Triangulated Categories and Kac-Moody Algebras*

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Dedicated to our teacher Shaoxue Liu for his 70th birthday

Abstract

By using the Ringel-Hall algebra approach, we find a Lie algebra arising in each triangulated category with $T^2 = 1$, where T is the translation functor. In particular, the generic form of the Lie algebras determined by the root categories, the 2-period orbit categories of the derived categories of finite dimensional hereditary associative algebras, gives a realization of all symmetrizable Kac-Moody Lie algebras.

1991 AMS Subject Classification. 16G20, 17B67; Secondary 16G10, 17B37.
Key Words and Phrases: triangulated category, Kac-Moody algebra, derived category, hereditary algebra.

Contents

1 Introduction

1.1 A Cartan datum is a pair $(I, (,))$ consisting of a finite set I and a symmetric bilinear form $(-,-)$ on the free abelian group $\mathbf{Z}[I]$ with values in $\mathbf{Z}$ such that the following conditions are satisfied.

(a) $(i,i) \in \{2,4,6,\cdots\}$ for any $i \in I$.

(b) $2(i,j)/(i,i) \in \{0,-1,-2,\cdots\}$ for any $i \neq j$ in I.

Then $C = (a_{ij})_{i,j\in I}$ with $a_{ij} = \frac{2(i,j)}{(i,i)}$ is just a symmetrizable generalized Cartan matrix. Without loss of generality, we may identify a Cartan datum $(I, (,))$ with the symmetrizable generalized Cartan matrix $C = (a_{ij})_{i,j\in I}$.

There are many well-developed mathematical disciplines corresponding to a given Cartan datum. In this article, we will consider the relations between the Kac-Moody algebras and the derived categories of representations of finite dimensional hereditary associative algebras.

*The research was supported in part by NSF of China (Peng and Xiao), by the fellowships from the Alexander von Homboldt Foundation (Peng) and by Trans-century training programme foundation for the talents,SECC (Xiao)

1.2 Let $C = (a_{ij})_{i,j \in I}$ be a symmetrizable generalized Cartan matrix. The associated Kac-Moody algebra $\mathbf{g}(C)$ is a complex Lie algebra on $3n$ generators e_i, f_i, h_i $(i \in I, n = |I|)$ and the following defining relations:

$$\begin{cases} [h_i, h_j] = 0 \ , \quad [e_i, f_i] = h_i \ , \quad [e_i, f_j] = 0 \quad \text{if } i \neq j \ , \\ [h_i, e_j] = a_{ij} e_j \ , \quad [h_i, f_j] = -a_{ij} f_j \ , \\ (\mathrm{ad} e_i)^{1-a_{ij}} e_j = 0 \ , \quad (\mathrm{ad} f_i)^{1-a_{ij}} f_j = 0 \quad \text{if } i \neq j \ , \end{cases}$$

where $\mathrm{ad}e_i = [e_i, -]$. Note that the Lie algebra $\mathbf{g}(C)$ defined here is the derived algebra of the Kac-Moody algebra given in the book [K1] and the two Lie algebras share a common structure and representation theory.

1.3 Given a field k, let A be a finite dimensional hereditary associative k-algebra, where "hereditary" means $\mathrm{Ext}^2_A(-,-) = 0$ everywhere. Let $\mathrm{mod}A$ be the category of finitely generated A-modules. And let $G(\mathrm{mod}A)$ be the Grothendieck group of $\mathrm{mod}A$, the factor group of the free abelian group generated by isomorphism classes $[X]$ of all A-modules X by the subgroup generated by the elements $[X] - [Y] + [Z]$ provided there exist exact sequences of form $0 \to X \to Y \to Z \to 0$ in $\mathrm{mod}A$. It is easy to see $G(\mathrm{mod}A) = \mathbf{Z}[I]$, here I is the set of isomorphism classes of simple A-modules. For $M \in \mathrm{mod}A$, by $\underline{\dim}M$ we denote the canonical image of M in $\mathbf{Z}[I]$, called the dimension vector of M. We have the symmetric Euler form defined on $\mathbf{Z}[I]$: for any $M, N \in \mathrm{mod}A$

$$\begin{aligned} (\underline{\dim}M, \underline{\dim}N) &= \dim_k \mathrm{Hom}_A(M,N) - \dim_k \mathrm{Ext}_A(M,N) \\ &+ \dim_k \mathrm{Hom}_A(N,M) - \dim_k \mathrm{Ext}_A(N,M). \end{aligned}$$

This is well-defined since $\mathrm{Ext}^2_A(-,-) = 0$.

We have the following observation (see [R8]).

If $(-,-)$ is the symmetric Euler form defined on the Grothendieck group $\mathbf{Z}[I]$ of $\mathrm{mod}A$, then the pair $(I, (-,-))$ is a Cartan datum. Conversely, any Cartan datum can be realized by a symmetric Euler form defined on the Grothendieck group of $\mathrm{mod}A$ for some finite dimensional hereditary k-algebra A.

1.4 According to the Gabriel theorem [G1] on representations of quivers and an extended version given by Dlab-Ringel[DR1], we know that finite dimensional hereditary k-algebras of finite representation type are given by Dynkin diagrams of type $A_n, D_n, B_n, C_n, E_6, E_7, E_8, F_4$ and G_2, and the map $\underline{\dim}$ induces a bijection between the isomorphism classes of all indecomposable modules over a hereditary algebra A of Dynkin type and the positive roots of the corresponding semisimple complex Lie algebra $\mathbf{g}$.

It is natural to ask whether it is possible to recover the Lie algebra structure of $\mathbf{g}$ in terms of the representations of A. This was succesfully attained by

Ringel [R2] and [R3] via the Ringel-Hall algebra construction. Let $K(\mathrm{mod}A)$ be the free abelian group with a basis indexed by the set of isomorphism classes of indecomposable A-modules. He showed that $K(\mathrm{mod}A)$ is the Chevalley $\mathbf{Z}$-form of $\mathbf{n}_+$ and hence $K(\mathrm{mod}A) \otimes_{\mathbf{Z}} \mathbf{C}$ and $\mathbf{n}_+$ are isomorphic Lie algebras, where $\mathbf{g} = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+$ is a triangular decomposition of $\mathbf{g}$.

Ringel consider the case A over finite fields and filtrations of finite A-modules. He proved the existence of Hall polynomials for the filtrations and, by using as structure constants the number of filtrations or evaluations of Hall polynomials at various valus, he got an associative algebra $\mathcal{H}_q(A)$ of A, which is called Hall algebra by him and now is called Ringel-Hall algebra often. The complex form $\mathcal{H}_1(A) \otimes_{\mathbf{Z}} \mathbf{C}$ of the degenerated Ringel-Hall algebra $\mathcal{H}_1(A)$ is just the universal enveloping algebra of $K(\mathrm{mod}A) \otimes_{\mathbf{Z}} \mathbf{C}$ and $\mathcal{H}_1(A)$ is the Kostant $\mathbf{Z}$-form of the universal enveloping algebra of $\mathbf{n}_+$. A similar investigation was also given later by Schofield [S2] and Riedtmann [Rie] who considered the Euler characteristic of corresponding varieties instead of the evaluation of the Hall polynomial at $q = 1$.

1.5 There is a lot of work after the Gabriel theorem to connect the representation theory of hereditary algebras of tame or wild type to Lie theory. A good approach is through representations of quivers. For example, Kac [K2] developed the Gabriel theorem to all infinite type cases and discovered a deep connectoin between root systems of Kac-Moody algebras and representations of quivers. Now the quiver-approach has become an important tool in the study of quantum groups and Lie theory (see, for example, [L2],[Na]). Of course, Ringel's Hall algebra approach is also effective for representation-infinite cases. However one should use the composition algebra to replace the Hall algebra. It is well-known now that the degenerate composition algebra of any finite dimensional hereditary algebra gives a realization of the universal enveloping algebra of the positive part of the corresponding Kac-Moody Lie algebra, and the Lie algebra can be recovered from this construction.

1.6 Another advance in the representation theory of finite dimensional algebras is the theory of derived category due to Happel [H1] and [H2]. The derived category of a finite dimensional algebra B, denoted by $D^b(B)$, is a triangulated category with the translation functor T. In particular, the category $D^b(B)$ was well-described when B is hereditary. Happel [H1] pointed out that, if A is hereditary and representation-finite, then the isomorphism classes of indecomposable objects in $D^b(A)/T^2$ correspond bijectively to the roots of $\mathbf{g}$ (not only positive roots). For this reason, $\mathcal{R} = D^b(A)/T^2$ is called the root category. In [K1] a fundamental question about the realization of non-affine Lie algebras was asked by Kac. Also Ringel [R4] asked whether it is possible to use representations of algebras to recover the whole Lie algebra $\mathbf{g}$ and not only $\mathbf{n}_+$. For the Ringel's question, the authors in [PX1] and [PX2] have given positive answers in the cases of semisimple complex Lie algebras and affine Lie algebras of type $\tilde{A}_{n-1}$, via the root categories of representation-finite hereditary algebras and cyclic quivers respectively.

In [PX1], it has been found that the category $\mathcal{R} = D^b(A)/T^2$ inherits a triangulated category structure from $D^b(A)$. According to Kac [K2] and Happel [H1], even when A is a representation-infinite hereditary algebra, the root category $\mathcal{R} = D^b(A)/T^2$ still provides a good framwork which is compatible with the root system of the corresponding Kac-Moody algebra. The aim of this article is to give a realization of all symmetrizable Kac-Moody algebras in a global way, via the root categories $\mathcal{R} = D^b(A)/T^2$ of finite dimensional hereditary algebras A.

1.7 In Section 2, we recall the definition and some elementary facts of a triangulated category. Section 3 is devoted to defining a $\mathbf{Z}/(q-1)$-Lie algebra from a 2-period triangulated category $\mathcal{C}$. The basis of this Lie algebra is given by the isomorphism classes of indecomposable objects in $\mathcal{C}$, the Cartan subalgebra is given by the Grothendieck group of $\mathcal{C}$, and the structure constants are given by the cardinality of the orbit spaces, which are defined by the group $\mathrm{Aut}X \times \mathrm{Aut}Y$ acting on the set $W(X, Y; L)$ of triangles with form $X \to L \to Y \to TX$. As a consequence, the translation functor provides the Chevalley involution and some easy properties of triangles provide a "Killing-form" on the Lie algebra. In Section 4, we consider the root category $\mathcal{R}$, a special 2-period triangulated category. By using the "generic" method to the Lie algebras constructed in Section 3, we get a transcendental Lie algebra. Our main theorem asserts that the complex form of the Lie subalgebra of the transcendental Lie algebra, generated by all exceptional objects in $\mathcal{R}$, is canonically isomorphic to the corresponding symmetrizable Kac-Moody Lie algebra. This gives a realization of every symmetrizable Kac-Moody Lie algebra in a global way. Here "global" means that the realization dosen't depend on the choice of the hereditary subcategory of $\mathcal{R}$ or equivalently on the concrete triangular decomposition of the Kac-Moody Lie algebra. However, if we restrict the definition to any hereditary subcategory of the root category $\mathcal{R}$, then the induced Lie subalgebra coincides with the Lie subalgebra of the degenerate Ringel-Hall algebra and so the calculations can be done locally. The purpose of Section 5 is to prove that the vectors of exceptional modules belong to the composition Lie algebra, Our proof is essentially due to Ringel in [R7] where the case of representation-finite was investigated. In Section 6, we replace the octahedral axiom of a triangulated category by a diagram of pushout and pullback. Proposition 6.1 gives a precise one to one correspondence between the orbit spaces of the pushout and the pullback. Section 7 is the most technical part of the present paper. After some complicated discussions, we get several formulae about the cardinalities of certain orbit spaces in 2-period triangulated categories. In Section 8, we prove Theorem 3.4, i.e. to check the Jacobi identity, based on the formulae in Section 7.

According to the definition of our Lie algebras, the Jacobi identity reflects a strong symmetry in 2-period triangulated categories (not only root categories). We find this symmetry by taking that q tends to 1. However the symmetry for ordinary q is not well-displayed at the moment.

2 Some basic facts in triangulated categories

2.1 Given a field k, the categories we consider in the present article are k-additive categories in the sense of Maclane [Ma]. In particular, our additive categories have finite direct sums. Moreover we assume throughout this article that (1) any object in the category can be decomposed into the direct sum of finitely many indecomposable ones, (2) the homomorphism space $\mathrm{Hom}(X,Y)$ for any two objects X and Y is finite dimensional over the central field k and (3) the endomorphism ring $\mathrm{End}(X)$ for any indecomposable object X is a finite dimensional local algebra over k.

We denote by fg the composition of morphisms $f : X \to Y$ and $g : Y \to Z$ in any category.

2.2 Let $\mathcal{C}$ be an additive category together with a translation T (an automorphism of $\mathcal{C}$) and a set Ω of some sextuples in $\mathcal{C}$ of form $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$. We call $(\mathcal{C}, T, \Omega)$ a triangulated category and the elements of Ω are then called triangles if the following axioms are satisfied.

(Tr1) Every sextuple isomorphic to a triangle is a triangle. Every morphism $u : X \to Y$ can be embedded into a triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$. The sextuple $X \xrightarrow{1_X} X \xrightarrow{0} 0 \xrightarrow{0} TX$ is a triangle, where 1_X is the identity morphism of X.

(Tr2) If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ is a triangle, then $Y \xrightarrow{v} Z \xrightarrow{w} TX \xrightarrow{-Tu} TY$ is a triangle.

(Tr3) Given two triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} TX'$ and morphisms $f : X \to X'$, $g : Y \to Y'$ such that $fu' = ug$, there exists a morphism $h : Z \to Z'$ such that (f, g, h) is a morphism from the first triangle to the second.

(Tr4) (The octahedral axiom) Given triangles

$$X \xrightarrow{u} Y \xrightarrow{i} Z' \xrightarrow{i'} TX,$$

$$Y \xrightarrow{v} Z \xrightarrow{j} X' \xrightarrow{j'} TY$$

and

$$X \xrightarrow{uv} Z \xrightarrow{k} Y' \xrightarrow{k'} TX,$$

there exist morphisms $f : Z' \to Y'$ and $g : Y' \to X'$ such that the following diagram commutes and the third row is a triangle.

$$
\begin{array}{ccccccccc}
T^{-1}Y' & \stackrel{T^{-1}k'}{\to} & X & \stackrel{1_X}{\to} & TX & & & & \\
\downarrow T^{-1}g & & \downarrow u & & \downarrow uv & & & & \\
T^{-1}X' & \stackrel{T^{-1}j'}{\to} & Y & \stackrel{v}{\to} & Z & \stackrel{j}{\to} & X' & \stackrel{j'}{\to} & TY \\
 & & \downarrow i & & \downarrow k & & \downarrow 1_{X'} & & \downarrow Ti \\
 & & Z' & \stackrel{f}{\to} & Y' & \stackrel{g}{\to} & X' & \stackrel{j'Ti}{\to} & TZ' \\
 & & \downarrow i' & & \downarrow k' & & & & \\
 & & TX & \stackrel{1_{TX}}{\to} & TX & & & &
\end{array}
$$

2.2 The following is basic (see, for example, [H2]):

Lemma 1 *Given a triangulated category, we have*
(a). *Let* $X \stackrel{u}{\to} Y \stackrel{v}{\to} Z \stackrel{w}{\to} TX$ *be a triange, then* $uv = vw = 0$.
(b). *Let* (f, g, h) *be a morphism between two triangles. If* f *and* g *are isomorphisms, then so is* h.
(c). *For any object* M *in the triangulated category,* $Hom(-, M)$ *and* $Hom(M, -)$ *are cohomological functors, i.e. for any triangle* $X \stackrel{u}{\to} Y \stackrel{v}{\to} Z \stackrel{w}{\to} TX$, *if set* $H = Hom(M, -)$ *and* $G = Hom(-, M)$, *then we have the long exact sequences*

$$\cdots \leftarrow H(T^iX) \stackrel{H(T^iu)}{\leftarrow} H(T^iY) \stackrel{H(T^iv)}{\leftarrow} H(T^iZ) \stackrel{H(T^iw)}{\leftarrow} H(T^{i+1}Z) \leftarrow \cdots$$

and

$$\cdots \leftarrow G(T^{i+1}X) \stackrel{G(T^iu)}{\leftarrow} G(T^iY) \stackrel{G(T^iv)}{\leftarrow} G(T^iZ) \stackrel{G(T^iw)}{\leftarrow} G(T^iZ) \leftarrow \cdots$$

.
(d). *Let* $X \stackrel{u}{\to} Y \stackrel{v}{\to} Z \stackrel{w}{\to} TX$ *and* $X' \stackrel{u'}{\to} Y' \stackrel{v'}{\to} Z' \stackrel{w'}{\to} TX'$ *be triangles. If* $g : Y \to Y'$ *is a morphism such that* $ugv' = 0$, *then there exists a morphism* (f, g, h) *from the first triangle to the second.*

We have also

Lemma 2 *Given a triangle of form*

$$M \stackrel{(f_1, f_2)}{\longrightarrow} N_1 \oplus N_2 \stackrel{\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}}{\longrightarrow} L \stackrel{h}{\longrightarrow} TM$$

If $f_2 = 0$, *then it is isomorphic to the triangle of form*

$$M \stackrel{(f_1, f_2)}{\longrightarrow} N_1 \oplus N_2 \stackrel{\begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}}{\longrightarrow} L_1 \oplus L_2 \stackrel{\begin{pmatrix} h_1 \\ 0 \end{pmatrix}}{\longrightarrow} TM$$

where $g_{22} : N_2 \to L_2$ *is an isomorphism and*

$$M \xrightarrow{f_1} N_1 \xrightarrow{g_{11}} L_1 \xrightarrow{h_1} TM$$

is a triangle.

