Hereditary Triangulated Categories

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ABSTRACT. We call a triangulated category *hereditary,* in case it is triangle equivalent to the bounded derived category $D^b(\mathcal{A})$ of a hereditary abelian category $\mathcal{A}$. Such a hereditary triangulated category has a very lucid structure: it is the additive closure of the shifted copies $\mathcal{A}[t]$ of $\mathcal{A}$, and there are non-zero homomorphisms from objects in $\mathcal{A}[t]$ only to objects in $\mathcal{A}[t]$ and $\mathcal{A}[t{+}1]$. We are going to show that these properties furnish a characterization of hereditary triangulated categories.

Our main interest lies in triangulated categories $\mathcal{C}$ where every object is a finite direct sum of indecomposable objects and where every idempotent splits. We will consider paths in such categories: they connect indecomposable objects and are given by a sequence of non-zero maps and of jumps from an object X to $X[1]$. In case $\mathcal{C}$ is hereditary, there does not exist a path from $X[1]$ to X, for any indecomposable object X. Conversely, we are going to show that if $\mathcal{C}$ is path-connected and if there exists at least one indecomposable object X with no path from $X[1]$ to X, then $\mathcal{C}$ is hereditary.

These characterizations of hereditary triangulated categories can be used in the study of piecewise hereditary (and thus of quasi-tilted) k-algebras, where k is a field. It follows from our considerations that a finite dimensional connected k-algebra A with a directing object in $D^b(\operatorname{mod} A)$ is derived equivalent to a finite dimensional hereditary algebra, in particular A is piecewise hereditary.

1. Triangles

We assume that $\mathcal{C}$ is a triangulated category. The triangles which we consider will always be distinguished ones. Let us recall from [H], 1.4, the following assertion:

Lemma 1. *Let $X \to Y \to Z \to$ be a triangle with maps u, v, w. The map u is split epi if and only if $v = 0$, if and only if w is split mono.*

Lemma 2. *Let $X \to Y \to Z \to$ be a triangle with maps u, v, w. Assume that there is given a decomposition $Y = Y_1 \oplus Y_2$, write $u = (u_1, u_2)$ and $v = (v_1, v_2)$, where $u_i \colon X \to Y_i$ and $v_i \colon Y_i \to Z$. If $u_2 = 0$, then v_2 is a split monomorphism.*

Proof: Consider the triangle $0 \to Y_2 \to Y_2 \to$ with maps $0, 1, 0$. There are

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commutative squares

$$\begin{array}{ccc} 0 & \longrightarrow & Y_2 \\ \downarrow & & \downarrow{\scriptstyle\iota} \\ X & \xrightarrow{u} & Y_1 \oplus Y_2 \\ \downarrow & & \downarrow{\scriptstyle\pi} \\ 0 & \longrightarrow & Y_2 \end{array}$$

where the right hand maps are just the canonical inclusion ι and the canonical projection π (thus the composition is the identity). We may complete these maps to maps of triangles:

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y_2 & \xrightarrow{1} & Y_2 & \longrightarrow & \\ \downarrow & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle h} & & \\ X & \xrightarrow{u} & Y_1 \oplus Y_2 & \xrightarrow{v} & Z & \longrightarrow & \\ \downarrow & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle h'} & & \\ 0 & \longrightarrow & Y_2 & \xrightarrow{1} & Y_2 & \longrightarrow & . \end{array}$$

Since the compositions on the left are isomorphisms, also $h'h$ is an isomorphism, thus h is split mono. On the other hand, $v_2 = v\iota = h$, thus v_2 is split mono.

Lemma 2*. *Let $X \to Y \to Z \to$ be a triangle with maps u, v, w. Assume that there is given a decomposition $Y = Y_1 \oplus Y_2$, write $u = (u_1, u_2)$ and $v = (v_1, v_2)$, where $u_i \colon X \to Y_i$ and $v_i \colon Y_i \to Z$. If $v_2 = 0$, then u_2 is a split epimorphism.*

Proof. The assertion is just the dual one to Lemma 2.

Lemma 3. *Let $X \to Y \to Z \to$ be a triangle with maps u, v, w. Assume that X, Y are indecomposable and that u is non-zero and not invertible. Also assume that there is given a decomposition $Z = \bigoplus_{i=1}^{t} Z_i$ with indecomposable objects Z_i and write $v = (v_1, \dots, v_t)$, $w = (w_1, \dots, w_t)$, with maps $v_i \colon Y \to Z_i$ and $w_i \colon Z_i \to X[1]$. Then $t \geq 1$ and all the maps v_i, w_i are non-zero and non-invertible.*

Proof: First of all, $t = 0$ would mean $Z = 0$, but then u has to be invertible. Assume that some $v_i = 0$. Then by Lemma 2, w_i is split mono. Since both Z_i and $X[1]$ are indecomposable, it follows that w_i is an isomorphism, thus w is split epi. But this implies that $u = 0$, by Lemma 1, a contradiction. This shows that all v_i are non-zero. By duality, also all w_i are non-zero. On the other hand, if some w_i is invertible, then w is split epi, and by Lemma 1, it follows that $u = 0$, a contradiction. Again, we use duality in order to see that also no map v_i can be invertible.

