Comparison of systems of stochastic partial differential equations

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Abstract

We provide a general comparison theorem for systems of stochastic partial differential equations, which is a powerful tool to study stability as well as stochastic invariance of the corresponding solution processes. An application showing stability properties of a predator-prey system is given.

KEY WORDS stochastic partial differential equation, comparison theorem

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1 Introduction

In contrast to the theory of ordinary stochastic differential equations, the theory of stochastic partial differential equations lacks of an important tool, namely, we have not a well-applicable Itô formula. One technique to handle the difficulties arising from the missing Itô formula is the comparison technique.

So, several authors, [3], [4], [5], [6], [8], proved comparison theorems for stochastic partial differential equations of the form

$$\frac{\partial}{\partial t}u(t,y) \;=\; Au(t,y) + f(t,y,u(t,y)) + g(t,y,u(t,y)) \cdot \dot{\eta}(t,y) \tag{1}$$

which are of special interest. Here A denotes a differential operator, f, g are real functions on $[0,T]\times\Omega\times\mathcal{O}\times\mathbb{R}$, $\mathcal{O}\subseteq\mathbb{R}^d$, and the gaussian noise $\dot{\eta}(t,y)$ is white in time resp. white or coloured in space. We mention that P. Kotelenez, [4], was the first one who considered a wide class of such equations with nontrivial diffusion coefficients. But in their recent paper [9], G. Tessitore and J. Zabczyk especially show that some arguments of the proof given in [4] are not satisfactory. The authors in [3], [6], [8] only investigate the case $d = 1$ under the additional assumption that A is the one-dimensional Laplacian with certain boundary conditions. T. Shiga resp. C. Mueller (cf. [8],[6]) approximate the Laplacian using a discretization procedure and we do not see how to apply this method in the case of more general differential operators. Differently the method used by C. Donati-Martin and E. Pardoux (cf. [3]), we feel to be applicable also in the case of an operator A which is the generator of a positivity preserving semigroup on $L^2(\mathcal{O})$, $\mathcal{O}\subseteq\mathbb{R}^d$, even in dimensions $d \geq 2$. However, this extension having not yet been done in [3] is now accomplished in [5] using a different method.

What we are doing in this paper is less to prove a better version of the comparison theorems given in [3] resp. [5] but more to apply them to <u>systems</u> of stochastic partial differential equations,

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that is f, g in (1) are not longer supposed to be real functions but, for example, n-dimensional vector fields. In this sense, our results extend the results in [1] for systems of ordinary stochastic differential equations to the more complicated case of stochastic partial differential equations, partly. We give remarks concerning the generality of the conditions we need.

Finally, in order to demonstrate the force of our comparison theorem, we study a hypothetical predator-prey system describing the phenomenon that the growth rate of the predator population depends on the ratio between prey and predator. At first, we discuss aspects of the stability of the basic deterministic system in a geometrical way which is interesting in itself and second, applying the same method, we show this stability properties also for the stochastic system obtained from the deterministic one by disturbing the coefficients, randomly.

Remark that we only deal with a more realistic model in a forthcoming paper because the difficulties arising from coefficients with singularities would not fit into the frame of this paper.

2 Notations

On a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ let be given m independent cylindrical Wiener processes $W_1(t), ..., W_m(t)$ in $\mathfrak{D}'(\mathcal{O})$ where $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$, $d \geq 1$, with sufficiently smooth boundary. Denote by $\mathbf{L}^2(\mathcal{O})$ the Hilbert space of vector fields $\mathbf{u} : \mathcal{O} \to \mathbb{R}^n$, $n \geq 2$, satisfying

$$\|\mathbf{u}\|_2^2 := \int_{\mathcal{O}} \|\mathbf{u}(y)\|_{\mathbb{R}^n}^2 \, \mathrm{d}y < \infty.$$

Here and in what follows, we place emphasis on n-dimensional objects by printing them as bold face letters. Clearly, n is the number of equations our system of stochastic partial differential equations consists of.

Define the multiplication operator

$$\boldsymbol{\mathcal{M}} : \mathbf{L}^2(\mathcal{O}) \to \mathfrak{L}(L^\infty(\mathcal{O}), \mathbf{L}^2(\mathcal{O}))$$

by

$$[\boldsymbol{\mathcal{M}}(\mathbf{u})\phi](y) = \phi(x)\mathbf{u}(y), \; x \in \mathcal{O},$$

and fix m bounded linear operators $B_1, ..., B_m : L^2(\mathcal{O}) \to L^\infty(\mathcal{O})$ with the property that

$$\|\boldsymbol{\mathcal{M}}(\mathbf{u}) \circ B_j\|_{\mathfrak{L}^2}^2 \leq C(B)\|\mathbf{u}\|_2^2, \; \mathbf{u} \in \mathbf{L}^2(\mathcal{O}), \; j = 1, ..., m, \tag{2}$$

where $\|\cdot\|_{\mathfrak{L}^2}$ denotes the Hilbert-Schmidt norm on $\mathfrak{L}(L^2(\mathcal{O}), \mathbf{L}^2(\mathcal{O}))$.

On $\mathbf{L}^2(\mathcal{O})$ we consider a separate operator $(\mathbf{A}, D(\mathbf{A}))$, that is

$$\mathbf{A}\boldsymbol{\phi} = \begin{pmatrix} A_1\phi_1 \\ \vdots \\ A_n\phi_n \end{pmatrix}, \quad \boldsymbol{\phi} \in D(\mathbf{A}),$$

which is assumed to be the generator of a strongly continuous semigroup $(\mathbf{S}(t))_{t \geq 0}$. Obviously, $\mathbf{S}(t)$ is also separated into n strongly continuous semigroups on $L^2(\mathcal{O})$:

$$\mathbf{S}(t)\boldsymbol{\phi} = \begin{pmatrix} S_1(t)\phi_1 \\ \vdots \\ S_n(t)\phi_n \end{pmatrix}, \quad \boldsymbol{\phi} \in \mathbf{L}^2(\mathcal{O}).$$

Using the Banach-Steinhaus theorem, for a fixed time horizon $T > 0$, we have

$$\sup_{t \leq T} \|\mathbf{S}(t)\|_{\mathfrak{L}(\mathbf{L}^2(\mathcal{O}))} \leq C = C(T) < \infty.$$

Now, let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a right-continuous filtration of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_0$ already contains all sets of $\mathbf{P}$-measure zero. Assume that the Wiener processes W_j are $\mathbb{F}$-adapted and that the increments $W_j(t) - W_j(s)$ are independent of $\mathcal{F}_s$ for all $t \geq s \geq 0$, $j = 1, ..., m$. Furthermore, let $\mathfrak{M}$ denote the σ-algebra of $\mathbb{F}$-progressively measurable subsets of $\mathbb{R}_+ \times \Omega$.

For $\mathfrak{M} \otimes \mathfrak{B}(\mathcal{O}) \otimes \mathfrak{B}(\mathbb{R}^n)$ – measurable functions $\mathbf{F}, \mathbf{G}_1, ..., \mathbf{G}_m : [0,T] \times \Omega \times \mathcal{O} \times \mathbb{R}^n \to \mathbb{R}^n$ we consider the mild form

$$(\star) \quad \mathbf{u}(t) \;=\; \mathbf{S}(t)\boldsymbol{\xi} + \int_0^t \mathbf{S}(t-s)\mathbf{F}(s,\mathbf{u}(s))\,\mathrm{d}s + \sum_{j=1}^m \int_0^t \mathbf{S}(t-s)\boldsymbol{\mathcal{M}}(\mathbf{G}_j(s,\mathbf{u}(s))) \circ B_j \,\mathrm{d}W_j(s)$$

of equations like (1) starting from an $\mathcal{F}_0$-measurable random variable $\boldsymbol{\xi} : \Omega \to \mathbf{L}^2(\mathcal{O})$.

Definition 1 An $\mathbf{L}^2(\mathcal{O})$-valued $\mathbb{F}$-progressively measurable process $\mathbf{u}$ is said to be a *solution of equation* $(\star)$ *starting from* $\boldsymbol{\xi}$ if $\mathbf{u}$ satisfies $(\star)$ a.s. for all $t \in [0,T]$.

Remark 1 a) The equality of elements of $\mathbf{L}^2(\mathcal{O})$ is seen as usual in the sense of equivalence classes.

b) The first integral on the right-hand side of $(\star)$ is understood as a Bochner integral while the remaining integrals are Hilbert space valued stochastic integrals (see for example [2]). Here, we omit the variable ω and write $\mathbf{F}(s,\mathbf{u}(s))$ resp. $\mathbf{G}_j(s,\mathbf{u}(s))$ for the random $\mathbf{L}^2$-elements $y \mapsto \mathbf{F}(s,\cdot,y,\mathbf{u}(s,y))$ resp. $y \mapsto \mathbf{G}_j(s,\cdot,y,\mathbf{u}(s,y))$, $j = 1, ..., m$.

c) For fixed j, the stochastic integral

$$\int_0^t \mathbf{S}(t-s)\boldsymbol{\mathcal{M}}(\mathbf{G}_j(s,\mathbf{u}(s))) \circ B_j \,\mathrm{d}W_j(s)$$

appears as

$$(3) \qquad \int_0^t \mathbf{S}(t-s)\boldsymbol{\mathcal{M}}(\mathbf{G}_j(s,\mathbf{u}(s)))\,\mathrm{d}W^Q(s)$$

where W^Q denotes a nuclear Wiener process with covariance Q if B_j equals $Q^{\frac{1}{2}}$. The additional properties $Q^{\frac{1}{2}} \in \mathfrak{L}(L^2(\mathcal{O}), L^\infty(\mathcal{O}))$ and (2) are satisfied if, for example, the eigenbasis $(e_k)_{k=1}^\infty$ of Q is uniformly bounded in $L^\infty(\mathcal{O})$, that is

$$\|e_k\|_{L^\infty(\mathcal{O})} \;\leq\; \text{const},\; k = 1, 2, ...$$

In general, the stochastic integral in (3) does not make sense for arbitrary nuclear covariance operators. But in special cases, if the semigroup $\mathbf{S}(t)$ generated by $\mathbf{A}$ has outstanding smoothing properties, the integral (3) even makes sense for $Q = Id$ and we could drop the introduction of these operators $B_1, ..., B_m$ as well as (2). The most famous example, well-studied in literature for $n = 1$, is the one-dimensional Laplacian, $i = 1, ..., n$. However, already in dimension $d = 2$ the Laplacian is no more an example. So, we essentially consider equations with a diffusion term given in $(\star)$, but we also give a corollary treating in the special situation of the one-dimensional Laplacian and $Q = Id$.

d) We do not assume solutions to be continuous in time with respect to $\|\cdot\|_2$. As a consequence, every version of a solution of $(\star)$ is also a solution of $(\star)$. In this sense a solution $\mathbf{u}$ of equation $(\star)$ starting from $\boldsymbol{\xi}$ is pathwise unique if and only if

$$\mathbf{u}(t) \;=\; \tilde{\mathbf{u}}(t) \quad \mathbf{P}\text{-a.s.},\; t \in [0,T],$$

for all other solutions $\tilde{\mathbf{u}}$ of $(\star)$ starting from $\boldsymbol{\xi}$. The comparison theorem we propose in the next section applies to <u>all</u> versions of the considered processes we do not differ in what follows.

e) The assumption on $(\mathbf{A}, D(\mathbf{A}))$ to be separated is quite natural even in the framework of comparison methods for systems of deterministic partial differential equations (cf. [10]). If $(\mathbf{A}, D(\mathbf{A}))$ is a differential operator, the property to be separated means that there is no interaction between the derivatives of different components of the considered system. Interaction may only appear between the components themselves.