Proof. See [PX1, Lemma 2.5]. $\square$

A triangle of form $M \overset{a}{\to} L \overset{b}{\to} N \overset{c}{\to} TM$ is said to be split if it is isomorphic to the triangle

$$M \overset{(1,0)}{\to} M \oplus N \overset{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}{\to} N \overset{0}{\to} TM$$

Then it is easy to see that $M \overset{a}{\to} L \overset{b}{\to} N \overset{c}{\to} TM$ is split if and only if a is a section, or b a retration, or $c = 0$.

The following seems to be well-known.

Lemma 3 *A triangle of the form* $M \overset{a}{\to} L \overset{b}{\to} N \overset{c}{\to} TM$ *is split if and only if* $L \simeq M \oplus N$.

Proof If $L \simeq M \oplus N$, then by applying $\mathrm{Hom}(-, M)$ to the triangle we get the long exact sequence

$$\cdots \leftarrow \mathrm{Hom}(M, M) \overset{a^*}{\leftarrow} \mathrm{Hom}(L, M) \overset{b^*}{\leftarrow} \mathrm{Hom}(N, M) \leftarrow \cdots$$

The sequence

$$0 \leftarrow \mathrm{Hom}(M, M) \overset{a^*}{\leftarrow} \mathrm{Hom}(L, M) \overset{b^*}{\leftarrow} \mathrm{Hom}(N, M) \leftarrow 0$$

must be exact since $\dim_k \mathrm{Hom}(L, M) = \dim_k \mathrm{Hom}(M, M) + \dim_k \mathrm{Hom}(N, M)$. Therefore a is a section and and so the triangle is split. The another hand is clear. $\square$

2.3 A triangulated category is called a 2-period triangulated category if the translation T satisfies $T^2 \simeq 1$.

Given a finite dimensional hereditary k-algebra A, or more generally a abelian hereditary category $\mathcal{A}$ such that every object has finite length, we have the derived category $D^b(A)$ or $D^b(\mathcal{A})$, which is a triangulated category with the translation T, obtained from the category of bounded complexes over A or in $\mathcal{A}$ by localizing with respect to the set of all quasi-isomorphisms (see [H1],[H2]). For our purpose we should consider the orbit category $\mathcal{R} = D^b(A)/T^2$ of $D^b(A)$ under the automorphism T^2. Let $F : D^b(A) \to \mathcal{R}$ be the canonical functor. Then F is a covering with the Galois group $\langle T^2 \rangle$ (in the sense of Gabriel [G2]). The translation T of $D^b(A)$ induces an automorphism of $\mathcal{R}$ of order 2, still denoted by T. In Appendix of [PX1] we proved that $\mathcal{R}$ is also a triangulated category with T as the translation and that the covering functor $F : D^b(A) \to \mathcal{R}$ is exact, i.e. F sends each triangle in $D^b(A)$ to a triangle in $\mathcal{R}$.

It is clear that the root category $\mathcal{R}$ is a 2-period triangulated category.

3 A Lie algebra over $\mathbf{Z}/(q-1)$ arising from a 2-period triangulated category

3.1 We assume in this section the central field k is a finite field and $q = |k|$, the cardinality of k.

Given a triangulated category $\mathcal{C}$, we consider the Grothendieck group $G(\mathcal{C})$ as usual. That is, $G(\mathcal{C})$ is the quotient of a free abelian group with a basis $\{[M]|M \in \mathcal{C}\}$, indexed by the isomorphism classes of all objects in $\mathcal{C}$, subject to the relations $[X] - [Y] + [Z]$ provided there exist triangles of form $X \to Y \to Z \to TX$. For any $M \in \mathcal{C}$, we denote by $\underline{\dim} M$ the canonical image of M in $G(\mathcal{C})$, called the dimension vector of M.

A triangulated category $\mathcal{C}$ is said finitary if the following two conditions are satisfied. (1) For any objects M and N in $\mathcal{C}$, $\mathrm{Hom}(M, N)$ is finite (equivalently finitely dimensional over k). (2) Given any vector $\underline{r}$, there are only finitely many objects $M \in \mathcal{C}$ such that $\underline{\dim} M = \underline{r}$. The Grothendieck group $G(\mathcal{C})$ of a triangulated category $\mathcal{C}$ is called proper, if for every nonzero indecomposable object $X \in \mathcal{C}$ the dimension vector of X in $G(\mathcal{C})$ is not zero, i.e, $\underline{\dim} X \neq 0$.

In this section we concentrate to consider the 2-period and finitary triangulated categories whose Grothendieck groups are proper.

3.2 Given $X, Y, L \in \mathcal{C}$, put

$$W(X, Y; L) = \{(f, g, h) \in \mathrm{Hom}(X, L) \times \mathrm{Hom}(L, Y) \times \mathrm{Hom}(Y, TX)|$$

$$X \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{h} TX \text{ is a triangle}\}$$

We can introduce the action of $\mathrm{Aut}X \times \mathrm{Aut}Y$ on $W(X, Y; L)$ as follows (see also [Rie]).

$$(a, c) \circ (f, g, h) = (af, gc^{-1}, ch(Ta)^{-1})$$

for any $(a, c) \in \mathrm{Aut}X \times \mathrm{Aut}Y$, $(f, g, h) \in W(X, Y; L)$ and denote the orbit space by

$$V(X, Y; L) = W(X, Y; L)/\mathrm{Aut}X \times \mathrm{Aut}Y.$$

For any $(f, g, h) \in W(X, Y; L)$, we set

$$(f, g, h)^\wedge = \{(af, gc^{-1}, ch(Ta)^{-1})|(a, c) \in \mathrm{Aut}X \times \mathrm{Aut}Y\}.$$

Then

$$V(X,Y;L)=\{(f,g,h)^\wedge|(f,g,h)\in W(X,Y;L)\}.$$

This is a finite set and so we have a non-negative integer

$$F^L_{YX}=|V(X,Y;L)|,$$

the cardinality of $V(X,Y;L)$. The number F^L_{YX} is very important and plays a role analogous to the Hall polynomial for Ringel-Hall algebras.

3.3 By ind$\mathcal{C}$ we denote the set of representatives of isomorphism classes of all indecomposable objects in $\mathcal{C}$. For any indecomposable object $X\in\mathcal{C}$, let $d(X)=\dim_k(\mathrm{End}X/\mathrm{radEnd}X)$.

For any $M\in\mathcal{C}$ we denote simply by h_M the dimension vector $\underline{\dim}M$ of M. Then the Grothendieck group $G(\mathcal{C})$ is generated by $\{h_M|M\in\mathcal{C}\}$ and $h_L=h_M+h_N$ if there exists a triangle of form $M\to L\to N\to TM$. So $h_M=-h_{TM}$. We may consider the rational extension $\mathbf{Q}\otimes_{\mathbf{Z}}G(\mathcal{C})$. Denote by $\mathbf{h}$ the subgroup of $\mathbf{Q}\otimes_{\mathbf{Z}}\mathbf{h}$ generated by $\frac{h_M}{d(M)}$, $M\in$ ind$\mathcal{C}$. Note that the Grothendieck group $G(\mathcal{C})$ is a subgroup of $\mathbf{h}$.

We define a symmetric bilinear function $(-|-)$ on $\mathbf{h}\times\mathbf{h}$ (also on $\mathbf{Q}\otimes_{\mathbf{Z}}\mathbf{h}\times\mathbf{Q}\otimes_{\mathbf{Z}}\mathbf{h}$) determined by

$$(h_X|h_Y)=\dim_k\mathrm{Hom}(X,Y)-\dim_k\mathrm{Hom}(X,TY)$$
$$+\dim_k\mathrm{Hom}(Y,X)-\dim_k\mathrm{Hom}(Y,TX)$$

for any X, $Y\in\mathcal{C}$. On $G(\mathcal{C})$ this is just the symmetric Euler form of $\mathcal{C}$. So it is well-defined and $(h_X|h_Y)=(h_Y|h_X)$.

Let $\mathbf{n}$ be a free abelian group with a basis $\{u_X|X\in\mathrm{ind}\mathcal{C}\}$. Note that the basis is bijectiovely corresponding to the set of the isomorphism classes of all indecomposable objects in $\mathcal{C}$. Let

$$\mathbf{g}=\mathbf{h}\oplus\mathbf{n},$$

a direct sum of $\mathbf{Z}$-modules. We shall consider the factor group

$$\mathbf{g}_{(q-1)}=\mathbf{g}/(q-1)\mathbf{g}.$$

By abuse of notions, for any $M\in\mathcal{C}$ we still use u_M,h_M to denote the corresponding residues.

3.4 For any $M,N,L\in\mathcal{C}$, we set $\gamma^L_{XY}=F^L_{XY}-F^L_{YX}$.

Now we define a bilinear operation $[-,-]$ on $\mathbf{g}_{(q-1)}$ as follows.

(a) For any two indecomposable objects $X, Y \in \mathcal{C}$

$$[u_X, u_Y] = \begin{cases} \sum\limits_{L \in \mathrm{ind}\mathcal{C}} \gamma_{XY}^L u_L & \text{if } Y \not\simeq TX \\ \frac{h_X}{d(X)} & \text{if } Y \simeq TX \end{cases}$$

(b) For any objects $X, Y \in \mathcal{C}$ with Y indecomposable

$$[h_X, u_Y] = -(h_X|h_Y)u_Y \text{ and } [u_Y, h_X] = -[h_X, u_Y]$$

(c) $[\mathbf{h}, \mathbf{h}] = 0$

We extend $[-,-]$ bilinearly onto the whole $\mathbf{g}_{(q-1)}$. This is a well-defined operation, since $(-|-)$ is a bilinear function on $\mathbf{h} \times \mathbf{h}$.

Theorem *Together with the operation $[-,-]$, $\mathbf{g}_{(q-1)}$ is a Lie algebra over $\mathbf{Z}/(q-1)$.*

The proof of Theorem needs a lot of preparation and so we put it off till Section 8.

The following is direct.

Corollary 1 *The translation T induces the Chevalley involution ω of $\mathbf{g}_{(q-1)}$, determined by $\omega(u_X) = u_{TX}$ for any indecomposable object $X \in \mathcal{C}$ and $\omega(h_Y) = h_{TY} = -h_Y$ for any object $Y \in \mathcal{C}$.* □

If we consider, in particular, the root category $\mathcal{R}$ of a finite dimensional hereditay and the corresponding Lie algebra $\mathbf{g}_{(q-1)}$ over $\mathbf{Z}/(q-1)$, then the triangles and the translation T in $\mathcal{R}$ provide an "invariant" "non-degenerate" symmetric bilinear form on $\mathbf{g}_{(q-1)}$ in a natural way. That is,

Corollary 2 *Let $\mathbf{g}_{(q-1)}$ be the Lie algebra over $\mathbf{Z}/(q-1)$ from the root category $\mathcal{R}$. Then there exists a "non-degenerate" symmetric bilinear $\mathbf{Z}/(q-1)$-valued form $(-|-)$ on $\mathbf{g}_{(q-1)}$ such that*
a) $(-|-)$ is "invariant" in the following sense:
(i) $-([h_X, w]|u_Y) = (h_X|[w, u_Y])d(Y)$
(ii) $d(X)([u_X, w]|u_Y) = (u_X|[w, u_Y])d(Y)$
for any indecomposable objects $X, Y \in \mathcal{R}$ and any $w \in \mathbf{g}_{(q-1)}$.
b) $(-|-)|_{\mathbf{h}}$ is defined by the symmetric Euler form in 3.3.
c) $(\mathbf{h}|\mathbf{n}) = 0$.
d) for any indecomposable objects $X, Y \in \mathcal{R}$,

$$(u_X|u_Y) = \begin{cases} 0 & \text{if } Y \not\simeq TX \\ 1 & \text{if } Y \simeq TX \end{cases}$$

The proof of Corollary 2 will be given in Section 8.

Remark It is easy to see that $(-|-)|_{\mathbf{h}}$ usually is not non-degenerate, except $\mathcal{R}$ finite type. However it is not difficult to extend the Cartan subalgebra to a larger one such that $(-|-)$ on it is no longer degenerate as did as in [K1].

4 Main theorem

4.1 Let k be a finite field with $q = |k|$, A a finite dimensional hereditary k-algebra and $\mathcal{R} = D^b(A)/T^2$ the root category. Then $\mathcal{R}$ is a finitary and 2-period triangulated category whose Grothendieck group is proper.

Let mod$A$ be the category of finitely generated A-modules. Given any modules $N_1, \cdots, N_t$ and M in mod$A$, let $\mathcal{F}^M_{N_1,\cdots,N_t}$ be the set of filtrations $M = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_t = 0$ such that V_{i-1}/V_i is isomorphic to N_i for $1 \leq i \leq t$. The cardinality of $\mathcal{F}^M_{N_1,\cdots,N_t}$ is denoted by $F^M_{N_1,\cdots,N_t}$. In particular, given any X, Y and M in mod$A$, F^M_{YX} is the number of submodules U of M such that $U \simeq X$ and $M/U \simeq Y$.

Let us recall the definition of Ringel-Hall algebra $\mathcal{H}(A)$. It is a free abelian group with the basis $\{u_{[M]} | M \in \text{mod}A\}$ indexed by the isomorphism classes of all finite dimensional A-modules. We also write u_M for $u_{[M]}$. The multiplication is defined by

$$u_{N_1} u_{N_2} = \sum_{[M], M \in \text{mod}A} F^M_{N_1 N_2} u_M$$

This is an associative ring with the unit $1 = u_0$.

Let ind$A$ be the set of representatives of isomorphism classes of all indecomposable modules in mod$A$ and $K(\text{mod}A)$ the subgroup of $\mathcal{H}(A)$ generated (freely) by $\{u_X | X \in \text{ind}A\}$. Then the proof of Proposition 5 in [R3] implies

Proposition *Over the residue ring* $\mathbf{Z}/(q-1)$, $K(modA)$ *is a Lie subalgebra of* $\mathcal{H}(A)$, *i.e. the factor group* $K(modA)/(q-1)K(modA)$ *is a Lie subalgebra of the factor ring* $\mathcal{H}(A)/(q-1)\mathcal{H}(A)$, *where the Lie bracket is the commutator* $[u_X, u_Y] = u_X u_Y - u_Y u_X$ $\square$

4.2 Note that mod$A$ can be embedded into $\mathcal{R}$ as a full subcategory of $\mathcal{R}$. In this case, similar to the discusion for the derived category $D^b(A)$ in [H1] we have that, for any $X, Y, M \in \text{mod}A$, there is an exact sequence $0 \to X \xrightarrow{f} M \xrightarrow{g} Y \to 0$ in mod$A$ if and only if there exists a triangle $X \xrightarrow{f} M \xrightarrow{g} Y \xrightarrow{h} TX$ in $\mathcal{R}$. Then one can easily see that $\mathcal{F}^M_{YX}$ and $V(X, Y; M)$ are isomorphic and so the values of F^M_{YX} from mod$A$ and from $\mathcal{R}$ coincide. This is the reason why we use the same notions to express the two concepts.

The root category $\mathcal{R}$ is divided by mod$A$ into two parts, mod$A$ and $T(\text{mod}A)$. We have defined $\mathbf{g}, \mathbf{n}$ and $\mathbf{h}$ for $\mathcal{R}$ in Section 3 and hence

$$\mathbf{n} = K(\text{mod}A) \oplus K(T(\text{mod}A)),$$

where $K(T(\text{mod}A))$ is the free abelian group with the basis $\{u_{TX} | X \in \text{ind}A\}$ and

$$\mathbf{g} = K(\text{mod}A) \oplus \mathbf{h} \oplus K(T(\text{mod}A)).$$

So the Lie algebra $K(\text{mod}A)/(q-1)K(\text{mod}A)$ is a subalgebra of $\mathbf{g}_{(q-1)}$ and

$$\mathbf{g}_{(q-1)} = K(\text{mod}A)/(q-1)K(\text{mod}A) \oplus \mathbf{h}/(q-1)\mathbf{h}$$

$$\oplus K(T(\text{mod}A))/(q-1)K(T(\text{mod}A))$$

is a triangular decomposition of $\mathbf{g}_{(q-1)}$.