2. Hereditary triangulated categories

Given an additive category $\mathcal{A}$ and classes $\mathcal{U}, \mathcal{V}$ of objects in $\mathcal{A}$, then we write $\mathrm{Hom}(\mathcal{U}, \mathcal{V}) = 0$ provided $\mathrm{Hom}(U, V) = 0$ for all $U \in \mathcal{U}$, and $V \in \mathcal{V}$.

We recall the following definition [BDD]: Given a triangulated category $\mathcal{C}$, a full subcategory $\mathcal{A}$ of $\mathcal{C}$ is said to be an *admissible abelian* subcategory provided first of all, $\mathcal{A}$ is abelian, second, $\operatorname{Hom}(\mathcal{A}[s],\mathcal{A}[t]) = 0$ for $s < t$, and third, given an exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, then the given maps $A \to B$ and $B \to C$ are the first two maps of a triangle $A \to B \to C \to$ in $\mathcal{C}$. Such a subcategory is said to be *closed under extensions* provided the following condition is satisfied: if $A \to B \to C \to$ is a triangle in $\mathcal{C}$, and if A, C belong to $\mathcal{A}$, then also B belongs to $\mathcal{A}$ and the given maps $A \to B$ and $B \to C$ are part of an exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$)

Given an abelian category $\mathcal{A}$, we denote by $D^b(\mathcal{A})$ the bounded derived category of finite complexes in $\mathcal{A}$, this is a triangulated category. If k is a field and A a finite dimensional k-algebra, let $\operatorname{mod} A$ be the category of all finite dimensional right A-modules; this is an abelian category, thus we may consider the bounded derived category $D^b(\operatorname{mod} A)$.

Theorem 1. *Let $\mathcal{C}$ be a triangulated category, let $\mathcal{A}$ be a full subcategory. The following conditions are equivalent:*

(i) *$\mathcal{A}$ is a hereditary abelian category, canonically embedded into its bounded derived category $\mathcal{C} = D^b(\mathcal{A})$.*

(ii) *$\mathcal{A}$ is an admissible abelian subcategory of $\mathcal{C}$ and closed under extensions; the additive closure of the union of the copies $\mathcal{A}[t]$ with $t \in \mathbb{Z}$ is $\mathcal{C}$; and $\operatorname{Hom}(\mathcal{A}[s],\mathcal{A}[t]) = 0$ for $t \notin \{s, s+1\}$.*

(iii) *The additive closure of the union of the copies $\mathcal{A}[t]$ with $t \in \mathbb{Z}$ is $\mathcal{C}$, and $\operatorname{Hom}(\mathcal{A}[s],\mathcal{A}[t]) = 0$ for $t < s$.*

Proof: The implication (i) $\implies$ (iii) is trivial.

(iii) $\implies$ (ii). First of all, we observe that the copies $\mathcal{A}[t]$ are pairwise disjoint, in the following sense: if A, B belong to $\mathcal{A}$ and if $A[s]$ and $B[t]$ are isomorphic, with different integers s, t, then $A = B = 0$. Here we use the condition that $\operatorname{Hom}(\mathcal{A}[s],\mathcal{A}[t]) = 0$ for $t < s$.

Now, let A, B be objects in $\mathcal{A}$, and consider a non-zero map $u\colon A[s] \to B[t]$ in order to show that $t \in \{s, s+1\}$. We may assume $t = 0$ and we have to conclude that $s = 0$ or $s = -1$. Of course, we know that $s \le 0$. Consider a corresponding triangle $A[s] \to B[0] \to C \to$, say with maps u, v, w. We decompose $C = C' \oplus C''$ where C' is a direct sum of objects in $\mathcal{A}[i]$ with $i \ge 0$ and C'' is a direct sum of objects in $\mathcal{A}[i]$ with $i < 0$. We can write $v = (v', v'')$ with $v'\colon B[0] \to C'$ and $v''\colon B[0] \to C''$, of course, $v'' = 0$. Similarly, we write $w = (w', w'')$, where $w'\colon C' \to A[s+1]$, and $w''\colon C'' \to A[s+1]$.

Since $v'' = 0$, Lemma 2 asserts that w'' is a split monomorphism, in particular C'' belongs to $\mathcal{A}[s+1]$.

First, we assume that $C'' \neq 0$. On the one hand, C'' belongs to $\mathcal{A}[s+1]$, on the other hand, it is a direct sum of objects in $\mathcal{A}[i]$ with $i < 0$. It follows that $s+1 < 0$. But this implies that the map $w'\colon C' \to A[s+1]$ is zero. Lemma 2* asserts that v' is split epi, thus there is a direct decomposition $B = B_1 \oplus B_2$ such that the restriction of v' to B_2 is zero, whereas the restriction of v' to B_1 is an isomorphism. Write $u = (u_1, u_2)$ with $u_i\colon A[s] \to B_i$, for $i = 1, 2$. Since the restriction of v to B_2 is zero (we know this for v' and we have $v'' = 0$ anyway), Lemma 2* asserts that $u_2\colon A[s] \to B_2$ is split epi. However, $s \le -2$, and $A[s]$ belongs to $\mathcal{A}[s]$, whereas B_2

belongs to $\mathcal{A}[0]$. This shows that $B_2 = 0$. As a conclusion, we see that $v = v'$ is split mono, thus Lemma 1 asserts that $u = 0$, a contradiction.

