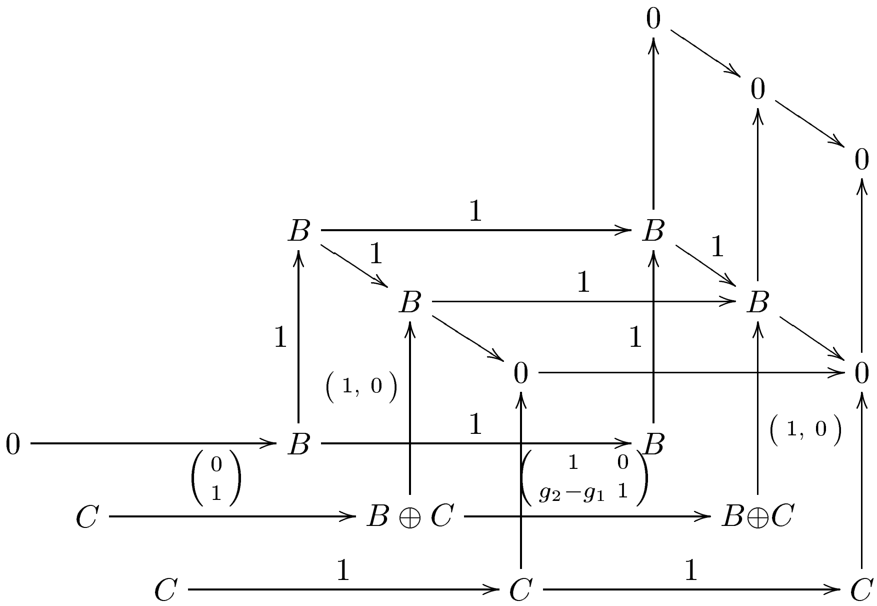
The 2-simplex

$$t\begin{pmatrix}
\begin{tikzcd}
B \arrow[r, "1"] \arrow[d, "\binom{1}{g_1}"] & B \arrow[r] \arrow[d, "\binom{1}{g_2}"] & 0 \arrow[d] \\
B \oplus C \arrow[r, "\begin{pmatrix} 1 & 0 \\ g_2 - g_1 & 1 \end{pmatrix}"] \arrow[d, "(g_1,\ -1)"] & B \oplus C \arrow[r] \arrow[d, "(g_2,\ -1)"] & 0 \arrow[d] \\
C \arrow[r, "1"] & C \arrow[r] & 0
\end{tikzcd}
\end{pmatrix}$$

provides a homotopy between (2.4) and the one-edge loop

$$\begin{tikzcd}
B \arrow[r, "1"] & B \arrow[r] & 0 \arrow[d] \\
B \oplus C \arrow[r, "\begin{pmatrix} 1 & 0 \\ g_2 - g_1 & 1 \end{pmatrix}"] & B \oplus C \arrow[r] & 0 \arrow[d] \\
C \arrow[r, "1"] & C \arrow[r] & 0
\end{tikzcd}$$

whose base point is $(B, B \oplus C, C)$. The 2-simplex



shows that the latter loop is contractible $\square$

**Corollary 2.2.** *Every loop in $G^{III}_{\cdot}\mathcal{A}$ is homotopic to a two-edge loop of the form (2.1).*

*Remark.* Recall that the $G$-construction was defined in [GG] by applying the combinatorial loop space functor $\omega$ (cf. [GG]) to the $S$-construction: $G.\mathcal{A} := \omega S.\mathcal{A}$. Gillet and Grayson proved that $|\omega S.\mathcal{A}| \sim \Omega |S.\mathcal{A}|$, but one cannot go further with $G$, i.e., $\omega G.\mathcal{A}$ does not reprsent the homotopy type of the loop space $\Omega|G.\mathcal{A}|$, because $G.\mathcal{A}$ does not satisfy the necessary condition:

If for a given pointed simplicial set $X$ one has $|\omega X| \sim \Omega |X|$, then every loop in $X$ is homotopic to a two-edge loop.

The corollary above shows that $G^{III}$ does satisfy this condition and one can therefore expect of $\omega G^{III}_{\cdot}\mathcal{A}$ to be at least a reasonable approximation to the loop space $\Omega K(\mathcal{A})$. We will discuss this question in a future paper.