4.3 Let E be a field extension of k and set $V^E = E \otimes_k V$ for any k-space V. Then A^E is a hereditary E-algebra and, for $M \in \text{mod}A$, M^E has a canonical A^E-module structure. Recall from [R5] that E is called conservative for $X \in \text{ind}A$ if $(\text{End}_A X/\text{radEnd}_A X)^E$ is a field again. In this case one has

$$(\text{End}_A X/\text{radEnd}_A X)^E \simeq \text{End}_{A^E} X^E/\text{radEnd}_{A^E} X^E.$$

So X^E is an indecomposable A^E-module.

Recall that an indecomposable A-module X is called exceptional if $\text{Ext}^1_A(X, X) = 0$. It follows from Ringel [R6] that

Proposition (Ringel[R6]) *If $X \in indA$ is exceptional, then $End_A X \simeq End_A S$ for some simple A-module S.* □

Let $\overline{k}$ be the algebraic closure of k and set

$$\Omega = \{E | k \subseteq E \subseteq \overline{k} \text{ is a finite field extension and conservative for}$$

$$\text{all exceptional } A\text{-modules}\}$$

Since there are only finitely many isomorphism classes of simple A-modules, as a consequence of Proposition we have

Corollary *The set Ω is an infinite set, i.e, there are infinitely many finite field extensions of k in $\overline{k}$ such that they are conservative for all exceptional A-modules.* □

4.4 For any $E \in \Omega$, $K(\mathrm{mod}A^E)/(|E|-1)K(\mathrm{mod}A^E)$ is a Lie subalgebra of $\mathcal{H}(A^E)/(|E|-1)\mathcal{H}(A^E)$ over the residue ring $\mathbf{Z}/(|E|-1)$. Consider the direct product of Lie algebras

$$\Pi = \prod_{E \in \Omega} K(\mathrm{mod}A^E)/(|E|-1)K(\mathrm{mod}A^E).$$

For $X \in \mathrm{mod}A$, by abuse of notion we still denote $u_X = (u_{X^E})_{E\in\Omega}$ in Π Let $S_1, \cdots, S_n$ be all non-isomorphic simple A-modules and $\mathcal{LC}(A)_1$ the Lie subalgebra of Π generated by $u_{S_1}, \cdots, u_{S_n}$, called the degenerate composition Lie algebra of A. Because Ω is an infinite set, it is easy to see that $\mathcal{LC}(A)_1$ is a torsion-free $\mathbf{Z}$-module. Note that each simple A-module is exceptional. On the other hand, We have the following.

Proposition *Let $X \in indA$ be exceptional. Then there exists a nonzero $a \in \mathbf{Z}$ such that $au_X \in \mathcal{LC}(A)_1$.*

The proof will be given in Section 5.

Let $\mathcal{LE}(A)_1$ be the Lie subalgebra of Π generated by all u_X with X exceptional A-modules. Then $\mathcal{LC}(A)_1 \subset \mathcal{LE}(A)_1$. However, in the case of rational extensions, by Proposition we directly have

Corollary *As Lie subalgebras over* $\mathbf{Q}$, $\mathbf{Q} \otimes_{\mathbf{Z}} \mathcal{LC}(A)_1 = \mathbf{Q} \otimes_{\mathbf{Z}} \mathcal{LE}(A)_1$. $\square$

4.5 Let us return to the root category $\mathcal{R}$ and take $\mathrm{mod}A$ as a fixed full subcategory of $\mathcal{R}$, where A is a finite dimensional hereditary k-algebra. An indecomposable object X in $\mathcal{R}$ is called exceptional if $\mathrm{Hom}_{\mathcal{R}}(X, TX) = 0$. Note that $\mathrm{Ext}^1_A(X,X) = \mathrm{Hom}_{\mathcal{R}}(X, TX)$ for any $X \in \mathrm{mod}A$. So $X \in \mathrm{ind}\mathcal{R}$ is exceptional if and only if $X \in \mathrm{ind}A$ or $TX \in \mathrm{ind}A$ is exceptional. Therefore

$$\begin{aligned}\Omega = \{E | k \subseteq E \subseteq \overline{k} \text{ is a finite field extension and conservative for} \\ \text{all exceptional objects in } \mathcal{R}\} \\ = \{E | k \subseteq E \subseteq \overline{k} \text{ is a finite field extension and conservative for} \\ \text{all exceptional } A\text{-modules }\}.\end{aligned}$$

Let $\mathcal{R}^E$ be the root category of A^E. Then as in 3.1 we have $\mathbf{h}^E$, $\mathbf{n}^E$, and $\mathbf{g}^E = \mathbf{h}^E \oplus \mathbf{n}^E$ and $\mathbf{g}^E_{(|E|-1)} = \mathbf{g}^E/(|E|-1)\mathbf{g}^E$, the latter is a Lie algebra over $\mathbf{Z}/(|E|-1)$ by Theorem in 3.4. We consider the direct product $\prod_{E\in\Omega} \mathbf{g}^E_{(|E|-1)}$ of Lie algebras and let $\mathcal{LE}(\mathcal{R})_1$ be the Lie subalgebra of $\prod_{E\in\Omega} \mathbf{g}^E_{(|E|-1)}$ generated by $\{u_X = (u_{X^E})_{E\in\Omega} | X \text{ is an exceptional object in } \mathcal{R}\ \}$.

Remark $\mathcal{LE}(\mathcal{R})_1$ is independent on the choice of the hereditary algebra A. Since, if B is another hereditary k-algebra of the same type as A, then by [H1] we have $D^b(B) = D^b(A)$ and so the root categories of B and of A are same.

4.6 Let $S_1, \cdots, S_n$ be the set of all non-isomorphic simple A-modules. Then the Cartan matrix $C = (a_{ij})_{n\times n}$ of A is given by

$$\begin{aligned} a_{ij} &= \frac{2(h_{S_i}|h_{S_j})}{(h_{S_i}|h_{S_i})} \\ &= l({}_{\mathrm{End}S_i}\mathrm{Hom}_{\mathcal{R}}(S_i, S_j)) - l({}_{\mathrm{End}S_i}\mathrm{Hom}_{\mathcal{R}}(S_i, TS_j)) \\ &+ l(\mathrm{Hom}_{\mathcal{R}}(S_j, S_i)_{\mathrm{End}S_i}) + l(\mathrm{Hom}_{\mathcal{R}}(S_j, TS_i)_{\mathrm{End}TS_i}) \end{aligned}$$

where $l({}_{\mathrm{End}S_i}(,))$ and $l((,)_{\mathrm{End}S_i})$ are the length of them as left EndS_i- and right EndS_i-modules respectively. So $a_{ii} = 2$ and $a_{ij} \leq 0$ are integers for $i \neq j$. Therefore $C = (a_{ij})_{n\times n}$ is a symmetrizable generalized Cartan matrix, since $d(S_i)a_{ij} = a_{ji}d(S_j)$ for $1 \leq i, j \leq n$, i.e, $(d(S_i))_{1\leq i\leq n}$ is the minimal symmetrization of C.

Remark The Cartan matrix $C = (a_{ij})_{n\times n}$ is independent on the hereditary algebra A. That is, if B is another hereditary algebra of the same type as A, then their Cartan matrices are identical up to a permutation on the index set. So we can call it the Cartan matrix of $\mathcal{R}$.

4.7 Given any symmetrizable generalized Cartan matrix $C = (a_{ij})_{n\times n}$, we have the complex Kac-Moody algebra ${}_{\mathbf{C}}\mathbf{g}(C)$ and the rational Kac-Moody algebra ${}_{\mathbf{Q}}\mathbf{g}(C)$ as defined as in 1.2. The following is the main theorem of this article.

Theorem *Let $\mathcal{R}$ be the root category of a finite dimensional hereditary k-algebra and $C = (a_{ij})_{n\times n}$ the symmetrizable generalized Cartan matrix of $\mathcal{R}$. Then we have the isomorphisms of Lie algebras ${}_{\mathbf{C}}\mathbf{g}(C) \simeq \mathbf{C} \otimes_{\mathbf{Z}} \mathcal{LE}(\mathcal{R})_1$ and ${}_{\mathbf{Q}}\mathbf{g}(C) \simeq \mathbf{Q} \otimes_{\mathbf{Z}} \mathcal{LE}(\mathcal{R})_1$.*

Proof let A be a hereditary algebra and $\mathcal{R}$ the root category of A. Then for any indecomposable object $X \in \mathcal{R}$, we have $X \in \mathrm{mod}A$ or $TX \in \mathrm{mod}A$. So X is exceptional in $\mathcal{R}$ if and only if X or TX is exceptional in mod$A$. Therefore, by Corollary in 4.4 $\mathbf{C} \otimes_{\mathbf{Z}} \mathcal{LE}(\mathcal{R})_1$ is the Lie subalgebra generated by u_{S_i} and u_{TS_i} for $i = 1, \cdots, n$, Where $S_1, \cdots, S_n$ are all non-isomorphic simple A-modules. Set $h_{S_i} = (h_{S_i^E})_{E\in\Omega}$, $i = 1, \cdots, n$, in $\prod\limits_{E\in\Omega} \mathbf{g}^E_{(|E|-1)}$. The following relations are easily verified.

(1) $[u_{S_i}, u_{TS_i}] = -\frac{h_{S_i}}{d(S_i)} \in \mathcal{LE}(\mathcal{R})_1$ for any i and $[u_{S_i}, u_{TS_j}] = 0$ if $i \neq j$

(2) $[h_{S_i}, h_{S_j}] = 0$ for $1 \leq i, j \leq n$

(3) $[\frac{h_{S_i}}{d(S_i)}, u_{S_j}] = a_{ij} u_{S_j}$, $[\frac{h_{S_i}}{d(S_i)}, u_{TS_j}] = -a_{ij} u_{TS_j}$ for $1 \leq i, j \leq n$.

Following Ringel [R2] and the explanation in 4.2, we have the Serre relations:

(4) $(ad u_{S_i})^{1-a_{ij}} u_{S_j} = 0$ and $(ad u_{TS_i})^{1-a_{ij}} u_{TS_j} = 0$ for $i \neq j$.

So the map $\Phi : {}_{\mathbf{C}}\mathbf{g}(C) \to \mathbf{C} \otimes_{\mathbf{Z}} \mathcal{LE}(\mathcal{R})_1$ given by $\Phi(e_i) = u_{S_i}$, $\Phi(f_i) = -u_{TS_i}$ and $\Phi(h_i) = -\frac{h_{S_i}}{d(S_i)}$, $1 \leq i \leq n$, is an epimorphism of Lie algebras. It is easy to see Φ is injective on the Cartan subalgebra. Because any Lie ideal of ${}_{\mathbf{C}}\mathbf{g}(C)$ intersecting the Cartan subalgebra trivially is trivial, the map Φ is an isomorphism of Lie algebras. It is clear that Φ also induces an isomorphism in the rational case. □

In this way and with the observation in 1.3, we get a realization of all symmetrizable Kac-Moody algebras from the root categories of all finite dimensional hereditary algebras.

Remark (1) Let us mention that the definition of $\mathcal{LE}(\mathcal{R})_1$ is given in a global way, this means that the Lie operation is given in terms of the root category and not depends on the particular choice of triangular decomposition and generators e_i, f_i, $1 \leq i \leq n$. (2) In the discussion above, mod$A$ can be replaced by any abelian hereditary category which is finitary in the sense of 3.1.

5 Vectors of Schur roots

The purpose of this section is to prove Proposition in 4.4. We will show that it is a consequence of the investigation of Ringel in Section 8 and 11-13 of [R7]. Let us recall something from [R7].

5.1 Let A be a finite dimensional hereditary k-algebra. The (extended) twisted Ringel-Hall algebra $\mathcal{H}_*(A)$ is defined as follows. It is a free $\mathbf{Q}[v, v^{-1}]$-module with the basis $\{K_\alpha, u_{[M]} | \alpha \in \mathbf{Z}[I], M \in \text{mod}A\}$. We again write u_M for $u_{[M]}$. The multiplication is given by

$$K_\alpha K_\beta = K_{\alpha+\beta} \quad K_\alpha u_M = v^{(\alpha, \underline{\dim} M)} u_M K_\alpha$$

$$u_M u_N = v^{\langle \underline{\dim} M, \underline{\dim} N \rangle} \sum_{[L], L \in \text{mod}A} F_{MN}^L u_L$$

for $\alpha, \beta \in \mathbf{Z}[I]$ and $M, N \in \text{mod}A$, where $\langle -, - \rangle$ is the Euler form and $(-, -)$ the symmetric Euler form defined on $G(\text{mod}A) = \mathbf{Z}[I]$. So $\langle \underline{\dim} M, \underline{\dim} N \rangle = \dim_k \text{Hom}_A(M, N) - dim_k \text{Ext}_A(M, N)$ and $(\underline{\dim} M, \underline{\dim} N) = \langle \underline{\dim} M, \underline{\dim} N \rangle + \langle \underline{\dim} N, \underline{\dim} M \rangle$.

A pair (X_1, X_2) is called orthogonal exceptional pair if X_1 and X_2 are indecomposable exceptional A-modules and $\mathrm{Hom}_A(X_1, X_2) = \mathrm{Hom}_A(X_2, X_1) = \mathrm{Ext}_A(X_2, X_1) = 0$. According to [S1] and [R7], we have the following.

Lemma *Let $X \in indA$ be non-simple exceptional. Then there exists orthogonal exceptional pair (X_1, X_2) and an exact sequence*

$$0 \to a_2X_2 \to X \to a_1X_1 \to 0$$

with integers $a_1, a_2 \geq 1$, where a_iX_i means the direct sum of a_i copies of X_i. $\square$

Let $\mathcal{C}(X_1, X_2)$ be the smallest full subcategory of mod$A$ which contains X_1, X_2 and is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms. By Crawley-Boevey [CB] and Ringel [R6], we know that $\mathcal{C}(X_1, X_2)$ is equivalent to the module category of a finite dimensional hereditary k-algebra with precisely two isomorphism classes of simple modules. Moreover, the canonical embedding functor from $\mathcal{C}(X_1, X_2)$ to mod$A$ is exact. Since X is exceptional, X lies in the preprojective component or in the preinjective component of $\mathcal{C}(X_1, X_2)$ according to Ringel [R1].

5.2 By [R1], there exists a k-species $\mathcal{S} = (k', k'', {}_{k'}M_{k''})$ given by a k'-k''-bimodule M with k' and k'' finite field extensions of k such that $\mathcal{C}(X_1, X_2) \simeq$ rep-$\mathcal{S}$, where rep-$\mathcal{S}$ is the category of all finite dimensional k-representations of $\mathcal{S}$. Therefore the subalgebra $\mathcal{H}_*(X_1, X_2)$ of $\mathcal{H}_*(A)$ generated by $\mathcal{C}(X_1, X_2)$, i.e. by $\{K_\alpha, u_Y | Y \in \mathcal{C}(X_1, X_2), \alpha = m_1\underline{\dim}X_1 + m_2\underline{\dim}X_2, m_1, m_2 \in \mathbf{Z}\}$, is naturally isomorphic to the (extended) twisted Ringel-Hall algebra $\mathcal{H}_*(\mathcal{S})$ of $\mathcal{S}$.

Let Δ be the (valued) quiver of $\mathcal{S}$. For a vertex i of Δ, we denote by $\sigma_i\Delta$ the (valued) quiver obtained from Δ by changing the orietation of all arrows which have i as starting vertex or end vertex and by $\sigma_i\mathcal{S}$ the k-species obtained from $\mathcal{S}$ by replacing ${}_{k'}M_{k''}$ by its k-dual. Then $\sigma_i\mathcal{S}$ is the k-species of type $\sigma_i\Delta$.

Let us assume that i is the sink for Δ. We denote by σ_i^+ the Bernstein-Gelfand-Ponomarev reflection functor (see [BGP],[DR2]) and then

$$\sigma_i^+ : \text{ rep-}\mathcal{S}\langle i\rangle \longrightarrow \text{rep-}\sigma_i\mathcal{S}\langle i\rangle$$

is an equivalence, where rep-$\mathcal{S}\langle i\rangle$ is the subcategory of all representations of $\mathcal{S}$ which do not have S_i as direct summand, here S_i is the simple representation corresponding to the vertex i.

So, we have the following (see Theorem 5 of [R7]).

