# DEGENERATION FOR PARABOLIC GROUP ACTIONS IN GENERAL LINEAR GROUPS

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*Dedicated to Helmut Lenzing on the occasion of his sixtieth birthday*

Abstract. Let $k$ be an algebraically closed field and $V$ a finite dimensional $k$-space. Let $\mathrm{GL}(V)$ be the general linear group of $V$ and $P$ a parabolic subgroup of $\mathrm{GL}(V)$. Now $P$ acts on its unipotent radical $P_u$ and on $\mathfrak{p}_u = \operatorname{Lie} P_u$, the Lie algebra of $P_u$, via the adjoint action. More generally, we consider the action of $P$ on the $l$-th member of the descending central series of $\mathfrak{p}_u$ denoted by $\mathfrak{p}_u^{(l)}$. All instances when $P$ acts on $\mathfrak{p}_u^{(l)}$ for $l \geq 0$ with a finite number of orbits are known. In this note we give a complete description of the closure relations among the $P$-orbits on $\mathfrak{p}_u^{(l)}$ in all these finite cases.

There is a canonical bijection between the set of $P$-orbits on $\mathfrak{p}_u^{(l)}$ and the set $\mathcal{F}(\Delta)(\mathbf{e})$ of isomorphism classes of $\Delta$–filtered modules of a particular dimension vector $\mathbf{e}$ of a certain quasi-hereditary algebra $\mathcal{A}(t,l)$. These isomorphism classes in turn are given by the orbits of the reductive group $G(\mathbf{e}) = \prod \mathrm{GL}(e_i)$ on the variety $\mathcal{R}(\Delta)(\mathbf{e})$ of all representations of the quasi-hereditary algebra $\mathcal{A}(t,l)$ with $\Delta$–filtration and dimension vector $\mathbf{e}$.

In our chief result, Theorem 1.1, we show that provided there is only a finite number of isomorphism classes of indecomposable modules in $\mathcal{F}(\Delta)$ the following three posets coincide: (1) the Bruhat-Chevalley order on the set of $P$-orbits on $\mathfrak{p}_u^{(l)}$; (2) the Bruhat-Chevalley order on the set of $G(\mathbf{e})$-orbits on $\mathcal{R}(\Delta)(\mathbf{e})$; (3) the poset opposite to the so called hom–order on the set of isoclasses of $\mathcal{F}(\Delta)(\mathbf{e})$.

The advantage of this hom–order is that it is given purely by discrete invariants and that it can be computed explicitly for any given finite case. Theorem 1.1 then in turn allows us to explicitly determine the closure relations for the $P$-orbits on $\mathfrak{p}_u^{(l)}$ with the aid of this hom–order. We present some examples of Hasse diagrams in an Appendix.

We introduce the notion of a global minimal degeneration in the category $\mathcal{F}(\Delta)$ as one which cannot be refined by passing to higher dimension vectors by taking direct sums of modules, and show that these particular degenerations can be obtained directly from the Auslander-Reiten quiver of $\mathcal{F}(\Delta)$.

## 1. Introduction

Throughout, $k$ is an algebraically closed field and $V$ a finite-dimensional $k$-vector space and $P$ is a parabolic subgroup of $\mathrm{GL}(V)$ with unipotent radical $P_u$. The Lie algebra of $P_u$ is denoted by $\mathfrak{p}_u$. The descending central series of $\mathfrak{p}_u$ is defined as usual by $\mathfrak{p}_u^{(0)} := \mathfrak{p}_u$ and $\mathfrak{p}_u^{(l)} := [\mathfrak{p}_u, \mathfrak{p}_u^{(l-1)}]$, for $l \geq 1$. The group $P$ acts on each $\mathfrak{p}_u^{(l)}$ via the adjoint action. The *Bruhat-Chevalley order* on the set of $P$-orbits on $\mathfrak{p}_u^{(l)}$ is given by the orbit closures: for two orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ we write $\mathcal{O}_1 \leq \mathcal{O}_2$ whenever $\mathcal{O}_1$ is contained in the (Zariski) closure of $\mathcal{O}_2$. The aim of this note is to

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give a complete description of this partial order provided the set of $P$-orbits on $\mathfrak{p}_u^{(l)}$ is finite. In [5], [6], and [2] precisely all those instances are determined.

A Levi subgroup $L$ of $P$ is (isomorphic to) a product of general linear groups, say $\mathrm{GL}(d_i)$, for $1 \leq i \leq t$ for some $t \in \mathbb{N}$, with $\dim V = \sum d_i$. The ordered tuple $\mathbf{d} = (d_1, \dots, d_t)$ determines the conjugacy class of $P$ in $\mathrm{GL}(V)$. To indicate this, we often write $P = P(\mathbf{d})$.

Given $\mathbf{d} \in \mathbb{N}^t$ we define the $t$-tuple $\mathbf{e} := \Sigma\mathbf{d}$ by $e_j = (\Sigma\mathbf{d})_j := \sum_{i=1}^{j} d_i$ for $1 \leq j \leq t$. For $P = P(\mathbf{d})$ there is a canonical bijection between the set of $P$-orbits on $\mathfrak{p}_u^{(l)}$ and the set of isomorphism classes of $\Delta$–filtered modules $\mathcal{F}(\Delta)(\mathbf{e})$ of dimension vector $\mathbf{e}$ of a certain quasi-hereditary algebra $\mathcal{A}(t,l)$ ([2], [5]). These isomorphism classes in turn are given by the orbits of the reductive group $G(\mathbf{e}) := \prod \mathrm{GL}(e_i)$ on the variety $\mathcal{R}(\Delta)(\mathbf{e})$ of all representations of the quasi-hereditary algebra $\mathcal{A}(t,l)$ with $\Delta$–filtration and dimension vector $\mathbf{e}$. In this context we may also consider the Bruhat-Chevalley order on the set of $G(\mathbf{e})$-orbits on $\mathcal{R}(\Delta)(\mathbf{e})$.

Let $X$ and $Y$ be in $\mathcal{F}(\Delta)(\mathbf{e})$. We write $X \leq_{\hom} Y$ provided $\dim \mathrm{Hom}(X, I) \leq \dim \mathrm{Hom}(Y, I)$ for every indecomposable module $I$ in $\mathcal{F}(\Delta)$. As in the case of Artin algebras it turns out that this induces a partial order on the set of isomorphism classes of $\mathcal{F}(\Delta)(\mathbf{e})$ to which we refer to as the hom–*order* on $\mathcal{F}(\Delta)$.

Our principal result concerning the action of $P$ on $\mathfrak{p}_u^{(l)}$ is

**Theorem 1.1.** *Let $t, l \in \mathbb{N}$, $\mathbf{d} \in \mathbb{N}^t$ and let $V$ be a finite-dimensional $k$-vector space such that $\dim V = \sum d_i$. Let $P = P(\mathbf{d})$ be the standard parabolic subgroup of $\mathrm{GL}(V)$ associated to $\mathbf{d}$. Set $\mathbf{e} = \Sigma\mathbf{d}$. Suppose that the number of $P$-orbits on $\mathfrak{p}_u^{(l)}$ is finite. Then the following posets coincide:*

(i) *the Bruhat-Chevalley order on the set of $P$-orbits on $\mathfrak{p}_u^{(l)}$;*
(ii) *the Bruhat-Chevalley order on the set of $G(\mathbf{e})$-orbits on $\mathcal{R}(\Delta)(\mathbf{e})$;*
(iii) *the poset opposite to the* hom*–order on the set of isoclasses of $\mathcal{F}(\Delta)(\mathbf{e})$.*

For our setting the advantage of working with the hom–order is due to the fact that it is given purely by discrete invariants, more specifically, by the ordered tuples $\underline{\mathrm{hdim}}(X)$ whose $j$-th entry consists of $\dim \mathrm{Hom}(X, I_j)$, where $I_j$ runs through a complete set of representatives of isomorphism classes of indecomposable modules in $\mathcal{F}(\Delta)$. Consequently, once the square matrix $(\dim \mathrm{Hom}(I_i, I_j))_{i,j}$ is computed, the hom–order on $\mathcal{F}(\Delta)(\mathbf{e})$ can be computed explicitly in any finite instance. Thus Theorem 1.1 allows us to explicitly determine the closure relations for the $P$-orbits on $\mathfrak{p}_u^{(l)}$ with the aid of this hom–order by computational means. We illustrate some examples in Appendix A.

While this method solves our initial problem of computing the Hasse diagram for any fixed choice of the parameters $t$, $l$ and $\mathbf{d}$, in Section 5 we take a more conceptual approach to degenerations which is independent of these parameters. The crucial idea in this context is the notion of a *global minimal degeneration*: by that we mean a pair of modules $X$ and $Y$ in $\mathcal{F}(\Delta)$ of the same dimension vector without common direct summand such that the difference $\underline{\mathrm{hdim}}(X) - \underline{\mathrm{hdim}}(Y)$ is a standard basis vector in $\mathbb{Q}^r$, where $r$ is the number of isomorphism classes of indecomposable modules in $\mathcal{F}(\Delta)$. In our principal result in Section 5 we show that such pairs correspond bijectively to almost split sequences in the Auslander-Reiten quiver of $\mathcal{F}(\Delta)$. The AR-quivers in turn can be computed explicitly in each of the relevant cases; for $l \geq 3$ these quivers can even be constructed in a uniform fashion. Ultimately, we obtain from these AR-quivers explicit formulae for the number of isomorphism classes of indecomposable modules, their dimension vectors, and the global minimal degenerations in $\mathcal{F}(\Delta)$.

The paper is structured as follows. In Section 2 we recall some relevant machinery from [5] and [2] necessary for translating our problem into the realm of quasi-hereditary algebras $\mathcal{A}$. In our main result there we show that the Bruhat-Chevalley order on the set of $P$-orbits on $\mathfrak{p}_u^{(l)}$ for $P = P(\mathbf{d})$ is equivalent to the poset given by the Bruhat-Chevalley order on the set of $G(\mathbf{e})$-orbits on $\mathcal{R}(\Delta)(\mathbf{e})$, where $\mathbf{e} = \Sigma\mathbf{d}$ (Theorem 2.5), thereby showing the equivalence of parts (i) and (ii) of Theorem 1.1. This result is valid independently of the representation type of $\mathcal{F}(\Delta)$.

In the following two sections we proceed to study arbitrary quasi-hereditary algebras. First we recall some basic facts and fundamental properties of these algebras and their $\Delta$–filtered modules. In the next section we show that the Bruhat-Chevalley order on the set of $G(\mathbf{e})$-orbits on $\mathcal{R}(\Delta)(\mathbf{e})$ is opposite to the poset given by the hom–order on the set of isoclasses of $\mathcal{F}(\Delta)(\mathbf{e})$ for each dimension vector $\mathbf{e}$ provided $\mathcal{A}$ is *$\Delta$–finite*, i.e., the subcategory $\mathcal{F}(\Delta)$ has only finitely many isomorphism classes of indecomposable modules (Theorem 4.1), thereby showing the equivalence of parts (ii) and (iii) of Theorem 1.1.

In Section 5 we introduce the notion of a *globally minimal degeneration* and show that these particular degenerations of modules can be obtained directly from the Auslander-Reiten quiver of $\mathcal{F}(\Delta)$. In Section 6 we compute the Auslander-Reiten quivers of $\mathcal{F}(t,l)$ in the instances when $t$ is maximal with respect to $l$ such that $\mathcal{A}(t,l)$ is still $\Delta$–finite. Here and later on we refer to these cases as the *extremal* finite ones. Along the way we present formulas for the number of $P$-orbits on $\mathfrak{p}_u^{(l)}$ for $P = P(\mathbf{d})$ and the number of indecomposable objects in $\mathcal{F}(\Delta)$.

In Appendix A we present some examples of Hasse diagrams of the Bruhat-Chevalley order on the set of $P$-orbits on $\mathfrak{p}_u^{(l)}$ in some finite cases; especially when $P$ is a Borel subgroup $B$ of $\mathrm{GL}(V)$; for the action of $B$ on $\mathfrak{b}_u$ these were first determined by V.V. Kashin [7] (see Figures 7.1 - 7.3).

Finally, in Appendix B we list the Auslander-Reiten quivers of $\mathcal{F}(t,l)$ in the extremal finite instances along with normal forms for the indecomposable modules.

## 2. Parabolic Groups and $\Delta$–filtered modules

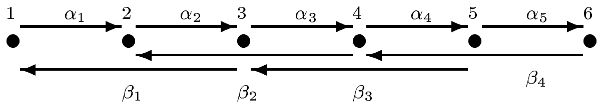
We maintain the notation from the Introduction. We fix integers $t, l \in \mathbb{N}$ and define a category $\mathcal{F}(t,l)$ as follows. The objects are pairs $(F, f)$, where $F$ is a flag $\{0\} = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_t = V$ of length $t$ of subspaces of some finite-dimensional $k$-vector space $V$, and $f$ is an endomorphism of $V$ which lies in $\mathfrak{p}_u^{(l)}$, where $P$ is the stabilizer of $F$ in $\mathrm{GL}(V)$. Consequently, $f(V_i) \subseteq V_{i-l-1}$ for $l+1 \leq i \leq t$. Let $(F, f)$ and $(F', f')$ be in $\mathcal{F}(t,l)$. A morphism $\varphi : (F, f) \to (F', f')$ is a linear map $\varphi : V \to V'$ such that $\varphi(V_i) \subseteq V_i'$ for $1 \leq i \leq t$ and $\varphi f = f'\varphi$. For $(F, f)$ in $\mathcal{F}(t,l)$ we set $d_i := \dim V_i - \dim V_{i-1}$ for $1 \leq i \leq t$ and call $\mathbf{d} = (d_1, \ldots, d_t)$ the *dimension vector* of $(F, f)$ and write $\underline{\dim} F = \mathbf{d}$. For $\mathbf{d} \in \mathbb{N}^t$ we denote by $\mathcal{F}(t,l)(\mathbf{d})$ the subcategory of $\mathcal{F}(t,l)$ of all objects of dimension vector $\mathbf{d}$.

If $\varphi : (F, f) \xrightarrow{\sim} (F', f')$ is an isomorphism in $\mathcal{F}(t,l)$, then $\mathbf{d} = \mathbf{d}'$ and so $\dim V = \dim V'$. After identifying $F$ with $F'$ we see that $\varphi$ lies in $P = P(\mathbf{d})$ and $f' = \varphi f \varphi^{-1}$, i.e., $f$ and $f'$ are endomorphisms in $\mathfrak{p}_u^{(l)}$ conjugate under the action of $P$.

**Lemma 2.1.** *For any $t, l \in \mathbb{N}$ and $\mathbf{d} \in \mathbb{N}^t$ the isomorphism classes of objects in $\mathcal{F}(t,l)(\mathbf{d})$ correspond bijectively to the $P$-orbits on $\mathfrak{p}_u^{(l)}$ for $P = P(\mathbf{d})$. This correspondence is induced by the map $(F, f) \mapsto P \cdot f$, where $P$ is the stabilizer of the flag $F$ and $f$ lies in $\mathfrak{p}_u^{(l)}$.*

*Proof.* It follows from the remarks above that there is a well-defined map from the set of isomorphism classes of objects in $\mathcal{F}(t,l)$ of fixed dimension vector $\mathbf{d}$ to the set of $P$-orbits on $\mathfrak{p}_u^{(l)}$ for $P = P(\mathbf{d})$. Clearly, it is onto, as for any $f$ in $\mathfrak{p}_u^{(l)}$, the pair $(F, f)$ lies in $\mathcal{F}(t,l)$ and its class maps to the $P$-orbit through $f$. Moreover, since for any two pairs $(F, f), (F, f')$ in $\mathcal{F}(t,l)$ and $\varphi \in P = P(\mathbf{d})$ satisfying $f' = \varphi f \varphi^{-1}$ the linear map $\varphi$ defines an isomorphism between $(F, f)$ and $(F, f')$, this correspondence is injective. $\square$

Instead of working directly with $\mathcal{F}(t,l)$, we pass to an equivalent category. For that purpose let $Q(t,l)$ be the quiver defined as follows: the set of vertices is simply $\{1, \ldots, t\}$ and the arrows of the quiver are $i \xrightarrow{\alpha_i} i+1$ for $i = 1, \ldots, t-1$ and $i+l+1 \xrightarrow{\beta_i} i$ for $i = 1, \ldots, t-l-1$. Let $J$ be the ideal in the path algebra $kQ(t,l)$ of this quiver given by the following relations: $\beta_1 \alpha_{l+1} = 0$ and $\beta_{i-l}\alpha_i = \alpha_{i-l-1}\beta_{i-l-1}$ for $l+1 < i < t$. Then we denote the finite-dimensional quotient algebra $kQ(t,l)/J$ by $\mathcal{A}(t,l)$. We illustrate the example $Q(6,1)$:




Let $\mathcal{A}(t,l)$–mod be the category of all finite-dimensional left $\mathcal{A}(t,l)$–modules and by $\mathcal{M}(t,l)$ we denote the full subcategory of modules $M$ in $\mathcal{A}(t,l)$–mod subject to the condition that all maps $M_{\alpha_i}$ are injective for $1 \leq i \leq t-1$.