## 3. Elements of $K_1$ via non-commutative diagrams

Double short exact sequences do not look very natural as categorical data. In this section we will consider non-commutative, i.e. not necessarily commutative diagrams of the form

$$d = \begin{array}{ccccc} A' & \longrightarrow & B' & \longrightarrow & C' \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ A'' & \longrightarrow & B'' & \longrightarrow & C'' \end{array} \tag{3.1}$$

consisting of six short exact sequences. We will write for short *n.c.d.* for such a diagram. These data seem to us a bit more natural than a d.s.e.s., and one can regard any d.s.e.s. $l = (A \overset{f_1}{\underset{f_2}{\rightrightarrows}} B \overset{g_1}{\underset{g_2}{\rightrightarrows}} C)$ as a particular case of n.c.d. by displaying it in the form

$$
d(l) = \begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "1"'] & B \arrow[r, "g_1"] \arrow[d, "1"'] & C \arrow[d, "1"'] \\
A \arrow[r, "f_2"'] \arrow[d] & B \arrow[r, "g_2"'] \arrow[d] & C \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}
$$

The same as one assigns a loop $\mu(l)$ in $G.\mathcal{A}$ to a d.s.e.s. $l$, we now assign a loop $\mu(d)$ in $G^{III}.\mathcal{A}$ to a n.c.d. $d$:

$$
\mu(d) = \begin{tikzcd}
(A',A,A'') \arrow[rr, "e(d)"] & & (B',B,B'') \\
& (0,0,0) \arrow[ul, "e(s_A)"] \arrow[ur, "e(s_B)"'] &
\end{tikzcd}
$$

Here $e(s_A)$ and $e(s_B)$ are the edges associated to the short exact sequences $s_A = (A' \to A \to A'')$ and $s_B = (B' \to B \to B'')$ (cf. (1.4)), and $e(d)$ is the edge in $G.^{III}\mathcal{A}$ obtained by forgetting the arrows of $s_A$ and $s_B$ in $d$ (cf. (1.2)). For any d.s.e.s. $l$, we obviously have

$$i(\mu(l)) = \mu(d(l)). \tag{3.2}$$

Let $m(d)$ denote the homotopy class of $\mu(d)$ in $K_1(\mathcal{A}) = \pi_1(G.^{III}\mathcal{A})$. We know from [Ne1] that every element of $K_1(\mathcal{A})$ can be represented as $m(l)$ for some d.s.e.s. $l$. It follows from (3.2) that every element of $K_1(\mathcal{A})$ can be represented as $m(d)$ for some n.c.d. $d$ and, moreover, as $m(d(l))$ for some $l$. A natural question now would be: what kind of relations in $K_1(\mathcal{A})$ do we have for the elements $m(d)$ in terms of diagrams? Below we state some relations which are either trivial or conjectural.

$(i)'$ If $d$ commutes, then $m(d) = 0$. For in this case the loop $\mu(d)$ is the contour of the 2-simplex $t(d)$ (cf. (1.6)).

$(ii)'$ (skew-symmetry) Let $d^t$ denote the transposed diagram. Then $m(d^t) = -m(d)$ for any n.c.d. $d$.

This relation is still conjectural in general, but for the diagrams of the form $d(l)$ it follows from Proposition 2.1. For, the loop $\mu(d(l)^t)$ is essentially the same as $\kappa(l)$ (cf. (2.1)) since the edge $e(d(l)^t)$ is degenerate, and we are done by (3.2).

$(iii)'$ (Jacobi identity) Suppose we are given a $3 \times 3 \times 3$ cubic diagram which consists of 27 objects and 27 short exact sequences and is not supposed to be subject to any commutativity condition, i.e. like (3.1) but three-dimensional. Numerate the directions 1,2,3. For any permutation $i, j, k$ of 1,2,3 consider the three "faces" of the diagram (including the one in the middle of the cube) orthogonal to the $i$-th direction. Each of them is a $3 \times 3$-diagram of the form (3.1). Let us think of the $j$-th direction as horizontal and of the $k$-th direction as vertical. Denote by $[i, j, k]$ the alternative sum of the elements $m(d)$ corresponding to these three diagrams, the element corresponding to the middle face being taken with minus. Then

$$[1,2,3] + [2,3,1] + [3,1,2] = 0 \tag{3.3}$$

in $K_1(\mathcal{A})$.