Thus, $C'' = 0$, and therefore C is a direct sum of objects in $\mathcal{A}[i]$ with $i \geq 0$. If w is non-zero, it follows that $s + 1 \geq 0$, thus $s \geq -1$, as we want to show. If $w = 0$, then u is a split monomorphism, thus $s = 0$. This completes the proof.

Finally, we have to show that $\mathcal{A}$ is an abelian admissible subcategory and closed under extensions. In order to do so, we may use the theory of t-categories, as introduced by Beilinson, Bernstein, Deligne [BBD] (but note that we deal with a very special case, that of a 'split t-category'; in this case the proof reduces to very few and easy considerations). Let

$$\mathcal{C}^{\geq n} = \operatorname{add}(\bigcup_{i\leq -n} \mathcal{A}[i]) \qquad \text{and} \qquad \mathcal{C}^{\leq n} = \operatorname{add}(\bigcup_{i\geq -n} \mathcal{A}[i])$$

(given a class $\mathcal{B}$ of objects of $\mathcal{C}$, we denote by $\operatorname{add}\mathcal{B}$ the smallest full subcategory of $\mathcal{C}$ containing $\mathcal{B}$ which is closed under finite direct sums and direct summands). We verify that the pair $\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}$ is, indeed, a t-category: Of course, $\mathcal{C}^{\leq n} = \mathcal{C}^{\leq 0}[-n]$ and $\mathcal{C}^{\geq n} = \mathcal{C}^{\geq 0}[-n]$, and we have $\mathcal{C}^{\leq 0} \subseteq \mathcal{C}^{\leq 1}$ as well as $\mathcal{C}^{\geq 0} \supseteq \mathcal{C}^{\geq 1}$. Also, we have $\operatorname{Hom}(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 1}) = 0$, since $\operatorname{Hom}(\mathcal{A}[i], \mathcal{A}[j]) = 0$ for $i \geq 0$ and $j \leq -1$. Finally, for every object X in $\mathcal{C}$, we need a triangle of the form (A, X, B) where A belongs to $\mathcal{C}^{\leq 0}$ and B belongs to $\mathcal{C}^{\geq 1}$. Since our objects are direct sums of indecomposable objects, it is sufficient to assume that X is indecomposable. Now any indecomposable object belongs to one of the subcategories $\mathcal{A}[t]$, with $t \in \mathbb{Z}$, thus either to $\mathcal{C}^{\leq 0}$ or to $\mathcal{C}^{\geq 1}$. If X belongs to $\mathcal{C}^{\leq 0}$, then take the triangle $(X, X, 0)$, otherwise the triangle $(0, X, X)$.

Now, $\mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$ is called the *heart* of the t-category, but this is just $\mathcal{A} = \mathcal{A}[0]$. According to Theorem 1.3.6 of [BBD], this is an abelian category and every exact sequence of the heart yields a triangle in $\mathcal{C}$, thus $\mathcal{A}$ is an admissible abelian subcategory of $\mathcal{C}$. Also, the heart of a t-category is closed under extension.

(ii) $\implies$ (i). In order to show that $\mathcal{A}$ is hereditary, we show that given an epimorphism $\epsilon\colon A' \to A$ in $\mathcal{A}$, any exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ is induced by ϵ. First of all, let A'' be the kernel of ϵ, thus we have also an exact sequence $0 \to A'' \to A' \to A \to 0$ in $\mathcal{A}$. The two exact sequences correspond to triangles $A \to B \to C \to A[1]$ with maps u, v, w and $A'' \to A' \to A \to A''[1]$ in $\mathcal{C}$ with maps μ, ϵ, δ. We apply $\operatorname{Hom}(C, -)$ to the triangle $A''[1] \to A'[1] \to A[1] \to A''[2]$, and obtain an exact sequence

$$\operatorname{Hom}(C, A'[1]) \xrightarrow{\epsilon[1]} \operatorname{Hom}(C, A[1]) \to \operatorname{Hom}(C, A''[2]),$$

but by assumption, $\operatorname{Hom}(C, A''[2]) = 0$. This shows that the map $\epsilon[1]$ is surjective. In particular, for $w \in \operatorname{Hom}(C, A[1])$, there is $w'\colon C \to A'[1]$ with $\epsilon[1]w' = w$. Let us complete w' to a triangle $A' \to B' \to C \to A'[1]$, with maps u', v', w'. Since $\mathcal{A}$ is closed under extensions, we see that B' belongs to $\mathcal{A}$ and that we have an exact sequence $0 \to A' \to B' \to C \to 0$ in $\mathcal{A}$ which induces the sequence $0 \to A \to B \to C \to 0$ under ϵ.