3 Main Result

Consider the equation $(\star)$ for different drift functions $\mathbf{F}^{(1)}$, $\mathbf{F}^{(2)}$ and diffusion coefficients $\mathbf{G}_1, ...,$ $\mathbf{G}_m$ satisfying

$$\|\mathbf{F}^{(l)}(t,\omega,y,\mathbf{x}) - \mathbf{F}^{(l)}(t,\omega,y,\mathbf{z})\|_{\mathbb{R}^n} \le C(T)\|\mathbf{x}-\mathbf{z}\|_{\mathbb{R}^n} \tag{4}$$

as well as

$$\|\mathbf{F}^{(l)}(t,\omega,y,\mathbf{x})\|_{\mathbb{R}^n} + \|\mathbf{G}_j(t,\omega,y,\mathbf{x})\|_{\mathbb{R}^n} \le C(T)(1+\|\mathbf{x}\|_{\mathbb{R}^n}) \tag{5}$$

for all $t \in [0,T]$, $\omega \in \Omega$, $y \in \mathcal{O}$, and $\mathbf{x},\mathbf{z} \in \mathbb{R}^n$, $l = 1,2$, $j = 1,...,m$. Furthermore, assume that

$$\sum_{j=1}^{m} |G_{ij}(t,\omega,y,\mathbf{x}) - G_{ij}(t,\omega,y,\mathbf{z})| \le C(T)|x_i - z_i| \tag{6}$$

for all $t \in [0,T]$, $\omega \in \Omega$, $y \in \mathcal{O}$, and $\mathbf{x},\mathbf{z} \in \mathbb{R}^n$, $i = 1,...,n$.

First remark that, by the standard procedure (cf. [2]), the conditions (4),(5),(6) ensure the existence of a pathwise unique solution of equation $(\star)$ (see Remark 1d)) corresponding to $\mathbf{F}^{(1)}$ resp. $\mathbf{F}^{(2)}$ for every square integrable $\mathbf{L}^2(\mathcal{O})$-valued initial distribution $\boldsymbol{\xi}$.

Fix $\mathbf{z} \in \mathbb{R}^n$ and define $\mathbf{F}^{(l)}_{\mathbf{z}} : [0,T] \times \Omega \times \mathcal{O} \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$F^{(l)}_{\mathbf{z},i}(t,\omega,y,\mathbf{x}) = F^{(l)}_i(t,\omega,y, \begin{pmatrix} z_1 \\ \vdots \\ x_i \\ \vdots \\ z_n \end{pmatrix}), \quad i = 1,...,n,\ l = 1,2, \tag{7}$$

we use to introduce our main condition on the drift coefficients:

$$(C_{1,2}) \qquad \begin{cases} \text{For any } t \in [0,T] \text{ we have} \\ \quad \mathbf{F}^{(1)}_{\mathbf{z}^{(1)}}(t,\cdot,y,\mathbf{x}) \le \mathbf{F}^{(2)}_{\mathbf{z}^{(2)}}(t,\cdot,y,\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^n,\ \mathbf{z}^{(1)} \le \mathbf{z}^{(2)}, \\ \text{for Lebesgue a.a. } y \in \mathcal{O}\ \mathbf{P}\text{-a.s.} \end{cases}$$

Here, $\mathbf{z}^{(1)}, \mathbf{z}^{(2)} \in \mathbb{R}^n$ are in the relation $\mathbf{z}^{(1)} \le \mathbf{z}^{(2)}$ if $z^{(1)}_i \le z^{(2)}_i$, $i = 1,...,n$. Furthermore, we write $\mathbf{u} \le \mathbf{v}$ for $\mathbf{u},\mathbf{v} \in \mathbf{L}^2(\mathcal{O})$ if $\mathbf{u}(y) \le \mathbf{v}(y)$ holds for Lebesgue-a.a. $y \in \mathcal{O}$ and demand that the semigroup $\mathbf{S}(t)$ is positivity preserving:

$$(C_{pp}) \qquad \forall\, t \in [0,T] : \quad \mathbf{S}(t)\boldsymbol{\phi} \ge \mathbf{0} \quad \text{for} \quad \boldsymbol{\phi} \in \mathbf{L}^2(\mathcal{O}) \quad \text{with} \quad \boldsymbol{\phi} \ge \mathbf{0}.$$

Theorem 1 *Let be given two $\mathcal{F}_0$-measurable square integrable random variables $\boldsymbol{\xi}^{(l)} : \Omega \to \mathbf{L}^2(\mathcal{O})$, $l = 1,2$, satisfying*

$$\boldsymbol{\xi}^{(1)} \le \boldsymbol{\xi}^{(2)} \quad \mathbf{P}\text{-a.s.}$$

Furthermore, assume that each $\boldsymbol{\xi}^{(l)}$ is product measurable as a function $(\omega,y) \mapsto \boldsymbol{\xi}^{(l)}(\omega)(y)$, $l = 1,2$. Suppose (4),(5),(6) and denote by $\mathbf{u}^{(l)}$ the pathwise unique solution of the equation $(\star)$ corresponding to $\mathbf{F}^{(l)}$ starting from $\boldsymbol{\xi}^{(l)}$, $l = 1,2$. Then the conditions (C_{pp}) and $(C_{1,2})$ imply that for all $t \in [0,T]$

$$\mathbf{u}^{(1)}(t) \le \mathbf{u}^{(2)}(t) \quad \mathbf{P}\text{-a.s.}$$

Remark 2 a) The condition on the initial values to be product measurable in $(\omega,y) \in \Omega \times \mathcal{O}$ which has at first appeared in [5] seems to be technical. But in the proof (cf. [5]) one feels that it could express some kind of regularity which is needed for the comparison of our infinite dimensional objects. Indeed, in the proof of Th.3.3.1, [5], appear $\mathbf{P}$-exceptional sets depending on $y \in \mathcal{O}$, and the product measurability in (ω, y) is needed to apply the Theorem of Fubini with the same reason as in our Remark 3 on page 11 with respect to (t, ω). However, we can drop this condition if the semigroup $\mathbf{S}(t)$ is strongly Feller and it means no restriction for deterministic initial values. We further mention that in our framework, by the Picard iteration we use to construct the solutions of our equations, a process which has an initial value being product measurable in (ω, y) is also product measurable in (ω, y) at each time $t > 0$.

b) We prove the theorem by approximating the solutions $\mathbf{u}^{(l)}$, $l = 1, 2$, and, in order to know whether we approximate the right solutions, we need the pathwise uniqueness of them ensured by the lipschitz conditions on the coefficients. Here the lipschitz condition (4) on the drift coefficients is not as restrictive because in applications less regular drift functions are often approximated by lipschitz functions, monotonously.

c) In addition to the lipschitz condition, (6) also implies that the i-th component of each function $\mathbf{G}_j(t, \omega, y, \cdot)$ only depends on the i-th component of the variable. An example of a system of ordinary stochastic differential equations given in [1], section 3, shows that this property cannot be droped, in general. This example may be applied in our situation if we especially assume that $\mathbf{A}$ vanishes and that B_m, $m = 1$, projects $L^2(\mathcal{O})$ to a certain one-dimensional subspace. We should mention that the condition (C_σ) discussed in [1], section 3, admits the Hölder continuity with exponent $\alpha \in (\frac{1}{2}, 1)$ of the diffusion coefficients. Unfortunately, there is no proof for the pathwise uniqueness of stochastic <u>partial</u> differential equations with Hölder continuous diffusion coefficients which are not lipschitz and we cannot weaken condition (6) in this sense.

d) The assumption $(C_{1,2})$ of the above comparison theorem for systems of stochastic partial differential equations is the well-known generalization of the condition

$$\mathbf{F}^{(1)}(t, \cdot, y, \mathbf{x}) \leq \mathbf{F}^{(2)}(t, \cdot, y, \mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \text{ for Lebesgue a.a. } y \in \mathcal{O} \ \mathbf{P}\text{-a.s., } t \in [0, T], \tag{8}$$

which is used in comparison theorems for only one equation, that is $n = 1$. There is an example with nonvanishing diffusion part again given in [1], section 3, which makes clear that, in the case $n \geq 2$, $(C_{1,2})$ cannot be replaced by the perhaps more natural condition (8), in general. If the condition $(C_{1,2})$ holds true for a function $\mathbf{F} = \mathbf{F}^{(1)} = \mathbf{F}^{(2)}$ then we call $\mathbf{F}$ *quasi-monotonously increasing*. For more references concerning this concept, which is also well-known in the deterministic theory of differential equations, see [10].

e) Clearly, for our theorem to be true in the special situation $\mathbf{F}^{(l)} \equiv \mathbf{0}$, $l = 1, 2$, and $\mathbf{G}_j \equiv \mathbf{0}$, $j = 1, ..., m$, it is necessary that the semigroup $\mathbf{S}(t)$ is positivity preserving. We presume that a strongly continuous semigroup on $\mathbf{L}^2(\mathcal{O})$ which is positivity preserving must be generated by a <u>separate</u> operator (see also Remark 1e)). But there is not yet any proof of this claim.

4 Proof of the Theorem and Corollaries

Proof of Theorem 1. We provide nice sequences of processes which approximate $\mathbf{u}^{(1)}$ resp. $\mathbf{u}^{(2)}$. The approximating processes are chosen to be solutions of separate systems of stochastic partial differential equations, that is the equation for one component does not depend on the other components. As a consequence, we may apply the comparison theorem proven in [5] to the components of the corresponding processes on each level of approximation finally leading to the comparison of the limit processes $\mathbf{u}^{(1)}$ resp. $\mathbf{u}^{(2)}$.

