Three Symmetries Groups.

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Introduction

Let $\Gamma = \mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2$ be a free product of three involutions s_1, s_2 and s_3. Denote by r the composition of these involutions $s_1 s_2 s_3$. Consider the discrete embedding $\Gamma \to PSL_2(\mathbf{R})$, that is injective homomorphism whose image G is a discrete group acting on hyperbolic plane L^2. We are interested in the embeddings such that G has a non-compact fundamental domain of finite volume. The image of s_i is the symmetry with respect to a point z_i. Since the volume is finite, the image of r is parabolic. In this paper we study Teihmuller and moduli spaces of these groups.We also find the modular group.

We would like to thank O. V. Schwarzman for constant attention to our work and useful conversations. We also thank E. B. Vinberg for his helpful comments and advices.

Three symmetries group

Let z_1, z_2 and z_3 be three distinct points on the hyperbolic plane; let G be a group generated by symmetries τ_1, τ_2, τ_3 with respect to these points, and let ρ be the composition of these symmetries $s_1 s_2 s_3$. In [1] it is proved that

1) if ρ is elliptic, then either G is non-discrete or its fundamental domain is compact.

2) if ρ is hyperbolic, then G is discrete, but its fundamental domain has infinite volume.

3) if ρ is parabolic, then G is a discrete group with non-compact fundamental domain of finite volume.

In the last two cases ρ has at least one fixed point at infinity. Hence, in both cases we have an ideal triangle whose sides m_1, m_2 and m_3 contain the points z_1, z_2 and z_3 respectively (see Fig. 1). We find an easy geometric way to understand if ρ is parabolic.

Theorem 1. *Let h_i, $(i = 1, 2, 3)$ be a line perpendicular to m_i and $z_i \in h_i$. The composition ρ is parabolic if and only if the lines h_i either have a common point or have a common perpendicular.*

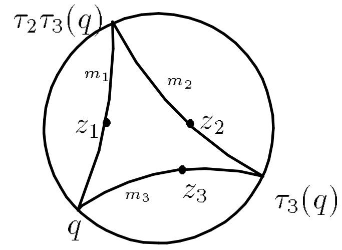

Figure 1.

Plan of the proof

By **altitude** of the triangle ABC we mean a line AH perpendicular to the side BC.

First, we prove that for any pair of points z_1, z_2 ($z_1 \in m_1, z_2 \in m_2$) there exists a unique point $z_3 \in m_3$ such that the composition ρ is parabolic (see lemma 1).

Then we prove that the line h_3 is orthogonal to a line $\overline{z_1 z_2}$ containing z_1 and z_2 (see lemma 2). Therefore, h_3 is an altitude of the triangle $z_1 z_2 z_3$. Similarly, h_1 and h_2 are altitudes of this triangle. Three altitudes of a hyperbolic triangle either have a common point (may be at infinity) or have a common perpendicular. In the Klein model, it means that the altitudes have a common point (though it may be outside the disc).

Thus, we have a necessary condition. The sufficiency is proved in lemma 3.

Remark. Let z_1, z_2 and z_3 be three distinct points on the hyperbolic plane. The proof of the theorem involves a way to understand if the composition of the symmetries $\tau_1\tau_2\tau_3$ is parabolic. Let h_i ($i = 1, 2, 3$) be a line containing z_i, $h_i \perp \overline{z_j z_k}$ (where i, j, k are different). Let s_i be a line through z_i and orthogonal to h_i. Then $\tau_1\tau_2\tau_3$ is parabolic if and only if the lines s_i compose an ideal triangle (moreover, it is easy to prove that $\tau_1\tau_2\tau_3$ is parabolic if s_1 intersects s_2 at infinity).

Equidistant line

Consider points z_1, z_2 and z_3 in the upper halfplane model.
Assume that $z_1 = i$, $z_2 = ki$, where $\ln k$ is the distance between z_1 and z_2. Let $z_3 = a + bi$. Then the symmetries τ_i are represented by matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sim \tau_1, \qquad \frac{1}{k}\begin{pmatrix} 0 & -k^2 \\ 1 & 0 \end{pmatrix} \sim \tau_2, \qquad \frac{1}{b}\begin{pmatrix} a & -a^2-b^2 \\ 1 & -a \end{pmatrix} \sim \tau_3$$

(up to the multiple $-E$).
We obtain $|tr\ \rho| = |\frac{a}{b}|\frac{k^2-1}{k}$.