Note that for a triangle $A \to B \to C \to$ in $\mathcal{C}$ with maps u, v, w, where all three objects belong to $\mathcal{A}$, the map w is uniquely determined by u and v, according to [BBD, 1.1.10 (ii)]. This we want to use in order to show that the identity functor on $\mathcal{A}$ extends to an equivalence of categories $D^b(\mathcal{A}) \to \mathcal{C}$. We define a functor

$$F\colon D^b(\mathcal{A}) \to \mathcal{C}$$

which extends the identity on $\mathcal{A}$. In order to distinguish objects in these two categories, let us denote the shift functors in $D^b(\mathcal{A})$ by $\langle - \rangle$. By definition, $\mathcal{C}$ is the additive closure of the disjoint union of the categories $\mathcal{A}[t]$ with $t \in \mathbb{Z}$; similarly, $D^b(\mathcal{A})$ is the additive closure of the disjoint union of the categories $\mathcal{A}\langle t \rangle$ with $t \in \mathbb{Z}$.

Of course, we want that F commutes with the shift functors $\langle - \rangle$ and $[-]$, this defines F on all the subcategories $\mathcal{A}\langle t \rangle$.

It remains to define F on the maps $A\langle t \rangle \to B\langle t+1 \rangle$, where A, B belong to $\mathcal{A}$. It is sufficient to do this for $t = 0$ and then to use the shift functors. Thus, let $w \colon A \to B\langle 1 \rangle$ be such a map. There exists a corresponding triangle $B \to C \to A \to$ in $D^b(\mathcal{A})$, say with maps u, v, w. Since $\mathcal{A}$ is extension closed in $D^b(\mathcal{A})$, and A, B belong to $\mathcal{A}$, also C belongs to $\mathcal{A}$ and we have the corresponding exact sequence $0 \to B \to C \to A \to 0$ in $\mathcal{A}$. This exact sequence gives rise to a unique map $\overline{w}$ in $\mathcal{C}$ such that $A \to B \to C \to$ with maps $u, v, \overline{w}$ is a triangle in $\mathcal{C}$. We want to put $F(w) = \overline{w}$. Our construction of $\overline{w}$ depends on the choice of C and the maps u, v. However, given another triangle $B \to C' \to A \to$ with maps u', v', w, we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \xrightarrow{u} & C & \xrightarrow{v} & A & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & B & \xrightarrow{u'} & C' & \xrightarrow{v'} & A & \longrightarrow & 0
\end{array}$$

in $\mathcal{A}$, and thus, in $\mathcal{C}$ a commutative diagram

$$\begin{array}{ccccccc}
B & \xrightarrow{u} & C & \xrightarrow{v} & A & \xrightarrow{\overline{w}} & B[1] \\
\| & & \downarrow & & \| & & \| \\
B & \xrightarrow{u'} & C' & \xrightarrow{v'} & A & \xrightarrow{\overline{w}} & B[1]
\end{array}$$

As we see, the two exact sequences $0 \to B \to C \to A \to 0$ and $0 \to B \to C' \to A \to 0$ give rise to the same map $\overline{w} \colon A \to B[1]$, thus it is well-defined to put $F(w) = \overline{w}$. The fact that $\overline{w}$ is uniquely determined by w also implies that our definition of F really yields a functor (one has to check that F behaves well with respect to composition of maps). Once the functoriality of F has been checked, it is straight-forward to see that F is an equivalence of categories.

Example. There does exist an abelian subcategory $\mathcal{A}$ of a triangulated category $\mathcal{C}$ such that $\mathcal{A}$ is hereditary, the additive closure of the union of the copies $\mathcal{A}[t]$ with $t \in \mathbb{Z}$ is $\mathcal{C}$, whereas the condition $\operatorname{Hom}(\mathcal{A}[s], \mathcal{A}[t]) = 0$ for $t \notin \{s, s+1\}$ is not satisfied.

Namely, let $\mathcal{C}$ be the bounded derived category $D^b(\operatorname{mod} A)$ where A is the path algebra of a quiver of type A_2 over a field k; this is the quiver with two vertices, say a and b, and a single arrow $a \to b$. Note that $\operatorname{mod} A$ has precisely three indecomposable modules, say S, I, T, where S is simple projective, I has length 2, and T is simple injective. Let $\mathcal{A}$ be the full subcategory with objects $S, T, I[1]$ (or better, the additive closure of this subcategory). Then this category is an abelian subcategory, it is hereditary, even semisimple: it is isomorphic to the category of H-modules, where $H = k \times k \times k$. Every indecomposable object of $\mathcal{C}$ can be shifted into $\mathcal{A}$, but there is a non-zero homomorphism $A[0] \to B[-1]$ where A, B belong to $\mathcal{A}$; just take $A = S, B = I[1]$. Note that $\mathcal{A}$ is not closed under extensions in $\mathcal{C}$, and, of course, that the categories $D^b(\operatorname{mod} A)$ and $D^b(\operatorname{mod} B)$ are not equivalent.

Remark 1. The case described in Theorem 1 seems to be one of the rare situations where the derived categories of a class of abelian categories can easily be characterized as special triangulated categories. Usually, the axioms of a triangulated category tend to be too broad for such an endeavour.

3. Paths in triangulated categories

Let $\mathcal{C}$ be a triangulated category. We will assume from now on that idempotents in $\mathcal{C}$ split and that every object in $\mathcal{C}$ is a (finite) direct sum of indecomposable objects.