Let $\mathbf{u}$ denote one of the solutions $\mathbf{u}^{(1)}$ resp. $\mathbf{u}^{(2)}$. We consider $\mathbf{u}$ to be the pathwise unique solution of equation $(\star)$ with $\boldsymbol{\xi}, \mathbf{F}$ standing for $\boldsymbol{\xi}^{(1)}, \mathbf{F}^{(1)}$ resp. $\boldsymbol{\xi}^{(2)}, \mathbf{F}^{(2)}$ simplifying the notation. For $N \in \mathbb{N}$ choose $t_1 = T/N$ and define

$$\mathbf{u}_{kt_1}(t) = \mathbf{u}(t + kt_1), \ t \in [0, t_1], \ k = 0, ..., N-1.$$

Fix $k \in \{0,...,N-1\}$. The $(\mathcal{F}_{t+kt_1})_{t\geq 0}$ – adapted process $\mathbf{u}_{kt_1}$ defined above is the pathwise unique solution of the equation

$$
\begin{aligned}
\mathbf{u}_{kt_1}(t) \;&=\; \mathbf{S}(t)\mathbf{u}(kt_1) + \int_0^t \mathbf{S}(t-s)\mathbf{F}(s+kt_1, \mathbf{u}_{kt_1}(s))\,\mathrm{d}s \;+ \\
&\qquad \sum_{j=1}^m \int_0^t \mathbf{S}(t-s)\boldsymbol{\mathcal{M}}(\mathbf{G}_j(s+kt_1, \mathbf{u}_{kt_1}(s))) \circ B_j \,\mathrm{d}W_j^k(s)
\end{aligned} \tag{9}
$$

with the <u>new</u> time horizon t_1. Here, W_j^k denotes the Wiener process $W_j(\cdot + kt_1) - W_j(kt_1)$ and the functions $\mathbf{F}(\cdot + kt_1, \cdot, \cdot, \cdot)$, $\mathbf{G}_j(\cdot + kt_1, \cdot, \cdot, \cdot)$, $j = 1, ..., m$, satisfy (4),(5),(6) with the same constant $C(T)$ for all $t \leq t_1, \omega \in \Omega, y \in \mathcal{O}$ and $\mathbf{x}, \mathbf{z} \in \mathbb{R}^n$.

Introduce the Banach space

$$
\begin{aligned}
\mathbb{B}_{kt_1} \;=\; \{\mathbf{v} = (\mathbf{v}(t))_{t\in[0,t_1]} |\; & \mathbf{v} \text{ is a } (\mathcal{F}_{t+kt_1})_{t\geq 0} \text{ – progressively measurable process with values in } \mathbf{L}^2(\mathcal{O}) \text{ such that} \\
& \|\mathbf{v}\|^2_{\mathbb{B}_{kt_1}} := \sup_{t\leq t_1} \mathbf{E}\|\mathbf{v}(t)\|_2^2 < \infty\}
\end{aligned}
$$

and identify elements of $\mathbb{B}_{kt_1}$ which are versions of each other. Given a square integrable $\mathcal{F}_{kt_1}$-measurable initial distribution $\boldsymbol{\xi} : \Omega \to \mathbf{L}^2(\mathcal{O})$, for various elements $\mathbf{v} \in \mathbb{B}_{kt_1}$ we consider the modified equation (9)

$$
\begin{aligned}
\mathbf{u}(t) \;&=\; \mathbf{S}(t)\boldsymbol{\xi} + \int_0^t \mathbf{S}(t-s)\mathbf{F}_{\mathbf{v}}(s+kt_1, \mathbf{u}(s))\,\mathrm{d}s \;+ \\
&\qquad \sum_{j=1}^m \int_0^t \mathbf{S}(t-s)\boldsymbol{\mathcal{M}}(\mathbf{G}_j(s+kt_1, \mathbf{u}(s))) \circ B_j \,\mathrm{d}W_j^k(s)
\end{aligned} \tag{10}
$$

where $\mathbf{F}_{\mathbf{v}}(s+kt_1, \mathbf{u}(s))$ understood as a Nemytskii operator (see also Remark 1b)) simplifies the notation $\mathbf{F}_{\mathbf{v}(s)}(s+kt_1, \mathbf{u}(s))$ introduced in (7).

Because of $\mathbf{v} \in \mathbb{B}_{kt_1}$, the standard procedure to construct solutions stimulated by (4),(5),(6) works although the linear growth of $\mathbf{F}_{\mathbf{v}}$ is now randomly disturbed and, hence, there exists a pathwise unique solution of the equation (10) starting from $\boldsymbol{\xi}$ we denote by $\Gamma_{k,\boldsymbol{\xi}}(\mathbf{v})$. As usual under the above assumptions, $\mathbf{E}\|\boldsymbol{\xi}\|_2^2 < \infty$ implies

$$
\sup_{t\leq t_1} \mathbf{E}\|\Gamma_{k,\boldsymbol{\xi}}(\mathbf{v})(t)\|_2^2 \;<\; \infty,
$$

that is

$$
\Gamma_{k,\boldsymbol{\xi}} : \mathbb{B}_{kt_1} \to \mathbb{B}_{kt_1}.
$$

Now we choose $\mathbf{v}^{(1)}, \mathbf{v}^{(2)} \in \mathbb{B}_{kt_1}$ to study contraction properties of $\Gamma_{k,\boldsymbol{\xi}}$. In a first step, from the strong continuity of $\mathbf{S}(t)$ and (2) follows

$$
\begin{aligned}
& \mathbf{E}\|\Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(1)})(t) - \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(2)})(t)\|_2^2 \\
\leq\;& 2\mathbf{E}\left(\int_0^t \|\mathbf{S}(t-s)[\mathbf{F}_{\mathbf{v}^{(1)}}(s+kt_1, \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(1)})(s)) - \mathbf{F}_{\mathbf{v}^{(2)}}(s+kt_1, \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(2)})(s))]\|_2 \,\mathrm{d}s\right)^2 + \\
& 4\sum_{j=1}^m \mathbf{E}\int_0^t \|\mathbf{S}(t-s)\boldsymbol{\mathcal{M}}(\mathbf{G}_j(s+kt_1, \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(1)})(s)) - \mathbf{G}_j(s+kt_1, \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(2)})(s))) \circ B_j\|^2_{\mathfrak{L}^2} \,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\leq\; 2C(T)^2\, t \int_0^t \mathbf{E}\|\mathbf{F}_{\mathbf{v}^{(1)}}(s+kt_1, \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(1)})(s)) - \mathbf{F}_{\mathbf{v}^{(2)}}(s+kt_1, \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(2)})(s))\|_2^2 \,\mathrm{d}s \;+ \\
&\quad\; 4C(T)^2 C(B) \sum_{j=1}^m \int_0^t \mathbf{E}\|\mathbf{G}_j(s+kt_1, \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(1)})(s)) - \mathbf{G}_j(s+kt_1, \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(2)})(s))\|_2^2 \,\mathrm{d}s \\
&=\; I_1 + I_2.
\end{aligned}
$$

By (7) we have the following equality

$$
I_1 \;=\; 2C(T)^2\, t \int_0^t \mathbf{E}\| \sum_{i=1}^n [F_i(s+kt_1, \begin{pmatrix} v_1^{(1)}(s) \\ \vdots \\ \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(1)})_i(s) \\ \vdots \\ v_n^{(1)}(s) \end{pmatrix}) - F_i(s+kt_1, \begin{pmatrix} v_1^{(2)}(s) \\ \vdots \\ \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(2)})_i(s) \\ \vdots \\ v_n^{(2)}(s) \end{pmatrix})]^2 \|_{L_1(\mathcal{O})} \,\mathrm{d}s.
$$

Using (4), we get for each component

$$
\begin{aligned}
&[F_i(s+kt_1, \begin{pmatrix} v_1^{(1)}(s) \\ \vdots \\ \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(1)})_i(s) \\ \vdots \\ v_n^{(1)}(s) \end{pmatrix}) - F_i(s+kt_1, \begin{pmatrix} v_1^{(2)}(s) \\ \vdots \\ \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(2)})_i(s) \\ \vdots \\ v_n^{(2)}(s) \end{pmatrix})]^2 \\
&\leq\; C(T)^2 \Big\{ (v_1^{(1)}(s) - v_1^{(2)}(s))^2 + \ldots \\
&\qquad + (\Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(1)})_i(s) - \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(2)})_i(s))^2 + \ldots + (v_n^{(1)}(s) - v_n^{(2)}(s))^2 \Big\} \\
&=\; C(T)^2 \Big\{ \|\mathbf{v}^{(1)}(s) - \mathbf{v}^{(2)}(s)\|_{\mathbb{R}^n}^2 \\
&\qquad + (\Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(1)})_i(s) - \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(2)})_i(s))^2 - (v_i^{(1)}(s) - v_i^{(2)}(s))^2 \Big\}
\end{aligned}
$$

and, summing up, we find

$$
\begin{aligned}
I_1 \;&\leq\; 2C(T)^4\, t \int_0^t \mathbf{E}\| \Big\{ n\, \|\mathbf{v}^{(1)}(s) - \mathbf{v}^{(2)}(s)\|_{\mathbb{R}^n}^2 + \|\Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(1)})(s) - \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(2)})(s)\|_{\mathbb{R}^n}^2 \\
&\qquad\qquad - \|\mathbf{v}^{(1)}(s) - \mathbf{v}^{(2)}(s)\|_{\mathbb{R}^n}^2 \Big\} \|_{L_1(\mathcal{O})} \,\mathrm{d}s \\
&=\; 2(n-1)C(T)^4\, t \int_0^t \mathbf{E}\|\mathbf{v}^{(1)}(s) - \mathbf{v}^{(2)}(s)\|_2^2 \,\mathrm{d}s \\
&\qquad\qquad + 2C(T)^4\, t \int_0^t \mathbf{E}\|\Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(1)})(s) - \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(2)})(s)\|_2^2 \,\mathrm{d}s \\
&\leq\; 2(n-1)C(T)^4\, t \int_0^t \mathbf{E}\|\mathbf{v}^{(1)}(s) - \mathbf{v}^{(2)}(s)\|_2^2 \,\mathrm{d}s \\
&\qquad\qquad + 2C(T)^4\, T \int_0^t \mathbf{E}\|\Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(1)})(s) - \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(2)})(s)\|_2^2 \,\mathrm{d}s.
\end{aligned}
$$

Using (6), I_2 is trivially estimated by

$$I_2 \le 4C(T)^4\, C(B) \int_0^t \mathbf{E}\|\Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(1)})(s) - \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(2)})(s)\|_2^2 \,\mathrm{d}s.$$

