A note on the best tridiagonal approximation for the inverse of a totally nonnegative Matrix

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Abstract

Let $\mathcal{T}$ be the set of the square totally nonnegative, $\mathcal{T}_{\mathcal{R}}$ of the tridiagonal, and $\mathcal{M}$ of the M-matrices. A result of M.Lewin in Ref.[5] shows that any two of the following three statements for a square regular matrix T imply the third one. $T^{-1} \in \mathcal{M}$, $T^{-1} \in \mathcal{T}_{\mathcal{R}}$, and $T \in \mathcal{T}$. An old presumption of a lot of researchers was: Given a matrix $T \in \mathcal{T}$. Is the solution of the problem $min_{X \in \mathcal{T}_{\mathcal{R}}} \|I - XT\|_F$ an M-matrix? Many numerical tests and theoretical considerations counted for a positive answer. We will prove in this article that this conjecture was wrong. But the numerical and theoretical observations leave a question open: Which further, probably *weak*, conditions do we need such that X is in $\mathcal{M}$?

Key words: Totally nonnegative matrices, minimizatition problems, tridiagonal M-matrices.

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1 Introduction

Given a nonsingular matrix $A \in L(\mathbb{R}^n)$ and a $b \in \mathbb{R}^n$. For a practical computation it is often useful or even necessary first to determine a proper preconditioner $X \in L(\mathbb{R}^n)$, such that one can reduce the error for the numerical solution of the system

$$Ax = b\,.$$

If the condition of XA is close to one the chances increase to get a good approximation for the exact solution, if one uses a direct numerical method for the computation of x. In advantageous cases one can even use iterative methods to solve $XAx = Xb$. To get an effective preconditioner, X should be sparse, so that the multiplications with A and b are cheap, second X should be easy to compute, and at last, to get a good condition number, XA should be close to the identity. So, one proper method could it be, to minimize under all tridiagonal matrices X the Frobenius-norm of $I - XA$. This leads to an interesting question of practical computing and matrix theory. Which structures has $X \in \mathcal{T}_{\mathcal{R}}$, if we know some structures of A? Here, if A is totally nonnegative, does it follow that X is an M-matrix?

Why do we ask this question? In Ref.[5] it is shown that if T is a regular totally nonnegative matrix, and T^{-1} is an M-matrix, then T^{-1} is tridiagonal. Also, if T^{-1} is a tridiagonal M-matrix, then T is totally nonnegative. Third, if T^{-1} is tridiagonal, and T totally nonnegative, then T^{-1} is an M-matrix. Further, it is easy to see, that if $T \in \mathcal{T}$ and regular, then T^{-1} is a checkerboard-matrix. But: Is the best tridiagonal approximation to the inverse of a totally nonnegative matrix, in the Frobenius-norm, an M-matrix?

This question was asked me by Mr. Michael Neumann during my stay at the University of Connecticut. Also some other researchers worked at this question. Generally, the inverse M-,Z-, and tridiadagonal matrix problems, are a main topic in matrix theory.

In this paper we will show that the abovementioned conjecture is wrong, but the practical experiments, we will show later on some examples, led *mostly* to X's which are M-matrices. So, the question which additional, *probably weak*, conditions for A we need, to get an M-matrix X, remains open.

Our main tool, to show that the conditions above are not sufficient, is a result of A.Ben-Tal and M.Teboulle in Ref.[1], which can also derived from more general theorem in L.Berg [2]. They show, that the least square solutions of rectangular systems lie in a convex combination of their square subsystems.

We continue with the precise definitions of the terms mentioned above and some notational conventions.

2 Preliminaries

We say a real square matrix is a Z-matrix, if it can be written as $sI - T$, where s is any real number and T is a nonnegative matrix. The subclass of M-matrices are those Z-matrices, where s is greater or equal to $\rho(T)$, the spectral radius of T.

