Note on Topological Hochschild Homology

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1 Introduction

In the appendix to [16] there is discussed a trace map $tr : K(R) \longrightarrow HH(R)$ from the algebraic K-theory of a simplicial ring to its Hochschild homology. In the main body of this paper a similar map from the algebraic K-theory $A(X)$ of a topological space X, as defined in [15], to stable K-theory is constructed, and one would like to interpret this in "algebraic" terms.

Informally one can view the K-theory of spaces as sort of a Quillen K - theory for rings up to homotopy. For an algebraic version of stable K - theory one would substitute the tensor products (over the integers) in the usual simplicial definition of Hochschild homology with a "tensor product" over the more basic ring up to homotopy $\Omega^\infty S^\infty$, i.e. with a suitable definition of the smash product of spectra.

Even for honest simplicial rings this view results in a nontrivial factorization of the trace map tr through a functor which – thanks to formal similarity – inherits the name "Topological Hochschild Homology" THH. There was also good reason to hope that $THH(R)$ would provide more information about $K(R)$ for classical rings than $HH(R)$.

A rigid construction of THH for FSP's – functors with smash product – was first described in Bökstedt's paper [1]. It involves explicit space level constructions for smash products of spectra. Since this smash product does not have the same formal properties as the tensor product on the nose, the definition of THH is not a verbatim translation of the one of HH.

In the meantime there have been constructed a number of symmetric monoidal smash product functors on suitable categories of spectra with the same formal properties as the classical tensor product [3], [5], [7], which allow for definitions of THH – for arbitrary A_∞ - rings – completely analogous to the one of HH. In the present paper we work in the setting of [3] and exploit this spectrum level view point of THH. We show that it leads to simpler, more conceptual proofs and extensions of early calculations of THH by Lindenstrauss [6], Pirashvili [11] and others who primarily used spectral sequence arguments.

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2 Categories of modules and algebras

Unless stated otherwise we use the terminology of [3].

2.1 Let R be a commutative $\mathbf{S}$-algebra and $\mathcal{M}od_R$ the category of R-module spectra. There is a commutative, associative and unital smash product functor $\wedge_R$ on $\mathcal{M}od_R$ making it a symmetric monoidal category [3, III. 3]. One has control over the homotopy type of $M \wedge_R N$ provided M or N is a cell R-module: there is a strongly convergent natural right half-plane spectral sequence of differential $\pi_*(R)$-modules

$$E^2_{p,q} = \mathrm{Tor}^{\pi_*(R)}_{p,q}(\pi_* M,\ \pi_* N) \Longrightarrow \pi_{p+q}(M \wedge_R N).$$

One can always resolve an R-module M: there is a cell R-module $\Gamma_R M$ and a weak equivalence $\Gamma_R M \longrightarrow M$. We say that $M \wedge_R N$ has the *correct homotopy type* if $\Gamma_R M \wedge_R N \to M \wedge_R N$ is a weak equivalence.

If R is a classical ring and M a right, N a left R-module, then the Eilenberg-MacLane spectra HM and HN can be represented as cell HR-module spectra and [3, IV. 2.1]

$$\pi_*(HM \wedge_{HR} HN) \cong \mathrm{Tor}^R_*(M, N).$$

More generally, if R is commutative and M is a CW HR-module spectrum there exists an associated cellular chain complex $C_*(M)$ of free R-modules such that $\pi_*(M) \cong H_*(C_*(M))$ [3, IV.2.3]. Moreover, if M and N are CW HR-module spectra there is a natural isomorphism

$$C_*(M \wedge_{HR} N) \cong C_*(M) \otimes_R C_*(N)$$

which is associative, commutative, and unital [3, IV.2.5] and [8, (2.4)].