All in all, we obtain

$$\begin{aligned}\mathbf{E}\|\Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(1)})(t) - \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(2)})(t)\|_2^2 \;&\le\; 2(n-1)C(T)^4\, t \int_0^t \mathbf{E}\|\mathbf{v}^{(1)}(s) - \mathbf{v}^{(2)}(s)\|_2^2 \,\mathrm{d}s \\ &+\; (2T + 4C(B))C(T)^4 \int_0^t \mathbf{E}\|\Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(1)})(s) - \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(2)})(s)\|_2^2 \,\mathrm{d}s\end{aligned}$$

for all $t \in [0, t_1]$. Then it is surely true that for all $t \in [0, t_1]$

$$\begin{aligned}\mathbf{E}\|\Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(1)})(t) - \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(2)})(t)\|_2^2 \;&\le\; 2(n-1)C(T)^4\, t_1^2\, \|\mathbf{v}^{(1)} - \mathbf{v}^{(2)}\|_{\mathbb{B}_{kt_1}}^2 \\ &+\; (2T + 4C(B))C(T)^4 \int_0^t \mathbf{E}\|\Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(1)})(s) - \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(2)})(s)\|_2^2 \,\mathrm{d}s\end{aligned}$$

holds. This implies

$$\begin{aligned}&\mathbf{E}\|\Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(1)})(t) - \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(2)})(t)\|_2^2 \\ &\qquad \le\; 2(n-1)C(T)^4\, t_1^2\, \|\mathbf{v}^{(1)} - \mathbf{v}^{(2)}\|_{\mathbb{B}_{kt_1}}^2 \exp\{(2T + 4C(B))C(T)^4 \cdot t\},\ t \le t_1, \end{aligned} \tag{11}$$

by using the Gronwall lemma if the function $s \mapsto \mathbf{E}\|\Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(1)})(s) - \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(2)})(s)\|_2^2$ is bounded. But the last is satisfied because $\Gamma_{k,\boldsymbol{\xi}}$ maps $\mathbb{B}_{kt_1}$ into $\mathbb{B}_{kt_1}$. From (11) immediately follows

$$\|\Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(1)}) - \Gamma_{k,\boldsymbol{\xi}}(\mathbf{v}^{(2)})\|_{\mathbb{B}_{kt_1}}^2 \;\le\; t_1^2\, C(n,T,B)\, \|\mathbf{v}^{(1)} - \mathbf{v}^{(2)}\|_{\mathbb{B}_{kt_1}}^2$$

where the last constant

$$C(n,T,B) \;=\; 2(n-1)C(T)^4 \exp\{(2T + 4C(B))C(T)^4 \cdot T\}$$

neither depends on $\boldsymbol{\xi}$ nor k. As a consequence, $\Gamma_{k,\boldsymbol{\xi}}$ is a <u>contraction</u> on $\mathbb{B}_{kt_1}$ for all k and $\boldsymbol{\xi}$ if we choose $t_1 = T/N$ sufficiently small.

Remember the pathwise unique solution $\mathbf{u}_{kt_1}$ of equation (9). Using the Fixpoint theorem of Banach we get that

$$\|\mathbf{u}_{kt_1} - \Gamma^{\nu}_{k,\mathbf{u}(kt_1)}(\mathbf{v})\|_{\mathbb{B}_{kt_1}} \to 0,\ \nu \to \infty, \tag{12}$$

because the limit in $\mathbb{B}_{kt_1}$ of the sequence $\mathbf{v}, \Gamma_{k,\mathbf{u}(kt_1)}(\mathbf{v}), \Gamma_{k,\mathbf{u}(kt_1)}(\Gamma_{k,\mathbf{u}(kt_1)}(\mathbf{v})), \ldots$, is a solution of (9) starting from $\mathbf{u}(kt_1)$ for each $\mathbf{v} \in \mathbb{B}_{kt_1}$.

Return to the solutions $\mathbf{u}^{(1)}$, $\mathbf{u}^{(2)}$ we want to compare. Clearly, for the proof of the theorem it is sufficient that for each $k \in \{0, \ldots, N-1\}$

$$\mathbf{u}^{(1)}_{kt_1}(t) \;\le\; \mathbf{u}^{(2)}_{kt_1}(t) \quad \mathbf{P}\text{-a.s.},\ t \in [0, t_1], \tag{13}$$

holds true. Fix $\mathbf{v} \in \mathbb{B}_{kt_1}$ additionally satisfying that $(\omega, y) \mapsto \mathbf{v}(t, \omega, y)$ is product measurable for each $t \in [0, t_1]$. If we can prove

$$\Gamma^{\nu}_{k,\mathbf{u}^{(1)}(kt_1)}(\mathbf{v})(t) \;\le\; \Gamma^{\nu}_{k,\mathbf{u}^{(2)}(kt_1)}(\mathbf{v})(t) \quad \mathbf{P}\text{-a.s.},\ t \in [0, t_1], \tag{14}$$

for every $\nu = 0, 1, 2, \ldots$, then (13) follows from assertion (12) since at least a subsequence of $\{\Gamma^{\nu}_{k,\mathbf{u}^{(l)}(kt_1)}(\mathbf{v})(t)\}_{\nu=0}^{\infty}$ converges to $\mathbf{u}^{(l)}_{kt_1}(t)$ in $\mathbf{L}^2(\mathcal{O})$ $\mathbf{P}$-a.s., $t \in [0, t_1]$, $l = 1, 2$. Simplifying the notation, we settle that $\Gamma_{k,\mathbf{u}^{(l)}(kt_1)}(\mathbf{v})$ corresponds to the solution of equation (10) starting from $\mathbf{u}^{(l)}(kt_1)$ with drift $\mathbf{F}^{(l)}_{\mathbf{v}}$ instead of $\mathbf{F}_{\mathbf{v}}$.

Without loss of generality, we only prove (14) in the case $k=0$. Indeed, if (14) is proven for $k=0$ then (13) especially says that

$$\mathbf{u}_0^{(1)}(t_1) \;=\; \mathbf{u}^{(1)}(t_1) \;\le\; \mathbf{u}^{(2)}(t_1) \;=\; \mathbf{u}_0^{(2)}(t_1) \quad \mathbf{P}\text{-a.s.}$$

and the situation in the case $k=1$ is exactly the same as in the case $k=0$ (see also Remark 2a)). Repeating this procedure, (14) can be proven for $k=0,...,N-1$ by the same method, successively.

Let be $k=0$. We prove (14) by induction. Of course,

$$\Gamma^0_{0,\boldsymbol{\xi}^{(1)}}(\mathbf{v})(t) \;=\; \mathbf{v}(t) \;\le\; \mathbf{v}(t) \;=\; \Gamma^0_{0,\boldsymbol{\xi}^{(2)}}(\mathbf{v})(t),$$

and (14) holds true for $\nu=0$. For a fixed $\nu\ge 0$ set

$$\mathbf{v}^{(l)} \;=\; \Gamma^{\nu}_{0,\boldsymbol{\xi}^{(l)}}(\mathbf{v}),\; l=1,2,$$

and suppose that we already would have proved

$$\mathbf{v}^{(1)}(t) \;\le\; \mathbf{v}^{(2)}(t) \quad \mathbf{P}\text{-a.s.},\; t\in[0,t_1].$$

We have to show

$$\Gamma_{0,\boldsymbol{\xi}^{(1)}}(\mathbf{v}^{(1)})(t) \;\le\; \Gamma_{0,\boldsymbol{\xi}^{(2)}}(\mathbf{v}^{(2)})(t) \quad \mathbf{P}\text{-a.s.},\; t\in[0,t_1],$$

that is under the assumptions of the theorem, we have to compare solutions of equation (10) with different initial values $\boldsymbol{\xi}^{(1)},\boldsymbol{\xi}^{(2)}$, different functions $\mathbf{F}^{(1)},\mathbf{F}^{(2)}$ and different processes $\mathbf{v}^{(1)},\mathbf{v}^{(2)}\in\mathbb{B}_0$ additionally satisfying

$$\mathbf{v}^{(1)}(t) \;\le\; \mathbf{v}^{(2)}(t) \quad \mathbf{P}\text{-a.s.},\; t\in[0,t_1],$$

as well as $(\omega,y)\mapsto\mathbf{v}^{(l)}(t,\omega,y)$ is product measurable for each $t\in[0,t_1]$, $l=1,2$.

In what follows we denote the solution of equation (10) starting from $\boldsymbol{\xi}^{(l)}$ with drift $\mathbf{F}^{(l)}_{\mathbf{v}^{(l)}}$ by $\mathbf{u}^{(l)}$, $l=1,2$, and show

$$\mathbf{u}^{(1)}(t) \;\le\; \mathbf{u}^{(2)}(t) \quad \mathbf{P}\text{-a.s.},\; t\in[0,t_1]. \tag{15}$$

This will prove the theorem as discussed above.

To start with, we clarify some special notation which will be used to prove (15). Consider the i-th component G_{ij} of an arbitrary diffusion coefficient $\mathbf{G}_j$. Then it follows from condition (6) that for fixed $t\in[0,T],\omega\in\Omega,y\in\mathcal{O}$ the inverse image

$$\{\mathbf{x}\in\mathbb{R}^n : G_{ij}(t,\omega,y,\mathbf{x})\in I\}$$

of an arbitrary open interval $I\subseteq\mathbb{R}$ has to be of the structure

$$\mathbb{R}\times\cdots\times\mathcal{O}_i\times\cdots\times\mathbb{R}$$

where $\mathcal{O}_i\subseteq\mathbb{R}$ is an open set. In other words, $G_{ij}(t,\omega,y,\cdot)$ is $\{\emptyset,\mathbb{R}\}\otimes\cdots\otimes \underset{\downarrow}{\overset{i\text{-th}}{}}\mathfrak{B}(\mathbb{R})\otimes\cdots\otimes\{\emptyset,\mathbb{R}\}$–mesurable and, consequently, for every $t\in[0,T],\omega\in\Omega,y\in\mathcal{O}$ there exists a function $g_{ij}(t,\omega,y,\cdot):\mathbb{R}\to\mathbb{R}$ such that

$$G_{ij}(t,\omega,y,\mathbf{x}) \;=\; g_{ij}(t,\omega,y,x_i),\; \mathbf{x}\in\mathbb{R}^n.$$

Hence, the functions

$$(t,\omega,y,x)\mapsto g_{ij}(t,\omega,y,x)$$

are especially $\mathfrak{M}\otimes\mathfrak{B}(\mathcal{O})\otimes\mathfrak{B}(\mathbb{R})$ – measurable. Furthermore, they are lipschitz in the fourth variable uniformly in $t\in[0,T],\omega\in\Omega,y\in\mathcal{O}$ by (6) and satisfy the growth condition

$$|g_{ij}(t,\omega,y,x)|\ \le\ C(T)(1+|x|),\ x\in\mathbb{R},\ t\in[0,T],\omega\in\Omega,y\in\mathcal{O},$$

by (5).

For $\mathbf{u}=(u_1,...,u_n)\in\mathbf{L}^2(\mathcal{O})$ denote by $\mathcal{M}(u_i)$ the i-th component of the multiplication operator $\boldsymbol{\mathcal{M}}(\mathbf{u})$ (see page 2).