Let T be an m by n real matrix, further let $\langle n\rangle = \{1, ..., n\}$, respectively $\langle m\rangle = \{1, ..., m\}$. Now, let $N \subset \langle n\rangle$ and $M \subset \langle m\rangle$ with $|N| = |M|$. We write T_{NM} for the square submatrix of T whose rows and columns are indexed by N and M, respectively. Matrices which fulfill for all N and M

$$\det(T_{NM}) \geq 0$$

are called totally nonnegative and we denote this class of matrices by $\mathcal{T}$. From Cramers rule it follows that, if a totally nonegative matrix is regular, then the inverse is a *checkerboard* matrix, i.e. $T_{ij} \geq 0$, if $i + j$ is even, and if $i + j$ is odd, then $T_{ij} \leq 0$.

Examples for totally nonnegative matrices, are inverses of tridiagonal M-matrices (cf. [5]). Also, *generalized Vandermonde matrices* are in $\mathcal{T}$, see Ref.[3]. $A \in L(\mathbb{R}^n)$ is a generalized Vandermonde matrix, if for some numbers

$$0 < a_1 < a_2 < ... < a_n \text{ and } \alpha_1 < \alpha_2 < ... < \alpha_n ,$$

$$A_{ij} = {a_i}^{\alpha_j} \text{ for all } 1 \leq i, j \leq n. \tag{1}$$

Such matrices arise, for example, in the analysis of interpolation problems.

As a further example, we mention that Gantmacher [3] showed that it is easy to construct totally nonnegative tridiagonal matrices and, also, to determine if a tridigonal matrix is in $\mathcal{T}$.

A result of Ben-Tal and Teboulle [1] deals with overdetermined rectangular systems. So, let A be an $m \times n$ real matrix with $m > n$ and $b \in \mathbb{R}^m$. Further, let

$$S = \{x \in \mathbb{R}^n : A_{N\langle n\rangle} x = b_N \,, \text{ with } N \subset \langle m\rangle \,, |N| = n, \text{ and } A_{N\langle n\rangle} \text{ regular}\}. \tag{2}$$

Then the least square solution(s) of

$$\min_{x\in\mathbb{R}^n} \|Ax - b\|_2^2 \tag{3}$$

lie(s) in the convex hull of S. We denoted in (2) the vector, where the indices of b, which do not lie in N, are deleted, by $b_N \in \mathbb{R}^n$. They also show that this result holds for all l_p-norms, and if rank $A = n$, one can even determine the convex factors of the unique solution x of (3):

$$x = \sum_{\substack{N\subset\langle m\rangle \\ \det(A_{N\langle n\rangle})\neq 0}} \frac{\det(A_{N\langle n\rangle})}{\det(A^t A)} x(N) \text{ ,where } A_{N\langle n\rangle} x(N) = b_N \,. \tag{4}$$

3 Numerical Experiments

In this paper we show only low dimensional examples, but many more and larger dimensional experiments were made and led to the same result, that the solution X of $\min_{X\in\mathcal{T}_\mathcal{R}} \|I - XT\|_F$ was allways an M-matrix, if $T \in \mathcal{T}$. To compute X, we just have to solve the least square problems

$$\min_{x_i\in\mathbb{R}^3} \|T^t_{\langle n\rangle\{i-1,i,i+1\}} x_i - e_i\|_2^2 \,,$$

for all $2 \le i \le n-1$, and for $i = 1, n$ the according two-dimensional systems. Then x_i^t are the nonzero elements in the i-th row of X, . If $T_{\{i-1,i,i+1\}\langle n\rangle} T^t_{\langle n\rangle\{i-1,i,i+1\}}$ is regular, then the solution is given by

$$x_i = (T_{\{i-1,i,i+1\}\langle n\rangle} T^t_{\langle n\rangle\{i-1,i,i+1\}})^{-1} T_{\{i-1,i,i+1\}\langle n\rangle} e_i \,, \tag{5}$$

where e_i denotes the i-th unit vector.

Example 1: First we choose a *generalized Vandermonde matrix*. For $\alpha_i = -\frac{1}{i}$ and $a_i = i$ for all $1 \le i \le 4$, one gets

$$T = \begin{pmatrix} 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ 0.5000 & 0.7071 & 0.7937 & 0.8409 \\ 0.3333 & 0.5774 & 0.6934 & 0.7598 \\ 0.2500 & 0.5000 & 0.6300 & 0.7071 \end{pmatrix}, \tag{6}$$

which is totally nonnegative and the (unique) best tridiagonal approximation to its inverse in the abovementioned sense is

$$X = \begin{pmatrix} 2.4404 & -18.0071 & 0 & 0 \\ -3.0833 & 77.8186 & -15.1562 & 0 \\ 0 & -62.6551 & 44.8807 & -6.3776 \\ 0 & 0 & -29.7202 & 7.6833 \end{pmatrix} \tag{7}$$

We checked that X is an M-matrix.

