BETTI NUMBERS FOR $p-$ STABLE IDEALS

Viviana Ene
Ovidius University
Department of Mathematics
Bd. Mamaia 124
8700 - Constantza,
Romania

Gerhard Pfister
Universität Kaiserslautern
Fachbereich Mathematik
Postfach 3049
67653 Kaiserslautern,
Germany

Dorin Popescu
Institute of Mathematics
University of Bucharest
P.O. Box 1-764
Bucharest 70700, Romania

Introduction

Let k be an infinite field of characteristic $p > 0$, $R = k[x_1, ..., x_n]$ the polynomial ring with the standard grading $R = \bigoplus_{d \geq 0} R_d$.
Let I $\subset R$ be a monomial ideal and G(I) a minimal set of monomials generating I. Let u be a monomial. Denote by $m(u)$ the maximum integer i such that $x_i|u$. If x_i^s divides u but x_i^{s+1} does not, we write $x_i^s \parallel u$. An ideal I is *stable* if it is monomial and for every monomial $u \in$ I and $i < m(u)$ it holds $x_i \left(u/x_{m(u)}\right) \in$ I. These ideals become well known after Eliahou - Kervaire described their minimal free resolutions (see[EK]). The generic initial ideals are Borel fixed ideals (see[Ei]), which are stable in characteristic zero. When $p = char\ k > 0$, they are not stable and they are called $p-$Borel ideals, their

study being very difficult (see [AH$_2$]). But the real analog of stable ideals in characteristic $p > 0$ are the $p-$ stable ideals which we introduce below:

We consider on $\mathbb{N}$ the following partial order: we say that $k \leq_p s$ if the binomial coefficient $\begin{pmatrix} s \\ k \end{pmatrix}$ is not divisible by p. Of course, $\leq_0$ is the usual order $\leq$ on $\mathbb{N}$. For a prime number p there is the following explicit description: $k \leq_p s$ if and only if $k_j \leq s_j$ for all $j \geq 0$, where k_j, s_j are defined by the $p-$ adic expansion $k = \sum_{j\geq 0} k_j p^j$, $s = \sum_{j\geq 0} s_j p^j$ of k, respectively s (see [Ei]).

Definition. A monomial ideal I is called $p-$ *stable* if $x_i^k \left(u/x_{m(u)}^k\right) \in$ I for all monomial $u \in$ I, $1 \leq i < m(u)$ and positive integers $k \leq_p s$, s being given by $x_{m(u)}^s \parallel u$.

In the first section we give some properties of $p-$ stable ideals and describe the saturated $p-$stable ideals. In the Section 2 we compute the Koszul homology of a subclass of $p-$ stable ideals, which are called *acceptable* $p-$ *stable* ideals (see Definition 2.1 and Theorem 2.5.). This was done applying the method used by Aramova and Herzog to compute the Koszul homology for Cohen - Macaulay $p-$ Borel principal ideals [AH$_2$]. As an application of Theorem 2.5 the Betti numbers, the Castelnuovo regularity and the Poincaré series of acceptable $p-$ stable ideals are computed. Our formula for the Castelnuovo regularity is on the line given by Pardue Conjecture (see [Pa]) and it might hold for all products of $p-$powers of stable ideals as we asked in Question 3.8.

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1. $p-$ stable ideals

As already mentioned in Introduction one considers the following partial order on $\mathbb{N}$:

$k \leq_p s$ if $\begin{pmatrix} s \\ k \end{pmatrix}$ is not divisible by p, which is equivalent with the condition: $k_j \leq s_j$ for all $j \geq 0$, where $k = \sum_{j\geq 0} k_j p^j$ and $s = \sum_{j\geq 0} s_j p^j$ are the $p-$ adic expansion of k, respectively s.

Definition 1.1. A monomial ideal I $\subset R$ is called $p-$ *stable* if every monomial $u \in$I satisfies the following condition:
$x_i^k \left(u/x_{m(u)}^k\right) \in$ I for any $1 \leq i < m(u)$ and $k \leq_p s$, s being given by
$x_{m(u)}^s \parallel u$ (shortly we say S_u^{I} holds).

Definition 1.2. *A* set $J \subset R_d$ of monomials is called $p-$ *stable* if for every monomial $u \in J$, S_u^J holds, that is:
$x_i^k \left(u/x_{m(u)}^k\right) \in J$ for any $1 \leq i < m(u)$ and $k \leq_p s$, s being given by
$x_{m(u)}^s \parallel u$.
In this case we say that $< J >$, the $k-$ space of R_d spanned by J, is $p-$ *stable* too. Clearly, a monomial ideal I is $p-$ stable if all its homogeneous components I_d have this property.

We start with a simple property of the above order on $\mathbb{N}$ which is very useful in the study of $p-$ stable ideals.