The following key observation is due to P. Gabriel.

**Lemma 2.2.** *The categories $\mathcal{F}(t,l)$ and $\mathcal{M}(t,l)$ are equivalent for all $t, l \in \mathbb{N}$.*

*Proof.* Let $(F, f)$ be in $\mathcal{F}(t,l)$ and define a representation $M = M(F, f)$ of the quiver $Q(t,l)$ above via the flag associated to $F$ as follows:

$$V_1 \xrightarrow{\alpha_1} V_2 \ \ldots \ V_{t-1} \xrightarrow{\alpha_{t-1}} V_t,$$

where the maps $M_{\alpha_i}$ are simply the inclusions $V_{i-1} \hookrightarrow V_i$ and the maps $M_{\beta_i}$ are the restrictions of $f$ to $V_{l+i+1}$. One easily checks that $M(F, f)$ satisfies the conditions above, i.e., $M(F, f)$ is in $\mathcal{M}(t,l)$, and that the map $(F, f) \mapsto M(F, f)$ defines an equivalence of the two categories. $\square$

Crucial to our approach is the following fact, cf. [3, §6 - 7], [2, §3]:

**Lemma 2.3.** *The subcategory $\mathcal{M}(t,l)$ of $\mathcal{A}(t,l)$*–mod *is precisely the category $\mathcal{F}(\Delta)$ of $\Delta$–filtered modules over the quasi-hereditary algebra $\mathcal{A}(t,l)$, for $t, l \in \mathbb{N}$.*

*Remark 2.4.* Let $t, l \in \mathbb{N}$. Then $\mathcal{F}(t,l)$ is a Krull-Schmidt category, i.e., every object in $\mathcal{F}(t,l)$ has a unique decomposition into a direct sum of indecomposable ones (up to the order of the summands). Being closed under taking direct summands, the subcategory $\mathcal{M}(t,l)$ inherits the Krull-Schmidt property from $\mathcal{A}(t,l)$–mod, whence so does $\mathcal{F}(t,l)$ by Lemma 2.2.

For $\mathbf{d} \in \mathbb{N}^t$ let $\mathbf{e} = \Sigma\mathbf{d}$ be the $t$-tuple defined by the partial sums $e_j := \sum_{i=1}^{j} d_i$ for $1 \leq j \leq t$. We denote by $\mathcal{F}(\Delta)(\mathbf{e})$ the subcategory of $\mathcal{F}(\Delta)$ of all modules of dimension vector $\mathbf{e}$. Observe that the collection of all $\mathcal{A}(t,l)$–modules in $\mathcal{F}(\Delta)$ of *fixed* dimension vector $\mathbf{e}$ together with a *fixed* set of $k$-spaces $V_i$ of dimension $e_i$, for $1 \leq i \leq t$, is an algebraic variety: We denote this variety by $\mathcal{R}(\Delta)(\mathbf{e})$. It is defined as the locally closed subvariety of the $k$-vector space

$$\mathcal{R}(\mathbf{e}) := \bigoplus_{\alpha_i} \mathrm{Hom}(V_i, V_{i+1}) \oplus \bigoplus_{\beta_j} \mathrm{Hom}(V_{j+l+1}, V_j)$$

of all possible linear maps corresponding to the arrows $\alpha_i$ and $\beta_j$ in the quiver $Q(t,l)$ satisfying the relations above (closed condition) and such that the collection of direct summands corresponding to the arrows $\alpha_i$ consist of injective linear maps (open condition). Since the injectivity of the linear maps $M_{\alpha_i}$ is preserved by isomorphisms, we infer that the natural action of the reductive group $G(\mathbf{e}) := \prod \mathrm{GL}(e_i)$ on $\mathcal{R}(\mathbf{e})$ leaves $\mathcal{R}(\Delta)(\mathbf{e})$ invariant. The action of an element $g = (g_1, g_2, \dots)$ in $G(\mathbf{e})$ on $\mathcal{R}(\Delta)(\mathbf{e})$ is given by base change in each of the spaces $V_i$, i.e., $g \cdot M(F,f) = (\dots, g_{i+1}M_{\alpha_i}g_i^{-1}, \dots, g_iM_{\beta_i}g_{i+l+1}^{-1}, \dots)$, where $g_i \in \mathrm{GL}(e_i)$ for each $i$.

**Theorem 2.5.** *Let $t, l \in \mathbb{N}$, $\mathbf{d} \in \mathbb{N}^t$ and let $V$ be a finite-dimensional $k$-vector space such that $\dim V = \sum d_i$. Let $P = P(\mathbf{d})$ be the standard parabolic subgroup of $\mathrm{GL}(V)$ associated to $\mathbf{d}$. Set $\mathbf{e} = \Sigma\mathbf{d}$. Then the following posets coincide:*

(i) *the Bruhat-Chevalley order on the set of $P$-orbits on $\mathfrak{p}_u^{(l)}$;*
(ii) *the Bruhat-Chevalley order on the set of $G(\mathbf{e})$-orbits on $\mathcal{R}(\Delta)(\mathbf{e})$.*

*Proof.* Observe that $\mathcal{R}(\Delta)(\mathbf{e})$ and $\mathfrak{p}_u^{(l)}(\mathbf{d})$ are *not* isomorphic varieties; $\mathfrak{p}_u^{(l)}(\mathbf{d})$ can be identified with the subvariety of $\mathcal{R}(\Delta)(\mathbf{e})$ consisting of all those representations $M$ of $Q(t,l)$ with a *fixed* set of injective maps $M_{\alpha_i}$. However, for our purpose it is enough to construct morphisms between these varieties in both directions preserving the orbit structure, as such morphisms do preserve orbit closures.

We fix vector spaces $V_i$ together with injections $V_i \to V_{i+1}$, where $\dim V_i = e_i$. Let $P$ be the stabilizer in $\mathrm{GL}(V)$ of the flag $\{0\} = V_0 \subseteq V_1 \subseteq V_2 \subseteq \ldots \subseteq V_{t-1} \subseteq V_t = V$. For $f$ in $\mathfrak{p}_u^{(l)}$ we define a representation $M(f) := \{M_{\alpha_i}, M_{\beta_j}\}$ for $1 \leq i < t$ and $1 \leq j < t-l$ in $\mathcal{F}(\Delta)(\mathbf{e})$ of $\mathcal{A}(t,l)$ by setting $M_{\beta_j} := f|_{V_{j+l+1}}$ and $M_{\alpha_i}$ is just the fixed injection in the flag. The map $f \mapsto M(f)$ obviously defines a morphism from $\mathfrak{p}_u^{(l)}(\mathbf{d})$ to $\mathcal{R}(\Delta)(\mathbf{e})$.

It remains to construct a morphism in the opposite direction. Let $\{M_{\alpha_i}, M_{\beta_j}\}$ be a representation in $\mathcal{F}(\Delta)(\mathbf{e})$ with vector spaces $V_1, \dots, V_t$ with $\dim V_i = e_i$, then $M_{\alpha_{t-1}}M_{\alpha_{t-2}} \cdots M_{\alpha_{t-l-1}}M_{\beta_{t-l-1}}$ is an endomorphism of $V_t$ lying in $\mathfrak{p}_u^{(l)}(\mathbf{d})$. Both morphisms preserve orbits which is easily deduced from the equivalences of the corresponding categories constructed above. $\square$

## 3. Quasi-hereditary algebras and $\Delta$–filtered modules

For representation-finite algebras it was recently shown by the last author that the geometric degeneration of modules can be characterized in terms of dimensions of morphism spaces [11]. Our aim is to obtain the analogous result for the category of $\Delta$–filtered modules of a quasi-hereditary algebra. This requires some additional arguments beyond the ones from *loc. cit.*; for instance, Proposition 3.4 ensures that the relation defined by comparing dimensions of spaces of homomorphisms yields a partial order for $\Delta$–filtered modules, the hom–order. Moreover, Proposition 3.4 is relevant in some arguments in Section 4. This also applies to some of the crucial results in Section 4 such as Proposition 4.2 and Lemmas 4.4 and 4.7.

In this section we recall some definitions and relevant properties of quasi-hereditary algebras. Unless stated otherwise we refer to [10] for proofs of the statements listed below.

Let $\mathcal{A}$ be a finite-dimensional algebra over $k$ and let $E(1), \dots, E(t)$ be a set of representatives of the isomorphism classes of simple $\mathcal{A}$–modules. For each $i$ we fix a projective cover $P(i)$ of $E(i)$ and denote by $\Delta(i)$ the maximal factor module of $P(i)$ with composition factors in $\{E(1), \dots, E(i)\}$. The $\Delta(i)$'s are called the *standard* modules of $\mathcal{A}$ and $\mathcal{F}(\Delta)$ denotes the category of all $\mathcal{A}$–modules $M$ which have a filtration $M = M_0 \supset \cdots \supset M_r = 0$ such that each factor $M_{i-1}/M_i$ belongs to $\{\Delta(1), \dots, \Delta(t)\}$. The algebra $\mathcal{A}$ is called *quasi-hereditary* provided $\operatorname{End}(\Delta(i)) = k$ and $P(i)$ belongs to $\mathcal{F}(\Delta)$ for each $i$.

We suppose from now on that $\mathcal{A}$ is quasi-hereditary. Then the full subcategory $\mathcal{F}(\Delta)$ of $\mathcal{A}$–mod is closed under direct summands and extensions. We call a module $X$ in $\mathcal{F}(\Delta)$ (relative) Ext-*projective* if $\operatorname{Ext}^1_{\mathcal{A}}(X, M) = 0$ for all $M \in \mathcal{F}(\Delta)$; likewise for Ext-*injective* modules in $\mathcal{F}(\Delta)$. The indecomposable Ext-projective modules in $\mathcal{F}(\Delta)$ are just the projective $\mathcal{A}$–modules $P(1), \dots, P(t)$, whereas the indecomposable Ext-injective modules in $\mathcal{F}(\Delta)$ are the so-called *characteristic* modules $T(1), \dots, T(t)$. The direct sum $T := \bigoplus T(i)$ is a *tilting* module [10].

We denote by rad the Jacobson radical of the category $\mathcal{A}$–mod, cf. [4, §3.2]. A (relative) *almost split sequence* in $\mathcal{F}(\Delta)$ is a non-split exact sequence

$$0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$$

in $\mathcal{A}-$mod with $X, Y, Z$ in $\mathcal{F}(\Delta)$ such that each radical morphism $\gamma \in \operatorname{rad}(M, Z)$ with $M \in \mathcal{F}(\Delta)$ factors through $\beta$ and each radical morphism $\delta \in \operatorname{rad}(X, M)$ with $M \in \mathcal{F}(\Delta)$ factors through $\alpha$. The category $\mathcal{F}(\Delta)$ *admits almost split sequences*; by that we mean: if $X$ is indecomposable and not Ext-injective in $\mathcal{F}(\Delta)$, then there exists an almost split sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{F}(\Delta)$. Moreover, $Z$ is determined uniquely up to isomorphism by $X$ and is often denoted by $\tau_\Delta^- X$. Conversely, if $Z$ is indecomposable and not Ext-projective in $\mathcal{F}(\Delta)$, then there exists an almost split sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{F}(\Delta)$; likewise, the module $X$ is determined by $Z$ up to isomorphism and is usually denoted by $\tau_\Delta Z$. The map $\tau_\Delta$ is called the *Auslander-Reiten translation* and the sequence above is also called an *Auslander-Reiten sequence.* The quiver whose vertices are the isomorphism classes of indecomposable modules in $\mathcal{F}(\Delta)$ and whose arrows are given by the maps in almost split sequences is called the *Auslander-Reiten quiver* of the underlying module category $\mathcal{F}(\Delta)$, see [4], [9].

For a finite-dimensional $\mathcal{A}$–module $M$ and an indecomposable module $Z$ we denote the multiplicity of $Z$ in $M$ as a direct summand by $\mu(M, Z)$. Observe that $\mu(M, Z)$ can be computed via

$$\mu(M, Z) = \dim \operatorname{Hom}(M, Z) - \dim \operatorname{rad}(M, Z).$$

**Lemma 3.1.** *Let $0 \to X \to Y \to Z \to 0$ be an almost split sequence in $\mathcal{F}(\Delta)$ and let $M$ be in $\mathcal{F}(\Delta)$. Then:*

(1) $\mu(M, Z) = \dim \operatorname{Hom}(M, X \oplus Z) - \dim \operatorname{Hom}(M, Y)$,
(2) $\mu(M, X) = \dim \operatorname{Hom}(X \oplus Z, M) - \dim \operatorname{Hom}(Y, M)$.

*Proof.* (1) From the definition of an almost split sequence we obtain an exact sequence

$$0 \to \mathrm{Hom}(M, X) \to \mathrm{Hom}(M, Y) \to \mathrm{rad}(M, Z) \to 0.$$

Thus, $\dim \mathrm{Hom}(M, X \oplus Z) - \dim \mathrm{Hom}(M, Y) = \dim \mathrm{Hom}(M, Z) - \dim \mathrm{rad}(M, Z) = \mu(M, Z)$. We obtain (2) by duality. $\square$

Up to Morita-equivalence we may assume that the algebra $\mathcal{A}$ is presented in the form $\mathcal{A} = kQ/J$ where $Q$ is a quiver and $J$ is an *admissible* ideal of the *path algebra* $kQ$ of $Q$ (cf. [4, §8]). The quiver $Q = (Q_0, Q_1, t, h)$ is described by its set of vertices $Q_0$, the set of arrows $Q_1$ and two maps $t, h : Q_1 \to Q_0$ which determine *tail* and *head* of each arrow.

An $\mathcal{A}$–module $M$ is a family $M = (M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1}$, where each $M_i$ is a finite-dimensional vector space and each $M_\alpha : M_{t\alpha} \to M_{h\alpha}$ is a $k$-linear map such that $M_\xi = 0$ for all $\xi \in J$, where $M_\xi$ is defined as follows: the element $\xi \in J$ is a linear combination of paths $w$ in $kQ$, say $\xi = \sum c_w w$ with $c_w \in k$; we set $M_w = M_{\alpha_m} \circ \cdots \circ M_{\alpha_1}$ for any path $w = \alpha_1 \ldots \alpha_m$ and define $M_\xi = \sum c_w M_w$.

For each $\mathcal{A}$–module $M$ we denote by $\underline{\dim}\, M \in \mathbb{N}^t$ the dimension vector of $M$. There are several equivalent descriptions of the dimension vector:

$$(\underline{\dim}\, M)_i = \dim M_i = \dim \mathrm{Hom}(P(i), M)$$

for $i = 1, \ldots, t$.

*Remark 3.2.* Note that for $M$ and $N$ in $\mathcal{F}(\Delta)$ we have $\underline{\dim}\, M = \underline{\dim}\, N$ precisely when $\dim \mathrm{Hom}(M, T(i)) = \dim \mathrm{Hom}(N, T(i))$ for each $i = 1, \ldots, t$.

Now fix $k$-spaces $M_i$ for each $i = 1, \ldots, t$, set $e_i = \dim M_i$ and $\mathbf{e} = (e_1, \ldots, e_t)$. The affine variety $\mathcal{R}(\mathbf{e})$ of $\mathcal{A}$–modules with dimension vector $\mathbf{e}$ consists of tuples of $k$-linear maps $(M_\alpha : M_{t\alpha} \to M_{h\alpha})_{\alpha \in Q_1}$ such that $M_\xi = 0$ for all $\xi \in J$. Clearly, this variety contains representatives of all isomorphism classes of $\mathcal{A}$–modules of dimension vector $\mathbf{e}$. Throughout, we identify $\mathcal{A}$–modules or representations of the associated quiver of dimension vector $\mathbf{e}$ with points in the variety $\mathcal{R}(\mathbf{e})$.