Now,using (7), the i-th component $u_i^{(l)}$ of the solution $\mathbf{u}^{(l)}$ satisfies the equation

$$\begin{aligned} u_i^{(l)}(t) &= S_i(t)\xi_i^{(l)}+\int_0^t S_i(t-s)F_i^{(l)}(s,\begin{pmatrix} v_1^{(l)}(s)\\ \vdots\\ u_i^{(l)}(s)\\ \vdots\\ v_n^{(l)}(s)\end{pmatrix})\,\mathrm{d}s\\ &\qquad +\sum_{j=1}^m\int_0^t S_i(t-s)\mathcal{M}(g_{ij}(s,u_i^{(l)}(s)))\circ B_j\,\mathrm{d}W_j(s)\end{aligned}$$

which does not depend on the other components $u_k^{(l)}$, $k\neq i$, that is, it is a separate stochastic partial differential equation. Here, $S_i(t)$ is a positivity preserving strongly continuous semigroup on $L^2(\mathcal{O})$, $(\omega,y)\mapsto\xi_i^{(l)}(\omega)(y)$ is product measurable, $l=1,2$, and it holds that

$$\xi_i^{(1)}\ \le\ \xi_i^{(2)}\quad \mathbf{P}\text{-a.s.}$$

as well as

$$\mathbf{E}\|\xi_i^{(l)}\|^2_{L^2(\mathcal{O})}<\infty,\ l=1,2.$$

The drift coefficients

$$f^{(l)}(s,\omega,y,x)\ :=\ F_i^{(l)}(s,\omega,y,\begin{pmatrix} v_1^{(l)}(s,\omega,y,)\\ \vdots\\ x\\ \vdots\\ v_n^{(l)}(s,\omega,y,)\end{pmatrix})$$

are real functions satisfying, on the one hand (4) together with a linear growth condition like (5) slightly disturbed by $\mathbf{v}^{(l)}\in\mathbb{B}_0$, and, on the other hand

(16) $\quad f^{(1)}(s,\cdot,y,x)\ \le\ f^{(2)}(s,\cdot,y,x),\quad x\in\mathbb{R}$, for Lebesgue a.a. $y\in\mathcal{O}$ $\mathbf{P}$-a.s., $s\in[0,t_1]$,

by $(C_{1,2})$ and

$$\mathbf{v}^{(1)}(s)\ \le\ \mathbf{v}^{(2)}(s)\quad \mathbf{P}\text{-a.s.},\ s\in[0,t_1].$$

All in all, the diffusion coefficients g_{ij} have already been discussed above, the assumptions of Th.3.3.1 in [5] are satisfied. Applying this theorem, we should only remark that its conclusion is true although our diffusion term is slightly more general than the corresponding one in the nuclear case considered in [5] (see also Remark 1c)); the extension of the result to coefficients also depending on $y\in\mathcal{O}$ with the above properties is completely trivial. So we get

$$u_i^{(1)}(t)\ \le\ u_i^{(2)}(t)\quad \mathbf{P}\text{-a.s.},\ t\in[0,t_1],$$

for $i=1,...,n$, that is (15). ■

Remark 3 From (16) immediately follows that

$$f^{(2)}(s,u_i^{(1)}(s)) - f^{(1)}(s,u_i^{(1)}(s)) \geq 0 \quad \mathbf{P}\text{-a.s., } s \in [0,t_1], \tag{17}$$

that is the **P**-exceptional set depends on s leading to complications. However, the conditions we suppose, imply that the $L^2(\mathcal{O})$-valued process $(f^{(2)}(s,u_i^{(1)}(s)) - f^{(1)}(s,u_i^{(1)}(s)))_{s\in[0,t_1]}$ is progressively measurable and this "regularity" allows to apply the Theorem of Fubini to

$$\begin{aligned} 0 &= \int_0^{t_1} \mathbf{E} \int_{\mathcal{O}} [(f^{(2)}(s,u_i^{(1)}(s)) - f^{(1)}(s,u_i^{(1)}(s)))(y)]^- \, \mathrm{d}y\mathrm{d}s \\ &= \mathbf{E} \int_0^{t_1} \int_{\mathcal{O}} [(f^{(2)}(s,u_i^{(1)}(s)) - f^{(1)}(s,u_i^{(1)}(s)))(y)]^- \, \mathrm{d}y\mathrm{d}s \end{aligned}$$

where $[...]^-$ denotes the negative part of the corresponding function. As a consequence, instead of (17) appears

$$f^{(2)}(s,u_i^{(1)}(s)) - f^{(1)}(s,u_i^{(1)}(s)) \geq 0 \quad \text{for Lebesgue-a.a. } s \in [0,t_1] \ \mathbf{P}\text{-a.s.}$$

we actually need.

Corollary 1 *Suppose that, except $(C_{1,2})$, all assumptions of Theorem 1 are satisfied and replace $(C_{1,2})$ by (8). If now one of the drift coefficients $\mathbf{F}^{(1)}$ resp. $\mathbf{F}^{(2)}$ is additionally quasi-monotonously increasing (see Remark 2d)) then*

$$\mathbf{u}^{(1)}(t) \leq \mathbf{u}^{(2)}(t) \quad \mathbf{P}\text{-}a.s.,\ t \in [0,T].$$

PROOF. Without restricting the generality, we choose $\mathbf{F}^{(2)}$ to be quasi-monotonously increasing and repeat the proof of Theorem 1. There is only one point where the condition $(C_{1,2})$ has been applied, namely, in order to cause (16). But, we know that $\mathbf{v}^{(1)}(s,\omega,y) \leq \mathbf{v}^{(2)}(s,\omega,y)$ implies

$$F_i^{(2)}(s,\omega,y,\begin{pmatrix} v_1^{(1)}(s,\omega,y) \\ \vdots \\ x \\ \vdots \\ v_n^{(1)}(s,\omega,y) \end{pmatrix}) \leq F_i^{(2)}(s,\omega,y,\begin{pmatrix} v_1^{(2)}(s,\omega,y) \\ \vdots \\ x \\ \vdots \\ v_n^{(2)}(s,\omega,y) \end{pmatrix}) = f^{(2)}(s,\omega,y,x),\ x \in \mathbb{R},$$

for Lebesgue a.a. $y \in \mathcal{O}$, **P**-a.a. $\omega \in \Omega$, $s \in [0,t_1]$, because $\mathbf{F}^{(2)}$ is quasi-monotonously increasing. So, additionally using (8), we get

$$f^{(1)}(s,\cdot,y,x) = F_i^{(1)}(s,\cdot,y,\begin{pmatrix} v_1^{(1)}(s,\cdot,y) \\ \vdots \\ x \\ \vdots \\ v_n^{(1)}(s,\cdot,y) \end{pmatrix}) \leq F_i^{(2)}(s,\cdot,y,\begin{pmatrix} v_1^{(1)}(s,\cdot,y) \\ \vdots \\ x \\ \vdots \\ v_n^{(1)}(s,\cdot,y) \end{pmatrix}) \leq f^{(2)}(s,\cdot,y,x),\ x \in \mathbb{R},$$

for Lebesgue a.a. $y \in \mathcal{O}$ **P**-a.s., $s \in [0,t_1]$, that is (16) is satisfied, in spite of it. ■

From Theorem 1 immediately follows that one can compare solutions of equation $(\star)$ with a fixed drift starting from different initial values if the fixed drift function is quasi-monotonously increasing. The next corollary shows that one can weaken this condition on the drift in special situations.

Corollary 2 *Let be given an $\mathcal{F}_0$-measurable square integrable random variable $\boldsymbol{\xi} : \Omega \to \mathbf{L}^2(\mathcal{O})$ satisfying*

$$\mathbf{0} \le \boldsymbol{\xi} \quad \mathbf{P}\text{-}a.s.,$$

which is product measurable, seen as a function $(\omega, y) \mapsto \boldsymbol{\xi}(\omega)(y)$. Suppose (4),(5),(6) for $\mathbf{F} = \mathbf{F}^{(1)} = \mathbf{F}^{(2)}$, $\mathbf{G}_j$, $j = 1, ..., m$, and denote by $\mathbf{u}$ the pathwise unique solution of the equation $(\star)$ starting from $\boldsymbol{\xi}$. Furthermore, we demand

$$\mathbf{F}(t, \cdot, y, \mathbf{0}) \;=\; \mathbf{0} \quad \text{for Lebesgue a.a. } y \in \mathcal{O} \ \mathbf{P}\text{-a.s.}, \ t \in [0, T],$$

as well as for each $j = 1, ..., m$

$$\mathbf{G}_j(t, \cdot, y, \mathbf{0}) \;=\; \mathbf{0} \quad \text{for Lebesgue a.a. } y \in \mathcal{O} \ \mathbf{P}\text{-a.s.}, \ t \in [0, T],$$

that is the process which is constantly equal to zero is the pathwise unique solution of $(\star)$ starting from $\mathbf{0}$. Then the conditions (C_{pp}) and

$$(C_{\mathbf{F}}) \qquad \begin{cases} \text{For any } t \in [0,T] \text{ we have} \\ \qquad \mathbf{F}_{\mathbf{z}}(t, \cdot, y, \mathbf{0}) \;\ge\; \mathbf{0}, \ \mathbf{z} \ge \mathbf{0}, \\ \text{for Lebesgue a.a. } y \in \mathcal{O} \ \mathbf{P}\text{-a.s.} \end{cases}$$

imply for all $t \in [0, T]$

$$\mathbf{u}(t) \;\ge\; \mathbf{0} \quad \mathbf{P}\text{-}a.s.$$

Proof. For the conclusion we want to show, again except $(C_{1,2})$, all assumptions needed to apply Theorem 1 are satisfied. As in the proof of Corollary 1 we only had to cause (16), but the more special situation that one of the processes we want to compare is identically zero implies, that we can weaken (16) to be true only for $x = 0$.

Indeed, remember that in the proof of Theorem 1 we have set

$$\mathbf{v}^{(1)} \;=\; \Gamma^{\nu}_{0,\boldsymbol{\xi}^{(1)}}(\mathbf{v})$$

at a certain level of approximation ν where $\mathbf{v}$ has been arbitrarily chosen, before.