2.2 The categories ${}_R\mathcal{M}od$, $\mathcal{M}od_R$, $\mathcal{A}lg_R$, and $\mathcal{CA}lg_R$ of left and right R-modules, R-algebras, and commutative R-algebras admit topological closed model category structures. Their weak equivalences are the weak equivalences of underlying spectra, their cofibrations, called q-cofibrations, are retractions of relative cell objects in their categories. Since retractions preserve weak equivalences it suffices to resolve an R-module M by a q-cofibrant one to obtain homotopical control of the smash product. For smash products of (commutative) algebras the argument is different because the forgetful functors

$$\mathcal{A}lg_R \longrightarrow \mathcal{M}od_R \qquad \mathcal{CA}lg_R \longrightarrow \mathcal{M}od_R$$

do not preserve q-cofibrant objects. Mandell's work [8] provides a remedy: if R is a classical commutative algebra and A and B are q-cofibrant commutative HR-algebras, then the isomorphism above induces a natural isomorphism of graded algebras

$$\pi_*(A \wedge_{HR} B) \cong H_*(C_*(A) \otimes_R C_*(B))$$

We will also need the following results.

2.3 [3, VII. 6.2, 6.5] (1) If A is a q-cofibrant R-algebra then the unit $R \to A$ is a q-cofibration of R-modules.

(2) If A and B are q-cofibrant (commutative) R-algebras then $A \wedge_R B$ has the correct homotopy type.

2.4 Lemma: If $R \to M$ is a q-cofibration of right R-modules then the functor $M \wedge_R -\colon {}_R\mathcal{M}od \to \mathcal{M}od_{\mathbf{S}}$ preserves weak equivalences. In particular, $M \wedge_R N$ has the correct homotopy type.

Proof: Since retractions preserve weak equivalences we may assume that $R \to M$ is a relative cell R-module. If $\phi : N_1 \to N_2$ is a weak equivalence of R-modules so is $R \wedge \phi : R \wedge_R N_1 \longrightarrow R \wedge_R N_2$. Now proceed as in the proof of [3, III. 3.8]. $\square$

Let $\mathcal{E}$ be any of the categories $\mathcal{M}od_R$, $\mathcal{A}lg_R$, $\mathcal{CA}lg_R$. We recall that $\mathcal{E}$ is topologically complete and cocomplete; in particular it is tensored and cotensored [3, VII § 2], [10].

2.5 Lemma: Let $i : A \to B$ be a q-cofibration in $\mathcal{E}$ and $K \subset L$ a CW-pair. Then $A \otimes L \cup B \otimes K \longrightarrow B \otimes L$ is a q-cofibration in $\mathcal{E}$.

Proof: Let $p : X \longrightarrow Y$ be a trivial q-fibration. Given a commutative square

$$\begin{tikzcd} A \otimes L \cup B \otimes K \arrow[r] \arrow[d] & X \arrow[d] \\ B \otimes L \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

we have to find the dotted lift. Passing to adjoints this is equivalent to finding a lift for

$$\begin{tikzcd} K \arrow[r] \arrow[d, phantom, "\cap"] & \mathcal{E}(B, X) \arrow[d, "(i^*, p_*)"] \\ L \arrow[r] & \mathcal{E}(A, X) \times_{\mathcal{E}(A,Y)} \mathcal{E}(B, Y) \end{tikzcd}$$

Since $\mathcal{E}$ is a topological closed model category (i^*, p_*) is a Serre-fibration and weak equivalence [3, VII § 4]. Hence the lift exists. $\square$

3 Topological Hochschild homology

3.1 Let K be a q-cofibrant commutative $\mathbf{S}$-algebra and R a (possibly commutative) K-algebra with multiplication $\mu : R \wedge_K R \to R$ and unit $\eta : K \to R$, and M an R-bimodule with structure maps $\xi_\ell : R \wedge_K M \to M$ and $\xi_r : M \wedge_K R \to M$. We write R^n for the n-fold smash product $R \wedge_K \ldots \wedge_K R$ if the ground ring spectrum K is clear from the context.