The reductive group $G(\mathbf{e}) := \prod \mathrm{GL}(e_i)$ acts on $\mathcal{R}(\mathbf{e})$ by conjugation in each vector space $M_i$ and the orbits are precisely the isomorphism classes of $\mathbf{e}$-dimensional modules. The modules in $\mathcal{F}(\Delta)$ with dimension vector $\mathbf{e}$ form a $G(\mathbf{e})$-stable subset $\mathcal{R}(\Delta)(\mathbf{e})$ of $\mathcal{R}(\mathbf{e})$. We consider the restriction of the Bruhat-Chevalley order to the orbits in $\mathcal{R}(\Delta)(\mathbf{e})$.

**Definition 3.3.** For $i = 1, 2$ let $N_i$ be in $\mathcal{F}(\Delta)(\mathbf{e})$ and let $\mathcal{O}_i$ be the $G(\mathbf{e})$-orbit of $N_i$ in $\mathcal{R}(\Delta)(\mathbf{e})$. Whenever $\mathcal{O}_1 \leq \mathcal{O}_2$ in the Bruhat-Chevalley order, by abuse of notation, we simply write $N_1 \leq N_2$ and call $N_1$ a *(geometric) degeneration* of $N_2$. As a word of caution, we should like to point out that in some literature this order is denoted by the *reversed* symbol, e.g., see [1], [8].

**Proposition 3.4.** *Let $M, N$ be in $\mathcal{F}(\Delta)(\mathbf{e})$. Then $M$ and $N$ are isomorphic provided* $\dim \mathrm{Hom}(M, X) = \dim \mathrm{Hom}(N, X)$ *for all $X \in \mathcal{F}(\Delta)$.*

*Proof.* We argue by induction on $\mathbf{e}$. Let $Z$ be a non-Ext-projective indecomposable module in $\mathcal{F}(\Delta)$ with almost split sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{F}(\Delta)$. By Lemma 3.1 we obtain

$$\begin{aligned} \mu(N, Z) &= \dim \mathrm{Hom}(N, X \oplus Z) - \dim \mathrm{Hom}(N, Y) \\ &= \dim \mathrm{Hom}(M, X \oplus Z) - \dim \mathrm{Hom}(M, Y) = \mu(M, Z). \end{aligned}$$

If $\mu(N,Z) > 0$, then $N = N' \oplus Z$, $M = M' \oplus Z$ and we apply induction to $N'$ and $M'$. On the other hand, if $\mu(N,Z) = 0 = \mu(M,Z)$ for *all* non-Ext-projective modules $Z$ in $\mathcal{F}(\Delta)$, then $M$ and $N$ are projective $\mathcal{A}$–modules of the same dimension vector. Hence they are isomorphic, as the dimension vectors $\underline{\dim}\, P(1), \dots, \underline{\dim}\, P(t)$ are linearly independent for any quasi-hereditary algebra. $\square$

**Definition 3.5.** Let $M$ and $N$ be in $\mathcal{F}(\Delta)(\mathbf{e})$ such that

$$\dim \operatorname{Hom}(M,X) \leq \dim \operatorname{Hom}(N,X) \text{ for all } X \in \mathcal{F}(\Delta).$$

Then we write $[M] \leq_{\text{hom}} [N]$, where $[Y]$ denotes the isomorphism class in $\mathcal{F}(\Delta)(\mathbf{e})$ of the module $Y$. According to Proposition 3.4, this defines a partial order on the set of isomorphism classes of $\mathcal{F}(\Delta)(\mathbf{e})$, called the hom*–order*. If $[N] \geq_{\text{hom}} [M]$, then, by abuse of notation, we also write $N \geq_{\text{hom}} M$; and we call $N$ a *Hom-degeneration* of $M$, or we also say that $N$ is a *degeneration of $M$ with respect to the* hom*–order*.

The following notion plays an important rôle in our further study.

**Definition 3.6.** A quasi-hereditary algebra $\mathcal{A}$ is called $\Delta$*-finite* provided the subcategory $\mathcal{F}(\Delta)$ of $\mathcal{A}$–mod has only finitely many isomorphism classes of indecomposable modules.

## 4. Degenerations of $\Delta$–filtered modules

The goal of this section is the following

**Theorem 4.1.** *Suppose that $\mathcal{A}$ is a $\Delta$–finite quasi-hereditary algebra. Then for each dimension vector $\mathbf{e}$ the following posets coincide:*

(i) *the Bruhat-Chevalley order on the set of $G(\mathbf{e})$-orbits on $\mathcal{R}(\Delta)(\mathbf{e})$;*
(ii) *the poset opposite to the* hom*–order on the set of isoclasses of $\mathcal{F}(\Delta)(\mathbf{e})$.*

It is easy to see that a degeneration in $\mathcal{F}(\Delta)(\mathbf{e})$ in the Bruhat-Chevalley order implies a degeneration in the hom–order: Let $M$ and $N$ be in $\mathcal{F}(\Delta)(\mathbf{e})$ and suppose $N \leq M$. It is well-known that this implies $\dim \operatorname{Hom}(N,X) \geq \dim \operatorname{Hom}(M,X)$ for each $\mathcal{A}$–module $X$ (cf. [8, §2]); this in particular applies to each module in $\mathcal{F}(\Delta)$; thus $N \geq_{\text{hom}} M$.

In order to show the reverse direction we need some preparations. Let $M$ and $N$ be in $\mathcal{F}(\Delta)(\mathbf{e})$. We first recall from [8, §3.4] that $M$ degenerates geometrically to $N$, i.e., $N \leq M$, provided there exists an exact sequence

$$0 \to N \to M \oplus Z \to Z \to 0$$

in $\mathcal{A}$–mod (in fact, the existence of such an exact sequence is equivalent to the existence of a geometric degeneration $N \leq M$, see [12]). Consequently, Theorem 4.1 follows, once we have established our next result.

**Proposition 4.2.** *Let $\mathcal{A}$ be a $\Delta$–finite quasi-hereditary algebra. Let $N \geq_{\text{hom}} M$ in $\mathcal{F}(\Delta)(\mathbf{e})$. Then there are modules $Z$, $Z'$ in $\mathcal{F}(\Delta)$ and exact sequences in $\mathcal{F}(\Delta)$*

$$0 \to N \to M \oplus Z \to Z \to 0 \qquad \textit{and} \qquad 0 \to Z' \to Z' \oplus M \to N \to 0.$$

In order to prove Proposition 4.2 we need a series of lemmas. First, we recall some notation from [8] which is used throughout this section. We set $[X,Y]$ for $\dim \operatorname{Hom}(X,Y)$ for any two $\mathcal{A}$–modules $X$ and $Y$. If $\Sigma :\ 0 \to D \to E \to F \to 0$ is an exact sequence in $\mathcal{A}$–mod and $M, N$ are $\mathcal{A}$–modules, we define additive, $\mathbb{Z}$–valued functions $\delta_{M,N}$, $\delta'_{M,N}$, $\delta_\Sigma$, and $\delta'_\Sigma$ on $\mathcal{A}$–modules $X$ as follows:

$$\begin{aligned}
\delta_{M,N}(X) &:= [N,X]-[M,X];\\
\delta'_{M,N}(X) &:= [X,N]-[X,M];\\
\delta_{\Sigma}(X) &:= \delta_{E,D\oplus F}(X)=[D\oplus F,X]-[E,X];\\
\delta'_{\Sigma}(X) &:= \delta'_{E,D\oplus F}(X)=[X,D\oplus F]-[X,E].
\end{aligned}$$

*Remark 4.3.* These definitions are motivated as follows. The function $\delta_\Sigma$ measures the "distance" between $E$ and $D\oplus F$: from the existence of the exact sequence $\Sigma$ we know that the $G(\mathbf{e})$-orbit passing through $E$ has the one of $D\oplus F$ in its closure, i.e., that $D\oplus F\leq E$. The two orbits coincide precisely when $\Sigma$ splits and in this case $\delta_\Sigma(X)=0$ for all $X$. However, if $\Sigma$ is an almost split sequence, we obtain $\delta_\Sigma(D)=1$ by Lemma 3.1, while $\delta_\Sigma(X)=0$ for all indecomposables $X\neq D$. Observe that $N\geq_{\text{hom}} M$ holds if and only if $\delta_{M,N}(X)\geq 0$ for all $X$ in $\mathcal{F}(\Delta)$.

Let $\mathcal{NI}$ and $\mathcal{NP}$ be complete sets of representatives of the isomorphism classes of non-Ext-injective and non-Ext-projective indecomposable modules in $\mathcal{F}(\Delta)$, respectively. Since we assume that $\mathcal{A}$ is $\Delta$–finite, both of these sets are finite. For each $X$ in $\mathcal{NI}$ we fix an almost split sequence $\Sigma(X)$ in $\mathcal{F}(\Delta)$ starting in $X$:

$$\Sigma(X): 0\to X\to E(X)\to \tau_\Delta^- X\to 0.$$

**Lemma 4.4.** *For $M,N$ in $\mathcal{F}(\Delta)$ with $\underline{\dim}\,M=\underline{\dim}\,N$ and any $X\in\mathcal{NI}$ we have*

$$\delta_{M,N}(X)=\delta'_{M,N}(\tau_\Delta^- X).$$

*Proof.* We define two functions $\varphi,\psi:\mathcal{NI}\to\mathbb{N}_0$ as follows:

$$\begin{aligned}
\varphi(X) &:= \max\{\delta_{M,N}(X),0\},\\
\psi(X) &:= \max\{-\delta_{M,N}(X),0\}.
\end{aligned}$$

Consider the following two direct sums of almost split sequences:

$$\bigoplus_{X\in\mathcal{NI}}\Sigma(X)^{\varphi(X)}: 0\to U'\to W'\to V'\to 0,$$

$$\bigoplus_{X\in\mathcal{NI}}\Sigma(X)^{\psi(X)}: 0\to U''\to W''\to V''\to 0.$$

Applying Lemma 3.1 we obtain the equalities

$$\begin{aligned}
\varphi(X) &= [U'\oplus V',X]-[W',X]=[\tau_\Delta^- X,U'\oplus V']-[\tau_\Delta^- W'],\\
\psi(X) &= [U''\oplus V'',X]-[W'',X]=[\tau_\Delta^- X,U''\oplus V'']-[\tau_\Delta^- W'']
\end{aligned}$$

for any $X\in\mathcal{NI}$. Hence, we get the equality

$$\delta_{M,N}(X)=[N,X]-[M,X]=[U'\oplus W''\oplus V',X]-[U''\oplus W'\oplus V'',X],$$

for any $X\in\mathcal{NI}$. Since $\underline{\dim}(U'\oplus W''\oplus V')=\underline{\dim}(U''\oplus W'\oplus V'')$, using Remark 3.2, we see that

$$[N,T(i)]-[M,T(i)]=0=[U'\oplus W''\oplus V',T(i)]-[U''\oplus W'\oplus V'',T(i)]$$

holds for any $i=1,\dots,n$. Hence $[N\oplus U''\oplus W'\oplus V'',X]=[M\oplus U'\oplus W''\oplus V',X]$ holds for any indecomposable $X\in\mathcal{F}(\Delta)$, so also for any $X\in\mathcal{F}(\Delta)$. Applying Proposition 3.4 we obtain an isomorphism

$$N\oplus U''\oplus W'\oplus V''\simeq M\oplus U'\oplus W''\oplus V'.$$

Finally, this implies the desired equality:

$$\begin{aligned}
\delta_{M,N}(X) &= [N,X]-[M,X]=\varphi(X)-\psi(X)\\
&= ([\tau_\Delta^- X, U'\oplus V']-[\tau_\Delta^- X, W'])-([\tau_\Delta^- X, U''\oplus V'']-[\tau_\Delta^- X, W''])\\
&= [\tau_\Delta^- X, U'\oplus W''\oplus V']-[\tau_\Delta^- X, U''\oplus W'\oplus V'']\\
&= [\tau_\Delta^- X, N]-[\tau_\Delta^- X, M]=\delta'_{M,N}(\tau_\Delta^- X).
\end{aligned}$$

□

We derive the following two consequences:

**Corollary 4.5.** *Let $M, N$ be in $\mathcal{F}(\Delta)$ with $\underline{\dim}\, M = \underline{\dim}\, N$ and $N \geq_{\hom} M$. Then*

$$\sum_{X\in\mathcal{NI}} \delta_{M,N}(X) = \sum_{X\in\mathcal{NP}} \delta'_{M,N}(X).$$

**Corollary 4.6.** *Let $M, N$ be in $\mathcal{F}(\Delta)$ with $\underline{\dim}\, M = \underline{\dim}\, N$. Then the following statements are equivalent:*

(1) $\delta_{M,N}(X) \geq 0$*, for all* $X \in \mathcal{F}(\Delta)$*;*
(2) $\delta'_{M,N}(X) \geq 0$*, for all* $X \in \mathcal{F}(\Delta)$*.*

Corollary 4.6 simply states that the hom–order on $\mathcal{F}(\Delta)$ introduced in 3.5 can also be defined in a dual fashion.

**Lemma 4.7.** *Let $M$ and $N$ be in $\mathcal{F}(\Delta)$ with $\underline{\dim}\, M = \underline{\dim}\, N$ and $N \geq_{\hom} M$. Let $U$ be an indecomposable module in $\mathcal{F}(\Delta)$. Then we have*

$$\mu(N,U)-\mu(M,U) \leq \delta_{M,N}(U)+\delta'_{M,N}(U).$$

*Proof.* Assume first that $U$ is not Ext-injective in $\mathcal{F}(\Delta)$, i.e., $U \in \mathcal{NI}$. Then applying Lemmas 3.1 and 4.4 to the sequence $\Sigma(U)$ we get

$$\begin{aligned}
\mu(N,U)-\mu(M,U) &= ([U\oplus\tau_\Delta^- U, N]-[U\oplus\tau_\Delta^- U, M])-([E(U),N]-[E(U),M])\\
&\leq ([U,N]-[U,M])+([\tau_\Delta^- U,N]-[\tau_\Delta^- U,M])\\
&= ([U,N]-[U,M])+([N,U]-[M,U])\\
&= \delta_{M,N}(U)+\delta'_{M,N}(U).
\end{aligned}$$

By duality, the inequality also holds if $U$ is not Ext-projective in $\mathcal{F}(\Delta)$. Thus we may assume that $U$ is both Ext-projective and Ext-injective in $\mathcal{F}(\Delta)$. Since $\underline{\dim}\, M = \underline{\dim}\, N$, we see that $[N,U]=[M,U]$ and $[U,N]=[U,M]$ by Remark 3.2. Hence, it remains to show that $\mu(N,U)\leq\mu(M,U)$. Consider the following direct sum of almost split sequences

$$\bigoplus_{X\in\mathcal{NI}} \Sigma(X)^{(\delta_{M,N}(X))} : 0 \to U' \to W' \to V' \to 0.$$

As in the proof of Lemma 4.4 we derive an isomorphism $N\oplus W' \simeq M\oplus U'\oplus V'$. Since $U$ is both Ext-projective and Ext-injective in $\mathcal{F}(\Delta)$, we see that $U$ is not a direct summand of $U'\oplus V'$. Hence, we obtain the desired inequality

$$\mu(N,U)\leq\mu(N\oplus W',U)=\mu(M\oplus U'\oplus V',U)=\mu(M,U).$$

□

We require the following well-known fact.