Lemma 1.3. *Let k, a, b be three positive integers such that $k \leq_p a+b$. Then there exists an integer c, $0 \leq c \leq k$, such that $k - c \leq_p a$ and $c \leq_p b$.*

Proof. It is enough to look at the coefficient of x^k in the expansion

$$(1+x)^{a+b} = (1+x)^a (1+x)^b . \qquad \blacksquare$$

Lemma 1.4. *Let $J \subset R_d$ be a $p-$ stable set of monomials. Then $J' := \{x_1, ..., x_n\}J \subset R_{d+1}$ is $p-$ stable.*

Proof. Let $v \in J$ be a monomial. Then S_v^J holds. We have to show that $S_{x_j v}^{J'}$ holds for every $1 \leq j \leq n$. Fix v, j and let s be given by
$x_{m(x_j v)}^s \parallel x_j v$, $i < m(x_j v)$ and $k \leq_p s$. If $m(v) > j$ then $m(x_j v) = m(v)$

and $x^s_{m(v)} \parallel v$. Since S^J_v holds it follows $x^k_i \left(v/x^k_{m(v)}\right) \in J$ and so

$$x^k_i \left(x_j v/x^k_{m(v)}\right) \in J^{'}.$$

If $m(v) < j$ then $m(x_j v) = j,\ s = k = 1$ and $x_i\,(x_j v/x_j) = x_i v \in J^{'}$. Now suppose $m(v) = j$. Then $m(x_j v) = j,\ x^{s-1}_j \parallel v$. Apply Lemma 1.3 for the case $a = s-1,\ b = 1$. Then $k \leq_p a+b$ and there exists c such that $c \leq_p 1$ and $k-c \leq_p s-1$. Thus $c = 0$ or $c = 1$. If $c = 0$ then $x^k_i \left(x_j v/x^k_j\right) = x_j\ x^k_i \left(v/x^k_j\right) \in x_j\ J \subseteq J^{'}$ because S^J_v holds. If $c = 1$ then $x^k_i \left(x_j v/x^k_j\right) = x_i x^{k-1}_i \left(v/x^{k-1}_j\right) \in x_i J \subseteq J^{'}$. Thus $S^{J^{'}}_{x_j v}$ holds. ■

Lemma 1.5. *Let I be a monomial ideal. The following conditions are equivalent:*

(i) I is $p-$ stable;

(ii) S^I_u holds for each monomial $u \in G(I)$, that is $x^k_i \left(u/x^k_{m(u)}\right) \in I$ for all monomial $u \in G(I)$, $1 \leq i < m(u)$ and positive integers $k \leq_p s$, s being given by $x^s_{m(u)} \parallel u$.

Proof. To show $(ii) = (i)$ (the other implication is trivial) it is enough to see that if $v \in \mathrm{I}$ is a monomial such that S^{I}_v holds then $S^{\mathrm{I}}_{x_j v}$ holds for every $1 \leq j \leq n$. But this follows from the proof of Lemma 1.4. ■

Lemma 1.6. *A finite product of $p-$ stable ideals is $p-$ stable. In particular a product of $p-$ powers of stable ideals is $p-$stable.*

Proof. Let I, J be two $p-$ stable ideals. We show that S^{IJ}_{uv} holds for u, v monomials from I, respectively J. Let s be defined by $x^s_{m(uv)} \parallel uv,\ i < m(uv)$ and $k \leq_p s$. If $m(u) \neq m(v)$, let us say $m(u) < m(v)$, then $m(uv) = m(v)$ and $x^k_i \left(uv/x^k_{m(uv)}\right) = u x^k_i \left(v/x^k_{m(v)}\right) \in \mathrm{I}J$ because S^J_v holds. If $m = m(u) = = m(v)$ then $m(uv) = m$ and let a, b be defined by $x^a_m \parallel u,\ x^b_m \parallel v$. Thus $s = a+b$ and by Lemma 1.3. we may find $0 \leq c \leq k$ such that $c \leq_p b,\ k-c \leq_p a$. Thus $x^k_i \left(uv/x^k_m\right) = x^{k-c}_i \left(u/x^{k-c}_m\right) x^c_i \left(v/x^c_m\right) \in \mathrm{I}J$ because $S^{\mathrm{I}}_u,\ S^J_v$ hold. Thus S^{IJ}_{uv}

holds. For the second statement we only note that a $p-$ power $T^{[p]}$ of a stable ideal T is $p-$ stable ($T^{[p^e]}$ denotes the ideal generated by f^{p^e} for $f \in T$). Let $u \in T$ and s an integer defined by $x_{m(u)}^s \parallel u$. Then $x_{m(u)}^{ps} \parallel u^p$. Let $i < m(u)$ and $k \leq_p ps$, in particular $k = pk'$ for a $k' \in \mathbb{N}$. Since T is stable we get by recurence $x_i^{k'} \left(u/x_{m(u)}^{k'}\right) \in T$ which gives $x_i^k \left(u^p/x_{m(u)}^k\right) \in T^{[p]}$. Thus $S_{u^p}^{T^{[p]}}$ holds . ■

Remark 1.7. In general one can show only that a $p-$ stable ideal is a union of products of $p-$powers of stable ideals. If $m = 3$, $p = 2$ and $I \subset R$ is the ideal generated by $\{x_1^2, x_1x_2, x_2^2, x_3^2\}$ then I is not a product of 2 - powers of stable ideals but $I = J \cup T^{[2]}$, for $J = \{x_1^2, x_1x_2\}$, $T = \{x_1, x_2, x_3\}$, stable ideals (even lex ideals).