It is well known that ρ is parabolic if and only if $|tr\ \rho| = 2$. This equation determines two rays on the halfplane; these rays compose the equidistant to the line $\overline{z_1 z_2}$. The distance d between this equidistant and $\overline{z_1 z_2}$ satisfies the following condition: $\cosh d = \frac{k^2+1}{k^2-1}$. If z_3 is located above the equidistant, then $|tr\ \rho| < 2$ and ρ is elliptic. Analogously, ρ is hyperbolic if z_3 lies under the equidistant.

In the unit disc model the equidistant is an arc (part of euclidean circle) containing "the ends" of the line.

Proof of theorem 1

Lemma 1. *Let pqr be an ideal triangle. For any points $z_1 \in pq, z_2 \in pr$ there exists a unique point $z_3 \in qr$ such that ρ (a product of symmetries with respect to these points) is parabolic.*

Proof. Consider points z_i in the upper halfplane model. Assume that $r = -1$, $q = 1$, and $p = \infty$. Then $\Re e\ z_1 = -1$, $\Re e\ z_2 = 1$, $z_3 = e^{i\varphi}$. Let $g = \Im m\ z_1$ and $h = \Im m\ z_2$. Then the symmetries τ_i are represented by matrices

$$\frac{1}{g}\begin{pmatrix} 1 & -g^2-1 \\ 1 & -1 \end{pmatrix} \sim \tau_1, \quad \frac{1}{h}\begin{pmatrix} 1 & h^2+1 \\ -1 & -1 \end{pmatrix} \sim \tau_2, \quad \frac{1}{\sin\varphi}\begin{pmatrix} \cos\varphi & -1 \\ 1 & -\cos\varphi \end{pmatrix} \sim \tau_3$$

(up to the multiple $-E$).

Therefore, $|tr\ \rho| = \frac{h^2+g^2+\cos\varphi(g^2-h^2)}{hg\sin\varphi}$ (since $h^2+g^2 > |\cos\varphi(g^2-h^2)|$). Given $|tr\ \rho| = 2$, this equation can be simplified to $2hg\sin\varphi = h^2+g^2+\cos\varphi(g^2-h^2)$. Replacing $\tan\frac{\varphi}{2}$ by t, we obtain $(ht-g)^2 = 0$, i.e. $ht = g$.

When φ runs through $[0, \pi]$, t runs through $[0,+\infty]$. Thus, having two points, we obtain the third in a unique way.

$\square$

Consider points z_i in the unit disc model.

Lemma 2. *The line h_3 is orthogonal to $\overline{z_1z_2}$.*

Proof. Let l be the equidistant to $\overline{z_1z_2}$. Since ρ is parabolic, $z_3 \in l$. Both l and qr are circle arcs in the unit disc. It follows from lemma 1 that they have a unique common point z_3. Hence, they are tangent at z_3 (see Fig. 2). Let h ($z_3 \in h$) be a line orthogonal to $\overline{z_1z_2}$. Since $h \perp \overline{z_1z_2}$, we have $h \perp l$, i.e. $h \perp qr$. Therefore, h coincides with h_3. Thus, z_3 belongs to the common perpendicular of $\overline{z_1z_2}$ and qr.

$\square$

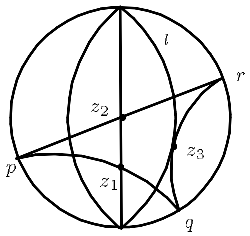

Figure 2.

Remark. Another proof of lemma 2 was suggested by O.V.Schwarzman:

Since ρ is parabolic, it is the product of two reflections with respect to the line pr and a certain line m (see Fig. 3). Denote these reflections by ϕ_{pr} and ϕ_m, i.e. $\rho = \phi_m\phi_{pr}$. Obviously, $\tau_3 = \phi_{h_3}\phi_{pr}$, where ϕ_{h_3} is the reflection with respect to h_3. We obtain $\phi_m\phi_{pr} = \rho = \tau_1\tau_2\phi_{h_3}\phi_{pr}$. Thus, we have $\phi_m\phi_{h_3} = \tau_1\tau_2$. Hence, $\tau_1\tau_2$ is decomposed into two reflections. Since $\tau_1\tau_2$ is hyperbolic with axis $\overline{z_1z_2}$, we can see that both h_3 and m are orthogonal to $\overline{z_1z_2}$. This completes the proof.

The necessity is proved. To prove the sufficiency consider the points in the Klein model.