Lemma *Let i be the sink of $\mathcal{S}$. The functor σ_i^+ yields a $\mathbf{Q}[v,v^{-1}]$-algebra isomorphism*

$$\sigma_i:\ \mathcal{H}_*(\mathcal{S})\langle i\rangle \longrightarrow \mathcal{H}_*(\sigma_i\mathcal{S})\langle i\rangle$$

such that $\sigma_i K_\alpha = K_{\sigma_i\alpha}, \sigma_i u_Y = u_{\sigma_i^+Y}$, *where* $\mathcal{H}_*(\mathcal{S})\langle i\rangle = \mathcal{H}_*(\mathcal{S})|rep\text{-}\mathcal{S}\langle i\rangle$ *and* $\sigma_i\alpha = \alpha - \frac{2(\alpha_i,\alpha)}{(\alpha_i,\alpha_i)}\alpha_i$, $\alpha_i = \underline{\dim} S_i$. $\square$

In the quantum groups and the Ringel-Hall algebras, the following notations and relations are often used.

$$[s]_\varepsilon = \frac{(v^\varepsilon)^s - (v^\varepsilon)^{-s}}{(v^\varepsilon)-(v^\varepsilon)^{-1}} = (v^\varepsilon)^{s-1} + (v^\varepsilon)^{s-3} + \cdots + (v^\varepsilon)^{-s+1},$$

$$[s]!_\varepsilon = \prod_{r=1}^{s}[r]_\varepsilon,$$

$$\left[\begin{array}{c} s \\ r \end{array}\right]_\varepsilon = \frac{[s]!_\varepsilon}{[r]!_\varepsilon[s-r]!_\varepsilon},$$

$$|s]_\varepsilon = \frac{(q^\varepsilon)^s - 1}{q^\varepsilon - 1} = (q^\varepsilon)^{s-1} + \cdots + q^\varepsilon + 1,$$

$$|s]!_\varepsilon = \prod_{r=1}^{s}|r]_\varepsilon,$$

$$\left|\begin{array}{c} s \\ r \end{array}\right]_\varepsilon = \frac{|s]!_\varepsilon}{|r]!_\varepsilon|s-r]!_\varepsilon},$$

where ε, s, r are non-negative integers and $r \le s$. Now set $q = v^2$. Then we have

$$\left|\begin{array}{c} s \\ r \end{array}\right]_m = v^{r(s-r)}\left[\begin{array}{c} s \\ r \end{array}\right]_m.$$

Let U be the Lusztig form of quantum group of type Δ. Its generators are E_i, F_i, K_α (see [BBP5]). Lusztig has proposed several braid group operations on U. In particuler, consider for any i the following operator $T''_{i,1}$ defined for the canonical generators E_i, F_i, K_α of U by

$$T''_{i,1}(E_i) = -F_iK_i^{\varepsilon_i}, \quad T''_{i,1}(F_i) = -K_i^{-\varepsilon_i}E_i,$$

$$T''_{i,1}(E_j) = \sum_{r+s=-a_{ij}} (-1)^r v^{-\varepsilon_i r} E_i^{(s)} E_j E_i^{(r)} \ \text{ for } j \neq i,$$

$$T''_{i,1}(F_j) = \sum_{r+s=-a_{ij}} (-1)^r v^{\varepsilon_i r} F_i^{(r)} F_j E_i^{(s)} \ \text{ for } j \neq i,$$

$$T''_{i,1}(K_\alpha) = K_{\sigma_i(\alpha)},$$

where $E_i^{(r)} = \frac{E_i^r}{[r]!_{\varepsilon_i}}$ and $(\varepsilon_i)_i$ is the minimal symmetrization.

Let $\mathcal{C}_*(\mathcal{S})$ be the composition subalgebra of $\mathcal{H}_*(\mathcal{S})$, which is the subalgebra of $\mathcal{H}_*(\mathcal{S})$ generated by u_{S_i} for S_i simple representations of $\mathcal{S}$. By Green's theorem [Gr] (or see [R8]), $\mathcal{C}_*(\mathcal{S})$ is canonically isomorphic to the Lusztig form of quantum group $U^+(\Delta)$, and then $\mathcal{C}_*(\mathcal{S})$ is independent on the orientation of Δ. Let $\mathcal{C}_*(\mathcal{S})\langle i\rangle = \mathcal{C}_*(\mathcal{S}) \cap \mathcal{H}_*(\mathcal{S})\langle i\rangle$, then

$$\sigma_i|_{\mathcal{C}_*(\mathcal{S})\langle i\rangle} : \quad \mathcal{C}_*(\mathcal{S})\langle i\rangle \longrightarrow \mathcal{C}_*(\mathcal{S})\langle i\rangle$$

is an automorphism.

5.3 The following very important observation is given by Ringel[R7]. His proof is only for finite type cases. So we need to generalize it to infinite type cases.

Proposition *Let i be the sink. The automorphism σ_i is the restriction of $T''_{i,1}$ to $\mathcal{C}_*(\mathcal{S})\langle i\rangle$.*

Proof (compare with 4.3 of [CX]). Let V_α, V_β be the two simple module in rep-$\mathcal{S}$.

First let α be the source. Then V_β is the simple projective and V_α the simple injective object. For simplity, we write

$$\langle \alpha, \beta\rangle = \dim_k \operatorname{Hom}(V_\alpha, V_\beta) - \dim_k \operatorname{Ext}(V_\alpha, V_\beta) < 0.$$

Let $m = -\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} = -\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$. Note that $\langle\beta,\alpha\rangle = 0$.

Given $0 \le t \le m$, for any exact sequence of form

$$0 \longrightarrow V_\beta \longrightarrow V \longrightarrow (m-t)V_\alpha \longrightarrow 0,$$

we have $V \simeq V_{\lambda_{m-t-a}} \oplus aV_\alpha$ for some $0 \le a \le m-t$, where $V_{\lambda_{m-t-a}}$ has the simple socle isomorphic to V_β So $V_{\lambda_{m-t-a}}$ is indecomposable and there exists non-split sequence

$$0 \longrightarrow V_\beta \longrightarrow V_{\lambda_{m-t-a}} \longrightarrow (m-t-a)V_\alpha \longrightarrow 0.$$

Note that for any $0 \le a \le m-t$, $V_{\lambda_{m-t-a}}$ exits and is isomorphic to some subobject of the indecomposable injective object V_{λ_m}. However such $V_{\lambda_{m-t-a}}$ usually is not unique up to isomorphism but their dimension verctors are same.

We use $u_{t\alpha}$ to denote u_{tV_α} and $F^\gamma_{\alpha,\beta}$ for $F^{V_\gamma}_{V_\alpha,V_\beta}$ and so on. Then

$$u_{(m-t)\alpha}u_\beta = v^{(m-t)\langle\alpha,\beta\rangle} \sum_{a=0}^{m-t} \sum_{\lambda_{m-t-a}} F^{\lambda_{m-t-a}\oplus a\alpha}_{(m-t)\alpha,\beta} u_{\lambda_{m-t-a}\oplus a\alpha},$$

where $\sum_{\lambda_{m-t-a}}$ goes through the isomorphism classes of all indecomposable objects of form $V_{\lambda_{m-t-a}}$. We will use the similar notion in the below. By the definitions of $V_{\lambda_{m-t-a}}$ we have $F^{\lambda_{m-t-a}\oplus a\alpha}_{(m-t)\alpha,\beta}=1$. Therefore

$$u_{(n-t)\alpha}u_\beta = v^{(n-t)\langle\alpha,\beta\rangle}\sum_{i=0}^{n-t}\sum_{\lambda_{m-t-a}} u_{\lambda_{m-t-a}\oplus a\alpha}.$$

Similarly, for $0\leq b\leq n-t$ we have

$$u_{(m-t-b)\alpha}u_\beta = v^{(m-t-b)\langle\alpha,\beta\rangle}\sum_{a=0}^{m-t-b}\sum_{\lambda_{m-t-b-a}} u_{\lambda_{m-t-b-a}\oplus a\alpha}.$$

Since any extension of the form

$$0\longrightarrow jV_\alpha\longrightarrow V'\longrightarrow V_{\lambda_{m-t-b-a}}\oplus iV_\alpha\longrightarrow 0$$

is split, it follows that

$$u_{\lambda_{m-t-b-a}\oplus a\alpha}u_{b\alpha} = v^{\langle\lambda_{m-t-b-a}+a\alpha,b\alpha\rangle}F^{\lambda_{m-t-b-a}\oplus(a+b)\alpha}_{\lambda_{m-t-b-a}\oplus a\alpha,b\alpha}u_{\lambda_{m-t-b-a}\oplus(a+b)\alpha}.$$

Because

$$\langle\lambda_{m-t-b-a}+a\alpha,b\alpha\rangle=\langle\beta+(m-t)\alpha-b\alpha,b\alpha\rangle=b(m-t-b)\langle\alpha,\alpha\rangle$$

and

$$F^{\lambda_{m-t-b-a}\oplus(a+b)\alpha}_{\lambda_{m-t-b-a}\oplus a\alpha,b\alpha}=F^{(a+b)\alpha}_{a\alpha,b\alpha}=\left|\begin{array}{c}a+b\\ b\end{array}\right]_{\varepsilon(\alpha)},$$

where $\varepsilon(\alpha)=\langle\alpha,\alpha\rangle=\frac{(\alpha,\alpha)}{2}$, we have

$$u_{\lambda_{m-t-b-a}\oplus a\alpha}u_{b\alpha}=\left(v^{\varepsilon(\alpha)}\right)^{b(m-t-b)}\left|\begin{array}{c}a+b\\ b\end{array}\right]_{\varepsilon(\alpha)}u_{\lambda_{m-t-b-a}\oplus(a+b)\alpha}.$$

Thus

$$u_{(m-t-b)\alpha}u_\beta u_{b\alpha}=v^{(m-t-b)\langle\alpha,\beta\rangle}\sum_{a=0}^{m-t-b}\sum_{\lambda_{m-t-b-a}}u_{\lambda_{m-t-b-a}\oplus a\alpha}u_{b\alpha}$$

$$=v^{(m-t-b)\langle\alpha,\beta\rangle}\left(v^{\varepsilon(\alpha)}\right)^{b(m-t-b)}\sum_{a=0}^{m-t-b}\left|\begin{array}{c}a+b\\ b\end{array}\right]_{\varepsilon(\alpha)}\sum_{\lambda_{m-t-b-a}}u_{\lambda_{m-t-b-a}\oplus(a+b)\alpha}$$

$$=\left(v^{\varepsilon(\alpha)}\right)^{-m(m-t-b)+b(m-t-b)}\sum_{a=0}^{m-t-b}\left(v^{\varepsilon(\alpha)}\right)^{ab}\left[\begin{array}{c}a+b\\ b\end{array}\right]_{\varepsilon(\alpha)}\sum_{\lambda_{m-t-b-a}}u_{\lambda_{m-t-b-a}\oplus(a+b)\alpha}.$$

It is easy to see that $u_\alpha^{(b)} = (v^{\varepsilon(\alpha)})^{b(b-1)} u_{b\alpha}$, where $u_\alpha^{(b)} = \frac{u_\alpha^b}{[b]!_{\varepsilon(\alpha)}}$. Denote

$$f = \sum_{b=0}^{m-t} (-1)^b (v^{\varepsilon(\alpha)})^{(m-t-b)(t+1)} u_\alpha^{(m-t-b)} u_\beta u_\alpha^{(b)}.$$

Then

$$f = \sum_{b=0}^{m-t} (-1)^b (v^{\varepsilon(\alpha)})^{(m-t-b)(t+1)} (v^{\varepsilon(\alpha)})^{(m-t-b)(m-t-b-1)} (v^{\varepsilon(\alpha)})^{b(b-1)} u_{(m-t-b)\alpha} u_\beta u_{b\alpha}$$

$$= \sum_{b=0}^{m-t} \sum_{a=0}^{m-t-b} (-1)^b (v^{\varepsilon(\alpha)})^{b(a+b-1)} \begin{bmatrix} a+b \\ b \end{bmatrix}_{\varepsilon(\alpha)} \sum_{\lambda_{m-t-b-a}} u_{\lambda_{m-t-b-a} \oplus (a+b)\alpha}$$

since $(m-t-b)(t+1) + (m-t-b)(m-t-b-1) + b(b-1) - m(m-t-b) + b(m-t-b) + ab = b(a+b-1)$. Let $s = i+j$. Note that

$$\sum_{b=0}^{m-t} \sum_{a=0}^{m-t-b} = \sum_{s=0}^{m-t} \sum_{b=0}^{s}.$$

Then

$$f = \sum_{s=0}^{m-t} \sum_{b=0}^{s} (-1)^b (v^{\varepsilon(\alpha)})^{b(s-1)} \begin{bmatrix} s \\ b \end{bmatrix}_{\varepsilon(\alpha)} \sum_{\lambda_{m-t-s}} u_{\lambda_{m-t-s} \oplus s\alpha}.$$

However

$$\sum_{b=0}^{s} (-1)^b (v^{\varepsilon(\alpha)})^{b(s-1)} \begin{bmatrix} s \\ b \end{bmatrix}_{\varepsilon(\alpha)} = 0 \quad \text{for} \quad s > 0.$$

So we have

$$f = \sum_{\lambda_{m-t}} u_{\lambda_{m-t}}.$$

That is,

$$\sum_{\lambda_{m-t}} u_{\lambda_{m-t}} = \sum_{b=0}^{m-t} (-1)^b (v^{\varepsilon(\alpha)})^{(m-t-b)(t+a)} u_\alpha^{(m-t-b)} u_\beta u_\alpha^{(b)}.$$

Now set

$$\langle \mathbf{e}_\beta + (m-t)\mathbf{e}_\alpha \rangle = (v^{-\varepsilon(\alpha)})^{(t+1)(m-t)} \sum_{\lambda_{m-t}} u_{\lambda_{m-t}}.$$

Then

$$\langle \mathbf{e}_\beta + (m-t)\mathbf{e}_\alpha \rangle = \sum_{b=0}^{m-t} (-1)^b (v^{-\varepsilon(\alpha)})^{(t+1)j} u_\alpha^{(m-t-b)} u_\beta u_\alpha^{(b)}.$$

On the other hand, if α is the sink, then we denote by $V_{\lambda'_t}$ the indecomposable object with the socle isomorphic to V_α and the top to tV_β. Similar to the case that α is the source, $V_{\lambda'_t}$ exits but it usually is not unique. In this case we set

$$\langle \mathbf{e}_\beta + t\mathbf{e}_\alpha \rangle = (v^{-\varepsilon(\alpha)})^{(t+1)(m-t)} \sum_{\lambda'_t} u_{\lambda'_t}.$$

By a dual discussion we have

$$\langle \mathbf{e}_\beta + t\mathbf{e}_\alpha \rangle = \sum_{b=0}^{t} (-1)^b (v^{-\varepsilon(\alpha)})^{(m-t+1)b} u_\alpha^{(b)} u_\beta u_\alpha^{(t-b)}.$$

Note that σ_α^+ induces a bijection from the set of isomorphim classes of all indecomposable objects of form $V_{\lambda'_t}$ to the set of those of form $V_{\lambda_{m-t}}$. So from the definitions we have

$$\sigma_\alpha \langle \mathbf{e}_\beta + t\mathbf{e}_\alpha \rangle = \langle \mathbf{e}_\beta + (n-t)\mathbf{e}_\alpha \rangle.$$

¿From the other side , Lusztig has shown , in [BBP5] 37.2.5, that

$$T''_{\alpha,1} \langle \mathbf{e}_\beta + t\mathbf{e}_\alpha \rangle = \langle \mathbf{e}_\beta + (n-t)\mathbf{e}_\alpha \rangle.$$

and, in [L1] 38.1.1 and 38.1.6, that

$$\langle \mathbf{e}_\beta + t\mathbf{e}_\alpha \rangle \quad \text{for} \quad 0 \le t \le n$$

is a generating set of $\mathcal{C}_*(\mathcal{S})\langle \alpha \rangle$. Therefore $T''_{i,1}$ and σ_i have the same effect on $\mathcal{C}_*(\mathcal{S})\langle i \rangle$. $\square$

5.4 Recall that a sequence $i_m, \cdots, i_1$ is called a sink sequence for Δ, provided i_m is a sink for Δ, and for any $1 \le t < m$, the vertex i_t is a sink for the orientation $\sigma_{i_{t+1}} \cdots \sigma_{i_m} \Delta$.

Now Let $X \in \text{ind}A$ be exceptional. And let $X \in$rep-$\mathcal{S}$ (see 5.1). Without loss of generality, we may assume that X is preinjective in rep-$\mathcal{S}$. Then there exists a sink sequence $i_m, \cdots, i_1$ for Δ such that

$$X = \sigma_{i_1}^+ \cdots \sigma_{i_m}^+ (S_{i_{m+1}}),$$

where $S_{i_{m+1}}$ is a simple object. Therefore as a consequence of Proposition in 5.3, we have

Proposition $v^{-\dim_k X + \langle \alpha_{i_{m+1}}, \alpha_{i_{m+1}} \rangle} u_X = T''_{i_1,1} \cdots T''_{i_m,1}(E_{i_m+1}).$

Now we can give the proof of Proposition in 4.4.