The following Lemma is inspired by [Ei,15.24]

Lemma 1.8. *Suppose that $I \subset R$ is a $p-$ stable ideal.*

Then $(I : x_n^\infty) = (I : (x_1, .., .x_n)^\infty)$.

The proof goes as in [Ei, 15.24.]. ■

Corollary 1.9. *Let I be a $p-$ stable ideal. I is saturated if and only if no generator from $G(I)$ involves x_n.*

Proof. By Lemma 1.8. the saturation, I^{sat}, of I is $(\mathrm{I} : x_n^\infty)$. Suppose $(\mathrm{I} : x_n^\infty) = \mathrm{I}$. If $G(\mathrm{I})$ contains a monomial u sucht hat $x_n^q \parallel u$ for some $q \geq 1$ then $u/x_n^q \in (\mathrm{I} : x_n^q) \setminus \mathrm{I}$, so $\mathrm{I} \subsetneq (\mathrm{I} : x_n^\infty)$, contradiction. Conversely, let $v \in (\mathrm{I} : x_n^\infty)$, that is $vx_n^s \in \mathrm{I}$ for some integer $s \geq 0$. Thus there exists $u \in G(\mathrm{I})$ such that $vx_n^s = uz$. Since $x_n \nmid u$, it results $x_n^s | z$ thus $v \in \mathrm{I}$, so $(\mathrm{I} : x_n^\infty) \subseteq \mathrm{I}$. The other inclusion is trivial, so $(\mathrm{I} : x_n^\infty) = \mathrm{I}$. ■

Remark 1.10. It is known that a stable ideal I has the property:

$(\mathrm{I} : x_n) = (\mathrm{I} : (x_1, ..., x_n))$ [Gr]. For $p-$ stable ideals this equality does not hold, though X_n is still a generic linear form for them by [Po]. This shows the following:

Example: Let $n = 2,\ p = 2$ and $\mathrm{I} = \left(x_1^2,\ x_2^2\right) \subset k[x_1, x_2]$. Then $x_2 \in (\mathrm{I} : x_2)$ but $x_2 \notin (\mathrm{I} : (x_1, x_2))$.

2. Koszul homology of acceptable $p-$ stable ideals

The homological study of $p-$ stable ideals is much easier if we restrict to a subclass of $p-$ stable ideals, which is defined below.

Definition 2.1. A monomial ideal $\mathrm{I} \subset R$ is called *acceptable $p-$ stable* if there exist some stable ideals $(\mathrm{I}_t)_{1 \le t \le s}$ of R and some integers $(r_t)_{1 \le t \le s}$, $r_s > ... > r_1 \ge 0$ such that

i) $\mathrm{I} = \prod_{t=1}^{s} \mathrm{I}_t^{[p^{r_t}]}$,

ii) I_t contains $x_{m_{t+1}}^{\left(p^{r_{t+1}-\ r_t}-1\right)}$ for each $1 \le t < s$, where $m_t = \max_{u \in G(\mathrm{I}_t)} m(u)$

Lemma 2.2. *Let J, I be two monomial ideals of R, r a positive integer and $m = \max_{u \in G(I)} m(u)$. If J is stable and $x_m^{p^r-1} \in J$ then*

i) $J \supset \{x_1, ..., x_m\}^{p^r\ -\ 1}$,

ii) $G(J\ I^{[p^r]}) = \{uv^{p^r} | u \in G(J),\ v \in G(I)\}$. *In particular* $\left|G(J\ I^{[p^r]})\right| = |G(J)|\,|G(I)|$.

Proof. i) follows easily by $x_m^{p^r-1} \in J$ because J is stable.

The set $T = \{uv^{p^r} | u \in G(J),\ v \in G(\mathrm{I})\}$ generates $J\mathrm{I}^{[p^r]}$ and it is enough to see that T is a minimal system of generators. Suppose that $uv^{p^r} \in (x)J\mathrm{I}^{[p^r]}$ for some $u \in G(J)$, $v \in G(\mathrm{I})$, that is

1) $uv^{p^r} = u' v'^{p^r} w$

for some $u' \in G(J),\ v' \in G(\mathrm{I})$ and w a monomial $\neq 1$. Let $1 \le i \le n$ and $\alpha_i, \alpha_i', \beta_i, \beta_i',\ \gamma_i \ge 0$ be the integers defined by $x_i^{\alpha_i} \parallel u,\ x_i^{\alpha_i'} \parallel u',\ x_i^{\beta_i} \parallel v,\ x_i^{\beta_i'} \parallel v',\ x_i^{\gamma_i} \parallel w$. If $v = v'$ then 1) gives $u \in (x)J$ which is not possible $(u \in G(J)!)$. Thus $v \neq v'$ and so there exist $1 \le j \le m$ such that $\beta_j < \beta_j'$, otherwise it follows $v \in (X)v' \subset (x)\mathrm{I}$ which contradicts $v \in G(\mathrm{I})$. By 1) we have $\alpha_j = \alpha_j' + p^r\left(\beta_j' - \beta_j\right) + \gamma_j$. Since our hypothesis in the form of i) gives $x_j^{p^r(\beta_j' - \beta_j)} \in (x)J$

we obtain $u \in \left(x_j^{\alpha_j}\right) \subset (x)J$ which contradicts $u \in G(\mathrm{I})$. Thus T is minimal. ■