We define *topological Hochschild homology* $THH^K(R;M)$ of R with coefficients in M over K to be the realization in $\mathcal{M}od_K$ (which coincides with the one in the category of all spectra) of the simplicial spectrum

$$[n] \longmapsto t_n^K(R;M) = R^n \wedge_K M$$

with the face and degeneracy maps

$$\begin{aligned} d_i &= \begin{cases} id^{n-1} \wedge \xi_\ell & i=0 \\ id^{n-i-1} \wedge \mu \wedge id^{i-1} \wedge id & 0<i<n \\ (id^{n-1} \wedge \xi_r) \circ \tau & i=n \end{cases} \\ s_i &= id^{n-i} \wedge \eta \wedge id^i \wedge id \qquad\qquad 0 \le i \le n \end{aligned}$$

where $\tau : R \wedge_K R^{n-1} \wedge_K M \cong R^{n-1} \wedge_K M \wedge_K R$ is the cyclic permutation. (Our notation differens from the one of [3] where $THH^K(R;M)$ and $t_n^K(R;M)$ are denoted by $thh^K(R;M)$ and $thh^K(R;M)_n$ respectively.) If $M = R$ we shorter write $THH^K(R)$ and $t_n^K(R)$. Also recall that $THH_*^K(R;M) = \pi_*(THH^K(R;M))$.

If R is a ring in the classical sense we use $THH(R)$ for $THH^{\mathbf{S}}(HR)$ in accordance with common practice. To obtain the correct homotopy type we have to assume that HR is resolved as a q-cofibrant $\mathbf{S}$-algebra. In view of (2.3) and [3, §VII.6] it does not matter whether HR is resolved as q-cofibrant $\mathbf{S}$-algebra or as q-cofibrant commutative $\mathbf{S}$-algebra as long as we are just interested in the homotopy type of the underlying spectrum of $THH(R)$. Both resolutions exist by [3, VII.4].

The "inclusion of the 0-skeleton" defines a map of K-modules

$$\xi : M \longrightarrow THH^K(A;M).$$

If R is a commutative K-algebra $t_\bullet^K(R)$ is a simplicial commutative K-algebra, hence so is $THH^K(R)$ by [10]. The map

$$\xi : R \longrightarrow THH^K(R)$$

makes it a commutative R-algebra with unit ξ.

3.2 If K is a commutative ring, R a K-algebra, and M an R-bimodule in the classical sense, the definition of Hochschild homology $HH^K(R,M)$ is exactly the same as the one of topological Hochschild homology if one replaces $\wedge_K$ in (3.1) by $\otimes_K$. Since $\wedge_{HK}$ is related to the total left derived of the tensor product over K, Hochschild homology $HH^K(R,M)$ coincides with topological Hochschild homology $THH^{HK}(HR,HM)$ under flatness conditions [3, IX.1.7]:
If K is a classical commutative ring, R is a K-flat K-algebra and M an R-bimodule, then

$$HH_*^K(R;M) \cong THH_*^{HK}(HR,HM)$$

as K-modules. If R is commutative this is an isomorphism of R-algebras.

3.3 Lemma: Let Q and R be K-algebras. Then

$$THH^K(Q \wedge_K R) \cong THH^K(Q) \wedge_K THH^K(R)$$

as K-modules. If Q and R are commutative K-algebras this is an isomorphism of commutative $(Q \wedge_K R)$-algebras.

Proof: By [3, X. 1.4] $THH^K(Q) \wedge_K THH^K(R)$ is the realization of the simplicial K-module

$$[n] \longmapsto Q^{n+1} \wedge_K R^{n+1} \cong (Q \wedge_K R)^{n+1}.$$

If Q and R are commutative, this is an isomorphism of $(Q \wedge_K R)$-algebras. $\square$

3.4 Remark: Let F be an FSP and Π a based monoid, i.e. a discrete monoid with a unit and a multiplicative 0 which serves as base point. Theorem 6.1 of Hesselholt and Madsen [4] states that

$$THH(F[\Pi]) \cong THH(F) \wedge |N^{cy}_{*\wedge}(\Pi)|$$

as spectra, where $N^{cy}_{*\wedge}(\Pi)$ is the cyclic nerve of Π formed using the smash product. This result is a special case of Lemma 3.3.

$$THH(F[\Pi]) \cong THH(F \wedge \mathbf{S}[\Pi]) \cong THH(F) \wedge_{\mathbf{S}} THH(\mathbf{S}[\Pi])$$