Proof of Proposition in 4.4: Note that the exact sequence in Lemma in 5.1 is a generic one. That is, for any conservative extension E of k in Ω,

$$0 \longrightarrow a_2 X_2^E \longrightarrow X^E \longrightarrow a_1 X_1^E \longrightarrow 0,$$

is an exact sequence and (X_1^E, X_2^E) is an orthogonal exceptional pair, also. And note that the set Ω of conservative extension of k is infinite. So we can assume that all formulae in 5.2 and 5.3 hold for generic v.

Since U is a Hopf algebra, we can define the adjoint representation of U on itself:

$$\mathrm{ad}(a)(b) = \sum a_i b \sigma(a_i') \quad \text{for } a, b \in U,$$

where $\Delta(a) = \sum a_i \otimes a_i'$ is the comultiplication and σ the antipode. Of course we have $\mathrm{ad}(a_1 a_2) = \mathrm{ad}(a_1)\mathrm{ad}(a_2)$. It is easy to see that

$$\mathrm{ad}(E_i)u = E_i u - K_i u K_i^{-1} E_i,$$

$$\mathrm{ad}(F_i)u = (F_i u - u F_i)K_i,$$

$$\mathrm{ad}(K_i)u = K_i u K_i^{-1}.$$

¿From the definition in 5.2 one can deduces that

$$T''_{i,1}(E_j) = (\mathrm{ad}(E_i^{(m)}))(E_j) \quad \text{for } i \neq j,$$

$$T''_{i,1}(F_j) = (-1)^m v^{m(m-1)\varepsilon_i} (\mathrm{ad}(F_i^{(m)}))(F_j) K_i^{-m\varepsilon_i} \quad \text{for } i \neq j,$$

$$T''_{i,1}(E_i) = -F_i K_i^{\varepsilon_i}$$

$$T''_{i,1}(F_i) = -K_i^{-\varepsilon_i} E_i,$$

where $m = -a_{ij}$. If we take the universal enveloping algebra of a symmetrizable Kac-Moody Lie algebra as the degeneration of its quantum group at $v = 1$, then we should identify $K_i = v^{h_i}$. Then under the limiting $v \to 1$, we have $\mathrm{ad}E_i \to [E_i, -]$ and $[m]! \to m!$. Therefore, in the case of Lie algebra, it holds that

$$T''_{i,1}(E_j) = \frac{(\mathrm{ad}(E_i))^m}{m!}(E_j) \quad \text{for } i \neq j,$$

$$T''_{i,1}(F_j) = \frac{(\mathrm{ad}(F_i))^m}{m!}(F_j) \quad \text{for } i \neq j,$$

$$T''_{i,1}(E_i) = -F_i,$$

$$T''_{i,1}(F_i) = -E_i.$$

As a consequence of Proposition above, we see that there exists an $0 \neq b \in \mathbf{Z}$ such that the vector bu_X belongs to the Lie subalgebra of $\mathcal{LC}(A)_1$ generated by u_{X_1} and u_{X_2}. By induction on $\underline{\dim}$, $a_1 u_{X_1}, a_2 u_{X_2} \in \mathcal{LC}(A)_1$ for some nonzero integers a_1, a_2. So there exits a nonzero integer a suth that $au_X \in \mathcal{LC}(A)_1$. The proof is finished. $\square$

6 Orbit spaces in triangulated categories

6.1 Let $\mathcal{C}$ be a triangulated category. Given any objects X, Y, Z, L, L' and M in $\mathcal{C}$ and for any $((f,g,h),(l,m,n)) \in W(X,Y;L) \times W(Z,L;M)$ we define

$$((f,g,h),(l,m,n))^\wedge = \{((afb^{-1}, bgc^{-1}, ch(Ta)^{-1}), (dl, mx^{-1}, xn(Td)^{-1}))|$$
$$a \in \mathrm{Aut}X, b, x \in \mathrm{Aut}L, c \in \mathrm{Aut}Y, d \in \mathrm{Aut}Z \text{ such that}$$
$$mx^{-1}bg = mg, fb^{-1}xn = fn \text{ and } m(x^{-1} - b^{-1}) \in mg\mathrm{Hom}(Y,L)\}.$$

It is routine to verify that the partition $((-,-,-),(-,-,-))^\wedge$ gives an equivalent relation among the elements of $W(X,Y;L) \times W(Z,L;M)$. So we have the orbit space

$$(W(X,Y;L) \times W(Z,L;M))^\wedge = \{((f,g,h),(l,m,n))^\wedge|$$
$$((f,g,h),(l,m,n)) \in W(X,Y;L) \times W(Z,L;M)\}.$$

Dually, for any $((l',m',n'),(f',g',h')) \in W(Z,X;L') \times W(L',Y;M)$ we define

$$((l',m',n'),(f',g',h'))^\wedge = \{((dl'x'^{-1}, x'm'a^{-1}, an'(Td)^{-1}), (b'f', g'c^{-1}, ch'(Tb')^{-1}))|$$
$$d \in \mathrm{Aut}Z, x', b' \in \mathrm{Aut}L', a \in \mathrm{Aut}X, c \in \mathrm{Aut}Y \text{ such that}$$
$$l'x'^{-1}b'f' = l'f', T^{-1}(h')b'^{-1}x'm' = T^{-1}(h')m' \text{ and } (b' - x')f' \in \mathrm{Hom}(L',Z)l'f'\}.$$

Then

$$(W(Z,X;L') \times W(L',Y;M))^\wedge = \{((l',m',n'),(f',g',h'))^\wedge|$$
$$((l',m',n'),(f',g',h')) \in W(Z,X;L') \times W(L',Y;M)\}.$$

Proposition *For any objects X, Y, Z and M in $\mathcal{R}$, we have the bijection*

$$\begin{array}{ccc} \bigcup_{[L],L\in\mathcal{C}} (W(X,Y;L) \times W(Z,L;M))^\wedge & \to & \bigcup_{[L'],L'\in\mathcal{C}} (W(Z,X;L') \times W(L',Y;M))^\wedge \\ ((f,g,h),(l,m,n))^\wedge & \mapsto & ((l',m',n'),(f',g',h'))^\wedge, \end{array}$$

where all morphisms fit into the following commutative diagram

$$\begin{array}{ccccccc}
 & & Z & \xrightarrow{1_Z} & Z & & \\
 & & \downarrow l' & & \downarrow l & & \\
T^{-1}Y & \xrightarrow{-T^{-1}h'} & L' & \xrightarrow{f'} & M & \xrightarrow{g'} & Y \\
\downarrow 1_{T^{-1}Y} & & \downarrow m' & & \downarrow m & & \downarrow 1_Y \\
T^{-1}Y & \xrightarrow{-T^{-1}h} & X & \xrightarrow{f} & L & \xrightarrow{g} & Y \\
 & & \downarrow n' & & \downarrow n & & \\
 & & TZ & \xrightarrow{1_{TZ}} & TZ & &
\end{array}$$

with all rows and columns triangles.

Proof Given any $((f,g,h),(l,m,n)) \in W(X,Y;L) \times W(Z,L;M)$, the axiom (Tr4) guarantees that there exist $L' \in \mathcal{C}$ and $((l',m',n'),(f',g',h')) \in W(Z,X;L') \times W(L',Y;M)$ such that the above diagram commutes, in which all rows and columns are triangles. If there exist another $L'' \in \mathcal{C}$ and $((l'',m'',n''),(f'',g'',h'')) \in W(Z,X;L'') \times W(L'',Y;M)$ such that the following diagram commutes, in which all rows and columns are triangles

$$\begin{array}{ccccccc}
 & & Z & \xrightarrow{1_Z} & Z & & \\
 & & \downarrow l'' & & \downarrow dl & & \\
T^{-1}Y & \xrightarrow{-T^{-1}h''} & L'' & \xrightarrow{f''} & M & \xrightarrow{g''} & Y \\
\downarrow 1_{T^{-1}Y} & & \downarrow m'' & & \downarrow mx^{-1} & & \downarrow 1_Y \\
T^{-1}Y & \xrightarrow{(T^{-1}c)(-T^{-1}h)a^{-1}} & X & \xrightarrow{afb^{-1}} & L & \xrightarrow{bgc^{-1}} & Y \\
 & & \downarrow n'' & & \downarrow xn(Td)^{-1} & & \\
 & & TZ & \xrightarrow{1_{TZ}} & TZ & &
\end{array}$$

where $a \in \mathrm{Aut}X$, $c \in \mathrm{Aut}Y$, $d \in \mathrm{Aut}Z$ and $b, x \in \mathrm{Aut}L$ such that $mx^{-1}bg = mg$, $fb^{-1}xn = fn$ and $m(x^{-1} - b^{-1}) \in mg\mathrm{Hom}(Y,L)$. Then $g'' = mx^{-1}bgc^{-1} = mgc^{-1} = g'c^{-1}$ and $n'' = afb^{-1}xn(Td)^{-1} = afn(Td)^{-1} = an'(Td)^{-1}$. By (Tr3) and Lemma 1 (b) in 2.2, $L'' \simeq L'$. Without loss of generality we may assume $L'' = L'$. Then we have the following morphisms of triangles

$$\begin{array}{ccccccc}
Z & \xrightarrow{l''} & L' & \xrightarrow{m''} & X & \xrightarrow{n''} & TZ \\
\downarrow d & & \downarrow x' & & \downarrow a & & \downarrow Td \\
Z & \xrightarrow{l'} & L' & \xrightarrow{m'} & X & \xrightarrow{n'} & TZ
\end{array}$$

and

$$\begin{array}{ccccccc}
L' & \xrightarrow{f''} & M & \xrightarrow{g''} & Y & \xrightarrow{h''} & TL' \\
\downarrow b' & & \downarrow 1_M & & \downarrow c & & \downarrow Tb' \\
L' & \xrightarrow{f'} & M & \xrightarrow{g'} & Y & \xrightarrow{h'} & TL'
\end{array}$$

So $l'' = dl'(x')^{-1}$, $m'' = x'm'a^{-1}$, $f'' = b'f'$ and $h'' = ch'(Tb')^{-1}$. Since $dl = l''f'' = dl'(x')^{-1}b'f'$ and $l = l'f'$, we have $l'(x')^{-1}b'f' = l'f'$. Also since $(T^{-1}c)(-T^{-1}h)a^{-1} = (-T^{-1}h'')m'' = -(T^{-1}c)(T^{-1}h')(b')^{-1}x'm'a^{-1}$ and $T^{-1}h = (T^{-1}h')m'$, we get $(T^{-1}h')(b')^{-1}x'm' = (T^{-1}h')m'$. By $f''mx^{-1} = m''afb^{-1}$, we see $b'f'mx^{-1} = x'm'fb^{-1}$. However $m(b^{-1} - x^{-1}) \in mg\mathrm{Hom}(Y,L)$. It follows that $m'fb^{-1} = f'mb^{-1} = f'mx^{-1}$ since $f'mg = f'g' = 0$. Theorefore $b'm'f = x'm'f$ and $b'f'm = x'f'm$. So $(b' - x')f' \in \mathrm{Hom}(L'',Z)l'f'$. By definition we have proved that $((l'',m'',n''),(f'',g'',h''))^\wedge = ((l',m',n'),(f',g',h'))^\wedge$. This means that the map in Proposition is well-defined.

On the other hand, by the symmetricity of the diagram in Proposition, we can naturally define a map from

$$\bigcup_{[L'],L'\in\mathcal{C}} (W(Z,X;L')\times W(L',Y;M))^\wedge$$

to

$$\bigcup_{[L],L\in\mathcal{C}} (W(X,Y;L)\times W(Z,L;M))^\wedge$$

which is the inverse one. so the proof is completed. $\square$

6.2 In fact, the definition of the orbit in $W(X,Y;L)\times W(Z,L;M)$ can be given in a simple way in case of Y indecomposable.

Lemma *Given $X,Y,Z,L,M\in\mathcal{C}$ with Y indecomposable and $((f,g,h),(l,m,n))\in W(X,Y;L)\times W(Z,L;M)$, we have*

$$((f,g,h),(l,m,n))^\wedge=\{((afb^{-1},bgc^{-1},ch(Ta)^{-1}),(dl,mb^{-1},bn(Td)^{-1}))|$$
$$a\in AutX, b\in AutL, c\in AutY, d\in AutZ\}.$$

Proof By definition,

$$((f,g,h),(l,m,n))^\wedge=\{((afb^{-1},bgc^{-1},ch(Ta)^{-1}),(dl,mx^{-1},xn(Td)^{-1}))|$$
$$a\in \mathrm{Aut}X, b,x\in \mathrm{Aut}L, c\in \mathrm{Aut}Y, d\in \mathrm{Aut}Z \text{ such that}$$
$$mx^{-1}bg=mg, fb^{-1}xn=fn \text{ and } m(x^{-1}-b^{-1})\in mg\mathrm{Hom}(Y,L)\}.$$

By setting $x'=b^{-1}x$, we have

$$((f,g,h),(l,m,n))^\wedge=\{((afb^{-1},bgc^{-1},ch(Ta)^{-1}),(dl,mx^{-1}b^{-1},bxn(Td)^{-1}))|$$
$$a\in \mathrm{Aut}X, b,x\in \mathrm{Aut}L, c\in \mathrm{Aut}Y, d\in \mathrm{Aut}Z \text{ such that}$$
$$mx^{-1}g=mg, fxn=fn \text{ and } m(1-x^{-1})\in mg\mathrm{Hom}(Y,L)\}.$$

Let $x\in\mathrm{Aut}L$ such that $mx^{-1}g=mg$, $fxn=fn$, $m(1-x^{-1})=mgt$ for some $t\in\mathrm{Hom}(Y,L)$.

(1) If $mg=0$, then $m=mx^{-1}$. Since $mxn=mn=0$, we have $xn=n(Td')^{-1}$ and $d'l=l$ for some $d'\in\mathrm{Aut}Z$ by (Tr3) and Lemma 1 (b) in 2.2.

(2) If $mg\neq 0$, since $mgtg=m(1-x^{-1})g=0$ and Y is indecomposable, then tg, and so gt, is nilpotent. Let $x'=1-gt\in\mathrm{Aut}L$. Then $mx^{-1}=mx'$ and $fx'=f$. Theorefore, by (Tr3) and Lemma 1 (b) in 2.2, $(x')^{-1}n=xn(Td')^{-1}$,

$d'l = l$ for some $d' \in \mathrm{Aut}Z$ and $(x')^{-1}g = g(c')^{-1}$, $c'h = h$ for some $c' \in \mathrm{Aut}Y$. Then

$$\begin{aligned}&((afb^{-1}, bgc^{-1}, ch(Ta)^{-1}), (dl, mx^{-1}b^{-1}, bxn(Td)^{-1}))\\ &= ((afb''^{-1}, b''gc''^{-1}, c''h(Ta)^{-1}), (d''l, mb''^{-1}, b''n(Td'')^{-1})),\end{aligned}$$

where $b'' = b(x')^{-1}$, $c'' = c(c')^{-1}$ and $d'' = d(d')^{-1}$. So we always have

$$((f,g,h),(l,m,n))^\wedge \subseteq \{((afb^{-1}, bgc^{-1}, ch(Ta)^{-1}), (dl, mb^{-1}, bn(Td)^{-1}))|$$

$$a \in \mathrm{Aut}X, b \in \mathrm{Aut}L, c \in \mathrm{Aut}Y, d \in \mathrm{Aut}Z\}.$$

The inclusion "$\supseteq$" is obvious. The proof is completed. $\Box$

6.3 Finally, we define the action of $\mathrm{Aut}Z$ on $W(Z, L; M)$ as follows: $d(l, m, n) = (dl, m, n(Td)^{-1})$ for any $(l, m, n) \in W(Z, L; M)$ and $d \in \mathrm{Aut}Z$. If we write

$$(l, m, n)^* = \{(dl, m, n(Td)^{-1})|d \in \mathrm{Aut}Z\},$$

then the orbit space is

$$W(Z, L; M)^* = \{(l, m, n)^*|(l, m, n) \in W(Z, L; M)\}.$$

7 Some relations for orbit spaces

7.1 Let $\mathcal{C}$ be a finitary and 2-period triangulated k-category.

¿From now on and until the end of this section, we fix objects $X, Y, Z, L, M \in \mathcal{C}$ and assume that X, Y, Z are indecomposable and $M \neq 0$.

For $\alpha = (l, m, n)^\wedge \in V(Z, L; M)$, we define

$$s(\alpha) = \dim_k n\mathrm{Hom}(TZ, L) = \dim_k\{nx|x \in \mathrm{Hom}(TZ, L)\}.$$

It is easy to see that $s(\alpha)$ only depends on the orbit α and not on the choice of the representative (l, m, n) of α.