Proposition 2.3. *Let* $I = \prod_{t=1}^{s} I_t^{[p^{r_t}]}$ *be an acceptable* $p-$ *stable ideal as in 2.1. Then* $G(\mathrm{I}) = \{\prod_{\tau=1}^{s} u_t^{p^{r_t}} \mid u_t \in G(I_t), 1 \leq t \leq s\}$. *In particular,* $|G(I)| = \prod_{t=1}^{s} |G(I_t)|$.

Proof. Apply induction on s. If $s = 1$ then clearly $G\left(\mathrm{I}_1^{[p^{r_1}]}\right) = \{u_1^{p^r} | u_1 \in \in G(\mathrm{I}_1)\}$. If $s > 1$ then denote $E = \prod_{t=2}^{s} \mathrm{I}^{[p^{r_t - r_2}]}$. By 2.1.ii) we see that $m(E) := \max_{v \in E} m(v) = m_2$ and $G\left(\mathrm{I}_1 E^{[p^{r_2 - r_1}]}\right) = \{u_1 v^{p^{r_2 - r_1}} | u_1 \in \mathrm{I}_1, v \in E\}$ by Lemma 2.2. Now apply the induction hypothesis and repeat the argument from case $s = 1$. ■

Lemma 2.4. *Let* I, J *be two monomial ideals of* R *such that* $\left(x_1^{p^r - 1}, ..., x_m^{p^r - 1}\right) \subset J$ *for a positive integer* r *and* $m = \max_{u \in I} m(u)$. *Let* F *be a free* $R-$ *module of basis* $(\varepsilon_u)_{u \in G(I)}$. *Then for each* $j \geq 2$ *there exists an exact sequence*

$$0 \to F \otimes_R H_j(x; R/J) \xrightarrow{\gamma} H_j\left(x; R/JI^{[p^r]}\right) \xrightarrow{\tau} H_j\left(x; R/I^{[p^r]}\right) \to 0$$

where $H_j(x; R/J)$ *denotes the Koszul homology defined by* $x = (x_1, ..., x_n)$ *on* R/J, τ *is canonical and* γ *is given by* $[z]\varepsilon_u \to [u^{p^r} z]$ *for a cycle* $z \in Z_j(x; R/J)$.

Proof. Here we may assume $k = \mathbb{Z}/p\mathbb{Z}$. Applying the Koszul homology for the exact sequence

$$0 \to \mathrm{I}^{[p^r]}/J\mathrm{I}^{[p^r]} \to R/J\mathrm{I}^{[p^r]} \to R/\mathrm{I}^{[p^r]} \to 0$$

we obtain the exact sequence

1) $$H_{j+1}(x; R/\mathrm{I}^{[p^r]}) \xrightarrow{\delta_{j+1}} H_j\left(x; \mathrm{I}^{[p^r]}/J\mathrm{I}^{[p^r]}\right) \xrightarrow{\rho_j} H_j(x; R/J\mathrm{I}^{[p^r]}) \to$$
$$\to H_j\left(x; R/\mathrm{I}^{[p^r]}\right) \xrightarrow{\delta_j} H_{j-1}\left(x; \mathrm{I}^{[p^r]}/J\mathrm{I}^{[p^r]}\right), \qquad 2 \leq j.$$

By [BH, Corollary 1.6.13] we have

$$H_j(x; R/\mathrm{I}) \cong H_j(x_1, ..., x_m; R/(\mathrm{I}, x_{m+1}, ..., x_n))$$

because $\{x_{m+1}, ..., x_n\}$ is a regular system on R/I. Thus a cycle $y \in Z_j(x; R/\mathrm{I})$

has the form

$$y = \sum_{\substack{\sigma=(i_1,\dots,i_j)\\ 1\le i_1<i_2<\dots<i_j\le m}} \hat{a}_\sigma e_\sigma, \quad e_\sigma = e_{i_1} \wedge \dots \wedge e_{i_j}, \quad a_\sigma \in R$$

and $\hat{a}_\sigma = \mathrm{cls}\, a_\sigma \mathrm{mod} \mathrm{I}$. By [$\mathrm{AH}_2$;Lemma 1.5] there exists a bijection

$$H_j(x;\ R/\mathrm{I}) \to H_j\left(x; R/\mathrm{I}^{[p]}\right)$$

acting on a basis by $[y] \to [y^p]$, where y^p denotes the cycle

$$\sum_{|\sigma|=j} \hat{a}_\sigma^p x_\sigma^{p-1} e_\sigma \in Z_j\left(x; R/\mathrm{I}^{[p]}\right),\ x_\sigma^{p-1} = \left(x_{i_1}\dots x_{i_j}\right)^{p-1}.$$