Let X, Y be two indecomposable objects in $\mathcal{C}$. A *path in $\mathcal{C}$ of length n* is a sequence $X_0, X_1, \dots, X_n$ of indecomposable objects in $\mathcal{C}$ such that for $1 \le i \le n$, we have $\operatorname{Hom}(X_{i-1}, X_i) \neq 0$ or $X_i = X_{i-1}[1]$; we say that the path *starts in* X_0 and *ends in* X_n, or that it is a path *from* X_0 *to* X_n.

Let $\mathcal{U}$ be a class of indecomposable objects in $\mathcal{C}$. We say that $\mathcal{U}$ is *path-closed* provided, first of all, $\mathcal{U}$ is closed under the translation functor and its inverse, and second, if X belongs to $\mathcal{U}$ and Y is indecomposable and $\operatorname{Hom}(X, Y) \neq 0$ or $\operatorname{Hom}(Y, X) \neq 0$, then also Y belongs to $\mathcal{U}$. Note that if $\mathcal{U}$ is path-closed, then also the complement $\mathcal{V}$ (consisting of all indecomposable objects of $\mathcal{C}$ which do not belong to $\mathcal{U}$) is path-closed.

A class $\mathcal{U}$ of indecomposable objects in $\mathcal{C}$ is said to be *path-connected* provided for any pair of objects X, Y in $\mathcal{U}$, there is a sequence $X = X_0, X_1, \dots, X_t = Y$ such that for $1 \le i \le t$, at least one of the three conditions $\operatorname{Hom}(X_{i-1}, X_i) \neq 0$, or $\operatorname{Hom}(X_i, X_{i-1}) \neq 0$, or $X_i = X_{i-1}[s]$ for some $s \in \mathbb{Z}$, is satisfied.

Lemma 4. *Let $\mathcal{U}$ be a path-closed class of indecomposable objects in $\mathcal{C}$ and let $\mathcal{V}$ be its complement. Let $\mathcal{C}_1 = \operatorname{add}\mathcal{U}$ and $\mathcal{C}_2 = \operatorname{add}\mathcal{V}$. Then both subcategories $\mathcal{C}_1$ and $\mathcal{C}_2$ are triangulated subcategories and $\mathcal{C}$ is their product.*

Proof. Of course, any object X in $\mathcal{C}$ is a direct sum $X = X_1 \oplus X_2$ with $X_1 \in \mathcal{C}_1$ and $X_2 \in \mathcal{C}_2$.

Now, let $f \colon X \to Y$ be a morphism in $\mathcal{C}_1$ and consider a triangle $X \to Y \to Z \to X[1]$ in $\mathcal{C}$, say with maps u, v, w. We can decompose $Z = Z_1 \oplus Z_2$, with $Z_1 \in \mathcal{C}_1$ and $Z_2 \in \mathcal{C}_2$. We apply the functor $\operatorname{Hom}(Z_2, -)$ to the triangle and obtain an exact sequence $\operatorname{Hom}(Z_2, Y) \to \operatorname{Hom}(Z_2, Z) \to \operatorname{Hom}(Z_2, X[1])$. But the left hand term and the right hand term both are zero (here we use that with X also $X[1]$ belongs to $\mathcal{C}_1$), thus also $\operatorname{Hom}(Z_2, Z) = 0$. But his implies that $Z_2 = 0$, since Z_2 is a direct summand of Z.

This shows that $\mathcal{C}_1$ has sufficiently many triangles, and the same is true for $\mathcal{C}_2$.

Finally, take an arbitrary triangle $X \to Y \to Z \to X[1]$ in $\mathcal{C}$, with maps u, v, w. We decompose $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$, $Z = Z_1 \oplus Z_2$. Clearly, the map u is of the form $u = u_1 \oplus u_2$ with $u_i \colon X_i \to Y_i$ (since $\operatorname{Hom}(X_1, Y_2) = 0 = \operatorname{Hom}(X_2, Y_1)$ and the same is true for v and w. Now, we consider triangles $X_i \to Y_i \to Z_i' \to X_i[1]$ with maps u_i, v_i', w_i' and with $Z_i' \in \mathcal{C}_i$. We form the direct sum, this is again a triangle, and the identity morphisms for X and Y yield a triangle morphism $(1, 1, h)$, where $h\colon Z_1 \oplus Z_2 \to Z_1' \oplus Z_2'$ has to be an isomorphism. Again, the non-existence of maps between $\mathcal{C}_1$ and $\mathcal{C}_2$ can be used: it follows that $h = h_1 \oplus h_2$ with isomorphisms $h_i\colon Z_i \to Z_i'$. This shows that the triangles in $\mathcal{C}$ are just the direct sums of triangles in $\mathcal{C}_1$ and triangles in $\mathcal{C}_2$.

The category $\mathcal{C}$ is called a *block* provided it is non-zero and not triangle-equivalent to the product category of two non-zero triangulated categories.