**Lemma 4.8.** *Consider the following two short exact sequences in $\mathcal{A}$–mod:*

$$\Sigma_1 : \quad 0 \to M_1 \xrightarrow{\left[\begin{smallmatrix} u_1 \\ f_1 \end{smallmatrix}\right]} M_2 \oplus N_1 \xrightarrow{[f_2, u_2]} N_2 \to 0, \quad \textit{and}$$

$$\Sigma_2 : \quad 0 \to M_2 \xrightarrow{\left[\begin{smallmatrix} v_1 \\ f_2 \end{smallmatrix}\right]} M_3 \oplus N_2 \xrightarrow{[f_3, v_2]} N_3 \to 0.$$

*Then the sequence*

$$\Sigma_3 : \ 0 \to M_1 \xrightarrow{\left[\begin{smallmatrix} v_1 u_1 \\ f_1 \end{smallmatrix}\right]} M_3 \oplus N_1 \xrightarrow{[\, f_3, \, -v_2 u_2 \,]} N_3 \to 0$$

*is also exact. Moreover, we have $\delta_{\Sigma_3} = \delta_{\Sigma_1} + \delta_{\Sigma_2}$.*

**Lemma 4.9.** *Let $\Sigma : 0 \to U \xrightarrow{f} W \to V \to 0$ be a nonsplittable exact sequence. Then $\delta_\Sigma(U) > 0$ and $\delta'_\Sigma(V) > 0$.*

*Proof.* The sequence $\Sigma$ induces an exact sequence

$$0 \to \operatorname{Hom}(V, U) \to \operatorname{Hom}(W, U) \xrightarrow{f^*} \operatorname{Hom}(U, U).$$

Assume that $f^*$ is an epimorphism. Then there is a homomorphism of $\mathcal{A}$–modules $h : W \to U$ such that $f^*(h) = hf = 1_U$. But then $f \notin \operatorname{rad}(U, W)$. This is a contradiction. Therefore, we must have $[V, U] - [W, U] + [U, U] > 0$, and consequently $\delta_\Sigma(U) > 0$. By duality we also obtain $\delta'_\Sigma(V) > 0$. $\square$

**Lemma 4.10.** *If $N < M$, then $\delta_{M,N}(N) > 0$ and $\delta'_{M,N}(N) > 0$.*

*Proof.* Suppose that $\delta'_{M,N}(N) = 0$. By Theorem 2.4 in [1], we know that if a module $U$ embeds into $N$ and $[U, N] = [U, M]$, then $U$ also embeds into M. Applying this fact to $U = N$, we see that $N$ embeds into $M$. But the modules $M$ and $N$ have the same dimension, and thus they are isomorphic. This contradicts the assumption that $N < M$. Hence, we must have $\delta'_{M,N}(N) > 0$ and $\delta_{M,N}(N) > 0$ by duality, as desired. $\square$

**Lemma 4.11.** *Let $M, N \in \mathcal{F}(\Delta)$ with $N >_{\hom} M$ and $\Sigma : 0 \to U \to W \to V \to 0$ an exact sequence in $\mathcal{F}(\Delta)$ such that $\delta_\Sigma \leq \delta_{M,N}$. Then there exists a short exact sequence $\Phi : 0 \to U \to Y \to Z \to 0$ in $\mathcal{F}(\Delta)$ such that $\delta_\Sigma \leq \delta_\Phi \leq \delta_{M,N}$ and $\delta_\Phi(Y) = \delta_{M,N}(Y)$.*

*Proof.* Let $\Sigma$ be as in the statement of the lemma. We choose a short exact sequence $\Phi : 0 \to U \to Y \xrightarrow{f} Z \to 0$ in $\mathcal{F}(\Delta)$ satisfying $\delta_\Sigma \leq \delta_\Phi \leq \delta_{M,N}$ such that $\delta_\Phi$ is maximal with respect to these inequalities. Since $\sum_{X \in \mathcal{NI}} \delta_{M,N}(X)$ is finite, such a sequence $\Phi$ exists. After splitting off direct summands of $\Phi$ of the form $0 \to 0 \to D \xrightarrow{\sim} D \to 0$, we may assume that $f \in \operatorname{rad}(Y, Z)$. Assume now that $Y = Y_1 \oplus Y_2$, where $Y_1$ is indecomposable with $\delta_\Phi(Y_1) < \delta_{M,N}(Y_1)$. Then $Y_1$ is non-Ext-injective, i.e., $Y_1 \in \mathcal{NI}$. Then there is an almost split sequence in $\mathcal{F}(\Delta)$ of the form

$$\Sigma(Y_1) : 0 \to Y_1 \xrightarrow{h} E(Y_1) \to \tau_\Delta^- Y_1 \to 0,$$

and of course we may realize $\Phi$ as

$$\Phi : 0 \to U \to Y_1 \oplus Y_2 \xrightarrow{f = [f_1, f_2]} Z \to 0.$$

Since $f_1 \in \operatorname{rad}(Y_1, Z)$, the push out of the almost split sequence $\Sigma(Y_1)$ is a splittable sequence. We obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \to & Y_1 & \xrightarrow{h} & E & \longrightarrow & \tau_\Delta^- Y_1 & \to 0 \\ & \downarrow f_1 & & \downarrow & & \| & \\ 0 \to & Z & \longrightarrow & \tau_\Delta^- Y_1 \oplus Z & \longrightarrow & \tau_\Delta^- Y_1 & \to 0. \end{array}$$

This implies that there exists a nonsplittable exact sequence in $\mathcal{F}(\Delta)$

$$\Psi : 0 \to Y_1 \xrightarrow{\left[\begin{smallmatrix} h \\ f_1 \end{smallmatrix}\right]} E \oplus Z \to \tau_\Delta^- Y_1 \oplus Z \to 0.$$

Applying Lemma 4.8 to $\Phi$ and $\Psi$ results in a new exact sequence in $\mathcal{F}(\Delta)$

$$\Theta : 0 \to U \to Y_2 \oplus E \to Z \oplus \tau_\Delta^- Y_1 \to 0.$$

Thus, by Lemmas 3.1 and 4.8, we see that for any $X \in \mathcal{F}(\Delta)$ we have

$$\delta_\Theta(X) = \delta_\Phi(X) + \delta_\Psi(X) = \delta_\Phi(X) + \delta_{\Sigma(Y_1)}(X) = \delta_\Phi(X) + \mu(X, Y_1).$$

Since $\delta_\Phi \leq \delta_{M,N}$ and $\delta_\Phi(Y_1) \leq \delta_{M,N}(Y_1) - 1$, we get $\delta_\Sigma \leq \delta_\Theta \leq \delta_{M,N}$. This contradicts our choice of the sequence $\Phi$. Hence, $\delta_\Phi(Y) = \delta_{M,N}(Y)$. This completes the proof of the lemma. □

**Lemma 4.12.** *If $N >_{\hom} M$, then $\delta_{M,N}(N) > 0$ and $\delta'_{M,N}(N) > 0$.*

*Proof.* We argue by induction on $\sum_{X \in \mathcal{NI}} \delta_{M,N}(X) > 0$. By Corollary 4.5 we have $\sum_{X \in \mathcal{NI}} \delta_{M,N}(X) = \sum_{X \in \mathcal{NP}} \delta'_{M,N}(X)$. Assume that $N >_{\hom} M$ and that $\delta_{M,N}(N) = 0$ or $\delta'_{M,N}(N) = 0$. By duality, we may assume that $\delta'_{M,N}(N) = 0$ and moreover, the modules $M$ and $N$ have no nonzero common direct summand. Consider an indecomposable direct summand $Y$ of $N$. By Lemma 4.7, we get

$$\mu(N, Y) = \mu(N, Y) - \mu(M, Y) \leq \delta_{M,N}(Y) + \delta'_{M,N}(Y) = \delta_{M,N}(Y).$$

So, the module $Y$ is not Ext-injective and consequently, $Y \in \mathcal{NI}$. Then there is an almost split sequence $\Sigma(Y)$ in $\mathcal{F}(\Delta)$. We define a new exact sequence in $\mathcal{F}(\Delta)$

$$\Sigma : 0 \to N \to E(N) \to \tau_\Delta^- N \to 0,$$

where $E(N) = \bigoplus_{Y \in \mathcal{NI}} E(Y)^{\mu(N,Y)}$ and $\tau_\Delta^- N = \bigoplus_{Y \in \mathcal{NI}} (\tau_\Delta^- Y)^{\mu(N,Y)}$. Applying Lemma 3.1, we obtain $\delta_\Sigma(Y) = \mu(N, Y) \leq \delta_{M,N}(Y)$, for any $Y \in \mathcal{NI}$. Consequently, $\delta_\Sigma \leq \delta_{M,N}$ and, by Lemma 4.11, there is an exact sequence

$$\Phi : 0 \to N \to W \to V \to 0$$

with $\delta_\Phi \leq \delta_{M,N}$ and $\delta_\Phi(W) = \delta_{M,N}(W)$. Then $M \oplus V \leq W$ and $\delta_{M \oplus V, W}(W) = 0$. Observe, that $\delta_{M,N} - \delta_{M \oplus V, W} = \delta_\Phi$ and, according to Lemma 4.9, we infer that $\delta_\Phi(N) > 0$. This leads to the inequality

$$\sum_{X \in \mathcal{NI}} \delta_{M \oplus V, W}(X) < \sum_{X \in \mathcal{NI}} \delta_{M,N}(X).$$

It follows from our inductive assumption that the modules $M \oplus V$ and $W$ are isomorphic. Thus, the sequence $\Phi$ has the form

$$\Phi : 0 \to N \to V \oplus M \to V \to 0,$$

and this implies that $N < M$, by Proposition 3.4 in [8]. Applying Lemma 4.10, we get $\delta'_{M,N}(N) > 0$, contradicting our assumption that $\delta'_{M,N}(N) = 0$. This completes the proof of the lemma. □

**Proof of Proposition 4.2.** Let $M, N \in \mathcal{F}(\Delta)$ with $N \geq_{\text{hom}} M$. We may assume that $N >_{\text{hom}} M$. Let $r(X) := \min\{\delta_{M,N}(X), \mu(N,X)\}$ for any indecomposable $X$ in $\mathcal{F}(\Delta)$. Let $\mathcal{I}$ be a set of representatives of all isomorphism classes of indecomposable modules in $\mathcal{F}(\Delta)$ and let $\mathcal{S} := \{X \in \mathcal{I} \mid r(X) > 0\}$. The set $\mathcal{S}$ does not contain Ext-injective modules from $\mathcal{F}(\Delta)$ and consequently, $\mathcal{S} \subseteq \mathcal{NI}$. Moreover, $\mathcal{S}$ is nonempty, by Lemma 4.12. Let $N' = \bigoplus_{X\in\mathcal{S}} X^{r(X)} = \bigoplus_{X\in\mathcal{I}} X^{r(X)}$ and $N'' = \bigoplus_{X\in\mathcal{I}} X^{\mu(N,X)-r(X)}$. Then $N = N' \oplus N''$. We define a new nonsplittable exact sequence as follows:

$$\Sigma : 0 \to \bigoplus_{X\in\mathcal{S}} X^{r(X)} \to \bigoplus_{X\in\mathcal{S}} E(X)^{r(X)} \to \bigoplus_{X\in\mathcal{S}} (\tau_\Delta^- X)^{r(X)} \to 0.$$

Applying Lemma 3.1 we obtain $\delta_\Sigma(X) = r(X) \leq \delta_{M,N}(X)$ for any indecomposable module $X$. Consequently, $\delta_\Sigma \leq \delta_{M,N}$ and, by Lemma 4.11, there is a nonsplittable exact sequence $\Phi : 0 \to N' \to W \to Z \to 0$ in $\mathcal{F}(\Delta)$ with $\delta_\Sigma \leq \delta_\Phi \leq \delta_{M,N}$ and $\delta_\Phi(W) = \delta_{M,N}(W)$. Then we have $N'' \oplus W \leq M \oplus Z$ and $\delta_{M\oplus Z, N''\oplus W}(W) = 0$. Let $N_1$ be any indecomposable direct summand of $N''$. Then $r(N_1) < \mu(N, N_1)$, and this leads to the equality $\delta_\Sigma(N_1) = r(N_1) = \delta_{M,N}(N_1)$. Hence we get

$$\delta_{M\oplus Z, N''\oplus W}(N_1) = \delta_{M,N}(N_1) - \delta_\Phi(N_1) = \delta_\Sigma(N_1) - \delta_\Phi(N_1) \leq 0.$$

So, $\delta_{M\oplus Z, N''\oplus W}(N_1) = 0$. This implies that $\delta_{M\oplus Z, N''\oplus W}(N'') = 0$ and furthermore, $\delta_{M\oplus Z, N''\oplus W}(N'' \oplus W) = 0$. Hence, we have $M \oplus Z \simeq N'' \oplus W$, by Lemma 4.12. Finally, $\Phi$ induces an exact sequence $0 \to N' \oplus N'' \to N'' \oplus W \to Z \to 0$, which has the form $0 \to N \to M \oplus Z \to Z \to 0$. In a similar fashion we obtain an exact sequence $0 \to Z' \to M \oplus Z' \to N \to 0$. This completes the proof of the proposition. $\square$

Finally, with Theorem 4.1 we have also completed the proof of Theorem 1.1.

## 5. Globally Minimal Degenerations of $\Delta$–filtered modules

Throughout this section, $\mathcal{A}$ is a $\Delta$–finite quasi-hereditary algebra. We maintain the notation from Section 3 and utilize the results of the previous sections.

Here we define the notion of a global minimal degeneration, virtual representations, and the associated Hasse diagram $\mathcal{H}$. Inside $\mathcal{H}$ each connected component $\mathcal{H}(\mathbf{e})$ of $\mathcal{H}$ contains the subposet $\mathcal{H}^+(\mathbf{e})$ of real representations for each dimension vector $\mathbf{e}$. These latter posets are precisely the ones we described in Theorem 1.1.

A *virtual representation* is a formal $\mathbb{Z}$-linear combination $\sum a_i I_i$, where $I_i$ runs through a complete set of representatives of indecomposable $\mathcal{A}$–modules in $\mathcal{F}(\Delta)$. The set of virtual representations admits a natural order given by the hom–order extended to virtual representations: so we write $\sum a_i I_i \leq \sum b_i I_i$ provided we have $\sum_i a_i \dim \operatorname{Hom}(I_i, I_j) \leq \sum_i b_i \dim \operatorname{Hom}(I_i, I_j)$ for each $I_j$. We denote the corresponding Hasse diagram of virtual representations by $\mathcal{H}$. Inside $\mathcal{H}$ we have convex subposets $\mathcal{H}(\mathbf{e})$ of virtual representations for each dimension vector $\mathbf{e}$; i.e., of those virtual representations $\sum a_i I_i$ with $\sum a_i \underline{\dim}\, I_i = \mathbf{e}$.

All these Hasse diagrams $\mathcal{H}(\mathbf{e})$ are isomorphic to each other for the various $\mathbf{e}$'s; in fact, $X \mapsto X \oplus Z$ defines an isomorphism of $\mathcal{H}(\mathbf{e})$ with $\mathcal{H}(\mathbf{e} + \underline{\dim}\, Z)$. Consequently, the group of virtual representations (whose group structure is given by taking direct sums) acts simply transitive on $\mathcal{H}$. In addition, since each module degenerates to the unique semisimple one with the same dimension vector, these posets $\mathcal{H}(\mathbf{e})$ are just the connected components of $\mathcal{H}$.

**Definition 5.1.** A degeneration $M$ to $N$ is called *minimal* provided $M$ and $N$ are adjacent in the associated Hasse diagram; it is called *globally minimal* provided $M$ and $N$ have no common direct summand and $M \oplus Z$ degenerates minimally to $N \oplus Z$ for any representation $Z$.

Note that these notions apply for a Bruhat-Chevalley poset, as well as a poset given by the hom–order.

The next result describes the globally minimal degenerations in terms of the Auslander-Reiten quiver of the category $\mathcal{F}(\Delta)$.

**Theorem 5.2.** *Let $\mathcal{A}$ be a $\Delta$–finite quasi-hereditary algebra and let $M$ and $N$ be in $\mathcal{F}(\Delta)$. Then $M$ degenerates globally minimal to $N$ precisely when there exists an Auslander-Reiten sequence*

$$0 \to N_1 \to M \to N_2 \to 0$$

*in $\mathcal{F}(\Delta)$ such that $N_1 \oplus N_2$ is isomorphic to $N$.*

*Moreover, any minimal degeneration $Y >_{\mathrm{hom}} Z$ in $\mathcal{H}$ is of the form $Y = N \oplus X$ and $Z = M \oplus X$ for some virtual representation $X$ and some globally minimal degeneration $N >_{\mathrm{hom}} M$. Thus we obtain a bijection between the set of pairs consisting of an Auslander-Reiten sequence (up to isomorphism) and a virtual representation and the set of all pairs consisting of adjacencies in $\mathcal{H}$, i.e., the set of all minimal degenerations for all dimension vectors, via*

$$(0 \to N_1 \to M \to N_2 \to 0, X) \mapsto (N_1 \oplus N_2 \oplus X, M \oplus X).$$

*Proof.* By definition of an Auslander-Reiten sequence a globally minimal Hom-degeneration $M$ to $N$ corresponds to an AR-sequence as stated in the theorem. By Theorem 4.1 the geometric and Hom-degenerations correspond. Thus the first claim follows. Given a minimal degeneration $Y >_{\mathrm{hom}} Z$ in $\mathcal{H}$, we can add a virtual representation $X$ such that $N = Y \oplus X$ and $M = Z \oplus X$ are representations without any common direct summand. Since $Y >_{\mathrm{hom}} Z$ is minimal, so is $N >_{\mathrm{hom}} M$ and by construction this degeneration is also globally minimal. $\square$

Observe that, according to Theorem 5.2, for a complete description of the Hasse diagrams $\mathcal{H}(\mathbf{e})$ for fixed $\mathbf{e}$ inside $\mathcal{H}$ we have to compute the Auslander-Reiten quiver of $\mathcal{F}(\Delta)$ in each instance. This is accomplished in the next section.