Since $(\mathbf{S}[\Pi])^n \cong (\mathbf{S} \wedge \Pi)^n \cong \mathbf{S} \wedge \Pi^n$ we obtain $THH(\mathbf{S}[\Pi]) \cong \Sigma^\infty |N^{cy}_{*\wedge}(\Pi)|$. Hence

$$THH(F[\Pi]) \cong THH(F) \wedge_{\mathbf{S}} \Sigma^\infty |N^{cy}_{*\wedge}(\Pi)| \cong THH(F) \wedge |N^{cy}_{*\wedge}(\Pi)|$$

3.5 Lemma: Let R be a K-algebra and B a commutative K-algebra. Then

$$THH^K(R) \wedge_K B \cong THH^B(R \wedge_K B)$$

as B-modules. If R is commutative this is an isomorphism of commutative $(R \wedge_K B)$-algebras.

Proof: $THH^B(R \wedge_K B)$ is the realization of the simplicial B-module

$$[n] \longmapsto (R \wedge_K B) \wedge_B \ldots \wedge_B (R \wedge_K B)$$

with B-action from the right on the last factor. This simplicial B-module is isomorphic as simplicial B-module to

$$[n] \longmapsto R^{n+1} \wedge_K B.$$

Since $- \wedge_K B$ commutes with realizations the result follows.

If R is commutative we have isomorphisms of $(R \wedge_K B)$-algebras. $\square$

3.6 Lemma: Let Q be a q-cofibrant commutative K-algebra and R a q-cofibrant (possibly commutative) Q-algebra. Then $t^K_\bullet(R)$ is proper as simplicial K-module if R is commutative or not. In the commutative case $t^K_\bullet(R)$ is also proper as simplicial commutative R-algebra.

Proof: This could be proved using generalized relative cell structures like in the proof of [3, VII. 7.5]. We take a different approach using the results of [12]. Recall from there that a map $f : X \to Y$ of module or algebra spectra is called a strong cofibration if each Hurewicz fibration in the category has the relative homotopy lifting property with respect to f. By assumption, the unit maps $K \to Q$ and $Q \to R$ are q-cofibrations and hence strong cofibrations in their categories. Is follows from [12] that $\eta : K \to R$ is a strong cofibration of K-modules (commutative K-algebras). Let sR^n denote the "union of the images" of the maps

$$id^{n-i-1} \wedge \eta \wedge id^i : R^{n-1} \longrightarrow R^n \qquad\qquad 0 \leq i \leq n-1$$

i.e. the appropriate iterated pushout in $\mathcal{M}od_K$ resp. $\mathcal{CA}lg_K$. By the pairing theorem for strong cofibrations the induced map $sR^n \longrightarrow R^n$ is a strong cofibration. Since $- \wedge_K R$ preserves iterated pushouts and strong cofibrations the result follows. □

3.7 Proposition: Let R be a q-cofibrant K-algebra. Then $\xi : R \longrightarrow THH^K(R)$ and its composite $K \to R \to THH^K(R)$ with the unit are q-cofibrations in $\mathcal{M}od_K$.

Proof: By [3, VII. 6.2] the unit $K \to R$ is a q-cofibration of K-modules. Hence (R, K) is a retract of a relative K-cell module $(\bar{R}, K)$. Let $\bar{\eta} : K \to \bar{R}$ be the inclusion. Form $st_n^K(R)$ and $st_n^K(\bar{R})$ as in the previous lemma. Then $st_n^K(\bar{R})$ is a subcomplex of $t_n^K(\bar{R})$ so that $st_n^K(\bar{R}) \longrightarrow t_n^K(\bar{R})$ is a q-cofibration of K-modules. Hence so is $st_n^K(R) \longrightarrow t_n^K(R)$. Let $THH^K(R)^{(n)}$ denote the n-skeleton of the realization in Mod_K. Then by (2.5) the map $THH^K(R)^{(n)} \longrightarrow THH^K(R)^{(n+1)}$ is a q-cofibration of K-modules and the statement follows. □

3.8 Proposition: Let R be a q-cofibrant commutative K-algebra. Then $THH^K(R)$ is q-cofibrant in $\mathcal{CA}lg_R$ and hence also in $\mathcal{CA}lg_K$.