Using the recurrence we see that $[y^{p^r}],\ y \in Z_j(x; R/\mathrm{I}),\ y^{p^r} = \sum_{|\sigma|=j} \hat{a}_\sigma^{p^r} x_\sigma^{p^r-1} e_\sigma$, form a basis in $H_j\left(x; R/\mathrm{I}^{[p^r]}\right)$. Since y is a cycle we have $\partial_j y = 0$, that is

$$\sum_{|\tau|=j-1} \left(\sum_{\substack{\sigma \supset \tau\\ |\sigma|=j}} (-1)^{\alpha(\sigma,\sigma\setminus\tau)} x_{\sigma\setminus\tau} \hat{a}_\sigma \right) e_\tau = 0,$$

where $\alpha(\sigma, q) = |\{\lambda \in \sigma | \lambda < q\}|$. Thus

2) $$\omega_\tau := \sum_{\substack{\sigma \supset \tau\\ |\sigma|=j}} (-1)^{\alpha(\sigma,\sigma\setminus\tau)} x_{\sigma\setminus\tau} a_\sigma \in \mathrm{I}.$$

It follows

$$\delta_j([y^{p^r}]) = [\partial_j(y^{p^r})] = \left[\sum_{|\tau|=j-1} \left(\sum_{\substack{\sigma\supset\tau\\ |\sigma|=j}} (-1)^{\alpha(\sigma,\sigma\setminus\tau)} x_{\sigma\setminus\tau}\ \bar{a}_\sigma^{p^r} x_\sigma^{p^r-1} \right) e_\tau \right]$$

$$= \left[\sum_{|\tau|=j-1} \bar{\omega}_\tau^{p^r} x_\tau^{p^r-1} e_\tau \right] = 0$$

("$-$" denotes the class modulo $J\mathrm{I}^{[p^r]}$), because by 2) and the assumption we have $\omega_\tau^{p^r} x_\tau^{p^r-1} \in J\mathrm{I}^{[p^r]},\ \tau = (i_1, \dots, i_{j-1})$,

$i_1 < \dots < i_{j-1} \le m,\ j \ge 2$ (here is necessary $j \ge 2$ because for $j = 1$, $x_\tau = 1 \notin J$!). Thus $\delta_j = 0$ for $j \ge 2$, that gives the following exact sequence from 1):

3) $0 \to H_j(x; \mathrm{I}^{[p^r]}/J\mathrm{I}^{[p^r]}) \xrightarrow{\rho_j} H_j(x; R/J\mathrm{I}^{[p^r]}) \to H_j(x; R/\mathrm{I}^{[p^r]}) \to 0,\ j \ge 2.$

Now we claim that

4) $\quad \mathrm{I}^{[p^r]}/J\mathrm{I}^{[p^r]} \cong F \otimes_R R/J.$

Let $R^{q_1} \overset{(c_{ij})}{\to} R^{q_0} \to \mathrm{I} \to 0,\ \ q_0 = |G(\mathrm{I})|$, be a minimal presentation of I, where the matrix (c_{ij}) has all entries in $(x_1, ..., x_m)$. Since the Frobenius homomorphism of R is a flat $k-$ endomorphism ($k = \mathbb{Z}/p\mathbb{Z}!$) of R we see by [Ku] that

5) $\quad R^{q_1} \overset{\left(c_{ij}^{p^r}\right)}{\to} R^{q_0} \to \mathrm{I}^{[p^r]} \to 0$

is exact. But $c_{ij}^{p^r} \in \left(x_1^{p^r}, ..., x_m^{p^r}\right) \subset J$ by hypothesis. Thus tensorizing 5) with $R/J \otimes_R -$ we get our claim. Moreover the isomorphism is given by $\varepsilon_u \to u^{p^r}$ and together with the canonical map ρ_j induce the wanted γ. ■

Theorem 2.5. *Let* $I = \prod_{t=1}^{s} I_t^{[p^{r_t}]}$ *be an acceptable* $p-$ *stable ideal as in 2.1. Then a basis in* $H_j(x; R/I)$, $j \geq 2$ *is given by the homology classes of the cycles of* $Z_j(x; R/I)$

$$(\star) \qquad \prod_{t>q} u_t^{p^{r_t}} f(\sigma; u_q)^{p^{r_q}},$$

where $1 \leq q \leq s, \quad |\sigma| = j-1, \quad \sigma = (i_1, ..., i_{j-1}), \quad i_1 < ... < i_{j-1} < m(u_q),\ f(\sigma; u_q) = (u_q/x_{m(u_q)})e_\sigma \wedge e_{m(u_q)},\ u_t \in G(\mathrm{I}_t).$

Proof. Apply induction on s. If $s = 1,\ r_1 = 0$ then $\mathrm{I} = \mathrm{I}_1$ is stable and by [AH_1, Proposition 2.0.1.], $H_j(x; R/\mathrm{I})$, $j \geq 2$, has a basis given by $f(\sigma; u_q)$. If $s = 1,\ r_1 > 0$ then $f(\sigma, u_q)^{p^{r_1}}$ form a basis as we have seen (apply [AH_2, 1.5]).