For $\beta = ((f, g, h), (l, m, n))^\wedge \in (W(X, Y; L) \times W(Z, L; M))^\wedge$, let

$$B' = \{b' \in n\mathrm{Hom}(TZ, L)|fb' = a'f \text{ for some } a' \in \mathrm{End}X\}.$$

This is a subspace of $n\mathrm{Hom}(TZ, L)$. We define $t(\beta) = \dim_k B'$. It is similar to see that $t(\beta)$ is well-defined.

7.2 We want to use F_{YX}^L defined in 3.2 to replace the Hall polynomial or the filtration number defined as for Ringel-Hall algebras. Those Hall polynomials or the filtration numbers provide the associativity of Ringel-Hall algebras in an easy way. But it is not the case here. We need some very careful observation.

Lemma *If $L \not\simeq TZ \oplus M$, then*
(1) $|W(Z, L; M)^*| = \sum_{\alpha \in V(Z,L;M)} |AutL|/q^{s(\alpha)}$
(2) $|V(X, Y; L)||W(Z, L; M)^*| = \sum_{\beta \in (W(X,Y;L) \times W(Z,L;M))^\wedge} |AutL|/q^{t(\beta)}$.

Proof (1) We set a map

$$\begin{array}{rccc} \varphi: & W(Z, L; M)^* & \rightarrow & V(Z, L; M) \\ & (l, m, n)^* & \mapsto & (l, m, n)^\wedge. \end{array}$$

Obviously it is a well-defined surjection. Then

$$\varphi^{-1}((l, m, n)^\wedge) = \{(l, mb^{-1}, bn)^* | b \in \mathrm{Aut}L\}$$

Let

$$G((l, m, n)^\wedge) = \{b \in \mathrm{Aut}L | (l, mb^{-1}, bn)^* = (l, m, n)^*\}.$$

By the definition we have $G((l, m, n)^\wedge) = \{b \in \mathrm{Aut}L | mb = m\}$. Therefore

$$1 - G((l, m, n)^\wedge) = \{b' \in \mathrm{End}L | mb' = 0 \ \text{ and } 1 - b' \in \mathrm{Aut}L\}$$

However if $mb' = 0$ for some $b' \in \mathrm{End}L$, then $b' = nt$ for some $t \in \mathrm{Hom}(TZ, L)$ by Lemma 1 (c) in 2.2. Note that Z is indecomposable and $L \not\simeq M \oplus TZ$. So tn and then $b' = nt$ are nilpotent. Hence $1 - b' \in \mathrm{Aut}L$. Then

$$1 - G((l, m, n)^\wedge) = \{b' \in \mathrm{End}L | mb' = 0\} = n\mathrm{Hom}(TZ, L)$$

and so $|G((l, m, n)^\wedge)| = q^{s(\alpha)}$ for $\alpha = (l, m, n)^\wedge \in V(Z, L; M)$. Therefore

$$|\varphi^{-1}((l, m, n)^\wedge)| = |\mathrm{Aut}L|/q^{s(\alpha)}.$$

By the surjectivity of φ, we have the formula in (1).

(2) The map

$$\begin{array}{rccc} \psi: & V(X, Y; L) \times W(Z, L; M)^* & \rightarrow & (W(X, Y; L) \times W(Z, L; M))^\wedge \\ & ((f, g, h)^\wedge, (l, m, n)^*) & \mapsto & ((f, g, h), (l, m, n))^\wedge \end{array}$$

is well-defined and surjective. By Lemma in 6.2, we have

$$\psi^{-1}(((f, g, h), (l, m, n))^\wedge) = \{((fb^{-1}, bg, h)^\wedge, (l, mb^{-1}, bn)^*) | b \in \mathrm{Aut}L\}.$$

Let

$$G(((f,g,h),(l,m,n))^\wedge) = \{b \in \mathrm{Aut}L | (fb^{-1}, bg, h)^\wedge = (f,g,h)^\wedge$$

$$\text{and } (l, mb^{-1}, bn)^* = (l,m,b)^*\}.$$

We claim that

$$1 - G(((f,g,h),(l,m,n))^\wedge) = \{b' \in n\mathrm{Hom}(TZ, L)|$$

$$a'f = fb' \text{ for some } a' \in \mathrm{End}X \text{ such that } 1-a' \in \mathrm{Aut}X\} \qquad (*)$$

In fact, if $b \in \mathrm{Aut}L$ such that $(fb^{-1}, bg, h)^\wedge = (f,g,h)^\wedge$ and $(l, mb^{-1}, bn)^* = (l,m,b)^*$, then $(fb^{-1}, bg, h) = (af, gc^{-1}, ch(Ta)^{-1})$ and $(l, mb^{-1}, bn) = (dl, m, n(Td)^{-1})$ for some $a \in \mathrm{Aut}X$, $c \in \mathrm{Aut}Y$ and $d \in \mathrm{Aut}Z$. Therefore $mb^{-1} = m$ and then $m(1-b^{-1}) = 0$. So $b' = 1-b^{-1} \in n\mathrm{Hom}(TZ, L)$ and $fb' = f(1-b^{-1}) = f - fb^{-1} = f - af = (1-a)f$. Then $a' = 1-a \in \mathrm{End}X$ such that $a'f = fb'$ and $1-a' \in \mathrm{Aut}X$. Since $G(((f,g,h),(l,m,n))^\wedge)$ is a group, we have the inclusion "$\subseteq$" for (*).

On the other hand, if $b' \in n\mathrm{Hom}(TZ, L)$ satisfies $a'f = fb'$ for some $a' \in \mathrm{End}X$ such that $1-a' \in \mathrm{Aut}X$, it is easily seen that $1-b' \in \mathrm{Aut}L$ since $L \not\cong TZ \oplus M$. Put $b^{-1} = 1-b'$. Then $fb^{-1} = f(1-b') = (1-a')f$. Let $a = 1-a'$. Because $(af, g, h(Ta)^{-1})$ is a triangle again and $afb = fb^{-1}b = f$, there is $c' \in \mathrm{Aut}Y$ such that $bg = gc'$ and $c'h = h(Ta)^{-1}$ by (Tr3) and Lemma 1 (b) in 2.2. Let $c = (c')^{-1}$. Therefore $(fb^{-1}, bg, h) = (af, gc^{-1}, ch(Ta)^{-1})$. Also $mb^{-1} = m(1-b') = m - mb' = m$ since $b' \in n\mathrm{Hom}(TZ, L)$. There exists $d \in \mathrm{Aut}Z$ such that $l = dl$, $bn = n(Td)^{-1}$ by (Tr3) and Lemma 1 (b) in 2.2. So $(l, mb^{-1}, bn) = (dl, m, n(Td)^{-1})$. We get the inclusion "$\supseteq$" for (*). Hence (*) holds.

So we have

$$1 - G(((f,g,h),(l,m,n))^\wedge) \subseteq \{b' \in n\mathrm{Hom}(TZ, L)|$$

$$a'f = fb' \text{ for some } a' \in \mathrm{End}X\}.$$

We claim that this is in fact an equality. It is obvious when $f = 0$; in this case both sides are equal to $n\mathrm{Hom}(TZ, L)$. Assume that $f \neq 0$. For $b' \in n\mathrm{Hom}(TZ, L)$ with $a'f = fb'$ for some $a' \in \mathrm{End}X$, b' is nilpotent since X is indecomposable. Then a' is nipotent and hence $1-a'$ is invertible. It follows that

$$1 - G(((f,g,h),(l,m,n))^\wedge) =$$

$$\{b' \in n\mathrm{Hom}(TZ, L) | a'f = fb' \text{ for some } a' \in \mathrm{End}X\}.$$

Therefore $|\psi^{-1}(\beta)| = |\mathrm{Aut}L|/q^{t(\beta)}$, where $\beta = ((f,g,h),(l,m,n))^\wedge$. So we have the formula in (2). $\square$

7.3 Now we consider the case $L \simeq M \oplus TZ$. For any $\beta \in (W(X,Y;M \oplus TZ) \times W(Z, M \oplus TZ; M))^\wedge$, by Lemma 3 in 2.2 and Lemma in 6.2, we can find a representative of the form $((f,g,h),(0,(1,0),\binom{0}{1})) \in \beta$, i.e, $\beta = ((f,g,h),(0,(1,0),\binom{0}{1}))^\wedge$ for some $(f,g,h) \in W(X,Y;M \oplus TZ)$. Put

$$\overline{B} = \{b' = \begin{pmatrix} 0 & 0 \\ b'_1 & b'_2 \end{pmatrix} \in \mathrm{End}(M \oplus TZ)|$$

$$b'_1 \in \mathrm{Hom}(TZ, M),\ b'_2 \in \mathrm{radEnd}TZ \text{ and } a'f = fb' \text{ for some } a' \in \mathrm{End}X\}$$

It is easy to see this is a vector space. Let $r(\beta) = \dim_k \overline{B}$. We claim that $r(\beta)$ is independent on the choice of (f,g,h) in the representative $((f,g,h),(0,(1,0),\binom{0}{1}))$ of β.

In fact, by Lemma in 6.2, any representative of β has the form $((afb^{-1}, bgc^{-1}, ch(Ta)^{-1}),(0,(1,0)b^{-1}, b\binom{0}{1}(Td)^{-1}))$ with $a \in \mathrm{Aut}X$, $b \in \mathrm{Aut}(M \oplus TZ)$, $c \in \mathrm{Aut}Y$ and $d \in \mathrm{Aut}Z$. If $(0,(1,0)b^{-1}, b\binom{0}{1}(Td)^{-1}) = (0,(1,0),\binom{0}{1})$, then b is of the form $b = \begin{pmatrix} 1 & 0 \\ b_1 & b_2 \end{pmatrix}$ with $b_1 \in \mathrm{Hom}(TZ, M)$ and $b_2 \in \mathrm{Aut}TZ$. It is easy to verify that

$$b\overline{B}b^{-1} = \{b'' = \begin{pmatrix} 0 & 0 \\ b''_1 & b''_2 \end{pmatrix} \in \mathrm{End}(M \oplus TZ)|$$

$$b''_1 \in \mathrm{Hom}(TZ, M),\ b''_2 \in \mathrm{radEnd}TZ \text{ and}$$
$$a''(afb^{-1}) = (afb^{-1})b'' \text{ for some } a'' \in \mathrm{End}X\}.$$

This implies our claim.

Lemma *(1) If $M \oplus TZ \not\simeq X \oplus Y$, then*

$$|V(X,Y;M \oplus TZ)| = \sum_\beta |Hom(TZ,M)||AutTZ|/q^{r(\beta)}$$

where β runs over $(W(X,Y;M \oplus TZ) \times W(Z, M \oplus TZ; M))^\wedge$.

(2) If $M \oplus TZ \simeq X \oplus Y$ but $M \not\simeq Y$ (so $M \simeq X$ and $TZ \simeq Y$), then

$$|V(X,Y;M \oplus TZ)| - 1 = \sum_\beta |Hom(TZ,M)||AutTZ|/q^{r(\beta)}$$

where β runs over

$$(W(X,Y;M \oplus TZ) \times W(Z, M \oplus TZ; M))^\wedge \setminus \{(((1,0),\begin{pmatrix}0\\1\end{pmatrix},0),(0,(1,0),\begin{pmatrix}0\\1\end{pmatrix}))^\wedge\}.$$

(3) If $M \oplus TZ \simeq X \oplus Y$ with $M \simeq X \simeq Y$ (so $M \simeq TZ$), then

$$|V(X,Y;M \oplus TZ)| - 1 = \sum_{\beta} |Hom(TZ,M)||AutTZ|/q^{r(\beta)} + |Hom(TZ,M)|$$

where β runs over

$$(W(X,Y;M \oplus TZ) \times W(Z, M \oplus TZ; M))^{\wedge} \backslash$$

$$\{(((1,0), \begin{pmatrix}0\\1\end{pmatrix}, 0), (0, (1,0), \begin{pmatrix}0\\1\end{pmatrix}))^{\wedge} \cup (((0,1), \begin{pmatrix}1\\0\end{pmatrix}, 0), (0, (1,0), \begin{pmatrix}0\\1\end{pmatrix}))^{\wedge}\}.$$

(4) If $M \oplus TZ \simeq X \oplus Y$ but $M \not\simeq X$ (so $M \simeq Y$ and $TZ \simeq X$), then

$$|(W(X,Y;M \oplus TZ) \times W(Z, M \oplus TZ; M))^{\wedge}| = 1$$

and

$$|V(X,Y;M \oplus TZ)| = |Hom(TZ,M)|.$$

Proof The map

$$\begin{array}{rccc} \Theta: & V(X,Y;M \oplus TZ) & \to & (W(X,Y;M \oplus TZ) \times W(Z, M \oplus TZ; M))^{\wedge} \\ & (f,g,h)^{\wedge} & \mapsto & ((f,g,h), (0,(1,0), \binom{0}{1}))^{\wedge}\} \end{array}$$

is well-defined. Note that all elements in $(W(X,Y;M \oplus TZ) \times W(Z, M \oplus TZ; M))^{\wedge}$ have the form $((f,g,h),(0,(1,0),\binom{0}{1}))^{\wedge}$. Hence the map Θ is surjective. Let

$$B = \{b = \begin{pmatrix} 1 & 0 \\ b_1 & b_2 \end{pmatrix} \in \mathrm{End}(M \oplus TZ) | b_1 \in \mathrm{Hom}(TZ,M), b_2 \in \mathrm{Aut}TZ\}.$$

So $b \in B$ if and only if $b \in \mathrm{Aut}(M \oplus TZ)$ and $(1,0)b = (1,0)$. For

$$\beta = ((f,g,h),(0,(1,0),\begin{pmatrix}0\\1\end{pmatrix}))^{\wedge} \in (W(X,Y;M \oplus TZ) \times W(Z, M \oplus TZ; M))^{\wedge},$$

it is easy to see $\Theta^{-1}(\beta) = \{(fb^{-1}, bg, h)^{\wedge} | b \in B\}$. Let

$$B((f,g,h)^{\wedge}) = \{b \in B | (fb^{-1}, bg, h)^{\wedge} = (f,g,h)^{\wedge}\}$$

Then

$$B((f,g,h)^{\wedge}) = \{b \in B | af = fb \text{ for some } a \in \mathrm{Aut}X\}$$

by (Tr3) and Lemma 1 (b) in 2.2. We denote $f = (f_1, f_2) : X \to M \oplus TZ$.

The first step. Note that there exists $\overline{a} \in \mathrm{Aut}X$ and $\overline{b} = \begin{pmatrix} 1 & 0 \\ \overline{b}_1 & \overline{b}_2 \end{pmatrix} \in B$ such that $\overline{a}f\overline{b} = (\overline{f}_1, \overline{f}_2)$ with $\overline{f}_2 = 0$ if and only if $f_2 = 0$. In this case, since Y is indecomposable, we must have that $f_1 : X \to M$ is an isomorphism and $X \oplus Y \simeq M \oplus TZ$ by Lemma 2 in 2.2. Then $Y \simeq TZ$. We may assume that $X = M$ and $Y = TZ$. So it is easy to see that $\beta = (((1,0), \binom{0}{1}, 0), (0, (1,0), \binom{0}{1}))^\wedge$ if and only if $f_2 = 0$ if and only if $(f,g,h)^\wedge = ((1,0), \binom{0}{1}, 0)^\wedge$. In this case $|\Theta^{-1}(\beta)| = 1$.