According to the proof of our corollary, $\mathbf{v}^{(1)}$ approximates the solution starting from $\boldsymbol{\xi}^{(1)} = \mathbf{0}$ which is identically zero, and, if we start with $\mathbf{v}$ to be chosen identically zero then $\mathbf{v}^{(1)}$ is also identically zero. Hence, we obtain

$$f^{(1)}(s, \omega, y, x) \;=\; F_i(s, \omega, y, \begin{pmatrix} 0 \\ \vdots \\ x \\ \vdots \\ 0 \end{pmatrix}).$$

From the proof of Th.3.3.1 given in [5] we know that (16) has essentially been used to provide for (17). As you know, $u_i^{(1)}$ vanishes identically and, because of our assumption on $\mathbf{F}(\cdot, \cdot, \cdot, \mathbf{0})$, it must be satisfied that

$$f^{(1)}(s, u_i^{(1)}(s)) \;=\; 0 \quad \mathbf{P}\text{-a.s.}, \ s \in [0, t_1].$$

That means, what we really need in order to prove our corollary is

$$F_i(s, \begin{pmatrix} v_1^{(2)}(s) \\ \vdots \\ 0 \\ \vdots \\ v_n^{(2)}(s) \end{pmatrix}) = f^{(2)}(s, u_i^{(1)}(s)) \geq 0 \quad \mathbf{P}\text{-a.s., } s \in [0, t_1],$$

if $\mathbf{v}^{(2)}(s) \geq \mathbf{0}$ $\mathbf{P}$-a.s., $s \in [0, t_1]$, which is an easy consequence of $(C_{\mathbf{F}})$. ■

Corollary 3 *Assume that either the assumptions of Theorem 1 or Corollary 1 or Corollary 2 are satisfied, but, consider the special situation $\mathcal{O} = [0,1]$, that is $d = 1$, and $A_1 = ... = A_n = \Delta$ where Δ denotes the Laplacian with Dirichlet boundary conditions. Furthermore, we drop condition (2) and work with the operators $B_1 = ... = B_m = Id$ (see also Remark 1c)). Then the conclusion of Theorem 1 resp. Corollary 1 resp. Corollary 2 remains true.*

PROOF. It suffices to consider this special situation with respect to Theorem 1. We follow the proof of Theorem 1 but now, in order to show (15), on page 10 we apply Th.2.1 in [3] instead of Th.3.3.1 in [5]. ■

5 Application

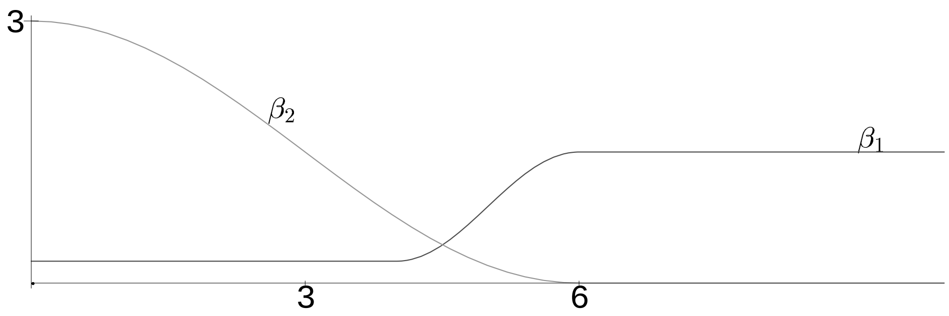
To start with, we consider a deterministic predator-prey system, that is $n = 2$, of the form

$$\begin{pmatrix} \frac{\partial}{\partial t} u_1 \\ \frac{\partial}{\partial t} u_2 \end{pmatrix} = \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix} + \boldsymbol{\mathcal{D}}(u_1, u_2) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{18}$$

Here, Δ denotes the Laplacian with Neumann boundary conditions in a bounded 3-dimensional domain $\mathcal{O}$ with smooth boundary, that is $d = 3$, and the matrix $\boldsymbol{\mathcal{D}}(u_1, u_2)$ looks like

$$\boldsymbol{\mathcal{D}}(u_1, u_2) = \begin{pmatrix} -\beta_1(|\frac{u_2}{u_1}|) & c\,\beta_2(|\frac{u_2}{u_1}|) \\ 0 & [\gamma - \beta_2(|\frac{u_2}{u_1}|)] \end{pmatrix}$$

where $c, \gamma > 0$ are constants and β_1, β_2 are smooth functions on $\mathbb{R}_+$. The following picture shows a possible type of functions we want to deal with:



Hence, the assumptions in the model (18) are: (i) The prey in the absence of any predation grows unboundedly in a Malthusian way; this is the γu_2 term. (ii) The prey's contribution to the predator's growth rate is $c\,\beta_2(|\frac{u_2}{u_1}|)u_2$; that is, it is proportional to the available prey if the ratio between prey and predator is well-balanced, but it becomes smaller if the prey starts to outnumber the predator's population and can withstand its attacks. (iii) The growth of the predator population is taken from the prey population; this is the $-\beta_2(|\frac{u_2}{u_1}|)u_2$ term. (iv) In the absence of any prey the predator's death rate results in exponential decay, that is the $-\beta_1(|\frac{u_2}{u_1}|)u_1$ term for $u_2 = 0$. The factor $-\beta_1(|\frac{u_2}{u_1}|)$ further says that this exponential decay even may increase if the prey population outnumbers the predators so much that it can "fight off" the predator's attacks, and the predator's habitat shrinks as a consequence.

Of course, this model lacks of reality and should only be seen as a jumping-off place for better models. One of the unrealistic assumptions in it, also appearing in the simple but well-known Volterra model, is that the prey growth is unbounded in the absence of predation. To prevent this, one could add a control term on the right-hand side of the equation for u_2 as decribed, for example, in section 3.3, [7]. We refer to a forthcoming paper for a more realistic model with coefficients which even have singularities.

Returning to (18), the main problem surely consists in finding the true parameter γ, β_1, β_2 which is certainly impossible. As a consequence, the matrix $\boldsymbol{\mathcal{D}}(u_1, u_2)$ includes some kind of uncertainty. A well-known possibility to take hold of such an uncertainty of an object is to add a noise to it, that is, for example, to consider

$$\boldsymbol{\mathcal{D}}(u_1, u_2) + \dot{\eta} \cdot I, \ I : \text{unitary matrix},$$

instead of $\boldsymbol{\mathcal{D}}(u_1, u_2)$ leading to the stochastic system

$$\begin{pmatrix} \frac{\partial}{\partial t} u_1 \\ \frac{\partial}{\partial t} u_2 \end{pmatrix} = \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix} + [\boldsymbol{\mathcal{D}}(u_1, u_2) + \dot{\eta} \cdot I] \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{19}$$

we understand in the sense of equation $(\star)$.

We will discuss the stability of the ratio $u_2(t)/u_1(t)$ with respect to the deterministic as well as the stochastic system for different initial values $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathbf{L}^2(\mathcal{O})$ where $\xi_1 \geq 0$ resp. $\xi_2 \geq 0$ describe the distribution of the predator resp. prey population at the beginning of our observation.

At first, for fixed $c > 0$ we introduce the function

$$\gamma = (1 + c\alpha)\beta_2(\alpha) - \beta_1(\alpha), \ \alpha \geq 0.$$

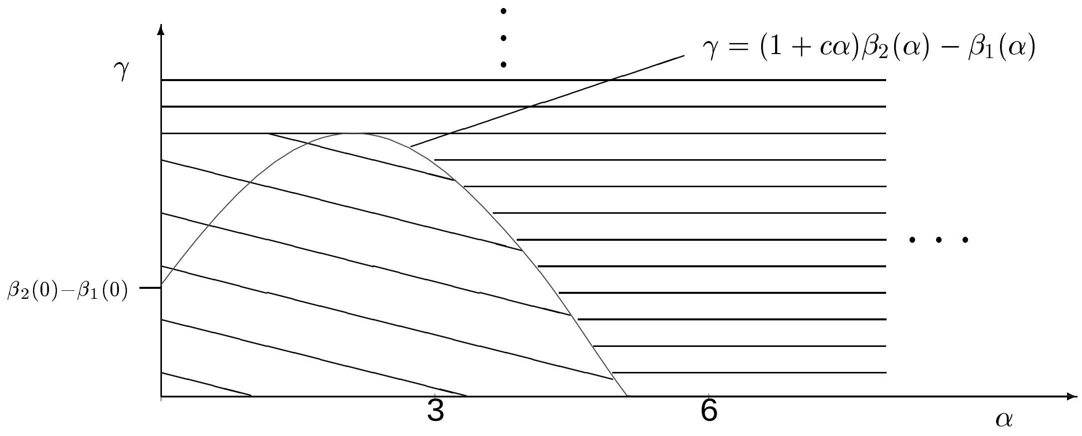
The graph

$$\mathcal{G} = \{(\alpha, \gamma) \in \mathbb{R}_+ \times \mathbb{R} : \gamma = (1 + c\alpha)\beta_2(\alpha) - \beta_1(\alpha)\}$$

of this function divides the area

$$\mathcal{P} = \{(\alpha, \gamma) : \alpha > 0, \gamma > 0\}$$

into parts; two of them we want to emphasize in the following picture based on the functions β_1, β_2 introduced in the last figure.



We define

$$\mathcal{P}_1' \quad : \quad \text{open subarea of } \mathcal{P} \text{ which is marked by diagonal lines}$$

respectively

$$\mathcal{P}_2 \quad : \quad \text{open subarea of } \mathcal{P} \text{ which is marked by horizontal lines}$$

and set

$$\mathcal{P}_1 \;=\; \mathcal{P}_1' \cup \{(\alpha, (1+c\hat{\alpha})\beta_2(\hat{\alpha}) - \beta_1(\hat{\alpha}) : \alpha \in [0, \hat{\alpha}]\}$$

where $\hat{\alpha}$ satisfies

$$(1+c\hat{\alpha})\beta_2(\hat{\alpha}) - \beta_1(\hat{\alpha}) = \max_{\alpha \geq 0}[(1+c\alpha)\beta_2(\alpha) - \beta_1(\alpha)].$$

Notice that

$$\mathcal{P} \setminus (\mathcal{P}_1 \cup \mathcal{P}_2) \;=\; \{(\alpha, \gamma) \in \mathcal{G} \cap \mathcal{P} : \alpha > \hat{\alpha}\}.$$

Here, we interpret γ to be the parameter appearing in $\boldsymbol{\mathcal{D}}(u_1, u_2)$ and α stands for a lower resp. upper bound of the ratio ξ_2/ξ_1. Clearly, each pair $(\alpha, \gamma) \in \mathcal{P}$ corresponds to a special situation of the model (18) with fixed parameters c, β_1, β_2. We want to investigate how the behavior in time of the ratio $u_2(t)/u_1(t)$ depends on (α, γ) if $\mathbf{u} = (u_1, u_2)$ is the solution of (18) with respect to (α, γ).

Before doing this, we still distinguish a further subarea of $\mathcal{P}$, namely, the open subarea which is surrounded by $\mathcal{G}$ and the axes:

$$\mathcal{P}_{\mathcal{G}} \;=\; \{(\alpha, \gamma) \in \mathcal{P}_1 : \gamma < (1+c\alpha)\beta_2(\alpha) - \beta_1(\alpha)\}.$$

Proposition 1 *For each $(\alpha, \gamma) \in \mathcal{P}$ define the mapping*

$$(\alpha, \gamma) \mapsto (\alpha_0, \gamma)$$

by

$$\alpha_0 \;=\; \inf\{\tilde{\alpha} \geq 0 : \gamma \leq (1+c\tilde{\alpha})\beta_2(\tilde{\alpha}) - \beta_1(\tilde{\alpha})\}$$

and settle $\inf \emptyset$ to be zero.