Proof: By [10, 4.5] $THH^K(R)$ is isomorphic as R-algebra to the internal realization of the simplicial R-algebra $t_\bullet^K(R)$ in $\mathcal{CA}lg_R$. We investigate this internal realization. The functor $R \wedge_K - : \mathcal{CA}lg_K \longrightarrow \mathcal{CA}lg_R$ preserves colimits and q-cofibration. Applying it to the pushout

$$\begin{array}{ccc} K & \longrightarrow & R \\ \downarrow & & \downarrow \\ R & \longrightarrow & R \wedge_K R \end{array}$$

in $\mathcal{CA}lg_K$ we obtain the pushout

$$\begin{array}{ccc} R & \longrightarrow & R \wedge_K R \\ \downarrow & & \downarrow \\ R \wedge_K R & \longrightarrow & R \wedge_K R \wedge_K R \end{array}$$

in $\mathcal{CA}lg_R$ showing that $t_2^K(R)$ is degenerate. By the same argument $t_n^K(R)$ is degenerate for all $n \geq 2$. Hence we have a pushout

$$\begin{tikzcd}
R \otimes I \cup_{R \otimes \partial I} (R \wedge_K R) \otimes \partial I \arrow[r] \arrow[d] & (R \wedge_K R) \otimes I \arrow[d] \\
R \arrow[r] & THH^K(R)
\end{tikzcd}$$

in $\mathcal{CA}lg_R$. Since the tensor has to be taken in $\mathcal{CA}lg_R$ and R is initial this pushout is isomorphic to the pushout

$$\begin{tikzcd}
(R \wedge_K R) \otimes \partial I \arrow[r] \arrow[d] & (R \wedge_K R) \otimes I \arrow[d] \\
R \arrow[r] & THH^K(R)
\end{tikzcd}$$

Since R is q-cofibrant in $\mathcal{CA}lg_K$ the object $R \wedge_K R$ is q-cofibrant in $\mathcal{CA}lg_R$. Hence $(R \wedge_K R) \otimes \partial I \longrightarrow (R \wedge_K R) \otimes I$ is a q-cofibration in $\mathcal{CA}lg_R$ by (2.5), which implies the result. $\square$

4 Main results and applications

For classical rings R and R-modules M let HR and HM again denote the associated Eilenberg-MacLane ring and module spectra. We will represent them as q-cofibrant objects in the appropriate categories of algebra or module spectra.

Our main observation is the following result. Proofs are deferred to Section 5.

4.1 Theorem: Let K be a commutative ring and R a K-algebra in the classical sense. Suppose there is a q-cofibrant commutative $\mathbf{S}$-algebra A, and a q-cofibrant A-algebra E such that
(1) HK can be represented as q-cofibrant commutative A-algebra
(2) there is a weak equivalence of HK-algebras $E \wedge_A HK \longrightarrow HR$, where HR is represented as q-cofibrant HK-algebra.
Then there is a homotopy equivalence of HK-module spectra

$$THH^A(HR) \simeq THH^{HK}(HR) \wedge_{HK} THH^A(HK).$$

If R is commutative, E a q-cofibrant commutative A-algebra and HR represented by a q-cofibrant commutative HK-algebra, this homotopy equivalence is a homotopy equivalence of commutative HK-algebra spectra. $\square$

By (3.7) and (3.8) the right side of the theorem has the correct homotopy type and the results of (2.1) and (2.2) apply. E.g. if K is a hereditary ring the higher Tor functors vanish and the hyperhomology spectral calculating $H_*(C_*(M) \otimes_K C_*(N))$ collapses to a short exact sequence [9, XII.12.3]. Since $C_*(M)$ is a complex of free K-modules, there is a non-natural splitting of this sequence by the Künneth Theorem. Using (3.2) we obtain

4.2 Corollary: Suppose that K is a hereditary ring, that R is K-flat, and that the assumptions of Theorem 4.1 hold. Then there is a non-natural isomorphism of graded K-modules

$$THH_n^A(HR) \cong \bigoplus_{p+q=n} HH_p^K(R) \otimes_K THH_q(K) \oplus \bigoplus_{p+q=n-1} \mathrm{Tor}_1^K(HH_p^K(R), THH_q(K)).$$

In the commutative case (2.2) has information about the multiplicative structure.