Lemma 5. *Let $\mathcal{C}$ be a non-zero triangulated category such that idempotents in $\mathcal{C}$ split and such that every object in $\mathcal{C}$ is a direct sum of indecomposable objects. Then: $\mathcal{C}$ is a block if and only if the class of indecomposable objects in $\mathcal{C}$ is path-connected.*

Proof. First assume that $\mathcal{C}$ is the product of two non-zero triangulated categories $\mathcal{C}_1$ and $\mathcal{C}_2$. Clearly, there is no path joining an indecomposable object from $\mathcal{C}_1$ with an indecomposable object from $\mathcal{C}_2$, and also no one in the opposite direction. On the other hand, all the objects in $\mathcal{C}_1$ and in $\mathcal{C}_2$ are direct sums of indecomposable objects, thus there do exist indecomposable objects in any of these categories. This shows that the class of indecomposable objects in $\mathcal{C}$ is not path-connected.

For the converse, start with an indecomposable object X in $\mathcal{C}$ and let $\mathcal{X}$ be the class of all indecomposable objects in $\mathcal{C}$ which can be connected to X by a sequence of paths and inverses of paths (thus $\mathcal{X}$ is the connected component with respect to path-connectivity in the class of indecomposable objects of $\mathcal{C}$). As above, let $\mathcal{Y}$ be the set of indecomposable objects of $\mathcal{C}$ which do not belong to $\mathcal{X}$. The decomposition of $\mathcal{C}$ as the product of $\operatorname{add}\mathcal{X}$ and $\operatorname{add}\mathcal{Y}$ shows that $\mathcal{Y}$ has to be empty.

Lemma 6. *Let $\mathcal{C}$ be a triangulated category which is a block. Either all non-zero maps in $\mathcal{C}$ are invertible, or else for any indecomposable object X in $\mathcal{C}$, there is a sequence $X = X_0, X_1, X_2, X_3 = X[1]$ with $\operatorname{Hom}(X_{i-1}, X_i) \neq 0$.*

This means the following: In case we consider paths $X_0, X_1, \dots, X_t$ in triangulated categories, we usually may assume (say after some refinement) that we have $\operatorname{Hom}(X_{i-1}, X_i) \neq 0$; the only exceptions are paths inside blocks where all non-zero maps are invertible. The typical example of the latter behaviour is the derived category $D^b(\operatorname{mod} k)$.

Proof. Let X be an indecomposable object in $\mathcal{C}$. First, assume that there exists an indecomposable object Y and a non-zero and non-invertible map $u\colon X \to Y$. There exists a triangle $X \to Y \to Z \to$ with maps u, v, w. Decompose $Z = \bigoplus_{i=1}^t Z_t$ with indecomposable objects Z_i and write $v = (v_1, \dots, v_t)$, and $w = (w_1, \dots, w_t)$. According to Lemma 3, we know that $t \geq 1$ and that all the maps v_i, w_i are non-zero and non-invertible. Thus $X_0 = X, X_1 = Y, X_2 = Z_1, X_3 = X[1]$ is a path as required.

Second, assume that for any indecomposable object Y, all non-zero homomorphisms $X \to Y$ are isomorphisms. Then we claim that also for any indecomposable object W, all non-zero homomorphisms $W \to X$ are isomorphisms. Let W be an indecomposable object in $\mathcal{C}$, and assume to the contrary that there exists a non-zero and non-invertible map $u'\colon W \to X$. Take a triangle $W \to X \to Y' \to$ with maps u', v', w' and decompose $Y' = \bigoplus_{i=1}^s Y_i$, and write $v' = (v_1', \dots, v_s')$. Again, using 1.3, we see that $s \geq 1$ and that $v_1'\colon X \to Y_1$ is non-zero and non-invertible, a contradiction. It follows that the set of all translates $X[r]$ with $r \in \mathbb{Z}$ is path-closed (and path-connected), thus these are all the objects of the block $\mathcal{C}$.

Lemma 7. *Let $\mathcal{C}$ be a triangulated category which is a block. Let X, Y be indecomposable objects in $\mathcal{C}$. Then there exists $n \geq 0$ and a path from X to $Y[n]$.*

Proof. By assumption, $\mathcal{C}$ is path-connected, thus there is a sequence $X = X_0, X_1, \dots, X_t = Y$ such that for $1 \leq i \leq t$, at least one of the conditions $\mathrm{Hom}(X_{i-1}, X_i) \neq 0$, or $\mathrm{Hom}(X_i, X_{i-1}) \neq 0$, or $X_i = X_{i-1}[s]$ for some $s \in \mathbb{Z}$, is satisfied. Inductively, we are going to adjust this sequence in order to obtain a path from X to some $Y[n]$ with $n \geq 0$.

First of all, if $X_i = X_{i-1}[-s]$ for some $s > 0$, then we delete the object X_i and replace X_j with $j > i$ be $X_j[s]$. Thus, we consider the new sequence $X_0, X_1, \dots, X_{i-1}, X_{i+1}[s], \dots, X_t[s]$. Note that the new pair $X_{i-1}, X_{i+1}[s]$ has the same properties as the old one $X_{i-1}[-s], X_{i+1}$.