We may describe $\mathcal{H}$ abstractly as follows: Each component $\mathcal{H}(\mathbf{e})$ is isomorphic to the *lattice poset* of rank $r - t$, where $r$ is the number of isomorphism classes of indecomposable objects in $\mathcal{F}(\Delta)$ and $t$ is the number of isomorphism classes of simple modules in $\mathcal{F}(\Delta)$, or equivalently of the injective indecomposable modules in $\mathcal{F}(\Delta)$. That is, $\mathcal{H}(\mathbf{e}) \simeq \mathbb{Z}^{r-t}$ with partial order $(a_i) \leq (b_i)$ whenever $a_i \leq b_i$ for all $i = 1, \dots, r-t$. This isomorphism is given by $X \mapsto \underline{\mathrm{hdim}}(X)_{\mathcal{NI}}$, where in forming this $(r-t)$-tuple we only compute homomorphism spaces with respect to indecomposables in $\mathcal{NI}$, i.e., ones which are *not* injective. Observe that each component $\mathcal{H}(\mathbf{e})$ is isomorphic to the Cayley graph of the free abelian group on $r-t$ generators associated to the standard basis vectors in $\mathbb{Q}^{r-t}$.

The main result of this section concerning the particular focus of this paper, namely when $\mathcal{A}$ is the path algebra $\mathcal{A}(t,l)$ from Section 2 above relates the Hasse diagrams $\mathcal{H}(\mathbf{e})$ to the Hasse diagrams of the Bruhat-Chevalley order on the set of orbits of the parabolic groups $P$ on $\mathfrak{p}_u^{(l)}$, where $P = P(\mathbf{d})$ and $\mathbf{e} = \Sigma\mathbf{d}$.

Inside each $\mathcal{H}(\mathbf{e})$ is the full subposet $\mathcal{H}^+(\mathbf{e})$ consisting of all *real* representations, i.e., the ones where all coefficients $a_i$ are non-negative. This subposet need *not* be convex, there may be minimal degenerations in $\mathcal{H}^+(\mathbf{e})$ which are not minimal in $\mathcal{H}(\mathbf{e})$. The posets $\mathcal{H}^+(\mathbf{e})$ are precisely the ones we study in the previous sections.

**Proposition 5.3.** *Fix $t$ and $l$ so that $\mathcal{A}(t,l)$ is $\Delta$–finite. Let $\mathbf{d}$ be in $\mathbb{N}^t$ and let $V$ be a finite-dimensional $k$-vector space such that* $\dim V = \sum d_i$. *Let $P = P(\mathbf{d})$ be the standard parabolic subgroup of* $\mathrm{GL}(V)$ *associated to* $\mathbf{d}$. *Then the Hasse diagram $\mathcal{H}^+(\Sigma\mathbf{d})$ for $\mathcal{F}(\Delta)$ is opposite to the Hasse diagram of the Bruhat-Chevalley order of the action of $P$ on $\mathfrak{p}_u^{(l)}$, where $P = P(\mathbf{d})$.*

*Proof.* Let $\mathbf{e} = \Sigma\mathbf{d}$. By definition of $\mathcal{H}(\mathbf{e})$ the subposet $\mathcal{H}^+(\mathbf{e})$ is exactly the Hasse diagram of the isomorphism classes of modules of dimension vector $\mathbf{e}$ with respect to the hom–order. By our main result Theorem 1.1, the hom–order is opposite to the Bruhat-Chevalley order of the action of $P$ on $\mathfrak{p}_u^{(l)}$ for $P = P(\mathbf{d})$. $\square$

## 6. Explicit calculations

We discuss further consequences of the results above for our setting of $\mathcal{F}(t,l)$, or the associated parabolic actions. Let $\mathcal{I}(t,l)$ be a complete set of representatives of isomorphism classes of indecomposable objects in $\mathcal{F}(t,l)$. According to [2] and [6], $\mathcal{I}(t,l)$ is a finite set precisely when either $l = 0$ and $t \leq 5$, or $l \geq 1$ and $t \leq 6 + 2l$. By $I_j$ we denote the $j$-th member of $\mathcal{I}(t,l)$ for $1 \leq j \leq r = |\mathcal{I}(t,l)|$. By Remark 2.4 $\mathcal{F}(t,l)$ is a Krull-Schmidt category, thus each object in $\mathcal{F}(t,l)$ has a unique decomposition as a direct sum of indecomposable ones (up to the order of the summands). The dimension vectors $\underline{\dim}\, I_j$ of the indecomposables $I_j$ are exactly the ones shown in the AR-quiver (see Figures 8.1, 8.4, and 8.7).

Now we present a formula for the number of orbits in all the finite instances. The proof follows at once from the results in [2], [6], and Lemma 2.1.

**Corollary 6.1.** *Fix $t$ and $l$ so that $\mathcal{A}(t,l)$ is $\Delta$–finite. Let $\mathbf{d}$ be in $\mathbb{N}^t$ and let $V$ be a finite-dimensional $k$-vector space such that* $\dim V = \sum d_i$. *Let $P = P(\mathbf{d})$ be the standard parabolic subgroup of* $\mathrm{GL}(V)$ *associated to* $\mathbf{d}$. *Set $r = |\mathcal{I}(t,l)|$. Then the number of $P$-orbits on $\mathfrak{p}_u^{(l)}$ equals the number of tuples $(a_1, \dots, a_r) \in \mathbb{N}_0^r$ such that*

$$\mathbf{d} = \sum_{i=1}^{r} a_i \,\underline{\dim}\, I_i.$$

*Remark 6.2.* Consider the case when $l = 0$, $t \leq 5$, and $\mathbf{d} \in \mathbb{N}^t$. Let $V$ be a $k$-vector space such that $\dim V = \sum d_i$. Let $\sigma$ be a permutation of $\{1, \dots, t\}$ and let $\mathbf{d}' := \sigma\mathbf{d}$ be the permutation of the $t$-tuple $\mathbf{d}$ by $\sigma$. Let $P = P(\mathbf{d})$ and $P' = P(\mathbf{d}')$ be the two parabolic subgroups in $\mathrm{GL}(V)$ associated to $\mathbf{d}$ and $\mathbf{d}'$, respectively. It was observed in [6, Cor. 4.7] that the number of orbits of $P$ on $\mathfrak{p}_u$ equals that of $P'$ on $\mathfrak{p}'_u$; independent of whether $P$ and $P'$ are conjugate in $\mathrm{GL}(V)$, they are conjugate precisely when $\mathbf{d} = \mathbf{d}'$, i.e., when $P = P'$. Although the number of orbits is the same, the Bruhat-Chevalley posets of these actions may be different. In Appendix A we illustrate this phenomenon by a small example (Figures 7.5 and 7.6).

For every fixed $l$, $t$, and $\mathbf{d} \in \mathbb{N}^t$, in each finite instance the data consisting of the orbits, their dimensions, and the incidence structure among them, i.e., the Hasse diagram, can be generated by machine calculations based on the formula from Corollary 6.1 and the $r \times r$-matrix $\mathbf{D} := (\dim \operatorname{Hom}(I_i, I_j))_{i,j}$. Given an $r$-tuple $\mathbf{a} := (a_1, \dots, a_r)$ from Corollary 6.1 such that $\mathbf{d} = \sum a_i \,\underline{\dim}\, I_i$, let $X_{\mathbf{a}}$ be the associated object in $\mathcal{F}(t, l)$, i.e., $X_{\mathbf{a}} := \sum a_i I_i$. Then we readily obtain the $r$-vector $\underline{\mathrm{hdim}}(X_{\mathbf{a}})$ whose $j$-th entry consists of $\dim \operatorname{Hom}(X_{\mathbf{a}}, I_j)$ simply by $\underline{\mathrm{hdim}}(X_{\mathbf{a}}) = \mathbf{a} \cdot \mathbf{D}$. Comparing these $r$-vectors in the resulting finite set obtained from all $r$-tuples $\mathbf{a}$ from Corollary 6.1 then allows us to determine the exact poset structure of the hom–order. And by Theorem 1.1 we obtain the desired poset of the Bruhat-Chevalley order simply by taking the opposite of the hom–poset. Our examples in Appendix A were generated in this fashion.

In order to compute the Auslander-Reiten quivers we require further details from the proof that $\mathcal{A}(t, l)$ is $\Delta$–finite precisely when $l$ and $t$ satisfy the conditions above. Following [2], we define an algebra $\mathcal{B}(t, l)$ as the quotient of $\mathcal{A}(t, l)$ by the ideal generated by the element $\alpha_{t-1}\beta_{t-l-1}$. Set $s := t - l - 1$. We may realize the algebra $\mathcal{B}(s, l)$ as the quotient of $\mathcal{A}(t, l)$ by the ideal generated by the last $l + 1$ vertices; that is the quiver of $\mathcal{B}(s, l)$ is obtained from the quiver of $\mathcal{A}(t, l)$ by deleting the last $l + 1$ vertices along with all arrows connected to these vertices and by taking the induced relations. In particular, the quiver of $\mathcal{B}(s, l)$ has again $s = t - l - 1$ vertices. The category of $\mathcal{B}(s, l)$–modules is equivalent to the one of all $\Delta$–filtered $\mathcal{A}(t, l)$–modules modulo morphisms which factor over the tilting module $T$; [3], [2].

Consequently, we can compute the AR-quiver of the category of $\Delta$–filtered $\mathcal{A}(t, l)$–modules by taking the Auslander-Reiten quiver of $\mathcal{B}(s, l)$ and inserting the direct summands of the tilting module $T$ at the appropriate places: these are readily determined, as some exact sequences in $\mathcal{B}(s, l)$–mod are not exact when viewed in $\mathcal{A}(t, l)$–mod. The direct summands of $T$ have to be inserted in such a way that the resulting AR-sequences are exact in $\mathcal{A}(t, l)$–mod. The appropriate spots in the AR-quivers in Appendix B are indicated by $*$'s. Unfortunately, we cannot compute the Auslander-Reiten quiver of $\mathcal{B}(s, l)$ directly; we have to pass to a $\mathbb{Z}$–covering of $\mathcal{B}(s, l)$ instead (see [2]). It is straightforward to compute the Auslander-Reiten quiver in that covering by the standard technique of *knitting* [9] (Figures 8.2, 8.5).

Observe that the Auslander-Reiten quivers for $l \geq 2$ are quite similar, they differ only in the part which is enclosed in dashed lines. The modules $M$ in this region have the property that all the maps $M_{\beta_j}$ are equal to zero. These representations have a particularly transparent structure. There are exactly $(t - l)(t - l - 1)/2$ indecomposable modules $M$ where all maps $M_\beta$ are equal to 0, and all but 12 of them lie in the dashed region. Since the apparent symmetry of $\mathcal{B}(s, l)$ and its covering translates to a similar one for the associated AR-quiver, we only present one half of the AR-quiver (Figures 8.2, 8.3, 8.5, and 8.6). We also compute the top and the socle of the modules in the AR-quiver (Figures 8.3 and 8.6).

Finally, we compute the Auslander-Reiten quivers of $\mathcal{A}(t, l)$ for the extremal finite cases, i.e., for $l = 0$ and $t = 5$ ([3], Figure 8.1), and for $l \geq 1$ and $t = 6 + 2l$ (Figures 8.4 and 8.7). Observe that it is enough to compute the Hasse diagrams for these extremal finite cases, as for a smaller value of $t$ we can canonically embed the corresponding smaller dimensional Lie algebra in the larger one by extending the representation trivially. The Auslander-Reiten quiver for $l = 0$ and $t \leq 5$ originally occur in [3]. Observe that in Appendix B we present the dimension vector $\mathbf{d}$ for $\mathcal{F}(t, l)$ rather than $\Sigma\mathbf{d}$ for $\mathcal{F}(\Delta)$.

It remains to compute normal forms for the indecomposable representations corresponding to each dimension vector occurring in the AR-quivers computed above. Most of them have a standard form: let $\mathbf{d}$ be a *thin* dimension vector (i.e., $d_i \leq 1$ for all $i = 1, \dots, t$) and assume that $d_j = 0$ for each $j = i-l, \dots, i-1, i+1, \dots, i+l$ whenever $d_i = 1$. Then we can define an element $f(\mathbf{d}) \in \mathfrak{p}_u^{(l)}$ via $f(\mathbf{d})_{i,j} = 1$, provided $d_i = d_j = 1$ and there is no such index between $i$ and $j$. This element defines an indecomposable object in $\mathcal{F}(t,l)$ of dimension vector $\mathbf{d}$. Note that these particular thin dimension vectors may appear more than once for $t \geq 4$ (e.g., see Figure 8.1: here $(1,1,1,1,1)$ appears five times). If necessary, we denote the standard representation by "$(s)$" and enumerate the other ones accordingly. It remains to compute normal forms for the non-standard representations. Note that such a normal form need not be unique, even in the case of a minimal number of non-zero entries (see Figure 4.1). Furthermore, observe that non-standard representations may arise from smaller dimension vectors by canonical embeddings, i.e., we extend the dimension vector with zero entries. The resulting endomorphism is the same as the original one, only that it now lies formally in a category $\mathcal{F}(t',l)$ associated to a larger parameter $t'$. We always present the normal forms for the minimal $t$ (e.g., see Figure 8.4); else, if a dimension vector arises from one of smaller length, i.e., if the corresponding indecomposable object is *induced by* one with a smaller dimension vector, we indicate this explicitly.

In our final result of this section we present the number $r$ of isomorphism classes of indecomposable modules in $\mathcal{F}(\Delta)$ in $\mathcal{A}(t,l)$–mod in the extremal finite cases. Thus we also obtain the maximal number of "lower" and "upper" adjacencies of a vertex in the Hasse diagram $\mathcal{H}(\mathbf{e})$; as this number equals $r-t$ (see the description of $\mathcal{H}(\mathbf{e})$ in the previous section).

**Proposition 6.3.** *The number of indecomposable objects in $\mathcal{F}(t,l)$ in each of the extremal finite cases is given in the following table:*

| $l$ | $t$ | $r = \lvert\mathcal{I}(t,l)\rvert$ |
|---|---|---|
| 0 | 5 | 45 |
| 1 | 8 | 99 |
| 2 | 10 | 95 |
| $l \geq 3$ | $6+2l$ | $56 + t + (t-l-1)(t-l)/2$ |

*Proof.* We obtain these numbers from the Auslander-Reiten quivers (Figures 8.1, 8.4, and 8.7). In the first three instances we simply count the number of indecomposable objects directly. Only the formula for $l \geq 3$ requires an argument: the number of indecomposable representations in the dashed region equals $(t-l)(t-l-1)/2-12$, the number of indecomposable direct summands of the tilting module $T$ equals $t$, and the number of the remaining ones is always 68. □

## 7. Appendix A: Some examples of Hasse Diagrams

In this section we present several examples of Hasse diagrams of the Bruhat-Chevalley order of the action of $P$ on $\mathfrak{p}_u^{(l)}$ in some finite cases. Each individual poset was computed by the method outlined in Section 6.

In the figures below the vertices indicate the orbits, the labels give their dimensions, and the edges represent the minimal degenerations in the sense of 5.1.