(i) Fix $(\alpha, \gamma) \in \mathcal{P}_{\mathcal{G}}$ and assume that $\xi_2/\xi_1 \leq \alpha$. Then it holds that

$$\forall n \in \mathbb{N}\; \exists\, t_n \geq 0\; \forall\, t \geq t_n :\; 0 \leq u_2(t)/u_1(t) \leq \alpha_0 + \frac{1}{n}.$$

(ii) Choose (α, γ) as in (i) and further assume that $\gamma \leq \beta_2(0) - \beta_1(0)$. Then it holds that

$$u_2(t, y)/u_1(t, y) \overset{t\to\infty}{\longrightarrow} 0, \quad \forall y \in \mathcal{O}.$$

(iii) Fix $(\alpha, \gamma) \in \mathcal{P}_1$ and assume that $\xi_2/\xi_1 \leq \alpha$, $\gamma > \beta_2(0) - \beta_1(0)$ as well as $\xi_2/\xi_1 \geq \varepsilon$ for some $\varepsilon > 0$. Then it holds that

$$u_2(t, y)/u_1(t, y) \overset{t\to\infty}{\longrightarrow} \alpha_0, \quad \forall y \in \mathcal{O}.$$

(iv) Fix $(\alpha, \gamma) \in \mathcal{P}_2$ and assume that $\xi_2/\xi_1 \geq \alpha$. Then it holds that

$$u_2(t, y)/u_1(t, y) \overset{t\to\infty}{\longrightarrow} \infty, \quad \forall y \in \mathcal{O}.$$

(v) If $(\alpha, \gamma) \in \mathcal{P} \cap \mathcal{G}$ and $\xi_2/\xi_1 = \alpha$ then

$$u_2(t)/u_1(t) \;=\; \alpha,\; t \geq 0,$$

even holds.

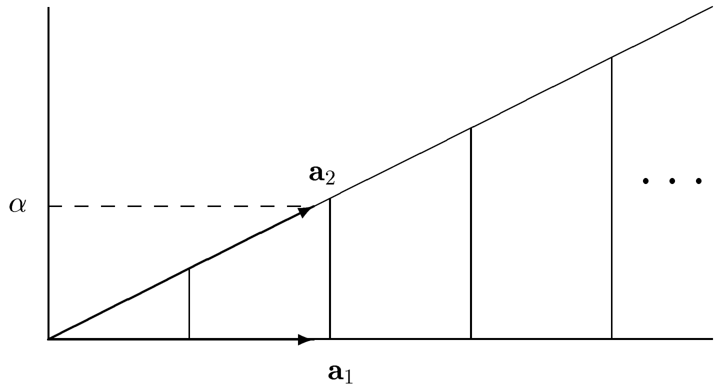
Remark 4 The above result depends on the special feature of the sets $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_{\mathcal{G}}$ which are closely related with β_1, β_2 given in the figure on page 13. But the following proof clears how to decompose $\mathcal{P}$ for other examples of β_1, β_2.

PROOF. Using the semigroup $S(t) = \exp\{t\Delta\}$, the solution $\mathbf{u} = (u_1, u_2)$ of (18) satisfies the equation

$$(18')\qquad \left\{\begin{array}{lcl} u_1(t) & = & S(t)\xi_1 + \int_0^t S(t-s)\left[-\beta_1(|\frac{u_2(s)}{u_1(s)}|)u_1(s) + c\,\beta_2(|\frac{u_2(s)}{u_1(s)}|)u_2(s)\right]\,\mathrm{d}s \\ u_2(t) & = & S(t)\xi_2 + \int_0^t S(t-s)\left[\gamma - \beta_2(|\frac{u_2(s)}{u_1(s)}|)\right]u_2(s)\,\mathrm{d}s \end{array}\right.$$

which is a special case of $(\star)$ without diffusion.

(i) Fix $n \in \mathbb{N}$ as well as $(\alpha, \gamma) \in \mathcal{P}_{\mathcal{G}}$, and introduce a cone in the 2-dimensional plane which is spanned by the vectors $\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{a}_2 = \begin{pmatrix} 1 \\ \alpha \end{pmatrix}$:



The assumption $\xi_2/\xi_1 \le \alpha$ on the initial distribution $\boldsymbol{\xi} \ge \mathbf{0}$ means that $\boldsymbol{\xi}$ lies in the above cone and, we have to prove that there exists a time t_n such that, for all $t \ge t_n$, $\mathbf{u}(t)$ lies in the cone spanned by $\mathbf{a}_1$ and, now, $\mathbf{a}_2(t_n) = \begin{pmatrix} 1 \\ \alpha_0 + \frac{1}{n} \end{pmatrix}$ instaed of $\mathbf{a}_2(0) = \mathbf{a}_2$. But the last assertion holds if and only if the coordinates of $\mathbf{u}(t)$ with respect to the coordinate system $\{\mathbf{a}_1, \mathbf{a}_2(t_n)\}$ of $\mathbb{R}^2$ are nonnegative for all $t \ge t_n$. In other words, if $\mathfrak{A}(t_n)$ represents the matrix with the columns $\mathbf{a}_1$ resp. $\mathbf{a}_2(t_n)$, that is

$$\mathfrak{A}(t_n) = \begin{pmatrix} 1 & 1 \\ 0 & \alpha_0 + \frac{1}{n} \end{pmatrix},$$

then we have to verify that

$$\mathfrak{A}^{-1}(t_n)\mathbf{u}(t) \ge \mathbf{0},\ t \ge t_n.$$

In order to show this, we introduce invertable matrices

$$\mathfrak{A}(t) = \begin{pmatrix} 1 & 1 \\ 0 & \alpha(t) \end{pmatrix}$$

where $t \mapsto \alpha(t) \in (0, \infty)$, $t \ge 0$, is a convex function satisfying

$$\alpha(0) = \alpha \quad \text{and} \quad \alpha(t) \le \alpha_0 + \frac{1}{n},\ t \ge t_n,$$

for a time t_n we specify later. If we can find such a function $t \mapsto \alpha(t)$, $t \geq 0$, that

$$\tilde{\mathbf{u}}(t) := \mathfrak{A}^{-1}(t)\mathbf{u}(t) \geq \mathbf{0},\ t \in [0,T], \tag{20}$$

for an arbitrary but fixed time T then (i) is proven.

But, an easy linear transformation of (18′) shows that $\tilde{\mathbf{u}}$ satisfies the equation

$$\tilde{\mathbf{u}}(t) = \mathbf{S}(t)\mathfrak{A}^{-1}(0)\boldsymbol{\xi} + \int_0^t \mathbf{S}(t-s)\mathbf{F}(s,\tilde{\mathbf{u}}(s))\,\mathrm{d}s$$

with (let $\alpha'_+(t)$ denote the right-hand derivative of $\alpha(t)$)

$$\mathbf{F}(t,\mathbf{x}) = \begin{pmatrix} -\beta_1(|\frac{\alpha(t)x_2}{x_1+x_2}|)(x_1+x_2) + \beta_2(|\frac{\alpha(t)x_2}{x_1+x_2}|)(1+c\alpha(t))x_2 - \gamma x_2 + \frac{\alpha'_+(t)}{\alpha(t)}x_2 \\ [\gamma - \beta_2(|\frac{\alpha(t)x_2}{x_1+x_2}|)]x_2 - \frac{\alpha'_+(t)}{\alpha(t)}x_2 \end{pmatrix}$$

and

$$\mathbf{S}(t) = \begin{pmatrix} \exp\{t\Delta\} \\ \exp\{t\Delta\} \end{pmatrix}.$$

We further mention that (C_{pp}), $\mathfrak{A}^{-1}(0)\boldsymbol{\xi} \in \mathbf{L}^2(\mathcal{O})$, $\mathfrak{A}^{-1}(0)\boldsymbol{\xi} \geq \mathbf{0}$, $\mathbf{F}(t,\mathbf{0}) = \mathbf{0}$, $t \in [0,T]$, and (4),(5) are satisfied. So, in order to apply Corollary 2, we only have to verify $(C_{\mathbf{F}})$ which is here equivalent to

$$\gamma \leq (1+c\alpha(t))\beta_2(\alpha(t)) - \beta_1(\alpha(t)) + \frac{\alpha'_+(t)}{\alpha(t)},\ t \in [0,T]. \tag{21}$$

Hence, the last condition is sufficient for (20) and, consequently, (i) is true if we find a time t_n and a convex function $t \mapsto \alpha(t) \in (0,\infty)$, $t \geq 0$, such that, on the one hand, the above condition holds true and, on the other hand,

$$\alpha(0) = \alpha \quad \text{and} \quad \alpha(t) \leq \alpha_0 + \frac{1}{n},\ t \geq t_n.$$

In what follows we construct t_n and $\alpha(t)$ with the last properties.

Remind $(\alpha,\gamma) \in \mathcal{P}_{\mathcal{G}}$. This yields

$$\gamma < (1+c\tilde{\alpha})\beta_2(\tilde{\alpha}) - \beta_1(\tilde{\alpha}),\ \forall \tilde{\alpha} \in (\alpha_0,\alpha].$$

Thus, if $\alpha \leq \alpha_0 + \frac{1}{n}$ then we may set

$$t_n = 0 \quad \text{and} \quad \alpha(t) = \alpha,\ t \geq 0,$$

and the desired properties are trivially satisfied. Otherwise, if $\alpha_0 + \frac{1}{n} < \alpha$ then we have

$$\gamma \leq (1+c\tilde{\alpha})\beta_2(\tilde{\alpha}) - \beta_1(\tilde{\alpha}) - c_n,\ \forall \tilde{\alpha} \in [\alpha_0 + \frac{1}{n},\alpha],$$

where

$$c_n := \min\{(1+c\tilde{\alpha})\beta_2(\tilde{\alpha}) - \beta_1(\tilde{\alpha}) - \gamma : \tilde{\alpha} \in [\alpha_0 + \frac{1}{n},\alpha]\}$$

is positive. Now we define

$$t_n = \frac{\ln\alpha - \ln(\alpha_0 + \frac{1}{n})}{c_n} \quad \text{and} \quad \alpha(t) = \begin{cases} \alpha \mathrm{e}^{-c_n t} & : \ t \leq t_n \\ \alpha \mathrm{e}^{-c_n t_n} & : \ t \geq t_n \end{cases}.$$

Clearly, this function $\alpha(t)$ satisfies (21) as well as $\alpha(0) = \alpha$, $\alpha(t) = \alpha_0 + \frac{1}{n}$, $t \geq t_n$, finishing the proof of (i).

(ii) If $\gamma \leq \beta_2(0) - \beta_1(0)$ then $\alpha_0 = 0$. Applying (i) we get

$$\forall n \in \mathbb{N}\ \exists\, t_n \geq 0\ \forall\, t \geq t_n :\ 0 \leq u_2(t,y)/u_1(t,y) \leq \frac{1}{n} \text{ for Lebesgue a.a. } y \in \mathcal{O}.$$

But, using the smoothing properties of $S(t) = \exp\{t\Delta\}$, $u_1(t,y)$ and $u_2(t,y)$ are continuous in $(t,y) \in (0,\infty) \times \mathcal{O}$ leading to

$$\forall n \in \mathbb{N}\ \exists\, t_n \geq 0\ \forall\, t \geq t_n\ \forall y \in \mathcal{O} :\ 0 \leq u_2(t,y)/u_1(t,y) \leq \frac{1}{n}$$

which is exactly the conclusion of (ii).