Second, assume that $\mathrm{Hom}(X_i, X_{i-1}) \neq 0$, say take a non-zero and non-invertible map $u\colon X_i \to X_{i-1}$. Let $X_i \to X_{i-1} \to Z \to X_i$ be a triangle with maps u, v, w. Let Z' be an indecomposable summand of Z, and $v'\colon X_{i-1} \to Z'$ and $w'\colon Z' \to X_i[1]$ the maps induced from v and w, respectively. According to Lemma 3, these maps v', w' are non-zero and non-invertible, thus we have a path $X_{i-1} \to Z' \to X_i[1]$. It follows that we can replace the given sequence by the new one $X = X_0, X_1, \dots, X_{i-1}, Z', X_i[1], \dots, X_t[1] = Y[1]$.

4. Hereditary triangulated categories which are blocks

Theorem 2. *Assume that $\mathcal{C}$ is a triangulated category which is a block. The following conditions are equivalent:*

(i) *$\mathcal{C} = D^b(\mathcal{H})$ for some hereditary abelian category $\mathcal{H}$.*
(ii) *If X is indecomposable in $\mathcal{C}$, then there is no path from $X[1]$ to X.*
(iii) *There exists an indecomposable object X in $\mathcal{C}$ with no path from $X[1]$ to X.*
(iv) *There are indecomposable objects X, Y in $\mathcal{C}$ with no path from Y to X.*

Proof. Given an object X in $\mathcal{C}$, let $[X, \to]$ be the class of all indecomposable objects U in $\mathcal{C}$ with a path from X to U.

The implication (i) $\Longrightarrow$ (ii) is obvious: Since X is indecomposable, it is of the form $X = A[n]$, where A is indecomposable in $\mathcal{H}$ and $n \in \mathbb{Z}$, and by induction on the length of paths, we see that $[X, \to] \subseteq \bigcup_{i \geq n} \mathcal{A}[i]$.

The implications (ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (iv) are trivial. The implication (iv) $\Longrightarrow$ (iii) follows from Lemma 7: Consider the given indecomposable objects X, Y such that there is no path from Y to X. As we have shown, there is a path from Y to $X[n]$ for some $n \geq 0$. By assumption, we must have $n \geq 1$. Clearly, there cannot exist a path from $X[1]$ to X, since otherwise we would have a path from $X[n]$ to X, thus from Y to X.

Finally, let us assume condition (iii): Let X be an indecomposable object of $\mathcal{C}$ with no path from $X[1]$ to X. We are going to construct an additive subcategory $\mathcal{A}$ which satisfies the conditions (iii) of Theorem 1. Let $\mathcal{U} = [X, \to]$, and let $\mathcal{V}$ be the complement of $\mathcal{U}$ in the class of all indecomposable objects of $\mathcal{C}$. Let

$$\mathcal{A} = \mathcal{U} \ \cap \ \mathcal{V}[1].$$

We want to show that any indecomposable object Y of $\mathcal{C}$ belongs to the union $\bigcup_{n\in\mathbb{Z}} \mathcal{A}[n]$.

First, let us assume that Y belongs to $\mathcal{U} = [X, \to]$. Thus, there is a sequence $X = X_0, X_1, \dots, X_t = Y$ such that for $1 \le i \le t$, we have $\mathrm{Hom}(X_{i-1}, X_i) \neq 0$, or $X_i = X_{i-1}[1]$. By induction, we can assume that X_{t-1} belongs to $\bigcup_{n\in\mathbb{Z}} \mathcal{A}[n]$, say $X_{t-1} = A[n]$ where A belongs to $\mathcal{U} \cap \mathcal{V}[1]$. The case $X_t = X_{t-1}[1]$ is trivial, thus we may assume that $\mathrm{Hom}(A[n], Y) \neq 0$.

Since A belongs to $\mathcal{U}$, there is a path from X to A, and by our assumption also from A to $Y[-n]$, thus $Y[-n]$ belongs to $\mathcal{U}$.

If $Y[-n]$ also belongs to $\mathcal{V}[1]$, then $Y[-n]$ belongs to $\mathcal{A}$, thus $Y \in \mathcal{A}[n]$.

Let us assume that $Y[-n]$ does not belong to $\mathcal{V}[1]$. This means that $Y[-n]$ belongs to $\mathcal{U}[1]$. We claim that in this case $Y[-n]$ also belongs to $\mathcal{V}[2]$. Assume not: then $Y[-n]$ belongs to $\mathcal{U}[2]$, thus there is a path from $X[2]$ to $Y[-n]$. Starting with a non-zero and non-invertible map $w\colon A[n] \to Y$, we obtain a triangle $Y[-1] \to W \to A[n] \to$ with maps u, v, w, and Lemma 3 asserts that there is a path from $Y[-1]$ to $A[n]$, thus from $Y[-n]$ to $A[1]$. In this way, we have obtained a path from $X[2]$ via $Y[-n]$ to $A[1]$, thus one from $X[1]$ to A. However, this means that A belongs to $\mathcal{U}[1]$ and thus not to $\mathcal{V}[1]$, a contradiction. Altogether, we have shown that $Y[-n]$ belongs to $\mathcal{U}[1]$ and to $\mathcal{V}[2]$, thus to $\mathcal{A}[1]$. Consequently, Y belongs to $\mathcal{A}[n+1]$.

Now, let Y be arbitrary. According to Lemma 7, there is $n \ge 0$ such that there exists a path from X to $Y[n]$, thus $Y[n]$ belongs to $\mathcal{U} = [X, \to]$. By the previous considerations, $Y[n]$ belongs to $\mathcal{A}[m]$ for some m, thus Y belongs to $\mathcal{A}[m-n]$.