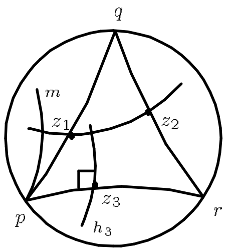

Figure 3.

Lemma 3. *Let pqr be an ideal triangle. Let PQR be a triangle with sides tangent to the disc at the points p, q, r ($p \in QR$, $q \in RP$, $r \in PQ$). Suppose that w is an inner point of PQR, and $\overline{wz_1}$, $\overline{wz_2}$, $\overline{wz_3}$ ($z_1 \in pq$, $z_2 \in rp$, $z_3 \in qr$) are perpendiculars to the sides of the triangle pqr. Then the composition of symmetries with respect to the points z_1, z_2, z_3 is parabolic.*

Proof. Consider the points z_1 and z_2 ($z_1 \in pq$, $z_2 \in rp$). Construct perpendiculars to the lines pr and pq containing z_1 and z_2 respectively (see Fig. 4). Let w be a common point of these perpendiculars (it may not be an inner point of the disc). Let $\overline{wz_3}$ ($z_3 \in qr$) be a perpendicular to qr. Take any point $z \in qr$ and construct a perpendicular h containing z. The point w belongs to the perpendicular h if and only if $z = z_3$. Therefore the necessary condition is satisfied by the point z_3 only. By lemma 1 the composition of the symmetries with respect to the points z_1, z_2, z_3 is parabolic.

□

Teihmuller space

Recall that G is the group generated by the symmetries with respect to the points z_1, z_2 and z_3. Move p, q and r to three fixed points isometrically. Consider the point $w = h_1 \bigcap h_2 \bigcap h_3$. Thus, any triple of generators of G corresponds to an inner point of $\triangle PQR$.

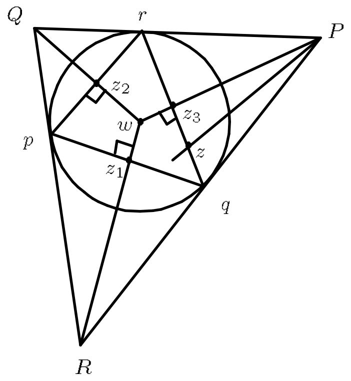

Figure 4.

It follows from lemma 3 that any inner point of $\triangle PQR$ determines a three symmetries group with a non-compact fundamental domain of finite volume. Therefore, the Teihmuller space of these groups is the triangle PQR (see Fig. 4).

Moduli space

Consider the moduli space of discrete three symmetries groups with unbounded fundamental domain of finite volume (i.e. the space of such groups with *any* triple of generators considered modulo the group of isometries of $\mathbf{H}^2$).

To find the moduli space we have to know which points of the Teihmuller space are equivalent under the action of the modular group, i.e. which triples generate the same group. For any group we choose a canonical triple of generators. Then we define an elementary transformation of a triple of generators and prove that any triple of generators can be obtained from the canonical triple via finitely many elementary transformations. This give us the way to find the modular group.

Theorem 2. *Moduli space is the triangle pqr. Modular group is a group generated by three elementary transformations α_1, α_2, α_3 with defining relations $\alpha_i^2 = e$, $i = 1, 2, 3$.*

Elementary transformation of generators

Let G be a group generated by symmetries with respect to the points z_1, z_2 and z_3 (see Fig. 5). Let $\hat{z}_2$ be a point that is symmetric to z_2 with respect to z_3. Then the symmetries with respect to the points z_1, $\hat{z}_2$ and z_3 generate the same group G. This exchange of the generators we call **an elementary transformation**.

Denote by $\hat{\tau}_2$ a symmetry with respect to the point $\hat{z}_2$. Note that the element $\tau_1\hat{\tau}_2\tau_3$ is parabolic (since $\hat{\tau}_2 = \tau_3\tau_2\tau_3$, and $|tr\tau_1\hat{\tau}_2\tau_3| = |tr\tau_1\tau_2\tau_3| = 2$).

Therefore, for any triple of generators there exist six elementary transformations (we can reflect any point with respect to two others). Triples z_1, $\hat{z}_2$, z_3 and $\hat{z}_1$, z_2, z_3 (where $\hat{z}_1 = \tau_3 z_1$, $\hat{\tau}_1 = \tau_3\tau_1\tau_3$) correspond to the same point of Teihmuller space. Hence, for any triple there exist only three distinct elementary transformations (these transformations are really different, since the triangles $z_1\hat{z}_2z_3$ and $\hat{z}_1z_2z_3$ are usually non-congruent).