This relation is still a conjecture as well as $(ii)'$, and we will now try to justify both of them. Note first that $(iii)'$ implies $(ii)'$. For consider the cub

$$\begin{tikzcd}
A' \arrow[rr] \arrow[dr] \arrow[ddd] & & A' \arrow[rr] \arrow[dr] \arrow[ddd] & & 0 \arrow[dr] \arrow[ddd] & & \\
& A' \arrow[rr] \arrow[dr] \arrow[ddd] & & B' \arrow[rr] \arrow[dr] \arrow[ddd] & & C' \arrow[dr] \arrow[ddd] & \\
& & 0 \arrow[rr] \arrow[ddd] & & C' \arrow[rr] \arrow[ddd] & & C' \arrow[ddd] \\
A \arrow[rr] \arrow[dr] \arrow[ddd] & & A \arrow[rr] \arrow[dr] \arrow[ddd] & & 0 \arrow[dr] \arrow[ddd] & & \\
& A \arrow[rr] \arrow[dr] \arrow[ddd] & & B \arrow[rr] \arrow[dr] \arrow[ddd] & & C \arrow[dr] \arrow[ddd] & \\
& & 0 \arrow[rr] \arrow[ddd] & & C \arrow[rr] \arrow[ddd] & & C \arrow[ddd] \\
A'' \arrow[rr] \arrow[dr] & & A'' \arrow[rr] \arrow[dr] & & 0 \arrow[dr] & & \\
& A'' \arrow[rr] \arrow[dr] & & B'' \arrow[rr] \arrow[dr] & & C'' \arrow[dr] & \\
& & 0 \arrow[rr] & & C'' \arrow[rr] & & C''
\end{tikzcd}$$

Let us always numerate the directions in the following way:

1. the vertical direction;
2. the horizontal direction on the plane of the figure;
3. the direction "orthogonal" to the plane of the figure.

The horizontal faces commute, hence $[1,2,3] = 0$, and we also have $[2,3,1] = -m(d)$ and $[3,1,2] = -m(d^t)$. It follows from (3.3) that $m(d) + m(d^t) = 0$.

Recall that we denote by $\mathcal{D}(\mathcal{A})$ the abelian group generated by the symbols $\langle l \rangle$ for all d.s.e.s's $l$ subject to relations $(i)$ and $(ii)$ of [Ne2, sect. 6]. Denote now $\mathcal{E}(\mathcal{A})$ the abelian group generated by the synbols $\langle d \rangle$ for all n.c.d's $d$ subject to relations $(i)'$ and $(iii)'$ posed on those symbols rather than on the elements $m(d)$.

**Lemma 3.1.** *There is a well-defined homomorphism $f : \mathcal{D}(\mathcal{A}) \to \mathcal{E}(\mathcal{A})$ given on generators by $f(\langle l \rangle) = \langle d(l) \rangle$.*

*Proof.* If $f_1 = f_2$ and $g_1 = g_2$ in $l$ (cf. (6.1) in [Ne2]), then $d(l)$ commutes and $\langle d(l) \rangle = 0$ in $\mathcal{E}(\mathcal{A})$ by virtue of $(i)'$. Thus $f$ is well-defined with respect to relation $(i)$.

Given a diagram

$$
\begin{array}{ccccc}
A' & \overset{a_1}{\underset{a_2}{\rightrightarrows}} & A & \overset{x_1}{\underset{x_2}{\rightrightarrows}} & A'' \\
f_1 \downdownarrows f_2 & & g_1 \downdownarrows g_2 & & h_1 \downdownarrows h_2 \\
B' & \overset{b_1}{\underset{b_2}{\rightrightarrows}} & B & \overset{y_1}{\underset{y_2}{\rightrightarrows}} & B'' \\
u_1 \downdownarrows u_2 & & v_1 \downdownarrows v_2 & & w_1 \downdownarrows w_2 \\
C' & \overset{c_1}{\underset{c_2}{\rightrightarrows}} & C & \overset{z_1}{\underset{z_2}{\rightrightarrows}} & C''
\end{array}
$$