The Hasse diagrams associated to the Bruhat-Chevalley poset of the actions of the Borel subgroup $B$ on $\mathfrak{b}_u$ in the finite instances already appeared in [7]. In our first three examples we present these cases for $\mathrm{GL}_3(k)$, $\mathrm{GL}_4(k)$, and $\mathrm{GL}_5(k)$; there are 5, 16, and 61 orbits, respectively (Figures 7.1 - 7.3). In Figure 7.4 we show the poset of the action of the parabolic subgroup $P(\mathbf{d})$ in $\mathrm{GL}_{10}(k)$ on $\mathfrak{p}_u(\mathbf{d})$, where $\mathbf{d} = (1,2,3,4)$. There are 151 orbits. Finally, we consider some instances when $l = 1$ (Figures 7.7 - 7.9).

We observe that if $P = B$ is the Borel subgroup in our examples, i.e., when $\mathbf{d} = (1, \dots, 1)$, then the posets of the Bruhat-Chevalley order of the action of $B$ on $\mathfrak{b}_u^{(l)}$ are *ranked*, in fact they are even *rank unimodal* and the length of a maximal chain equals $\dim \mathfrak{b}_u^{(l)}$ (Figures 7.1 - 7.3 and 7.8, 7.9). Our other examples show that this does not hold in general.

In Figures 7.5 and 7.6 we present an instance of the phenomenon described in Remark 6.2. There we consider two non-conjugate parabolic subgroups $P(\mathbf{d})$ in $\mathrm{GL}_5(k)$. While in the case $\mathbf{d} = (2,1,2)$, the poset is rank unimodal, its counterpart for the permuted tuple $\mathbf{d} = (2,2,1)$ is not even ranked. This example illustrates that, although here we only interchange two adjacent coefficients, the resulting posets are already quite different even for such rather small dimension vectors.

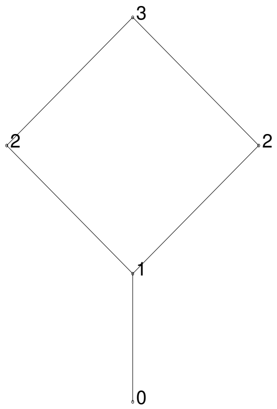



Figure 7.1: The Bruhat-Chevalley poset of the action of the Borel subgroup $B$ in $\mathrm{GL}_3(k)$ on $\mathfrak{b}_u$; here $\mathbf{d} = (1, 1, 1)$.

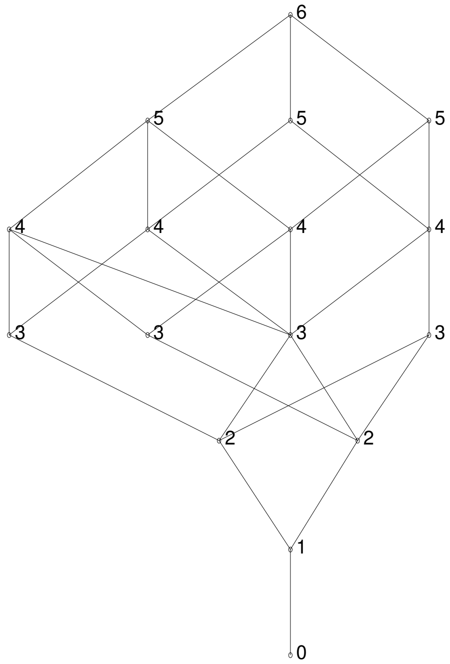



Figure 7.2: The Bruhat-Chevalley poset of the action of the Borel subgroup $B$ in $\mathrm{GL}_4(k)$ on $\mathfrak{b}_u$; here $\mathbf{d} = (1, 1, 1, 1)$.

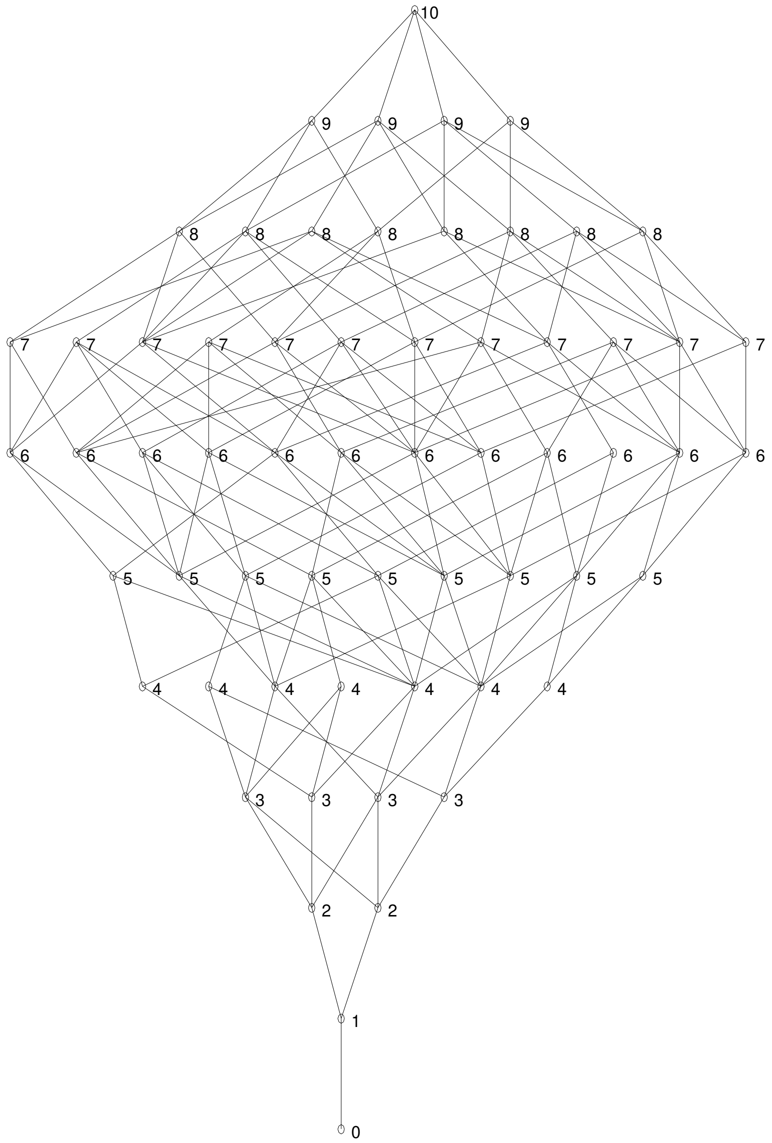



Figure 7.3: The Bruhat-Chevalley poset of the action of the Borel subgroup $B$ in $\mathrm{GL}_5(k)$ on $\mathfrak{b}_u$, i.e., here $\mathbf{d} = (1,1,1,1,1)$.

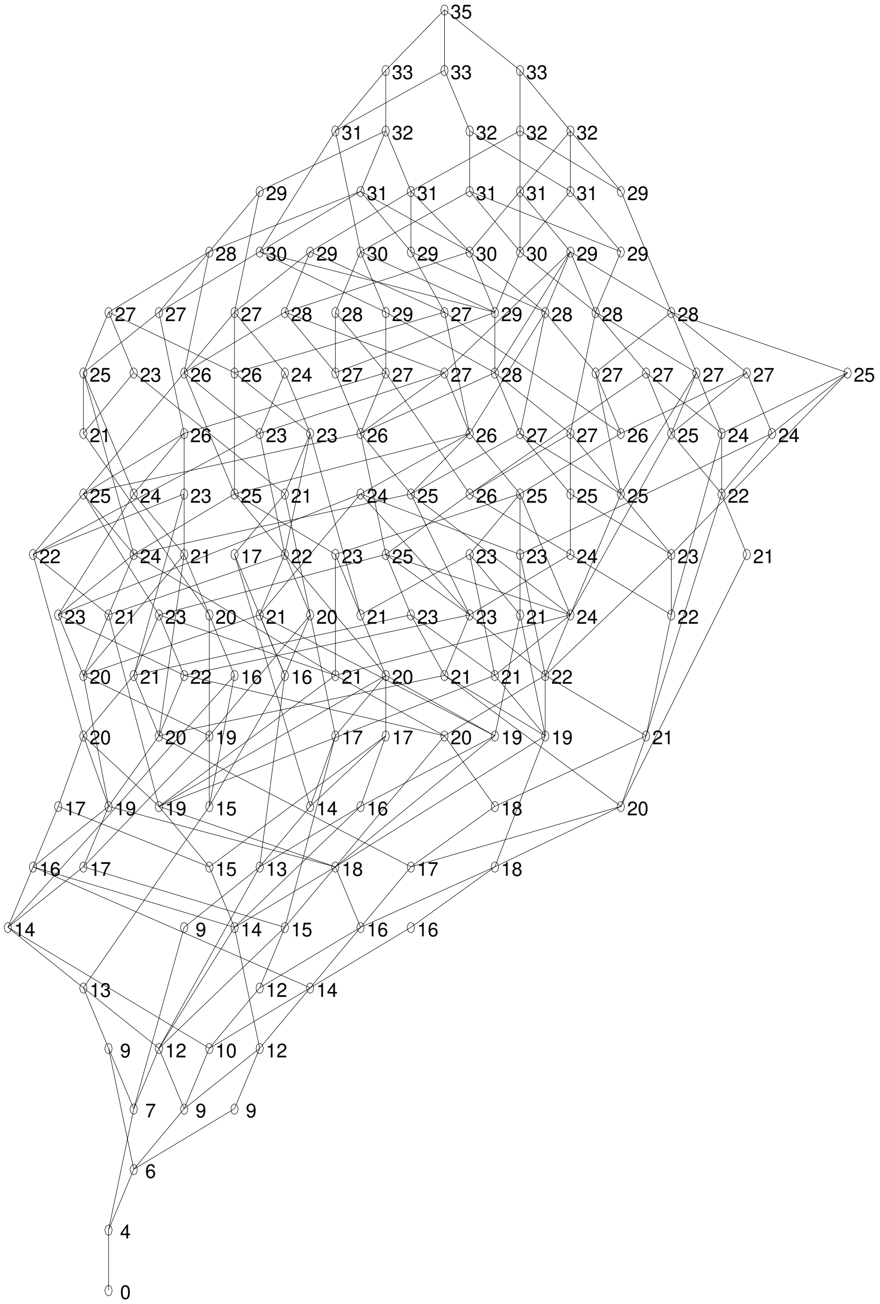



Figure 7.4: The Bruhat-Chevalley poset of the action of the parabolic subgroup $P(\mathbf{d})$ in $\mathrm{GL}_{10}(k)$ on $\mathfrak{p}_u(\mathbf{d})$, where $\mathbf{d} = (1, 2, 3, 4)$.

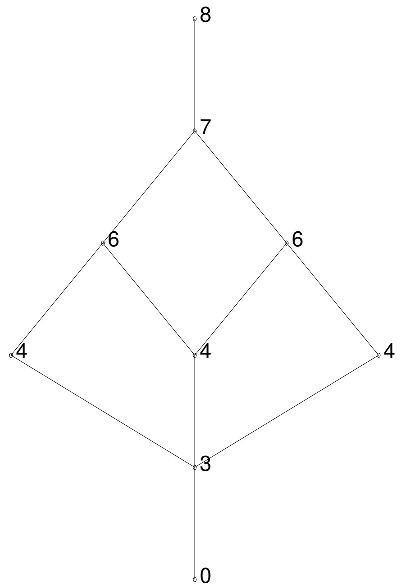



Figure 7.5: The Bruhat-Chevalley poset of the action of the parabolic subgroup $P(\mathbf{d})$ in $\mathrm{GL}_5(k)$ on $\mathfrak{p}_u(\mathbf{d})$, where $\mathbf{d} = (2, 1, 2)$.

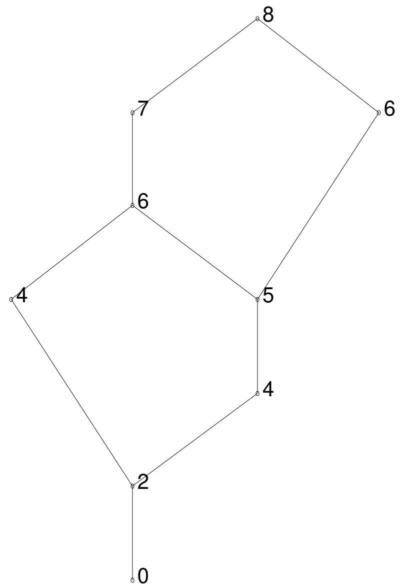



Figure 7.6: The Bruhat-Chevalley poset of the action of the parabolic subgroup $P(\mathbf{d})$ in $\mathrm{GL}_5(k)$ on $\mathfrak{p}_u(\mathbf{d})$, where $\mathbf{d} = (2, 2, 1)$.

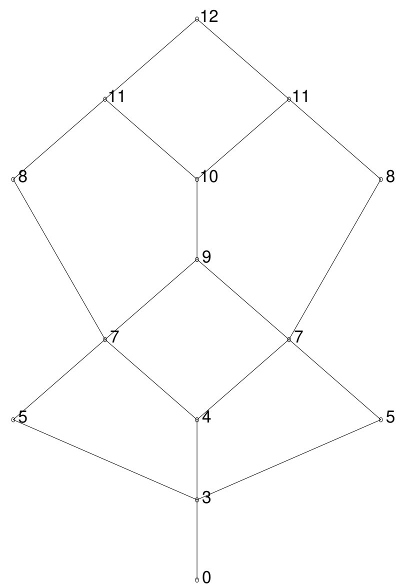



Figure 7.7: The Bruhat-Chevalley poset of the action of the parabolic subgroup $P(\mathbf{d})$ in $\mathrm{GL}_8(k)$ on $\mathfrak{p}_u^{(1)}(\mathbf{d})$, where $\mathbf{d} = (2, 2, 2, 2)$.

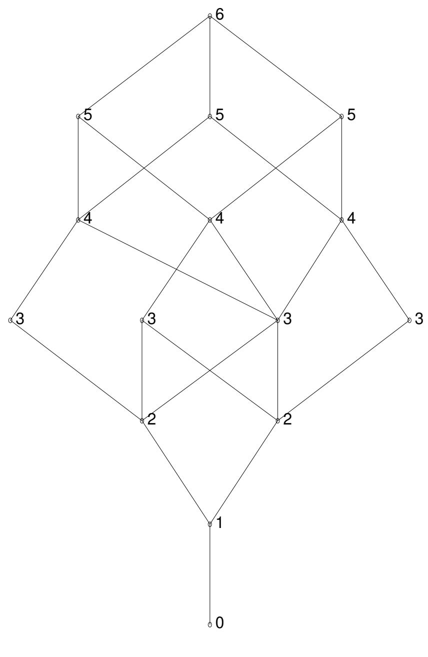



Figure 7.8: The Bruhat-Chevalley poset of the action of the Borel subgroup $B$ in $\mathrm{GL}_5(k)$ on $\mathfrak{b}_u^{(1)}(\mathbf{d})$, where $\mathbf{d} = (1, 1, 1, 1, 1)$.