Let $\hat{z}_2$ be a point symmetric to z_2 with respect to z_3. Then the replacement of the triple $z_1z_2z_3$ by the triple $z_1\hat{z}_2z_3$ is called **an elementary transformation at** z_3.

If G is generated by symmetries with respect to the points z_1, z_2, z_3, $\triangle z_1z_2z_3$ is called **generator triangle** .

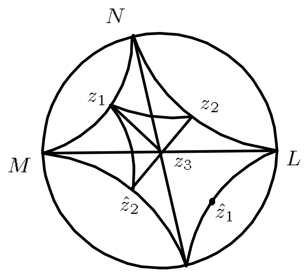

Figure 5.

Elementary transformation of the quotient space

The triangle $\triangle LMN$ is a fundamental domain of G (see Fig. 5), where the ray z_1M is glued together with z_1N, the ray z_2N is glued together with z_2L, and z_3L is glued together with z_3M. Hence, the quotient is a sphere whose hyperbolic metric has one cusp $\overline{M} = \overline{N} = \overline{L}$ and three cone singularities at the points $\bar{z}_1$, $\bar{z}_2$, $\bar{z}_3$ with angle π.

Fixed triple of generators of the group G corresponds to fixed cuts $\overline{M}\bar{z}_1, \overline{M}\bar{z}_2$ and $\overline{M}\bar{z}_3$ of the quotient. At the same time, the generators are determined by the cuts of the quotient.

Consider the elementary transformation at the point z_3. Then the set of cuts transforms as shown in Fig. 6b.

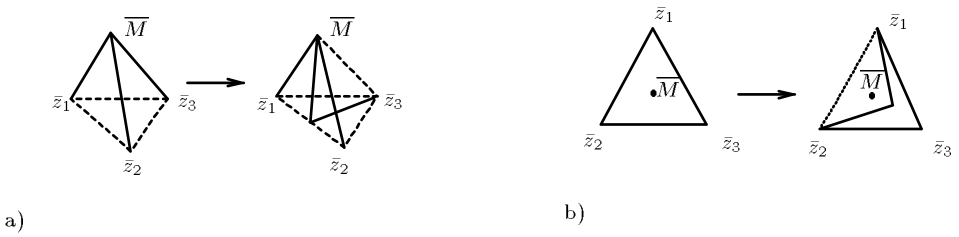

Figure 6.

Canonical generators

Choose a triple of generators such that the perimeter of the generator triangle is minimal. Such a generator triangle is called **minimal**. It is proved below that there exists a unique minimal triangle up to isometry.

Lemma 4. *There exists a minimal triangle.*

Proof. Suppose that there is no minimal triangle. Then we can find an infinite sequence of the generator triangles $\triangle z_1^i z_2^i z_3^i$ whose perimeters strictly decrease. Let P be the perimeter of the first triangle.

Let z_1 be equivalent to z_1^i modulo G. Let $\varphi_i \in G$ be isometries such that $\varphi(z_1^i) = z_1$. Suppose $\hat{z}_2^i = \varphi(z_2^i)$ and $\hat{z}_3^i = \varphi(z_3^i)$. Since perimeters decrease, all such points $\hat{z}_2^i$ and $\hat{z}_2^i$ belong to the disc of radius P centered in z_1. Obviously, symmetries with respect to the points $\hat{z}_2^i$ and $\hat{z}_3^i$ belong to G. This contradicts to the discretness of G.

□

Lemma 5. *There exist at most two minimal triangles. If there are two distinct minimal triangles T_1 and T_2, then they are right-angled congruent triangles, and in this case T_1 can be obtained from T_2 by an elementary transformation.*

Proof. Consider a shortest path $\bar{z}_1\bar{z}_2$ on the quotient. It cannot be selfintersecting, otherwise there exists a shorter path. Moreover, it does not contain $\bar{z}_3$, since $z_3 \notin \overline{z_1 z_2}$ for any triple of generators.

Consider a shortest path from $\bar{z}_2$ to $\bar{z}_3$. Let us prove that it does not intersect the path $\bar{z}_1\bar{z}_2$. Really, let $\bar{w}$ be a common point of these paths, the farthest from $\bar{z}_2$ ($\bar{z}_2 \neq \bar{w}$). Denote by $[\bar{z}_2\bar{w}]_{\bar{z}_1}$ the segment $[\bar{z}_2\bar{w}]$ of the path $\bar{z}_2\bar{z}_1$, and by $[\bar{z}_2\bar{w}]_{\bar{z}_3}$ the segment $[\bar{z}_2\bar{w}]$ of the path $\bar{z}_2\bar{z}_3$.