in which the first arrows commute with the first ones and the second arrows commute with the second ones (cf. (5.1) and (6.2) in [Ne2]), consider the cube

$$
\begin{tikzcd}[column sep=small]
A' \arrow[rrr, "a_1"] \arrow[dr, "f_1"] \arrow[ddd, "1"'] & & & A \arrow[rrr, "x_1"] \arrow[dr, "g_1"] \arrow[ddd, "1"] & & & A'' \arrow[dr, "h_1"] \arrow[ddd, "1"] & & \\
& B' \arrow[rrr, "b_1"] \arrow[dr, "u_1"] \arrow[ddd, "1"] & & & B \arrow[rrr, "y_1"] \arrow[dr, "v_1"] \arrow[ddd, "1"] & & & B'' \arrow[dr, "w_1"] \arrow[ddd, "1"] & \\
& & C' \arrow[rrr, "c_1"] \arrow[ddd, "1"] & & & C \arrow[rrr, "z_1"] \arrow[ddd, "1"] & & & C'' \arrow[ddd, "1"] \\
A' \arrow[rrr, "a_2"] \arrow[dr, "f_2"] \arrow[ddd] & & & A \arrow[rrr, "x_2"] \arrow[dr, "g_2"] \arrow[ddd] & & & A'' \arrow[dr, "h_2"] \arrow[ddd] & & \\
& B' \arrow[rrr, "b_2"] \arrow[dr, "u_2"] \arrow[ddd] & & & B \arrow[rrr, "y_2"] \arrow[dr, "v_2"] \arrow[ddd] & & & B'' \arrow[dr, "w_2"] \arrow[ddd] & \\
& & C' \arrow[rrr, "c_2"] \arrow[ddd] & & & C \arrow[rrr, "z_2"] \arrow[ddd] & & & C'' \arrow[ddd] \\
0 \arrow[rrr] \arrow[dr] & & & 0 \arrow[rrr] \arrow[dr] & & & 0 \arrow[dr] & & \\
& 0 \arrow[rrr] \arrow[dr] & & & 0 \arrow[rrr] \arrow[dr] & & & 0 \arrow[dr] & \\
& & 0 \arrow[rrr] & & & 0 \arrow[rrr] & & & 0
\end{tikzcd}
$$

The horizontal faces commute, hence $[1, 2, 3] = 0$. We have

$$
\begin{aligned}
[2,3,1] &= \langle d(l')\rangle - \langle d(l)\rangle + \langle d(l'')\rangle \\
[3,1,2] &= \langle d(l_A)^t\rangle - \langle d(l_B)^t\rangle + \langle d(l_C)^t\rangle \\
&= -\langle d(l_A)\rangle + \langle d(l_B)\rangle - \langle d(l_C)\rangle
\end{aligned}
$$

i.e., $(iii)'$ implies $\langle d(l')\rangle - \langle d(l)\rangle + \langle d(l'')\rangle = \langle d(l_A)\rangle - \langle d(l_B)\rangle + \langle d(l_C)\rangle$. Thus $f$ is well-defined with respect to relation $(ii)$■

If the conjectural relations $(ii)'$ and $(iii)'$ are true for the elements $m(d)$, then we have a well-defined map

$$\mathcal{E}(\mathcal{A}) \to K_1(\mathcal{A}), \quad \langle d\rangle \mapsto m(d)$$

and, as a consequence, the map $f : \mathcal{D}(\mathcal{A}) \to \mathcal{E}(\mathcal{A})$ is injective since the composite map $\mathcal{D}(\mathcal{A}) \to K_1(\mathcal{A})$ is an isomorphism [Ne3].

*Conjecture.* The map $f : \mathcal{D}(\mathcal{A}) \to \mathcal{E}(\mathcal{A})$ is an isomorphism for any exact category $\mathcal{A}$.

*Remark.* We do not know how to produce a non-trivial d.s.e.s. out of a n.c.d. in order to define an inverse map $\mathcal{E}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$.

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