Example 2: Gantmacher, Ref.[3], showed that for a tridiagonal matrix T to be a totally nonnegative matrix, it is sufficient that all the principal minors are nonnegative. This is the case for

$$T = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}. \tag{8}$$

Then

$$X = \begin{pmatrix} 0.5714 & -0.2143 & 0 & 0 & 0 & 0 \\ -0.3333 & 0.7333 & -0.2667 & 0 & 0 & 0 \\ 0 & -0.2000 & 0.6000 & -0.2000 & 0 & 0 \\ 0 & 0 & -0.2000 & 0.6000 & -0.2000 & 0 \\ 0 & 0 & 0 & -2.6667 & 0.7333 & -0.3333 \\ 0 & 0 & 0 & 0 & -0.2143 & 0.5714 \end{pmatrix}, \tag{9}$$

which is, because of its diagonal dominance, in $\mathcal{M}$. It surprises that X does not preserve the symmetry of T, but its *point symmetry*.

4 Result

These and many other experiments led to the presumption that X is an M-matrix, whenever $T \in \mathcal{T}$.

But, let us assume the first column of T is zero, then, using the result of Ben-Tal and Teboulle, the (possible) nonzero elements in the second row of X are a convex combination of the transposed, uniquely solvable solutions of the systems

$$T^t_{N\{1,2,3\}} x = e_{2\,N} \quad \text{for all } N \subset \langle n \rangle \text{ with } |N| = 3, \tag{10}$$

However, only those N which contain the second row of $T^t_{N\{1,2,3\}}$, i.e. $2 \in N$, are matching a part of e_2, which is not zero. Further, all the systems with $1 \in N$ are singular. Therefore, we have only to observe those 3×3 subsystems of $T^t_{N\{1,2,3\}}$ which contain its second row as its first row, and we have allways $e_{2\,N} = (1\ 0\ 0)^t$.

Since T is totally nonnegative, all 3×3 subsystems of $T^t_{N\{1,2,3\}}$ are totally nonnegative and their inverses are checkerboard matrices, i.e. the solutions for the regular systems in (10) are of the form

$$x = \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{11}$$

where $+$, $-$ stand for nonnegative, resp. nonpositive elements. Therefore, X^t's second columns sign pattern are

$$\begin{pmatrix} + \\ - \\ + \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{12}$$

and, if there is for only one N a regular subsystem $T^t_{N\{1,2,3\}}$, X can not be in $\mathcal{M}$ anymore.

Of course we assumed that T is singular. But X as a function of T is continious. Therefore, to get an example, which shows that this result holds also for regular $T \in \mathcal{T}$, we just multiply the first column of a $T \in \mathcal{T}$ with a *comparably* small positive factor. Because of the multilinearity of determinants the new matrix is again totally nonnegative. Let us do this with example (6). We divided its first column by $10,000$:

The matrix

$$T = \begin{pmatrix} 0.0001 & 1.0000 & 1.0000 & 1.0000 \\ 0.00005 & 0.7071 & 0.7937 & 0.8409 \\ 0.0000\overline{3} & 0.5774 & 0.6934 & 0.7598 \\ 0.000025 & 0.5000 & 0.6300 & 0.7071 \end{pmatrix} \tag{13}$$

leads to

$$X = \begin{pmatrix} 0.0000 & -0.0000 & 0 & 0 \\ 0.0726 & -0.2414 & 0.1716 & 0 \\ 0 & -0.62229 & 1.7458 & -1.1352 \\ 0 & 0 & -0.0175 & 0.0198 \end{pmatrix}, \tag{14}$$

which is obviously not a Z- and, therefore also not an M-matrix.

The question now is: Which further conditions do we need for a totally nonnegative matrix, such that the best tridiagonal approximtion to its inverse, in the sense described above, is an M-matrix.

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