(iii) Fix $(\alpha,\gamma) \in \mathcal{P}_1$. Under the assumptions of (iii) we will show that

$$\forall n \in \mathbb{N}\ \exists\, t_n \geq 0\ \forall\, t \geq t_n :\ \alpha_0 - \frac{1}{n} \leq u_2(t)/u_1(t) \leq \alpha_0 + \frac{1}{n}.$$

Then the conclusion of (iii) follows as in the proof of (ii).

Fix $n \in \mathbb{N}$. If $(\alpha,\gamma) \in \mathcal{P}_{\mathcal{G}}$ then we can immediately apply (i). If $(\alpha,\gamma) \in \mathcal{P}_1 \setminus \mathcal{P}_{\mathcal{G}}$ then $\alpha \leq \alpha_0$ and with it $\xi_2/\xi_1 \leq \alpha_0$. Now, the proof of (i) makes clear that

$$u_2(t)/u_1(t) \leq \alpha_0,\ t \geq 0,$$

because $(\alpha,\gamma) \in \mathcal{P}_1 \setminus \mathcal{P}_{\mathcal{G}}$ also implies

$$\gamma\ =\ (1 + c\alpha_0)\beta_2(\alpha_0) - \beta_1(\alpha_0).$$

All in all, by (i),

$$\exists\, t_n \geq 0\ \forall\, t \geq t_n :\ \alpha_0 - \frac{1}{n} \leq u_2(t)/u_1(t)$$

remains to show we do in an analogous way as described in the proof of (i): We will find a time t_n such that, for all $t \geq t_n$, $\mathbf{u}(t)$ lies in the cone spanned by the vectors $\begin{pmatrix} 0 \\ \alpha_0 - \frac{1}{n} \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \alpha_0 - \frac{1}{n} \end{pmatrix}$ proving (iii), finally.

Remark that $(\alpha,\gamma) \in \mathcal{P}_1$ and $\gamma > \beta_2(0) - \beta_1(0)$ together yield $\alpha_0 > 0$ and we may assume $\alpha_0 > \varepsilon$ without restricting the generality. If $\alpha_0 - \frac{1}{n} \leq 0$ then we define

$$t_n = 0 \quad \text{and} \quad \alpha(t) = \varepsilon,\ t \geq 0.$$

Otherwise, if $\alpha_0 - \frac{1}{n} > 0$ then, again without restricting the generality, we may assume $\varepsilon < \alpha_0 - \frac{1}{n}$ and define

$$t_n = \frac{\ln(\alpha_0 - \frac{1}{n}) - \ln\varepsilon}{c_n} \quad \text{and} \quad \alpha(t) = \begin{cases} \varepsilon e^{c_n t} & : \ t \in [0,t_n] \\ \varepsilon e^{c_n t_n} & : \ t \geq t_n \end{cases}$$

where

$$c_n\ :=\ \min\{\gamma - (1 + c\tilde{\alpha})\beta_2(\tilde{\alpha}) + \beta_1(\tilde{\alpha}) : \tilde{\alpha} \in [0, \alpha_0 - \frac{1}{n}]\}$$

is positive. Obviously, $t \mapsto \alpha(t)$, $t \geq 0$, is a convex function satisfying

$$\alpha(0) = \varepsilon \quad \text{and} \quad \alpha(t) \geq \alpha_0 - \frac{1}{n},\ t \geq t_n.$$

We introduce the matrices
$$\mathfrak{A}(t) \;=\; \begin{pmatrix} 0 & 1 \\ \alpha(t) & \alpha(t) \end{pmatrix}$$
which are invertable and consider the process $\tilde{\mathbf{u}}(t) = \mathfrak{A}^{-1}(t)\mathbf{u}(t)$, $t \in [0,T]$, T arbitrarily chosen. By linear transformation of (18′), $\tilde{\mathbf{u}}$ satisfies the equation
$$\tilde{\mathbf{u}}(t) \;=\; \mathbf{S}(t)\mathfrak{A}^{-1}(0)\boldsymbol{\xi} \;+ \int_0^t \mathbf{S}(t-s)\mathbf{F}(s,\tilde{\mathbf{u}}(s))\,\mathrm{d}s$$
with
$$\mathbf{F}(t,\mathbf{x}) \;=$$
$$= \begin{pmatrix} \beta_1(|\frac{\alpha(t)(x_1+x_2)}{x_2}|)x_2 - \beta_2(|\frac{\alpha(t)(x_1+x_2)}{x_2}|)(1+c\alpha(t))(x_1+x_2) + \gamma(x_1+x_2) - \frac{\alpha'_+(t)}{\alpha(t)}(x_1+x_2) \\ -\,\beta_1(|\frac{\alpha(t)(x_1+x_2)}{x_2}|)x_2 + c\,\beta_2(|\frac{\alpha(t)(x_1+x_2)}{x_2}|)\alpha(t)(x_1+x_2) \end{pmatrix}.$$
As in the proof of (i), all assumptions of Corollary 2 are satisfied. We only mention that $(C_{\mathbf{F}})$ is here equivalent to
$$\gamma \geq (1+c\alpha(t))\beta_2(\alpha(t)) - \beta_1(\alpha(t)) + \frac{\alpha'_+(t)}{\alpha(t)},\; t \in [0,T],$$
which is true by the definition of $\alpha(t)$. Consequently, from Corollary 2 follows
$$\tilde{\mathbf{u}}(t) \;\geq\; \mathbf{0},\; t \in [0,T],$$
finishing the proof of (iii).

(iv) If $(\alpha,\gamma) \in \mathcal{P}_2$ then
$$\gamma \;>\; (1+c\tilde{\alpha})\beta_2(\tilde{\alpha}) - \beta_1(\tilde{\alpha}),\; \forall \tilde{\alpha} \geq \alpha,$$
and
$$c \;:=\; \min\{\gamma - (1+c\tilde{\alpha})\beta_2(\tilde{\alpha}) + \beta_1(\tilde{\alpha}) : \tilde{\alpha} \geq \alpha\}$$
is a positive constant. We define
$$\alpha(t) \;=\; \alpha \mathrm{e}^{ct},\; t \geq 0,$$
and obtain
$$\tilde{\mathbf{u}}(t) \;\geq\; \mathbf{0},\; t \in [0,T],\; T \text{ arbitrarily chosen},$$
by repeating the part of the proof of (iii) which follows the introduction of the matrices $\mathfrak{A}(t)$. This yields
$$u_2(t)/u_1(t) \;\geq\; \alpha \mathrm{e}^{ct},\; t \geq 0,$$
and we end the proof of (iv) in the same way as (ii) has been derived from (i).

(v) If $(\alpha,\gamma) \in \mathcal{P} \cap \mathcal{G}$ then
$$\gamma \;=\; (1+c\alpha)\beta_2(\alpha) - \beta_1(\alpha).$$
We set
$$\alpha(t) \;=\; \alpha,\; t \geq 0,$$

and remark that, by assumption, $\xi_2/\xi_1 \leq \alpha$ as well as $\xi_2/\xi_1 \geq \alpha$, holds true. Consequently, on the one hand, following the proof of (i) we have

$$u_2(t)/u_1(t) \ \leq \ \alpha, \ t \geq 0,$$

and, on the other hand, from the proof of (iii) follows

$$u_2(t)/u_1(t) \ \geq \ \alpha, \ t \geq 0.$$

So (v) and with it the proposition is proven. ■

Although there are other methods in the deterministic theory to conclude the above result, we would like to emphasize, that our method also applies to the stochastic system (19) and we may prove the same result.

Proposition 2 *Let $\mathbf{u} = (u_1, u_2)$ be the pathwise unique solution of the system*

$$\begin{aligned} u_1(t) &= S(t)\xi_1 + \int_0^t S(t-s)\left[-\beta_1(|\tfrac{u_2(s)}{u_1(s)}|)u_1(s) + c\,\beta_2(|\tfrac{u_2(s)}{u_1(s)}|)u_2(s)\right] \mathrm{d}s \\ &\qquad\qquad + \int_0^t S(t-s)\mathcal{M}(u_1(s)) \circ B\,\mathrm{d}W(s) \\ u_2(t) &= S(t)\xi_2 + \int_0^t S(t-s)\left[\gamma - \beta_2(|\tfrac{u_2(s)}{u_1(s)}|)\right] u_2(s)\,\mathrm{d}s + \int_0^t S(t-s)\mathcal{M}(u_2(s)) \circ B\,\mathrm{d}W(s) \end{aligned}$$

starting from an $\mathbf{L}^2(\mathcal{O})$-valued square integrable initial distribution $\boldsymbol{\xi} = (\xi_1, \xi_2) \geq \mathbf{0}$ where $S(t) = \exp\{t\Delta\}$, $W(t)$ is a cylindrical Wiener process in $\mathfrak{D}'(\mathcal{O})$, $\boldsymbol{\mathcal{M}}(\mathbf{u}) = (\mathcal{M}(u_1), \mathcal{M}(u_2))$ and $B : L^2(\mathcal{O}) \to L^\infty(\mathcal{O})$ satisfies (2).

Then the assertions of Proposition 1 hold almost surely true.

Proof. We repeat the proof of Proposition 1 and only remark that, first, the process $\tilde{\mathbf{u}}$ now satisfies the equation

$$\tilde{\mathbf{u}}(t) \ = \ \mathbf{S}(t)\mathfrak{A}^{-1}(0)\boldsymbol{\xi} + \int_0^t \mathbf{S}(t-s)\mathbf{F}(s, \tilde{\mathbf{u}}(s))\,\mathrm{d}s + \int_0^t \mathbf{S}(t-s)\boldsymbol{\mathcal{M}}(\tilde{\mathbf{u}}(s)) \circ B\,\mathrm{d}W(s),$$

correspondingly, and, second, the $\mathbf{P}$-exceptional set in the assertions of (i)-(v) may be chosen independent from t because it is well-known in our special situation that there exists a version of the random field $(\omega \mapsto \mathbf{u}(t, \omega, y))_{t>0, y\in\mathcal{O}}$ which is continuous in (t, y). We do not need that the mapping $(\omega, y) \mapsto \boldsymbol{\xi}(\omega)(y)$ is product measurable since the semigroup generated by the Laplacian is strongly Feller (see also Remark 2a)). ■

Consequently, in difference to the behavior of the singular components $u_1(t)$ resp. $u_2(t)$, the stability of the ratio $u_2(t)/u_1(t)$ does not change if we disturb the coefficients of the deterministic system (18) by an additive noise.

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