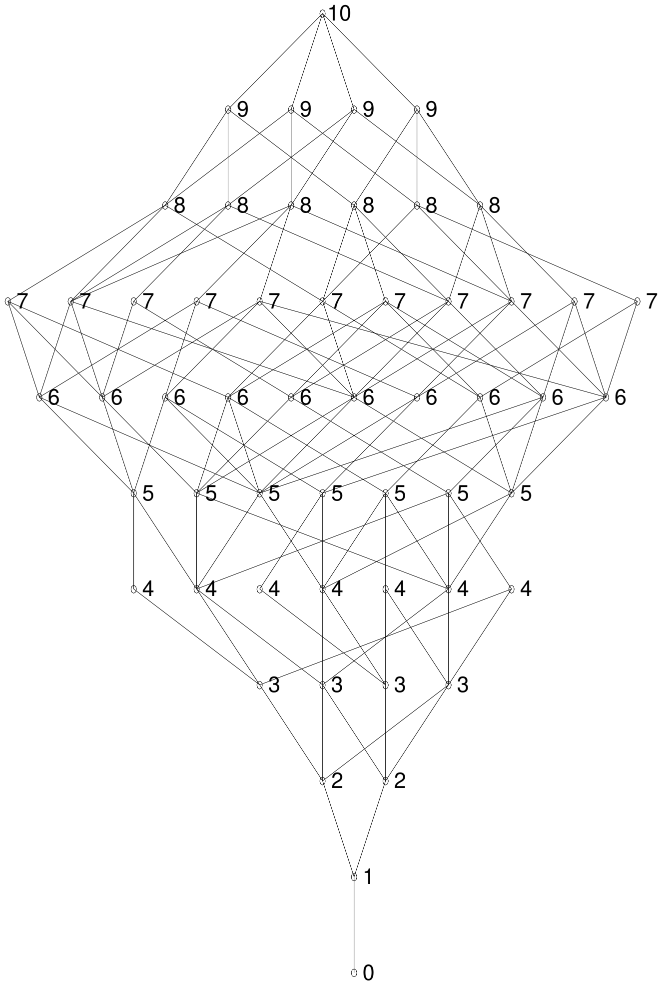
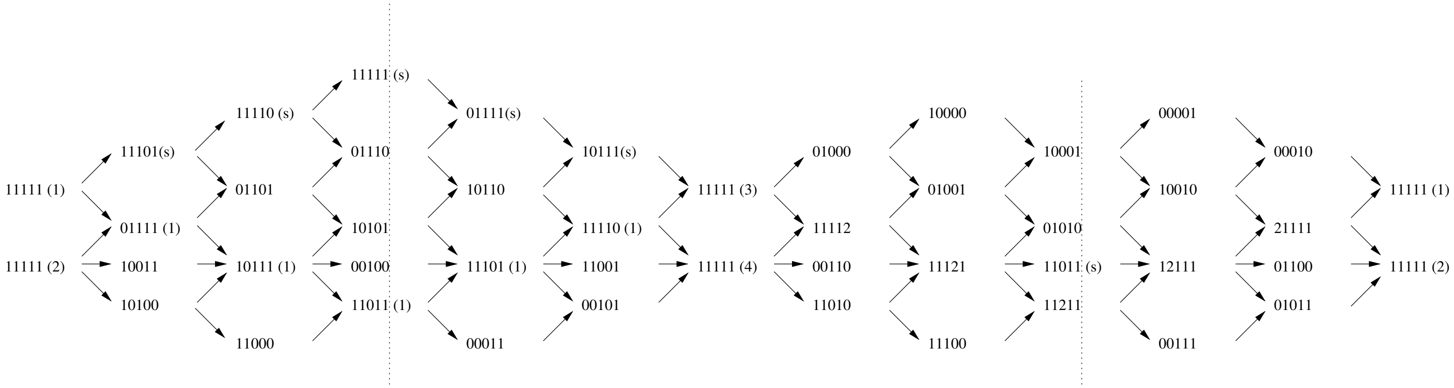



Figure 7.9: The Bruhat-Chevalley poset of the action of the Borel subgroup $B$ in $\mathrm{GL}_6(k)$ on $\mathfrak{b}_u^{(1)}(\mathbf{d})$, where $\mathbf{d} = (1,1,1,1,1,1)$.

# 8. Appendix B: The Auslander-Reiten quivers of $\mathcal{F}(t,l)$

Figure 8.1: The Auslander-Reiten quiver of F(5,0) (see [3]).
(the dotted lines indicate the symmetry of the AR-quiver)




the normal forms of indecomposable matrices:

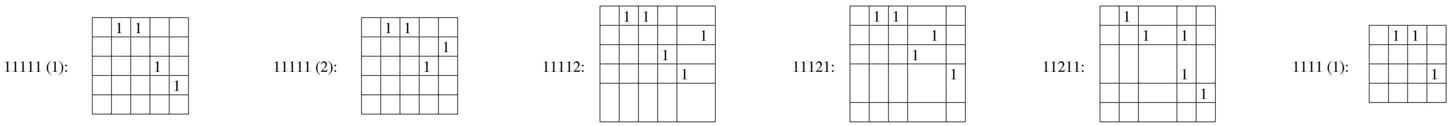
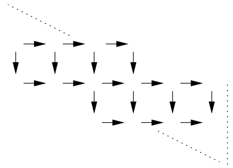


Figure 8.2: Part of the Auslander Reiten quiver of the covering of B(6,1); shape of the quiver:
(dotted lines indicate the symmetries of the AR-quiver)



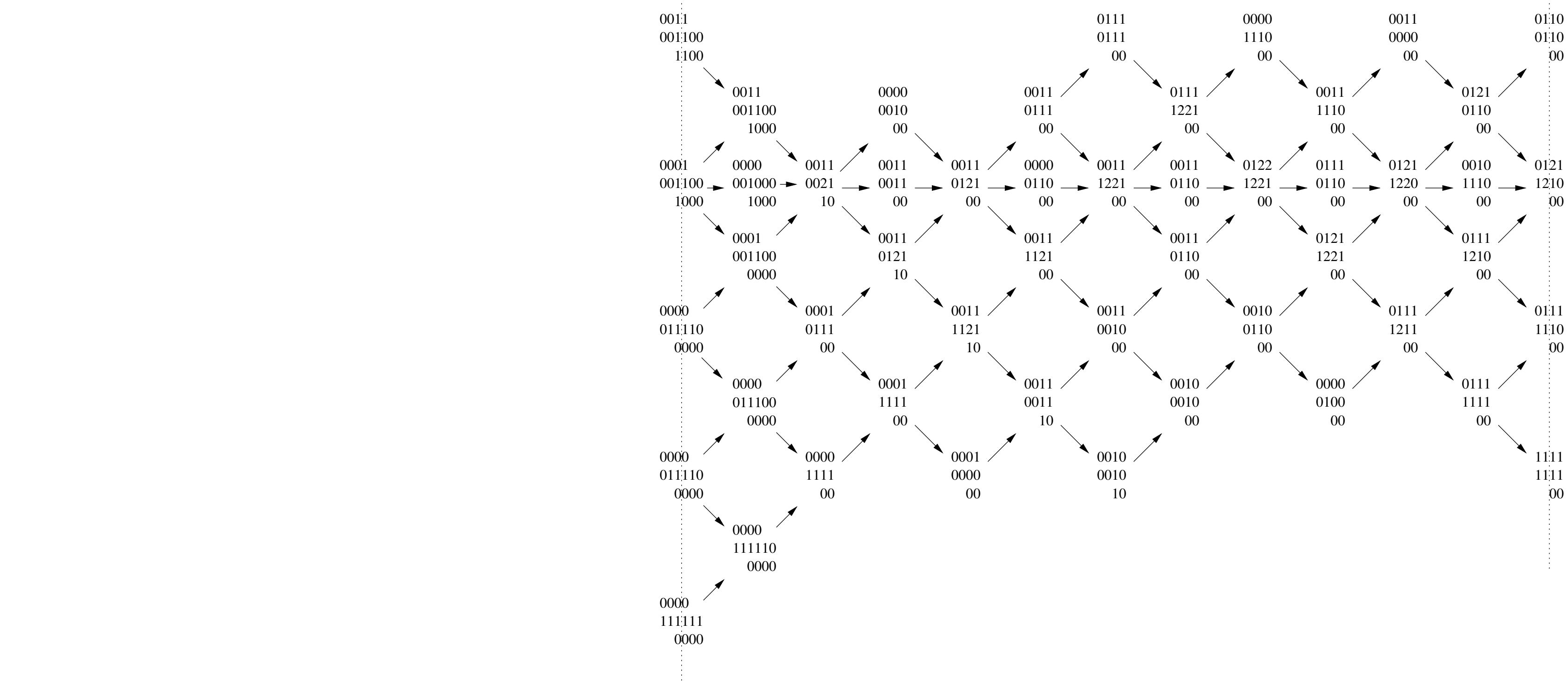


Figure 8.3: half of the Auslander-Reiten quiver of B(6,1) with top and socle
first row: dimension vector of the top
second row: dimension vector
third row: dimension vector of the socle

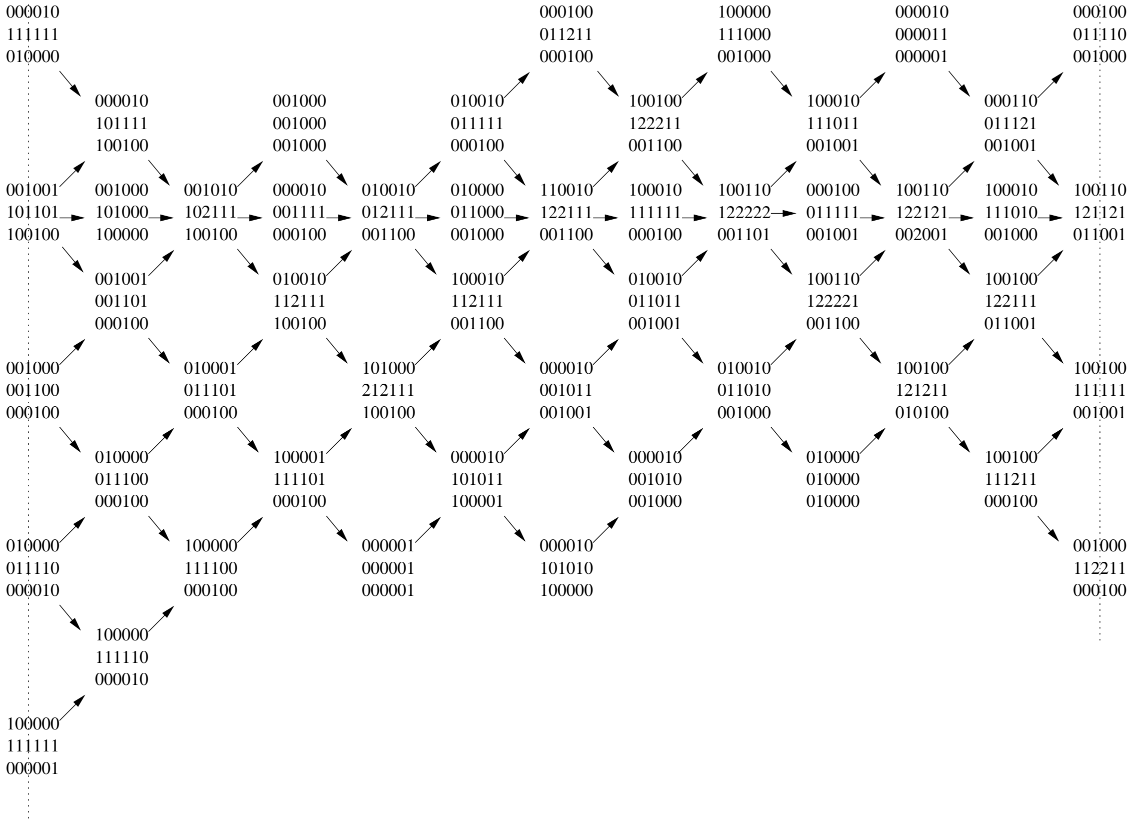


Figure 8.4: Auslander-Reiten quiver of F(8,1)
first row: dimension vector of the corresponding representation in B(6,1)
second row: dimension vector of the extension by direct summands of the characteristic module
third row: dimension vector of the flag or the endomorphism respectively

(parallel solid lines denote the symmetry of the AR -quiver)
(the dotted lines are for orientation and correspond to the dotted lines in Figures 8.2 and 8.3)

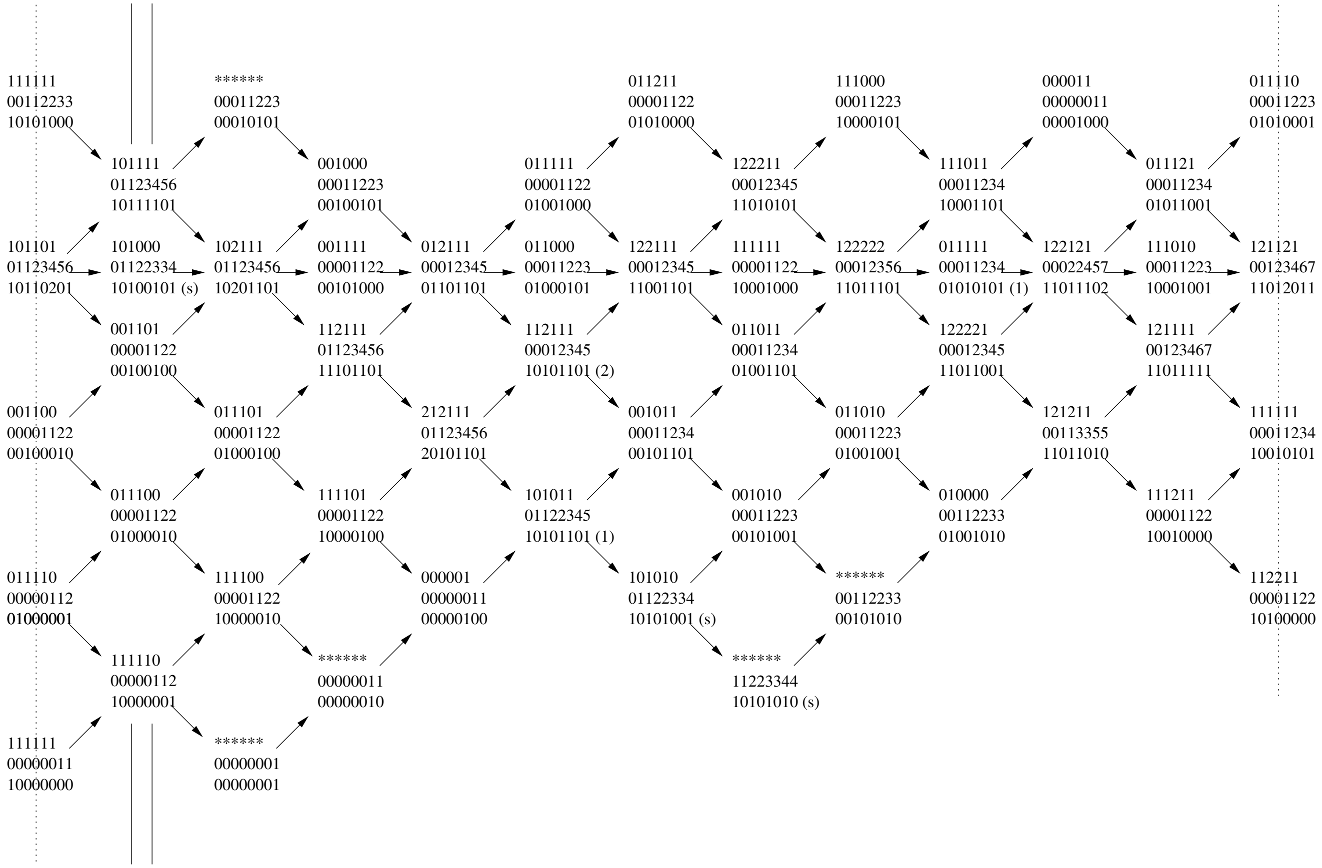
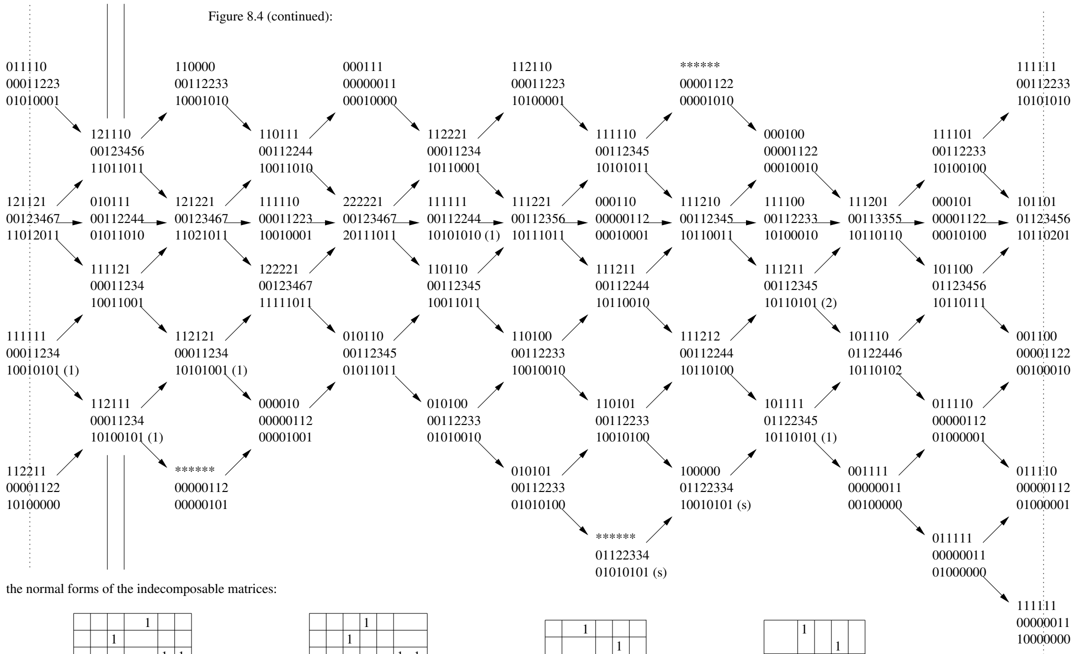
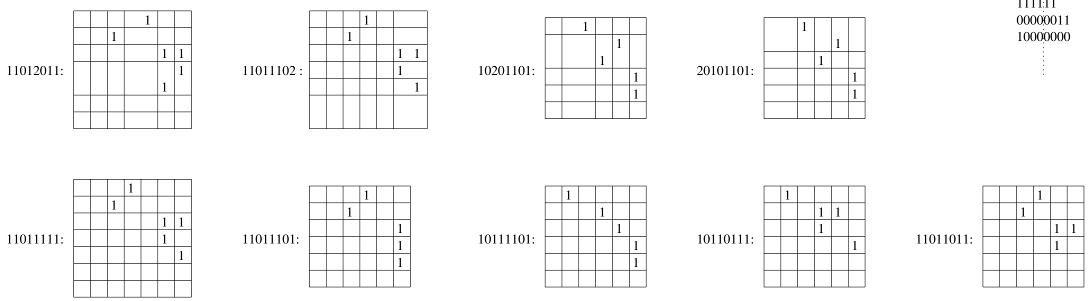