Suppose $s > 1$. By [AH_2, 1.5] we may assume $r_1 = 0$ as before. Apply Lemma 2.4. for $J = \mathrm{I}_1$, $\tilde{\mathrm{I}} = \prod_{t=2}^{s} \mathrm{I}_t^{p^{r_t - r_2}}$. We have the following exact sequence

$$0 \to \bigoplus_{v \in G(\tilde{\mathrm{I}})} H_j(x; R/J)\varepsilon_v \overset{\gamma}{\to} H_j(x; R/J\tilde{\mathrm{I}}^{[p^{r_2}]}) \to H_j(x; R/\tilde{\mathrm{I}}^{[p^{r_2}]}) \to 0.$$

By induction hypothesis the cycles of type $(\star)$ with $q > 1$ induce a basis in $H_j(x; R/\tilde{\mathrm{I}}^{[p^{r_2}]})$ and it remains to show that the cycles of type $(\star)$ with $q = 1$ induce a basis in $\mathrm{Im}\gamma$. Since $\{f(\sigma; u_1)\}_{\sigma, u_1}$ induce a basis in $G(J)$ as in case $s = 1$ and $G(\tilde{\mathrm{I}}) = \{\prod_{t=2}^{s} u_t^{p^{r_t - r_2}} | u_t \in G(\mathrm{I}_t)\}$, by Proposition 2.3, we see that indeed the cycles of type $(\star)$ with $q = 1$ have the form $\{v^{p^{r_2}} f(\sigma; u_1)\}_{v \in G(\tilde{\mathrm{I}}), \sigma, u_1}$ which corresponds bijectively by γ with the basis, $\{f(\sigma; u_1)\varepsilon_v\}_{v \in G(\tilde{\mathrm{I}}), \sigma, u_1}$, of

$\bigoplus_{v\in G(\bar{\mathrm{I}})} H_j(x;R/J)\varepsilon_v$. ■

The following Corollary is an extension of [AH_2, Theorem 1.6.].

Corollary 2.6. *Let* $(q_t)_{1\le t\le s}$, $(r_t)_{1\le t\le s}$, $(a_t)_{1\le t\le s}$ *be some integers such that* $1 \le q_s \le ... \le q_1 \le n$, $0 \le r_1 < ... < r_s$, $0 \le a_t < p^{r_{t+1}-r_t}$ *for* $1 \le t < s$, $a_s \ge 0$, *and let* $I = \prod_{t=1}^{s} ((x_1,...,x_{q_t})^{a_t})^{[p^{r_t}]}$. *Then a basis in* $H_j(x;R/I)$, $j \ge 2$ *is given by the homology classes of the cycles of* $Z_j(x;R/I)$

$$\prod_{t>c} u_t^{p^{r_t}} f(\sigma;u_c)^{p^{r_c}},$$

where $1 \le c \le s$, $\sigma = (i_1,...,i_{j-1})$, $1 \le i_1 < ... < i_{j-1} < m(u_c)$,

$u_t \in \{x_1,...,x_{q_t}\}^{a_t}$, $f(\sigma;u_c) = (u_c/x_{m(u_c)})e_\sigma \wedge e_{m(u_c)}$.

Corollary 2.7. *Let* $(q_t)_{1\le t\le s}$, $(a_t)_{1\le t\le s}$ *be some integers such that* $1 \le q_s \le q_{s-1} \le ... \le q_1 \le n$, $0 \le a_t < p$ *and* I *the smallest* $p-$ *stable ideal containing* $\prod_{t=1}^{s} x_{q_t}^{p^{t-1}a_t}$. *Then a basis in* $H_j(x;R/I)$, $j \ge 2$ *is given by the homology classes of the cycles of* $Z_j(x;R/I)$

$$\prod_{t>c} u_t^{p^{t-1}} f(\sigma;\ u_c)^{p^{c-1}},$$

where $1 \le c \le s$, $\sigma = (i_1,...,i_{j-1})$, $1 \le i_1 < ... < i_{j-1} < m(u_c)$,

$u_t \in \{x_1,...,x_{q_t}\}^{a_t}$, $f(\sigma,u_c) = (u_c/x_{m(u_c)})e_\sigma \wedge e_{m(u_c)}$.

For the proof note that the smallest $p-$stable ideal containing $x_{q_t}^{a_t}$ is $(x_1,...,x_{q_t})^{a_t}$ and so $\mathrm{I} = \prod_{t=1}^{s} ((x_1,...,x_{q_t})^{a_t})^{[p^{t-1}]}$. Now we may apply 2.6.