The second step. Similarly note that there exist $\overline{a} \in \mathrm{Aut}X$ and $\overline{b} = \begin{pmatrix} 1 & 0 \\ \overline{b}_1 & \overline{b}_2 \end{pmatrix} \in B$ such that $\overline{a}f\overline{b} = (\overline{f}_1, \overline{f}_2)$ with $\overline{f}_1 = 0$ if and only if $f_2 : X \to TZ$ is an isomorphism, by Lemma 2 in 2.2. In this case we must have $X \oplus Y \simeq M \oplus TZ$. Then $Y \simeq M$. We may assume that $X = TZ$ and $Y = M$. So it is easy to see that $\beta = (((0,1), \binom{1}{0}, 0), (0, (1,0), \binom{0}{1}))^\wedge$ if and only if f_2 is an isomorphism. In this case we have a bijection $B((f,g,h)^\wedge) \to \mathrm{Aut}X (= \mathrm{Aut}TZ)$ which sends $b = \begin{pmatrix} 1 & 0 \\ b_1 & b_2 \end{pmatrix} \in B((f,g,h)^\wedge)$ to $a \in \mathrm{Aut}X$ with relations $a(f_1, f_2) = (f_1, f_2)\begin{pmatrix} 1 & 0 \\ b_1 & b_2 \end{pmatrix}$, i.e, $af_1 = f_1 \oplus f_2 b_1$ and $af_2 = f_2 b_2$. So

$$|\Theta^{-1}(\beta)| = |B|/|B((f,g,h)^\wedge)| = |\mathrm{Hom}(TZ, M)||\mathrm{Aut}X|/|\mathrm{Aut}X| = |\mathrm{Hom}(TZ, M)|.$$

The third step. If $X \oplus Y \simeq M \oplus TZ$ but $M \not\simeq X$, then f_2 is an isomorphism by Lemma 2 in 2.2. So we have (4) by the second step. For (1), (2) and (4), by the discussions in the first and second step, it remains to consider the following situation for $(f,g,h)^\wedge \in V(X, Y; M \oplus TZ)$: for any $\overline{a} \in \mathrm{Aut}X$ and $\overline{b} = \begin{pmatrix} 1 & 0 \\ \overline{b}_1 & \overline{b}_2 \end{pmatrix} \in B$, denoting $\overline{a}f\overline{b} = (\overline{f}_1, \overline{f}_2)$, we always have $\overline{f}_1 \neq 0$ and $\overline{f}_2 \neq 0$. In this case, for $b = \begin{pmatrix} 1 & 0 \\ b_1 & b_2 \end{pmatrix} \in B$ if there is $a \in \mathrm{Aut}X$ such that $af = fb$, then $af_1 = f_1 + f_2 b_1$ and $af_2 = f_2 b_2$, i.e, $(a-1)f_1 = f_2 b_1$ and $(a-1)f_2 = f_2(b_2 - 1)$. We claim that $b_2 - 1$ is nilpotent. Otherwise $b_2 - 1$ is invertible since $\mathrm{End}TZ$ is a local ring. But $f_2 \neq 0$. So $a - 1$ is invertible too. However

$$(a-1)(f_1, f_2)\begin{pmatrix} 1 & 0 \\ -(b_2-1)^{-1}b_1 & 1 \end{pmatrix} = ((a-1)f_1 - f_2 b_1, f_2(b_2-1)) = (0, f_2(b_2-1))$$

a contradiction to the assumption. Therefore $b_2 - 1$ and $a - 1$ are nilpotent. Then

$$1 - B((f,g,h)^\wedge) = \{b' = \begin{pmatrix} 0 & 0 \\ b'_1 & b'_2 \end{pmatrix} \in \mathrm{End}(M \oplus TZ)|$$

$b'_1 \in \mathrm{Hom}(TZ, M),\ b'_2 \in \mathrm{radEnd}TZ$ such that $a'f = fb'$ for some $a' \in \mathrm{End}X\}$

Therefore

$$|\Theta^{-1}(\beta)| = |B|/|B((f,g,h)^\wedge)| = |\mathrm{Hom}(TZ,M)||\mathrm{Aut}TZ|/q^{r(\beta)}.$$

The proof is completed. □

Remark For $(f,g,h) \in W(X,Y;L)$, if we define

$$(f,g,h)^{**} = \{(afb^{-1}, bgc^{-1}, ch(Ta)^{-1} | a \in \mathrm{Aut}X, b \in \mathrm{Aut}L \text{ and } c \in \mathrm{Aut}Y\}$$

and define the orbit space

$$W(X,Y;L)^{**} = \{(f,g,h)^{**} | (f,g,h) \in W(X,Y;L)\}$$

then Lemma essentially gives the relation between $|V(X,Y;L)|$ and $|W(X,Y;L)^{**}|$ when L is decomposable.

7.4 In the following discussion, we consider our equations in the residue ring $\mathbf{Z}/(q-1)$. We take for granted that, for $u,v,w \in \mathbf{Z}$ with $v \neq 0$, putting $v = q^s v'$ such that v' is not divided by q, if $u = v'w$ then $\frac{u}{v}$ is to represent the residue class of w in $\mathbf{Z}/(q-1)$.

Proposition 1 *Let X, Y, Z, M and L in $\mathcal{C}$ with X, Y and Z indecomposable and $M \neq 0$. Then we have the following relations in $\mathbf{Z}/(q-1)$.*

(1) If $L \not\simeq TZ \oplus M$, then

$$F_{YX}^L F_{LZ}^M = |(W(X,Y;L) \times W(Z,L;M))^\wedge|$$

(2) If $M \oplus TZ \not\simeq X \oplus Y$, then

$$F_{YX}^{M\oplus TZ}/|AutTZ| = |(W(X,Y;M\oplus TZ) \times W(Z, M\oplus TZ; M))^\wedge|$$

(3) If $M \oplus TZ \simeq X \oplus Y$ but $M \not\simeq Y$, then

$$(F_{YX}^{M\oplus TZ} - 1)/|AutTZ| = |(W(X,Y;M\oplus TZ) \times W(Z, M\oplus TZ; M))^\wedge| - 1$$

(4) If $M \oplus TZ \simeq X \oplus Y$ with $M \simeq X \simeq Y$, then

$$(F_{YX}^{M\oplus TZ} - |Hom(TZ,M)| - 1)/|AutTZ|$$
$$= |(W(X,Y;M\oplus TZ) \times W(Z, M\oplus TZ; M))^\wedge| - 2$$

(5) If $M \oplus TZ \simeq X \oplus Y$ but $M \not\simeq X$, then

$$|(W(X,Y;M\oplus TZ) \times W(Z, M\oplus TZ; M))^\wedge| = 1 \text{ and } F_{YX}^{M\oplus TZ} = 1.$$

Proof These are direct consequences of Lemmas in 7.2 and in 7.3. $\square$

Dually, we have the following.

Proposition 2 *Let X, Y, Z, M and L' in $\mathcal{C}$ with X, Y and Z indecomposable and $M \neq 0$. Then we have the following relations in $\mathbf{Z}/(q-1)$.*

(1) If $L' \not\simeq T^{-1}Y \oplus M$, then

$$F_{YL'}^{M} F_{XZ}^{L'} = |(W(Z,X;L') \times W(L',Y;M))^{\wedge}|$$

(2) If $T^{-1}Y \oplus M \not\simeq X \oplus Z$, then

$$F_{XZ}^{T^{-1}Y\oplus M}/|AutT^{-1}Y| = |(W(Z,X;T^{-1}Y \oplus M) \times W(T^{-1}Y \oplus M, Y; M))^{\wedge}|$$

(3) If $T^{-1}Y \oplus M \simeq X \oplus Z$ but $M \not\simeq Z$, then

$$(F_{XZ}^{T^{-1}Y\oplus M} - 1)/|AutT^{-1}Y| = |(W(Z,X;T^{-1}Y \oplus M) \times W(T^{-1}Y \oplus M, Y; M))^{\wedge}| - 1$$

(4) If $T^{-1}Y \oplus M \simeq X \oplus Z$ with $M \simeq X \simeq Z$, then

$$(F_{XZ}^{T^{-1}Y\oplus M} - |Hom(M, T^{-1}Y)| - 1)/|AutT^{-1}Y|$$

$$= |(W(Z,X;T^{-1}Y \oplus M) \times W(T^{-1}Y \oplus M, Y; M))^{\wedge}| - 2$$

(5) If $T^{-1}Y \oplus M \simeq X \oplus Z$ but $M \not\simeq X$, then

$$|(W(Z,X;T^{-1}Y \oplus M) \times W(T^{-1}Y \oplus M, Y; M))^{\wedge}| = 1 \text{ and } F_{XZ}^{T^{-1}Y\oplus M} = 1.$$

$\square$

The following conclusion is well-known for Ringel-Hall algebra. It also plays a role in our situation.

Lemma *Let X, Y and $L \in \mathcal{C}$ with X and Y indecomposable.*

(1) If L is decomposable and $L \not\simeq X \oplus Y$, then F_{XY}^{L} is divided by $q-1$.

(2) The number $F_{XY}^{X\oplus Y} - F_{YX}^{X\oplus Y}$ is divided by $q-1$.

(3) If $X \simeq TY$ then $F_{XY}^{0} = 1$; and $F_{XY}^{0} = 0$ otherwise.

Proof (1) By Lemma in 7.3 and since $|\mathrm{Aut}TZ|$ is divided by $q-1$. (2) By Lemma (2),(3) and (4) in 7.3 and since $|\mathrm{Aut}TZ|$ is divided by $q-1$. (3) This is easy. $\square$

7.5 Unlike as in Ringel-Hall algebras, there is no associativity in our situation. However the following result can be viewed as a weak associativity which is crucial to verify the Jacobi identity in our Lie algebras.

Proposition *Let X, Y, Z and $M \in \mathcal{C}$ with X, Y and Z indecomposable and $M \neq 0$. Then in $\mathbf{Z}/(q-1)$ we have the following.*

(1) If $X \not\simeq TY, X \not\simeq TZ$ and $Y \not\simeq TZ$, then

$$\sum_{[L], L\in\mathcal{C}} F_{YX}^{L} F_{LZ}^{M} + F_{YX}^{M\oplus TZ}/|AutZ| = \sum_{[L'], L'\in\mathcal{C}} F_{YL'}^{M} F_{XZ}^{L'} + F_{XZ}^{M\oplus TY}/|AutY|$$

(2) If $Z \not\simeq X$ and $Z \not\simeq TX$, then

(a) when $M \not\simeq Z$ we have

$$\sum_{[L], L\in\mathcal{C}} F_{TXX}^{L} F_{LZ}^{M} + F_{TXX}^{M\oplus TZ}/|AutZ| = \sum_{[L'], L'\in\mathcal{C}} F_{TXL'}^{M} F_{XZ}^{L'} + F_{XZ}^{M\oplus X}/|AutX|$$

(b) when $M \simeq Z$ we have

$$\sum_{[L], L\in\mathcal{C}} F_{TXX}^{L} F_{LZ}^{Z} + F_{TXX}^{Z\oplus TZ}/|AutZ| = \sum_{[L'], L'\in\mathcal{C}} F_{TXL'}^{Z} F_{XZ}^{L'}$$

(3) If $Y \not\simeq X$ and $Y \not\simeq TX$, then

(a) when $M \not\simeq Y$ we have

$$\sum_{[L], L\in\mathcal{C}} F_{YX}^{L} F_{LTX}^{M} + F_{YX}^{M\oplus X}/|AutX| = \sum_{[L'], L'\in\mathcal{C}} F_{YL'}^{M} F_{XTX}^{L'} + F_{XTX}^{M\oplus TY}/|AutY|$$

(b) when $M \simeq Y$ we have

$$\sum_{[L], L\in\mathcal{C}} F_{YX}^{L} F_{LTX}^{Y} = \sum_{[L'], L'\in\mathcal{C}} F_{YL'}^{Y} F_{XTX}^{L'} + F_{XTX}^{Y\oplus TY}/|AutY|$$

(c) when $M \not\simeq X$ we have

$$\sum_{[L], L\in\mathcal{C}} F_{YX}^{L} F_{LTY}^{M} + F_{YX}^{M\oplus Y}/|AutY| = \sum_{[L'], L'\in\mathcal{C}} F_{YL'}^{M} F_{XTY}^{L'} + F_{XTY}^{M\oplus TY}/|AutY|$$

(d) when $M \simeq X$ we have

$$\sum_{[L], L\in\mathcal{C}} F_{YX}^{L} F_{LTY}^{X} + (F_{YX}^{X\oplus Y} - 1)/|AutY| = \sum_{[L'], L'\in\mathcal{C}} F_{YL'}^{X} F_{XTY}^{L'} + (F_{XTY}^{X\oplus TY} - 1)/|AutY|$$

Proof By Proposition in 6.1 we have

$$\sum_{[L], L\in\mathcal{C}} |(W(X, Y; L) \times W(Z, L; M))^{\wedge}| = \sum_{[L'], L'\in\mathcal{C}} |(W(Z, X; L') \times W(L', Y; M))^{\wedge}.|$$

(1) If we have the assumption of (1), then $M\oplus TZ\not\simeq X\oplus Y$ and $M\oplus TY\not\simeq X\oplus Z$. According to Proposition 1 (1)(2) and Proposition 2 (1)(2) in 7.4, we have

$$\sum_{[L],L\not\simeq M\oplus TZ} F^L_{YX}F^M_{LZ}+F^{M\oplus TZ}_{YX}/|\mathrm{Aut}Z| = \sum_{[L'],L'\not\simeq M\oplus TY} F^M_{YL'}F^{L'}_{XZ}+F^{M\oplus TY}_{XZ}/|\mathrm{Aut}Y|$$

here we note that $T^{-1}=T$ and $\mathrm{Aut}|T^{-1}Y|=|\mathrm{Aut}TY|=|\mathrm{Aut}Y|$. However $F^{M\oplus TZ}_{YX}$ and $F^{M\oplus TY}_{XZ}$ are divided by $q-1$ according to Lemma (1) in 7.4. So we have the equation in (1).
(2) Assume that $Y\simeq TX$ but $Z\not\simeq X$ and $Z\not\simeq TX$. If $M\not\simeq Z$ then $M\oplus TZ\not\simeq X\oplus TX$ and $M\oplus X\not\simeq X\oplus Z$. Similarly as in (1) we have (2) (a). If $M\simeq Z$, then we have

$$\sum_{[L],L\not\simeq Z\oplus TZ} F^L_{TXX}F^Z_{LZ}+|(W(X,TX;Z\oplus TZ)\times W(Z,Z\oplus TZ;Z))^\wedge|$$

$$=\sum_{[L'],L'\not\simeq Z\oplus TY} F^Z_{TXL'}F^{L'}_{XZ}+|(W(Z,X;Z\oplus TY)\times W(Z\oplus TY,Y;Z))^\wedge|$$

by Proposition 1 (1) and 2 (1) in 7.4. It is easy to see $F^Z_{TX,X\oplus Z}=1$, and according to Proposition 2 (5) in 7.4,

$$F^{X\oplus Z}_{XZ}=|(W(Z,X;X\oplus Z)\times W(X\oplus Z,TX;Z))^\wedge|=1$$

where $X\oplus Z\simeq Z\oplus TY$. By Proposition 1 (2) in 7.4

$$F^{Z\oplus TZ}_{TXX}/|\mathrm{Aut}Z|=|(W(X,TX;Z\oplus TZ)\oplus W(Z,Z\oplus TZ;Z))^\wedge|$$

since $Z\oplus TZ\not\simeq X\oplus TX$. Therefore we have (2) (b).

It is similar for the proof of (3) when $Z\simeq TX$ but $Y\not\simeq X$ and $Y\not\simeq TX$. □

8 Proof of Theorem 3.4

8.1 In this section we also assume that $\mathcal{C}$ is a finitary and 2-period triangulated k-category whose Grothendieck group is proper.

We need the following two lemmas.

Lemma 1 *Let $X\in\mathcal{C}$ be an indecomposable object. Then*

$$|AutX|=|EndX|-|radEndX|=|radEndX|(q^{d(X)}-1)$$

Lemma 2 *Let $X,Y\in\mathcal{C}$ be indecomposable and $X\not\simeq Y$. Then*

$$F^{X\oplus Y}_{XY}=|Hom(Y,X)|=q^{\dim_k Hom(Y,X)}.$$

Proof See [R3]. $\square$

8.2 Now we can give

Proof of Theorem 3.4 All equations below are verified over $\mathbf{Z}/(q-1)$. It is easy to see the rule

$$[x,y]=-[y,x] \text{ for any } x,y\in \mathbf{g}_{(q-1)} \tag{1}$$

from the definition in 3.4.