Figure 8.4 (continued):

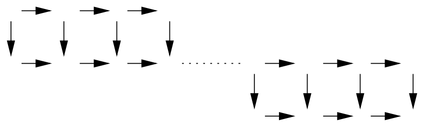



the normal forms of the indecomposable matrices:




10101010 (1), 10100101 (1), 00101101, 10101001 (1), 10001101, 01011001, 01011010, 10011001 are induced by 1111 (1)
10110110, 10110101 (2), 10110011, 01011011, 11011001,10101011 are induced by 11111 (2)
10110101 (1) is induced by 11111 (1)

Figure 8.5: part of the Auslander-Reiten quiver of the covering of B(5+l,l) for l > 1;
shape of the quiver:



(dotted line denotes the symmetry
part in broken lines contains the wing of modules concentrated on a vertical line
the module in the box on lower left corner appears only for l=2
modules which are not in the box and not in the wing are concentrated in the three squares)

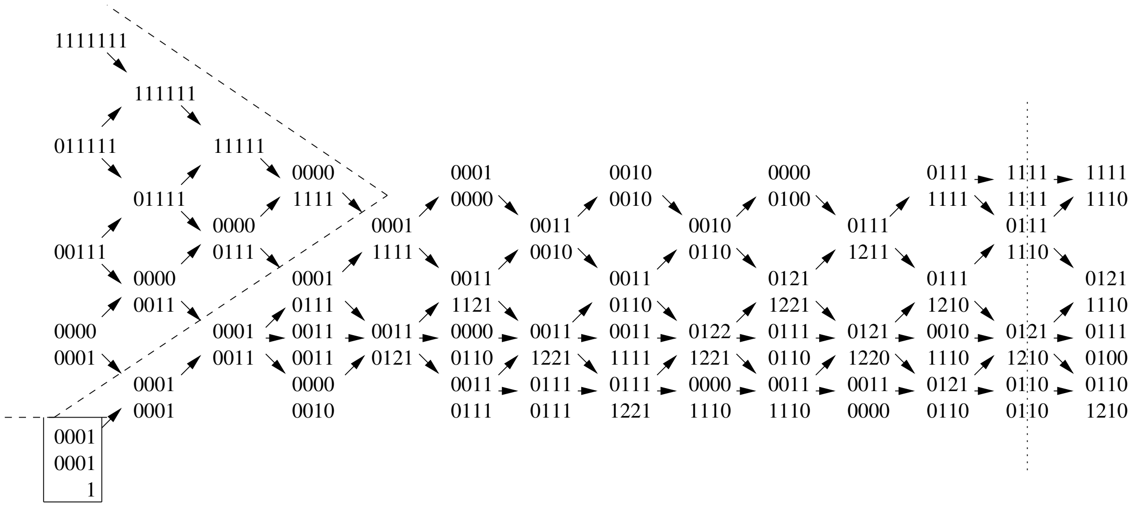


Figure 8.6: part of theAuslander-Reiten quiver of B(7,2) with top and socle
first row: dimension vector of the top
second row: dimension vector
third row: dimension vector of the socle

(dotted line denotes the symmetry
part in broken lines denotes the wing of the "line"-modules)

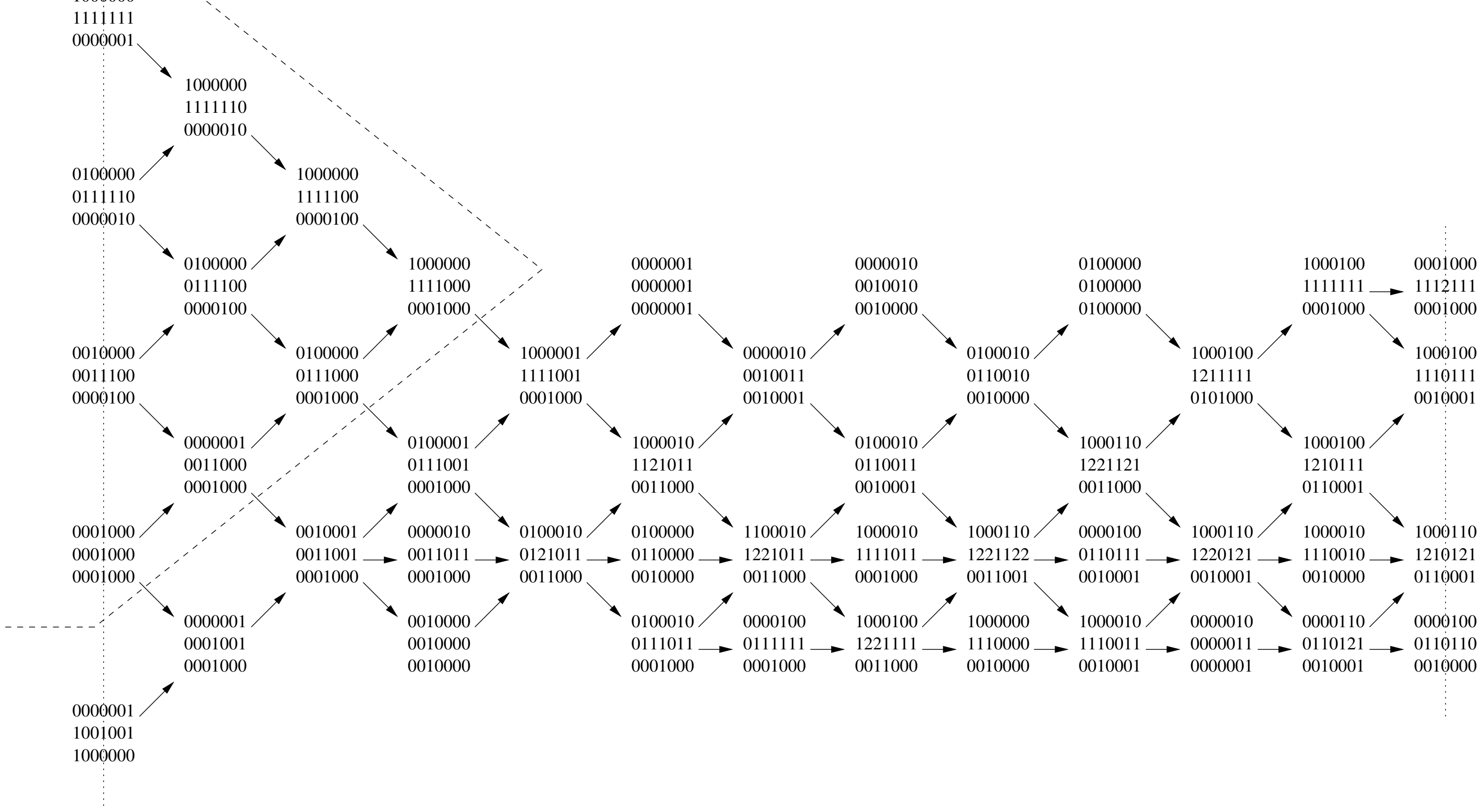


Figure 8.7: Auslander-Reiten quiver of A(10.2) and F(10,2)
first row: dimension vector in B(7,2)
second row: dimension vector of the extension by direct summands of the characteristic module
third row: dimension vector of the flag

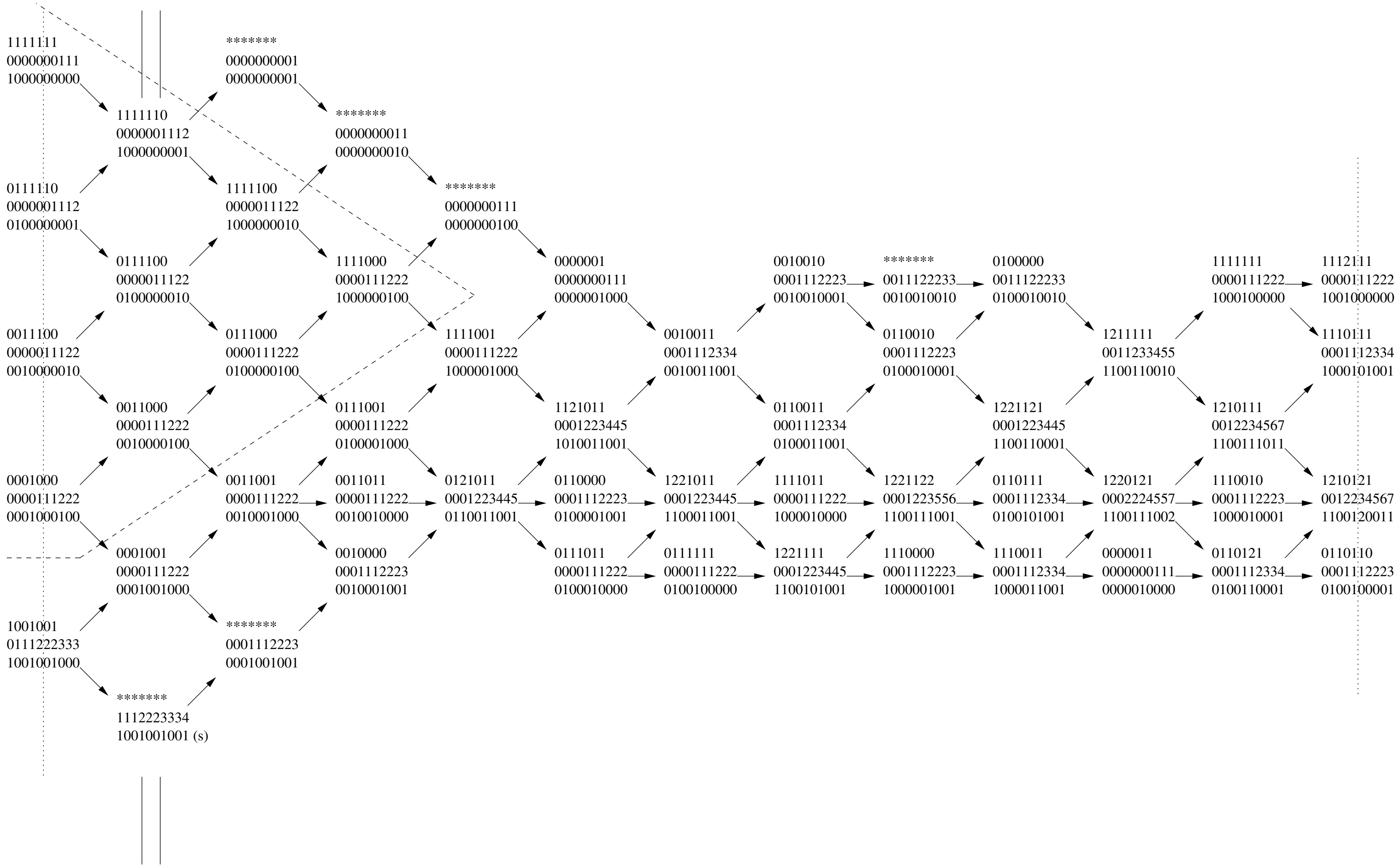
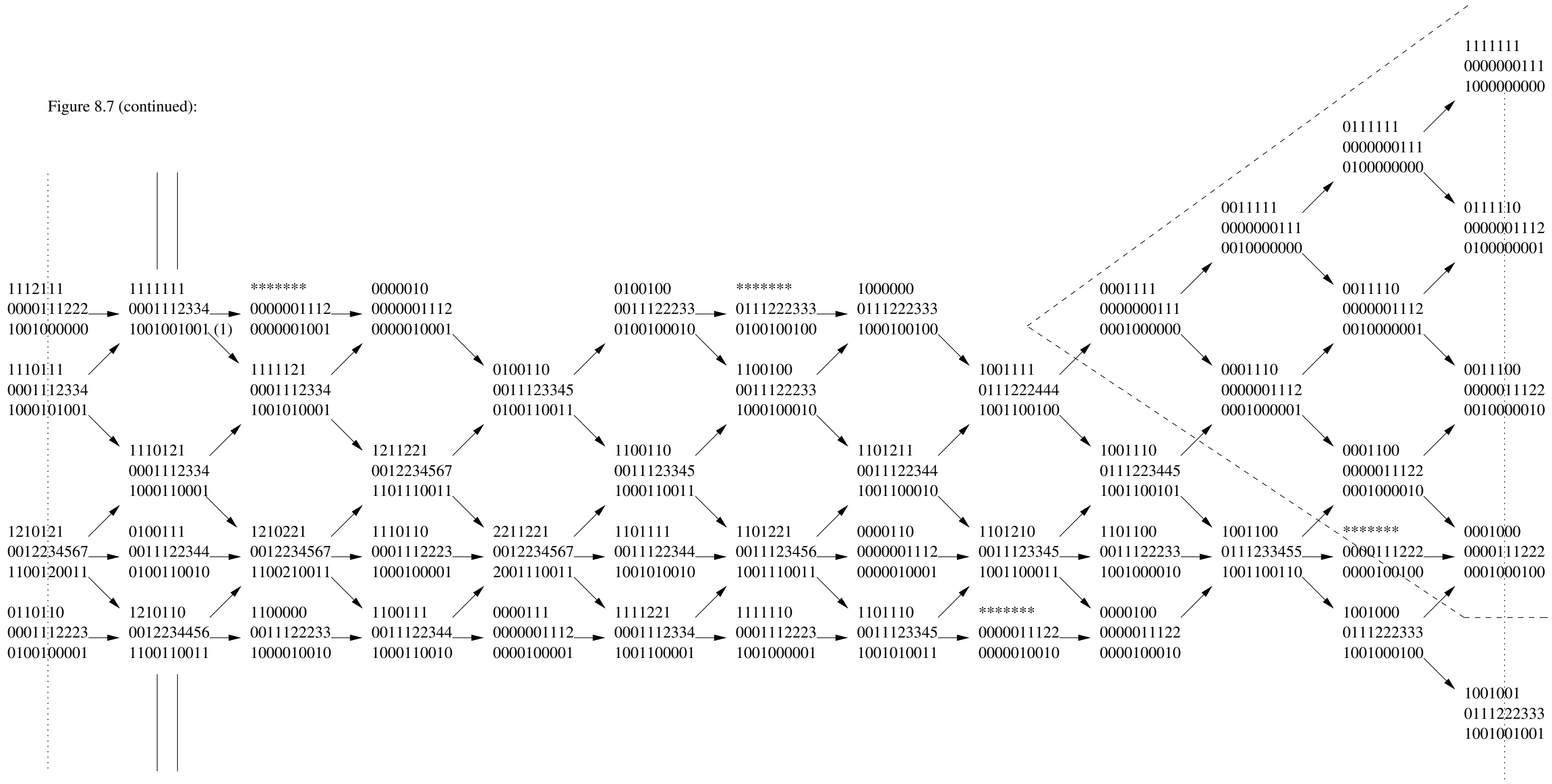


Figure 8.7 (continued):




1001001001 (1) and all other four-dimensional representations are induced by 1111 (1)
all five-, six-, and seven-dimensional representations are induced by normal forms from Figure 8.4

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