Remark 2.8. If in Corollary 2.7, $q_1 = ... = q_s = n$ and $a = \sum_{i=1}^{s} a_i p^{i-1}$ then we obtain a basis in $H_j(x;R/\mathrm{I})$, where I is the smallest $p-$ stable ideal containing x_n^a. Clearly I is a $p-$ Borel ideal and so Corollary 2.7 coincides in this case with [AH_2, Theorem 1.6].

3. Applications of Theorem 2.5

In this section $\mathrm{I} = \prod_{t=1}^{s} \mathrm{I}_t^{[p^{r_t}]}$ is an acceptable $p-$ stable ideal as in 2.1. In particular, I_t is stable and contains $\left(x_1,...,x_{m_{t+1}}\right)^{(p^{r_{t+1}-r_t}-1)}$,

$m_t = \max_{u_t \in G(\mathrm{I}_t)} m(u_t)$. Let $\beta_j(R/\mathrm{I}),\ j \in \mathbb{N},$ be the j^{-th} Betti number of R/I over R, that is

$$\beta_j(R/\mathrm{I}) = dim_k Tor_j^R(k, R/\mathrm{I}) = dim_k H_j(x; R/\mathrm{I}).$$

Theorem 3.1.

$$\beta_j(R/I) = \sum_{q\,=\,1}^{s} \sum_{u_q \in G(I_q)} \left(\prod_{t>q} |G(I_t)| \right) \begin{pmatrix} m(u_q)-1 \\ j-1 \end{pmatrix},\ j \geq 2.$$

For the proof note that if we fix $u_q \in G(\mathrm{I}_q),\ 1 \leq q \leq s$ there exist exactly $\prod_{t>q} |G(\mathrm{I}_t)| \begin{pmatrix} m(u_q)-1 \\ j-1 \end{pmatrix}$ cycles of type $(\star)$ from Theorem 2.5.

Corollary 3.2. $pd_R(R/\mathrm{I}) = m_1 = pd_R R/\mathrm{I}_1,$

$depth_R(R/\mathrm{I}) = n - m_1 = depth_R R/\mathrm{I}_1.$

For the proof note that $\beta_j(R/\mathrm{I}) = 0$ if and only if $j > m_1$ since $m_1 \geq m_2 \geq ... \geq m_s$.

Theorem 3.3. *Let* $d_t = \max_{u_t \in G(I_t)} \deg u_t$ *and* $reg(I)$ *be the Castelnuovo regularity of* I. *Then*

$$reg\ I\ =\ \max_{1\leq q\leq s} \{ \sum_{t>q}^{s} p^{r_t} d_t + \max_{u_q \in G(I_q)} [\, p^{r_q} \deg u_q + (p^{r_q}-1)(m(u_q)-1)] \}$$

Proof. Let a_j be the largest integer for which $H_j(x; R/\mathrm{I})_{a_j} \neq 0$.

Thus a_j is the biggest degree possible in the cycles of type $(\star)$ from Theorem 2.5. Note that $\deg\ f(\sigma; u_q)^{p^{r_q}} = \deg \left(u_q / x_{m(u_q)} e_\sigma \wedge e_{m(u_q)} \right)^{p^{r_q}} =$

$= p^{r_q}(\deg u_q - 1) + \deg \left(x_\sigma^{p^{r_q}-1} x_{m(u_q)}^{p^{r_q}-1} e_\sigma \wedge e_{m(u_q)} \right) =$

$= p^{r_q}(\deg u_q - 1) + (p^{r_q}-1)j + j,\ \ j \leq m(u_q)$, and so

$$a_j = \max_{1\leq q\leq s} \{ \sum_{t>q} p^{r_t} d_t + p^{r_q} \max_{\substack{u_q \in G(\mathrm{I}_q) \\ j \leq m(u_q)}} (\deg u_q - 1) + (p^{r_q}-1)j + j \}.$$

Since $reg\ \mathrm{I} = \max_j \{a_j - j + 1\}$ we are done. ■

Remark 3.4. i) Trying to simplify the formula of $reg\ \mathrm{I}$, one might believe that $reg\ \mathrm{I}$ is reached on the cycle of type $(\star)$ with $q = 1$. This is not the case, let $s = 2,\ r_1 = 0,\ r_2 = 1,\ \mathrm{I}_1 = \mathrm{I}_2 = (x_1, ..., x_n)$. Then the degree of the cycles of type $(\star)$ for $q = 1$ is $p + 1$ and the degree of $f((1, ..., n-1); x_n)^p$ is

$p+(p-1)(n-1)$. Thus we have to keep first "max" in the formula of reg I.

ii) Also $\max\limits_{u_q \in G(\mathrm{I}_q)} [p^{r_q} \deg u_q + (p^{r_q}-1)(m(u_q)-1)]$ in the formula of reg I is not necessarily $p^{r_q} d_q + (p^{r_q}-1)(m_q-1)$.

Indeed, if $\mathrm{I}_q = (x_1^2, x_1x_2, x_1x_3, x_2^3)$, $r_q = 1$, then the maximum is reached for $u_q = x_2^3$, namely $4p-1$.