Finally, we have to show the following: $\mathrm{Hom}(A[s], A'[s']) \neq 0$ for A, A' in $\mathcal{A}$ and integers s, s' implies $s \le s'$. We may assume that $s = 0$ and have to show that $s' \ge 0$. Since A belongs to $\mathcal{U}$ and $\mathrm{Hom}(A, A'[s']) \neq 0$, we see that also $A'[s']$ belongs to $\mathcal{U}$. Now assume that $s' < 0$, thus $s'+1 \le 0$. By assumption, $A'[s']$ belongs to $\mathcal{V}[s'+1]$, and since $s'+1 \le 0$, $\mathcal{V}[s'+1] \subseteq \mathcal{V}$. But $A'[s'] \in \mathcal{V}$ means that $A'[s']$ does not belong to $\mathcal{U}$, a contradiction. This completes the proof.

Remark 2. One may be surprised to see the equivalence of the conditions (ii) and (iii): The existence of a single indecomposable object with a special property forces all the indecomposable objects to have this property! This indicates a rather unusual character of homogeneity.

Remark 3. Note that in Theorem 2, we cannot expect that $\mathcal{H}$ is unique up to equivalence. For example, let $\mathcal{C} = D^b(\operatorname{mod} A(n))$, where $A(n)$ is the path algebra of the quiver with two vertices a and b, and n arrows $a \to b$. In case $n \ge 3$, there are precisely three equivalence classes of hereditary categories $\mathcal{H}$ such that $\mathcal{C}$ is triangle equivalent to $D^b(\mathcal{H})$ and these categories behave rather differently: two of them have no non-zero projective or injective objects, in contrast to $\mathcal{H} = \operatorname{mod} A(n)$. In case $n = 2$, there are even uncountably many equivalence classes of such categories $\mathcal{H}$.

Let us call a path $X_0, X_1, \dots, X_n$ *proper* provided for $1 \le i \le n$, there exists a non-zero and non-invertible map $X_{i-1} \to X_i$ or else $X_i = X_{i-1}[1]$. An indecomposable object X in a triangulated category $\mathcal{C}$ will be said to be *directing* provided there is no proper path of length at least 1 starting and ending in X.

Corollary 1. *Assume that $\mathcal{C}$ is a triangulated category which is a block. If $\mathcal{C}$ contains a directing object, then $\mathcal{C} = D^b(\mathcal{H})$ for some hereditary abelian category $\mathcal{H}$.*

Proof. Let X be a directing object in $\mathcal{C}$. Any path from $X[1]$ to X could be composed with the path $X, X[1]$, and (maybe after deleting some repetitions) we obtain a proper path from X to X of length at least 1. Thus, no path from $X[1]$ to X exists, and can we apply Theorem 2.

Remark 3. Dealing with abelian categories, the corresponding notion of a directing object has been found very useful [Ri] (the paths to be considered here are of the form $X_0, X_1, \dots, X_n$ such that for $1 \le i \le n$, there exists a non-zero and non-invertible map $X_{i-1} \to X_i$). We note the following: *Let $\mathcal{H}$ be a hereditary abelian category. An indecomposable object X of $\mathcal{H}$ is directing in $\mathcal{H}$ if and only if X is directing when considered as an object of the triangulated category $D^b(\mathcal{H})$.*

5. Piecewise hereditary algebras

Let k be a field. The finite dimensional k-algebras A, B are said to be *derived equivalent*, provided the triangulated categories $D^b(\operatorname{mod} A)$ and $D^b(\operatorname{mod} B)$ are triangle-equivalent.

Corollary 2. *Let A be a finite dimensional connected k-algebra and assume that $D^b(\operatorname{mod} A)$ contains a directing object. Then A is derived-equivalent to a finite-dimensional hereditary algebra.*

Recall [HRS2] that a finite dimensional k-algebra R is said to be *piecewise hereditary* provided the bounded derived category $D^b(\operatorname{mod} R)$ is triangle-equivalent to the bounded derived category $D^b(\mathcal{H})$ of some hereditary abelian category $\mathcal{H}$. (An important class of piecewise hereditary algebras are the quasi-tilted algebras introduced in [HRS1].) Under the additional assumption on R to be piecewise hereditary, the assertion of Corollary 2 is the main result of the paper [HR] by Happel and Reiten and our contribution is the verification that the existence of a directing object in $D^b(\operatorname{mod} A)$ forces A to be piecewise hereditary, see Corollary 1. In this way, we answer a corresponding question of I. Reiten [Re] (and the author is indebted to her helpful comments concerning the presentation of the paper).

Note that Theorem 2 furnishes the following characterization of piecewise hereditary algebras:

Corollary 3. *Let A be a finite dimensional connected k-algebra. The following conditions are equivalent:*
(i) *A is piecewise hereditary.*
(ii) *For any indecomposable object X in $D^b(\operatorname{mod} A)$, there is no path from $X[1]$ to X.*
(iii) *There exists an indecomposable object X in $D^b(\operatorname{mod} A)$ with no path from $X[1]$ to X.*

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