The first case. Let $X,Y,Z\in \mathrm{ind}\mathcal{C}$ we should verify the following Jacobi identity holds.

$$[[u_X,u_Y],u_Z]=[[u_X,u_Z],u_Y]+[u_X,[u_Y,u_Z]] \tag{2}$$

(i) If $X\not\simeq TY, X\not\simeq TZ$ and $Y\not\simeq TZ$, we have $\gamma_{XY}^L=\gamma_{XZ}^L=\gamma_{YZ}^L=0$ for L decomposable or $L=0$ by Lemma in 7.4. Therefore it follows from Proposition (1) in 7.5 that for any $M\in\mathrm{ind}\mathcal{C}$ we have

$$\sum_{[L],L\in\mathrm{ind}\mathcal{C}}\gamma_{XZ}^L\gamma_{LY}^M+\sum_{[L],L\in\mathrm{ind}\mathcal{C}}\gamma_{XL}^M\gamma_{YZ}^L$$

$$=\sum_{[L],L\in\mathcal{C}}\gamma_{XZ}^L\gamma_{LY}^M+\sum_{[L],L\in\mathcal{C}}\gamma_{XL}^M\gamma_{YZ}^L$$

$$=\sum_{[L],L\in\mathcal{C}}(F_{XZ}^LF_{LY}^M-F_{ZX}^LF_{LY}^M-F_{YL}^MF_{XZ}^L+F_{YL}^MF_{ZX}^L)$$

$$+\sum_{[L],L\in\mathcal{C}}(F_{XL}^MF_{YZ}^L-F_{XL}^MF_{ZY}^L-F_{YZ}^LF_{LX}^M+F_{ZY}^LF_{LX}^M)$$

$$=F_{ZY}^{M\oplus TX}/|\mathrm{Aut}X|-F_{XZ}^{M\oplus TY}/|\mathrm{Aut}Y|+F_{YZ}^{M\oplus TX}/|\mathrm{Aut}X|-F_{ZX}^{M\oplus TY}/|\mathrm{Aut}Y|$$

$$+\sum_{[L],L\in\mathcal{C}}(F_{XL}^MF_{YZ}^L+F_{ZY}^LF_{LX}^M-F_{ZX}^LF_{LY}^M-F_{YL}^MF_{XZ}^L)$$

$$=\sum_{[L],L\in\mathcal{C}}F_{XY}^LF_{LZ}^M+F_{XY}^{M\oplus TZ}/|\mathrm{Aut}Z|+\sum_{[L],L\in\mathcal{C}}F_{ZL}^MF_{YX}^L+F_{YX}^{M\oplus TZ}/|\mathrm{Aut}Z|$$

$$-\sum_{[L],L\in\mathcal{C}}F_{ZL}^MF_{XY}^L-F_{XY}^{M\oplus TZ}/|\mathrm{Aut}Z|-\sum_{[L],L\in\mathcal{C}}F_{YX}^LF_{LZ}^M-F_{YX}^{M\oplus TZ}/|\mathrm{Aut}Z|$$

$$=\sum_{[L],L\in\mathcal{C}}(F_{XY}^LF_{LZ}^M+F_{ZL}^MF_{YX}^L-F_{ZL}^MF_{XY}^L-F_{YX}^LF_{LZ}^M)$$

$$=\sum_{[L],L\in\mathcal{C}}\gamma_{XY}^L\gamma_{LZ}^M=\sum_{[L],L\in\mathrm{ind}\mathcal{C}}\gamma_{XY}^L\gamma_{LZ}^M$$

This implies that the Jacobi identity (2) holds.

(ii) If $Y \simeq TX$ but $Z \not\simeq X$ and $Z \not\simeq TX$, we have $\gamma_{XZ}^{L} = \gamma_{YZ}^{L} = 0$ for L decomposable or $L = 0$ by Lemma in 7.4. Let

$$S(M) = \sum_{[L], L \in \mathrm{ind}\mathcal{C}} \gamma_{XZ}^{L} \gamma_{LY}^{M} + \sum_{[L], L \in \mathrm{ind}\mathcal{C}} \gamma_{XL}^{M} \gamma_{YZ}^{L}.$$

Then

$$\begin{aligned} S(M) &= \sum_{[L], L \in \mathcal{C}} \gamma_{XZ}^{L} \gamma_{LY}^{M} + \sum_{[L], L \in \mathcal{C}} \gamma_{XL}^{M} \gamma_{YZ}^{L} \\ &= \sum_{[L], L \in \mathcal{C}} (F_{XZ}^{L} F_{LY}^{M} - F_{ZX}^{L} F_{LY}^{M} - F_{YL}^{M} F_{XZ}^{L} + F_{YL}^{M} F_{ZX}^{L}) \\ &+ \sum_{[L], L \in \mathcal{C}} (F_{XL}^{M} F_{YZ}^{L} - F_{XL}^{M} F_{ZY}^{L} - F_{YZ}^{L} F_{LX}^{M} + F_{ZY}^{L} F_{LX}^{M}) \end{aligned}$$

If $M \not\simeq Z$, by Proposition (2) (a) and (3) (a)(c) in 7.5, it is similar to show that

$$S(M) = \sum_{[L], L \in \mathcal{C}} \gamma_{XY}^{L} \gamma_{LZ}^{M}.$$

¿From Lemma in 7.4 and the proper Grothendieck group of $\mathcal{C}$, we have $\gamma_{XY}^{L} = \gamma_{XTX}^{L} = 0$ for any $L \in \mathcal{C}$. So $S(M) = 0$. If $M \simeq Z$, by Proposition (2)(b) and (3) (b) (d) in 7.5 we have

$$\begin{aligned} S(Z) &= (F_{ZTX}^{Z \oplus TX} - 1)/|\mathrm{Aut}X| - (F_{XZ}^{Z \oplus X} - 1)/|\mathrm{Aut}X| \\ &+ (F_{TXZ}^{Z \oplus TX} - 1)/|\mathrm{Aut}X| - (F_{ZX}^{Z \oplus X} - 1)/|\mathrm{Aut}X| \\ &+ \sum_{[L], L \in \mathcal{C}} (F_{XL}^{Z} F_{TXZ}^{L} + F_{ZTX}^{L} F_{LX}^{Z} - F_{ZX}^{L} F_{LTX}^{Z} - F_{TXL}^{Z} F_{XZ}^{L}) \\ &= (F_{ZTX}^{Z \oplus TX} - 1)/|\mathrm{Aut}X| - (F_{XZ}^{Z \oplus X} - 1)/|\mathrm{Aut}X| \\ &+ (F_{TXZ}^{Z \oplus TX} - 1)/|\mathrm{Aut}X| - (F_{ZX}^{Z \oplus X} - 1)/|\mathrm{Aut}X| \\ &+ \sum_{[L], L \in \mathcal{C}} F_{XTX}^{L} F_{LZ}^{Z} + F_{XTX}^{Z \oplus TZ}/|\mathrm{Aut}Z| + \sum_{[L], L \in \mathcal{C}} F_{ZL}^{Z} F_{TXX}^{L} + F_{TXX}^{Z \oplus TZ}/|\mathrm{Aut}Z| \\ &- \sum_{[L], L \in \mathcal{C}} F_{ZL}^{Z} F_{XTX}^{L} - F_{XTX}^{Z \oplus TZ}/|\mathrm{Aut}Z| - \sum_{[L], L \in \mathcal{C}} F_{TXX}^{L} F_{LZ}^{Z} - F_{TXX}^{Z \oplus TZ}/|\mathrm{Aut}Z| \\ &= (F_{ZTX}^{Z \oplus TX} - 1)/|\mathrm{Aut}X| - (F_{XZ}^{Z \oplus X} - 1)/|\mathrm{Aut}X| \\ &+ (F_{TXZ}^{Z \oplus TX} - 1)/|\mathrm{Aut}X| - (F_{ZX}^{Z \oplus X} - 1)/|\mathrm{Aut}X| + \sum_{[L], L \in \mathcal{C}} \gamma_{XTX}^{L} \gamma_{LZ}^{Z} \end{aligned}$$

Note that $\gamma_{XTX}^{L} = 0$. Then by Lemma 1 and 2 in 8.1 we have

$$S(Z) = (q^{\dim_k \mathrm{Hom}(TX,Z)} - 1)/(q^{d(X)} - 1) - (q^{\dim_k \mathrm{Hom}(Z,X)} - 1)/(q^{d(X)} - 1)$$

$$+(q^{\dim_k \mathrm{Hom}(Z,TX)}-1)/(q^{d(X)}-1)-(q^{\dim_k \mathrm{Hom}(X,Z)}-1)/(q^{d(X)}-1)$$

$$=\dim_k \mathrm{Hom}(TX,Z)/d(X)-\dim_k \mathrm{Hom}(Z,X)/d(X)$$

$$+\dim_k \mathrm{Hom}(Z,TX)/d(X)-\dim_k \mathrm{Hom}(X,Z)/d(X)$$

$$=-(\frac{h_X}{d(X)}|h_Z)$$

Therefore the Jacobi identity (2) holds.

(iii) If $Y\simeq TX$ and $Z\simeq X$ or $Z\simeq TX$, by the rule (1), it is easy to see that the Jacobi identity (2) holds.

Note the rule (1). The above discussions show that the Jacobi identity (2) holds for any X,Y and Z in $\mathrm{ind}\mathcal{C}$.

The second case Let $X,Y,Z\in\mathrm{ind}\mathcal{C}$. We should verify the following Jacobi identity holds.

$$[[h_X,u_Y],u_Z]=[[h_X,u_Z],u_Y]+[h_X,[u_Y,u_Z]] \tag{3}$$

In fact, $[[h_X,u_Y],u_Z]=-(h_X|h_Y)[u_Y,u_Z]$ and

$$[[h_X,u_Z],u_Y]=-(h_X|h_Z)[u_Z,u_Y]=(h_X|h_Z)[u_Y,u_Z]$$

If $Z\simeq TY$, then $[u_Y,u_Z]=\frac{h_Y}{d(Y)}$ and $h_Z=-h_Y$. Hence

$$[h_X,[u_Y,u_Z]]=\frac{1}{d(Y)}[h_X,h_Y]=0.$$

So the Jacobi identity (3) holds. Assume that $Z\not\simeq TY$. Then $[u_Y,u_Z]=\sum\limits_{[L],L\in\mathrm{ind}\mathcal{C}}\gamma_{YZ}^L u_L$. If $[u_Y,u_Z]=0$, it is clear that the Jacobi identity (3) holds. If $\gamma_{YZ}^L\neq 0$, then $h_L=h_Y+h_Z$. Therefore

$$[h_X,[u_Y,u_Z]]=\sum_{[L],L\in\mathrm{ind}\mathcal{C}}\gamma_{YZ}^L(-(h_X|h_L)u_L)$$

$$=(-(h_X|h_Y)-(h_X|h_Z))\sum_{[L],L\in\mathrm{ind}\mathcal{C}}\gamma_{YZ}^L u_L$$

$$=-(h_X|h_Y)[u_Y,u_Z]-(h_X|h_Z)[u_Y,u_Z]$$

This implies the Jacobi identity (3) holds.

The third case. Finally it is easy to see the Jacobi identity

$$[[h_X,h_Y],u_Z]=[h_X,[h_Y,u_Z]]+[[h_X,u_Z],h_Y] \tag{4}$$

holds.

Together with $[\mathbf{h},\mathbf{h}]=0$, we finish the proof of Theorem 3.4. □

8.3 The following lemma is necessary for the proof of Corollary 3.4.

Lemma *Let $X, Y, Z \in \mathcal{R}$ be indecomposable. (1) If there exists a hereditary subcategory $\mathcal{A}$ of $\mathcal{R}$ such that $X, Z \in \mathcal{A}$, then*

$$|V(X,Z;Y)||AutX||AutZ| = |W(X,Z;Y)|.$$

(2) If there is no hereditary subcategory of $\mathcal{R}$ containing both X and Z, then

$$|V(X,Z;Y)||AutX||AutZ|/|Hom(X,TZ)| = |W(X,Z;Y)|.$$

Proof We have (1) since the action of $\mathrm{Aut}X \times \mathrm{Aut}Z$ on $W(X,Z;Y)$ is free

in this case. To show (2), first we work in the derived category $D^b(A)$. We claim that, if $X', Y', Z' \in D^b(A)$ are indecomposable, then

$$|V(X',Z';Y'||\mathrm{Aut}X'||\mathrm{Aut}Z'|/|\mathrm{Hom}(X',T^{-}Z')| = |W(X',Z';Y')| \qquad (*)$$

In fact, if $W(X',Z';Y')$ is empty, then $(*)$ holds clearly. Assume that $W(X',Z';Y')$ is not empty. Given $(f,g,h) \in W(X',Z';Y')$, we denote

$$D = \{(\alpha,\beta) \in \mathrm{Aut}X' \times \mathrm{Aut}Z' | \alpha f = f, g\beta^{-1} = g, \beta h (T\alpha)^{-1} = h\}$$

and $D_{X'} = \{\alpha \in \mathrm{Aut}X' | \alpha f = f\}$. Then it is easy to see that D is a subgroup of $\mathrm{Aut}X' \times \mathrm{Aut}Z'$ and $D_{X'}$ a subgroup of $\mathrm{Aut}X'$. Clearly the map $\varphi : D \to D_{X'}$ by sending (α,β) to α is a homomorphism of groups. By Lemma 1 (d) in 2.2 we see that φ is surjective. Now consider $\ker\varphi = \{(1,\beta) | \beta \in \mathrm{Aut}Z', g\beta^{-1} = g, \beta h = h\}$. Then $\ker\varphi \simeq \{\beta \in \mathrm{Aut}Z' | g\beta^{-1} = g, \beta h = h\}$. There is a bijection from $\{\beta \in \mathrm{Aut}Z' | g\beta^{-1} = g, \beta h = h\}$ to $\mathrm{Hom}(TX', Y')$. Hence $|\ker\varphi| = |\mathrm{Hom}(TX',Y')| = |\mathrm{Hom}(X',T^{-}Y')|$. Now consider $D_{X'}$. There is a bijection from $D_{X'}$ to $\mathrm{Hom}(X',T^{-}Z')/\mathrm{Hom}(X',T^{-}Y')(-T^{-}g)$. However, by $\mathrm{Hom}(X',T^{-}X') = 0$ we have a bijection from $\mathrm{Hom}(X',T^{-}Y')$ to $\mathrm{Hom}(X',T^{-}Y')(-T^{-}g)$ which sends $w \in \mathrm{Hom}(X',T^{-}Y')$ to $w(-T^{-}g)$. So $|D_{X'}| = |\mathrm{Hom}(X',T^{-}Z')|/|\mathrm{Hom}(X',T^{-}Y')|$. Therefore $|D| = |D_{X'}||\ker\varphi| = |\mathrm{Hom}(X',T^{-}Z')|$. By the definition of $V(X',Z';Y')$ we see that $(*)$ holds.

Now we return to the root category $\mathcal{R}$. Of course we may assume that $W(X,Z;Y)$ is not empty. Let $F : D^b(A) \to \mathcal{R}$ be the canonical covering functor (see 2.3). Take $X', Y', Z' \in D^b(A)$ with $FX' = X$, $FY' = Y$ and $FZ' = Z$. We can take the above X', Y', Z' such that $\mathrm{Hom}(X',Y') \neq 0$ and $\mathrm{Hom}(Y',Z') \neq 0$ since X, Y, Z is indecomposable and $W(X,Z;Y)$ is not empty. It is easy to see that F induces the isomorphisms $\mathrm{Aut}X' \simeq \mathrm{Aut}X$ and $\mathrm{Aut}Z' \simeq \mathrm{Aut}Z$, and the bijections $W(X',Z';Y') \to W(X,Z;Y)$ and $V(X',Z';Y') \to V(X,Z;Y)$. Naturally $\mathrm{mod}A$ can be viewed as a full subcategory of $D^b(A)$. Without loss of generality we may assume that $X' \in \mathrm{mod}A$. Then $Z' \notin \mathrm{mod}A$ and so $Z' \in T(\mathrm{mod}A)$. Hence $\mathrm{Hom}(X', T^{2i-1}Z') = 0$ for any $0 \neq i \in \mathbf{Z}$.

Therefore F induces the isomorphism $\mathrm{Hom}(X', T^{-}Z') \simeq \mathrm{Hom}(X, TZ)$. By $(*)$ we obtain the formula in (2). $\square$

Proof of Corollary 3.4 We can define the symmetric bilinear $\mathbf{Z}/(q-1)$-valued form $(-|-)$ according to b), c) and d) of Corollary 3.4. Only a) of Corollary 3.4 should be verified. For (i) of a), it suffices to show that

$$-([h_X, u_Y]|u_{TY}) = (h_X|[u_Y, u_{TY}])d(Y).$$

This is obvious by definition. For (ii) of a), we may assume that there exist indecomposable objects X, Y and Z in $\mathcal{R}$ and a triangle of the form $X \to Z \to Y \to TX$. Then we have a triangle of the form $Y \to TX \to TZ \to TY$. ¿From the above lemma, it follows that there exists $t \in \mathbf{Z}$ such that

$$q^t|\mathrm{Aut}X||V(X,Y;Z)| = |V(Y,TZ;TX)||\mathrm{Aut}Z|$$

holds in $\mathbf{Z}$ since $|W(X,Y;Z)| = |W(Y,TZ;TX)|$. Because

$$|\mathrm{Aut}X| = |\mathrm{radEnd}X|(q-1)(q^{d(X)-1} + q^{d(X)-2} + \cdots + 1)$$

and

$$|\mathrm{Aut}Z| = |\mathrm{radEnd}Z|(q-1)(q^{d(Z)-1} + q^{d(Z)-2} + \cdots + 1)$$

we have

$$q^t|\mathrm{radEnd}X|(q^{d(X)-1} + q^{d(X)-2} + \cdots + 1)|V(X,Y;Z)|$$
$$= |V(Y,TZ;TX)||\mathrm{radEnd}Z|(q^{d(Z)-1} + q^{d(Z)-2} + \cdots + 1)$$

holds in $\mathbf{Z}$. Therefore

$$d(X)|V(X,Y;Z)| = |V(Y,TZ;TX)|d(Z)$$

holds in $\mathbf{Z}/(q-1)$. This implies that (ii) of a) holds. $\square$

Acknowledgement

Some part of this work was done while the both authors visited Universität Bielefeld. They would like to thank Prof. Dr. C.M.Ringel for the encouragement and hospitality.

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