4.3 Corollary: If R is a commutative K-algebra, the assumptions of Theorem 4.1 for the commutative case and of Corollary 4.2 hold, and if $HH_p^K(R)$ is K-flat for all p, then

$$THH_*^A(HR) \cong HH_*^K(R) \otimes_K THH_*(K)$$

as graded K-algebras.

Applications:

Take a classical commutative ring R and a based monoid Π. Let $R[\Pi]$ be its associated monoid ring, i.e. the usual monoid ring with $0 \in R$ identified $0 \in \Pi$. We will deduce from Theorem 4.1

4.4 Theorem: If R is a hereditary ring, then, as R-modules,

$$THH_n(R[\Pi]) \cong$$

$$\bigoplus_{p+q=n} HH_p^R(R[\Pi]) \otimes_R THH_q(R) \oplus \bigoplus_{p+q=n-1} \mathrm{Tor}_1^K(HH_p^R(R[\Pi]), THH_q(R)).$$

If Π is the based monoid $\Pi = \{0, 1, x, x^2, \ldots, x^{n-1}\}$ with the multiplication given by the rule $x^n = 0$, then $\mathbb{Z}[\Pi] \cong \mathbb{Z}[X]/(X^n)$, and hence

4.5 Corollary: $THH_k(\mathbb{Z}[X]/(X^n)) \cong$

$$\bigoplus_{p+q=k} HH_p^{\mathbb{Z}}(\mathbb{Z}[X]/(X^n)) \otimes_{\mathbb{Z}} THH_q(\mathbb{Z}) \oplus \bigoplus_{p+q=k-1} \mathrm{Tor}_1^{\mathbb{Z}}(HH_p^{\mathbb{Z}}(\mathbb{Z}[X]/(X^n)), THH_q(\mathbb{Z}))$$

as abelian groups.

If G is a discrete group and $\Pi = G_+$, i.e. G with an additional multiplicative 0, then $\mathbb{Z}[\Pi]$ is the usual group ring of G over $\mathbb{Z}$ whose classical Hochschild homology is known in terms of group homology, e.g. see [17, 9.7.5]. We obtain

4.6 Corollary: Let Λ be a set of representatives of conjugacy classes of elements of G and $C_G(x)$ the centralizer subgroup of $x \in G$. Then, as abelian groups,

$$THH_k(\mathbb{Z}[G]) \cong$$
$$\left(\bigoplus_{x\in\Lambda}\bigoplus_{p+q=k} H_p(C_G(x);\mathbb{Z}) \otimes_{\mathbb{Z}} THH_q(\mathbb{Z})\right) \oplus \left(\bigoplus_{x\in\Lambda}\bigoplus_{p+q=k-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_p(C_G(x);\mathbb{Z}), THH_q(\mathbb{Z}))\right)$$

where $H_p(C_G(x);\mathbb{Z})$ is the usual group homology with trivial $\mathbb{Z}$-coefficients.
In particular, if G is abelian of order n then

$$THH_k(\mathbb{Z}[G]) \cong \bigoplus_{p+q=k} H_p(G;\mathbb{Z})^n \otimes_{\mathbb{Z}} THH_q(\mathbb{Z}) \oplus \bigoplus_{p+q=k-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_p(G;\mathbb{Z})^n, THH_q(\mathbb{Z}))$$

.

(Commutative) ring spectra can be localized in their categories [3, Chap. VIII] (see also [14]). From Corollary 4.3 we obtain

4.7 Theorem: Let R be a subring of the rationals $\mathbb{Q}$. Then, as graded algebras,

$$THH_*(R) \cong R \otimes THH_*(\mathbb{Z}).$$

Of course, this is a special case of the more general result that THH^K commutes with localizations [13].

Away from the characteristic it is possible to adjoin roots of unity to a ring spectrum [14]. We have

4.8 Theorem: Let $R \subset \mathbb{Q}$ be a subring and ζ a primitive n-th root of unity such that the prime factors of n are invertible in R. Then

$$THH_*(R(\zeta)) \cong HH_*^{\mathbb{Z}}(R(\zeta)) \otimes_{\mathbb{Z}} THH_*(\mathbb{Z})$$

as graded rings.