Suppose that $[\bar{z}_2\bar{w}]_{\bar{z}_1}$ is shorter than $[\bar{z}_2\bar{w}]_{\bar{z}_3}$ (write $[\bar{z}_2\bar{w}]_{\bar{z}_1} < [\bar{z}_2\bar{w}]_{\bar{z}_3}$). Then replace the segment $[\bar{z}_2\bar{w}]_{\bar{z}_3}$ of the path $\bar{z}_2\bar{z}_3$ by the segment $[\bar{z}_2\bar{w}]_{\bar{z}_1}$. We obtain a path, which is shorter than the original one. Therefore, the segment $[\bar{z}_2\bar{w}]_{\bar{z}_1}$ of the path $\bar{z}_2\bar{z}_1$ and the segment $[\bar{z}_2\bar{w}]_{\bar{z}_3}$ of the path $\bar{z}_2\bar{z}_3$ have the same length. These paths are the segments of the straight lines on the development. Hence, they intersect each other transversely. Let $\bar{u} \in [\bar{z}_2\bar{w}]_{\bar{z}_1}$ and $\bar{v} \in \bar{w}\bar{z}_3$ be points close to $\bar{w}$ such that the triangle $\bar{u}\bar{w}\bar{v}$ does not contain any singularities. Then $\bar{u}\bar{v} < \bar{u}\bar{w} + \bar{w}\bar{v}$. Consider the path $\bar{z}_2\bar{u}\bar{v}\bar{z}_3$. It is shorter than the path $[\bar{z}_2\bar{w}]_{\bar{z}_1}\bar{z}_3$ (see Fig. 7), i.e. it is shorter than the original minimal path $\bar{z}_2\bar{z}_3$. Since this is impossible, the minimal paths $\bar{z}_1\bar{z}_2$, $\bar{z}_2\bar{z}_3$ and $\bar{z}_3\bar{z}_1$ do not intersect each other.

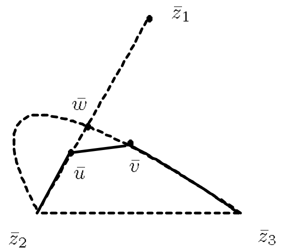

Figure 7.

Let $\bar{z}_1\bar{z}_2$, $\bar{z}_2\bar{z}_3$ and $\bar{z}_3\bar{z}_1$ be minimal paths. Suppose that there is another minimal path from $\bar{z}_1$ to $\bar{z}_2$. Since it can intersect nothing, it is either homotopic to the path $\bar{z}_1\bar{z}_2$ or travels as shown in Fig. 8. The former is impossible, otherwise there are two distinct straight lines containing the points z_1 and z_2 on the development. The latter means that there is an elementary transformation preserving the perimeter of the generator triangle. It is possible if and only if the angle $z_1z_3z_2$ is right. Really, suppose that it is either acute or obtuse. Then we have either $z_1z_2 < z_1\hat{z}_2$ or $z_1z_2 > z_1\hat{z}_2$. Since z_1z_3 is preserved and $z_2z_3 = \hat{z}_2z_3$, the perimeter is changed (see Fig. 5).

Since any triangle has at most one right angle, only one of the distances z_1z_2, z_2z_3 and z_3z_1 can be reached by two distinct paths. Therefore for any group G there exist at most two minimal triangles.

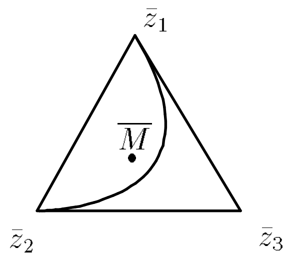

Figure 8.

$\square$

Corollary 2.1. *The canonical triple is well defined.*

Proof. Any two different minimal triangles are symmetric (with respect to the line $\overline{z_1z_3}$ in Fig. 5).

$\square$

By **acute-angled triangle** we mean a triangle having no obtuse angles.

Lemma 6. *A triangle that has an obtuse angle cannot be minimal.*

Proof. Suppose that the angle z_3 is obtuse in $\triangle z_1z_2z_3$. The perimeter of the generator triangle can be decreased by the elementary transformation at the point z_3 ($z_1z_3 = z_1z_3$, $z_2z_3 = z_2'z_3$ and $z_1z_2 > z_1z_2'$, since the angle z_3 is obtuse). Contradiction.