Using this result we give a simple proof of

4.9 Theorem: Let $f \in \mathbb{F}_p[X]$ be an arbitrary monic polynomial. Then

$$THH_*(\mathbb{F}_p[X]/(f)) \cong HH_*^{\mathbb{F}_p}(\mathbb{F}_p[X]/(f)) \otimes THH_*(\mathbb{F}_p)$$

as graded algebras.

4.10 Remark: The following results allow the explicit computation of topological Hochschild homology groups:
(1) Let $f \in R[X]$ be a monic polynomial over a commutative ring R. Then by [2]

$$HH_k^R(R[X]/(f)) = \begin{cases} R[X]/(f) & \text{if } k = 0 \\ R[X]/(f, f') & \text{if } k \text{ odd} \\ Ann(f') \text{ in } R[X]/(f) & \text{if } k \text{ even} \end{cases}$$

In particular, $HH_*^{\mathbb{Z}}(R(\zeta))$ in (4.8) is additively free.

(2) Bökstedt's calculations:

$$THH_k(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}/i & k=2i-1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \qquad \begin{array}{l} THH_*(\mathbb{F}_p) \cong \mathbb{F}_p[x] \\ \text{with degree } (x)=2 \end{array}$$

(3) Corollary 4.5 and Theorem 4.9 together with (1) and (2) completely determine $THH_k(\mathbb{Z}[X]/(X^n))$ and $THH_k(\mathbb{F}_p[X]/(f))$ and hence retrieve Lindenstrauss's calculations of [6]. Theorem 4.4 is essentially Theorem 6.1 of Hesselholt and Madsen [4] and Proposition 2.2 of Pirashvili [11]. Corollary 4.6 covers Lindenstrauss's calculation for $\mathbb{Z}[X]/(X^n-1)$.

5 Remaining proofs

Proof of Theorem 4.1: Since E is a q-cofibrant (commutative) A-algebra $E \wedge_A HK$ is a q-cofibrant (commutative) HK-algebra. Hence the weak equivalence $E \wedge_A HK \longrightarrow HR$ is a genuine h-equivalence in $\mathcal{A}lg_{HK}$ (resp. $\mathcal{CA}lg_{HK}$). The induced map

$$t_\bullet^A(E \wedge_A HK) \longrightarrow t_\bullet^A(HR)$$

is dimensionswise an h-equivalence of HK-modules (resp. commutative HK-algebras). Since both simplicial objects are proper by (3.6) we obtain an h-equivalence in $\mathcal{M}od_{HK}$ (resp. $\mathcal{CA}lg_{HK}$)

$$THH^A(E \wedge_A HK) \simeq THH^A(HR).$$

Similarly, we obtain a genuine h-equivalence

$$THH^{HK}(E \wedge_A HK) \simeq THH^{HK}(HR).$$

Now

$$\begin{aligned} THH^A(E \wedge_A HK) &\cong THH^A(E) \wedge_A THH^A(HK) && \text{by (3.3)} \\ &\cong THH^A(E) \wedge_A HK \wedge_{HK} \; THH^A(HK) \\ &\cong THH^{HK}(E \wedge_A HK) \wedge_{HK} THH^A(HK) && \text{by (3.5).} \end{aligned}$$

□

Proof of Theorem 4.4: Let E be a q-cofibrant resolution of $\Sigma^\infty(G_+)$ in the category of non-commutative $\mathbf{S}$-algebras. Let HR and $\widetilde{HR}$ be q-cofibrant resolutions of the Eilenberg-MacLane spectrum of R as associative and commutative $\mathbf{S}$-algebra, respectively. Then $E \wedge_{\mathbf{S}} \widetilde{HR}$ is a resolution of $H(R[G])$ as q-cofibrant $\widetilde{HR}$-algebra: by (2.3) and (2.4) we have weak equivalences

$$E \wedge_{\mathbf{S}} \widetilde{HR} \to E \wedge_{\mathbf{S}} HR \to \Sigma^\infty(G_+) \wedge_{\mathbf{S}} HR \cong HR \wedge G_+ \cong H(R[G]).$$

Hence Theorem 4.1 applies. □

Proof of Theorem 4.8: Let $n = p_1^{r_1} \cdot \ldots \cdot p_k^{r_k}$ be the prime factorization of n and ζ_i a p^{r_i}-th root of unity. Then $\zeta = \zeta_1 \cdot \ldots \cdot \zeta_k$ is a primitive n-th root of unity. Let $M = \{p_1, \ldots, p_k\}$ and E' be a q-cofibrant resolution of the localization $\mathbf{S}[M^{-1}]$ of the sphere spectrum $\mathbf{S}$. By [14] we can successively adjoin $\zeta_1, \ldots, \zeta_k$ to E' to obtain $E'(\zeta)$. Let E be a q-cofibrant resolution of $E'(\zeta)$ as commutative E'-algebra. Now apply Theorem 4.1 with $K = R$, $R = R(\zeta)$, and $A = E'$. Finally use that THH and HH commute with localizations. □

Proof of Theorem 4.9: We need an algebraic lemma.

5.1 Lemma: If $f \in \mathbb{F}_p[X]$ is irreducible then $\mathbb{F}_p[X]/(f^n) \cong (\mathbb{F}_p[X]/(f))[Y]/(Y^n)$.

Proof: $A = \mathbb{F}_p[X]/(f^n)$ is a local ring with maximal ideal $\mathbf{m} = (\bar{f})$ and residue field $R = \mathbb{F}_p[X]/(f)$, where $\bar{g}$ denotes the image of $g \in \mathbb{F}_p[X]$ in A. Let $\pi : A \to R$ be the projection. $\pi(\bar{X})$ is a simple root of f in R. Since $\mathbf{m}$ is nilpotent, Hensel's lemma applies. Hence there is a $z \in A$ such that $\pi(z) = \pi(\bar{X})$ and $f(z) = 0$. So the map

$$\mathbb{F}_p[X] \to A, \qquad X \mapsto z$$

induces a ring homomorphism $\phi : R \to A$. Since R is a field ϕ is injective. We extend ϕ to a ring homomorphism

$$\theta : R[Y]/(Y^n) \to A, \qquad Y \mapsto \bar{f}.$$

θ is injective:

$$0 = \theta\left(\sum_{i=k}^{n-1} r_i Y^i\right) = \sum_{i=k}^{n-1} \phi(r_i)\bar{f}^i \quad \text{implies} \quad 0 = \left(1 + \sum_{i=1}^{n-k-1} \phi(r_k^{-1}) r_{k+i} \bar{f}^i\right) \cdot \bar{f}^k.$$

if $r_k \neq 0$. Since the first factor is not in $\mathbf{m}$ it is a unit in A. Hence $\bar{f}^k = 0$ in A, a contradiction. Hence all $r_k = 0$. Since source and target of θ have the same dimension as $\mathbb{F}_p$-vect or spaces, θ is bijective. □

Suppose we can construct a cofibrant $\mathbf{S}$-algebra $E(f)$ for an irreducible polynomial f in $\mathbb{F}_p[X]$ such that $E(f) \wedge_{\mathbf{S}} H\mathbb{F}_p \simeq H(\mathbb{F}_p[X]/(f))$ and Π is the based monoid $\{0, 1, x, \ldots, x^{n-1}\}$ with multiplication determined by $x^n = 0$, then $E(f)[\Pi]$, made cofibrant, is the $\mathbf{S}$-algebra required for f^n and we are done.
If f is irreducible, $\mathbb{F}_p[X]/(f)$ is a cyclotomic extension $\mathbb{F}_p(\zeta)$ of $\mathbb{F}_p$ where ζ is an appropriate n-th primitive root of unity with n and p coprime. The $\mathbf{S}$-algebra$E(f)$ is the spectrum constructed in the proof of Theorem 4.8, for which $E(f) \wedge_{\mathbf{S}} H\mathbb{Z} \simeq H(\mathbb{Z}[L^{-1}](\zeta))$, where L is the set of primes dividing n. □

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