□

Lemma 7. *Any generator triangle can be transformed to an acute-angled triangle via finitely many elementary transformations.*

Proof. If the original triangle has no obtuse angles, there is nothing to prove. Suppose that it has an obtuse angle z_3. Consider the elementary transformation at the point z_3. If we obtain a triangle without obtuse angles, everything is proved. Otherwise we make the elementary transformation at the vertex of the obtuse angle again, and so on. The perimeter of the generator triangle strictly decreases at each step. By lemma 4 this procedure will be finished after finitely many steps.

□

Moduli space

It was proved that no minimal triangle has obtuse angle. Now we find a domain D of the Teihmuller space such that any point of D determines a triangle having no obtuse angle.

Lemma 8. *$D = \triangle pqr$. The boundary points of $\triangle pqr$ determine right-angled triangles.*

Proof. Let w be the common point of h_1, h_2 and h_3. We have three cases:

1) If w is located outside $\triangle pqr$, then it is located outside $\triangle z_1z_2z_3$. Hence, the triangle $z_1z_2z_3$ has an obtuse angle.

2) If w belongs to the boundary of pqr, then w coincides with one of the vertices z_i of the triangle $z_1z_2z_3$. Hence, the triangle $z_1z_2z_3$ has a right angle.

3) Let w be an inner point of the triangle pqr (see Fig. 4). Then $z_2 = Qw \bigcap pr$, and $z_3 = Pw \bigcap rq$ (where Q and w belong to distinct half-planes with respect to the line pr, and P and w belong to distinct half-planes with respect to rq). Hence, z_1 and z_2 belong to distinct half-planes with respect to Rw, i.e. the line h_1 intersects the segment z_2z_3. Therefore, both angles $z_1z_2z_3$ and $z_1z_3z_2$ are acute. Similarly, angle $z_2z_1z_3$ is acute.

□

Lemma 9. *Any acute-angled triangle is a minimal triangle of some group.*

Proof. Suppose that there is a point t_1 of the Teihmuller space corresponding to non-minimal acute-angled triangle T_1. Let $t_{1,min}$ be a point corresponding to the minimal triangle $T_{1,min}$ of this group. Each of t_1 and $t_{1,min}$ determines three centers of symmetry on the sides of the ideal triangle pqr. Since the triangle pqr is fundamental domain, there exist only one triple of centers on the sides of $\triangle pqr$. Therefore t_1 coincides with $t_{1,min}$. This proves the lemma.

□

Thus, distinct points of $\triangle pqr$ determine different groups and any group corresponds to a point of pqr. Therefore the moduli space is the triangle pqr.

Proof of theorem 2

It follows from lemmas 4 and 6, that for any group there exists a minimal generator triangle having no obtuse angles. It was proved in lemmas 9 and 5, that distinct acute-angled triangles correspond to different groups. It follows from lemma 7 that any triangle can be transformed to acute-angled triangle via finitely many elementary transformations. Thus, the modular group is generated by three elementary transformations.

Suppose that they are not independent, i.e. there is a closed sequence of triangles T_i, $(i = 1, ..., n)$ such that T_{i+1} can be obtained from T_i by an elementary transformation (the first triangle can be obtained from the last). Assume that T_{i-1} does not coincide with T_{i+1} for any i (otherwise we can reduce this cycle to $T_1, ..., T_{i-1}, T_{i+2}, ..., T_n$). Let T_k $(k \in 1, ..., n)$ be a triangle of maximal perimeter. Then elementary transformations leading to T_{k-1} and T_{k+1} cannot increase the perimeter of the generator triangle. Therefore, these transformations cannot be transformations at the acute angles. Since no triangle has two non-acute angles and $T_{i-1} \neq T_{i+1}$, this is impossible.

Remark. The elementary transformations α_i preserve some points of the Teihmuller space. These points correspond to two distinct minimal triangles, i.e. the points contained in the boundary of the triangle pqr (see lemma 5). Thus, topologically, the action of α_i on the Teihmuller space is a reflection (see Fig. 9).

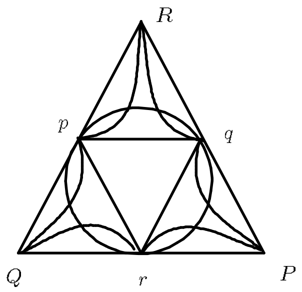

Figure 9.

References

[1] Beardon, A.F., The geometry of discrete groups, Springer-Verlag, 1983.