Corollary 3.4. *Suppose* $m_s = 1$ *and let* $T = \prod\limits_{t=1}^{s-1} I_t^{[p^{r_t}]}$.
Then $reg\ I = p^{r_s} d_s + reg\ T$.

Proof. Denote

$$D_q(\mathrm{I}) = \sum_{t>q}^{s} p^{r_t} d_t + \max_{u_q \in G(\mathrm{I}_q)} [p^{r_q} \deg u_q + (p^{r_q}-1)(m(u_q)-1)],$$

$1 \le q \le s$. Since $m_s = 1$ we have $D_s(\mathrm{I}) = p^{r_s} d_s < D_q(\mathrm{I})$ for all $q < s$. Thus $reg\ \mathrm{I} = \max\limits_{1 \le q < s} D_q(\mathrm{I}) = p^{r_s} d_s + \max\limits_{1 \le q \le s-1} D_q(T) = p^{r_s} d_s + regT$. ■

Remark 3.5. The above Corollary is inspired by a result from [Pa] :
" Let $v = \prod\limits_{i=1}^{n} x_i^{\mu_i}$ and $J(v)$ the $p-$ Borel ideal generated by v, that is the smallest $p-$ Borel ideal containing v. Then $regJ(v) = \mu_1 + regJ(\prod\limits_{i=2}^{n} x_i^{\mu_i})$". Using this Corollary we may assume from now on $m_s \ge 2$.

Corollary 3.6. *Let* $(m_t)_{1 \le t \le s}$, $(r_t)_{1 \le t \le s}$, $(a_t)_{1 \le t \le s}$ *be some integers such that* $2 \le m_s \le ... \le m_1 \le n$, $0 \le r_1 < ... < r_s$, $0 \le a_t < p^{r_{t+1} - r_t}$ *for* $1 \le t < s$, $a_s \ge 0$, *and let* $I = \prod\limits_{t=1}^{s} ((x_1, ..., x_{m_t})^{a_t})^{[p^{r_t}]}$.
Then $reg\ I = \max\limits_{1 \le q \le s} D_q(I)$, *where* $D_q(I) = \sum\limits_{t=q+1}^{s} p^{r_t} a_t + (p^{r_q}-1)(m_q-1)$.

Remark 3.7. Suppose $r_t = p^t$, $1 \le t \le s$. Then $a_t < p$ and I is the smallest $p-$ stable ideal (also $p-$ Borel), which contains the monomial $v = \prod\limits_{t=1}^{s} x_{m_t}^{p^t a_t}$ and $(D_q(\mathrm{I}))_q$ coincide with the integers (D_{m_t}) associated to v by Pardue (see[Pa], or [AH_2,Conjecture 3.1.]. Corollary 3.6 says in particular that the Pardue Conjecture holds for monomials of type $v(m_s \le ... \le m_1!)$. Moreover, if $m_1 = ... = m_s$ then I is a $p-$ Borel principal Cohen - Macaulay ideal (see [Pa]) for which the Pardue conjecture follows also by [AH_2]. Hoping in an extension

of the Pardue Conjecture for $p-$stable ideals, we may ask the following:

Question 3.8. Let $\mathrm{I} = \prod_{t=1}^{s} \mathrm{I}_t^{[p^{r_t}]}$ be a product of $p-$powers of stable ideals, which is not necessarily acceptable. Then is regI exactly $\max_{1\le q\le s} D_q(I)$, where

$D_q(\mathrm{I}) = \sum_{t>q}^{s} p^{r_t} d_t + \max_{u_q\in G(\mathrm{I}_q)} \left[p^{r_q} \deg u_q + (p^{r_q}-1)(m(u_q)-1)\right],$
$1 \le q \le s$, where $d_t = \max_{u_t\in G(I_t)} \deg u_t$?

Now, let $P(R/\mathrm{I}, T)$ be the Poincaré series of R/I.

Theorem 3.9.

$$P(R/I,T) = \tfrac{1}{(1-T)^n} + \tfrac{1}{(1-T)^n} \sum_{q=2}^{s} \prod_{t\ge q} \Big(\sum_{u_t\in G(I_t)} T^{p^{r_t}\deg u_t}\Big) -$$
$$- \sum_{q=1}^{s} \sum_{u_q\in G(I_q)} \prod_{t>q} \Big(\sum_{u_t\in G(I_t)} T^{p^{r_t}\deg u_t}\Big) \tfrac{T^{p^{r_q}\deg u_q}}{(1-T)^{n-p^{r_q}(m(u_q)-1)}}$$

Proof. Let $L_* : \ldots \to L_j \multimap L_{j-1} \multimap \ldots \multimap L_1 \multimap L_0 \to R/\mathrm{I} \multimap 0$

be the minimal free resolution of R/I over R. By exactness of the resolution it results:

$$P(R/\mathrm{I},T) = \sum_{j\ge 0} (-1)^j P(L_j,T).$$

Clearly, $L_0 \simeq R$, so $P(L_0,T) = \frac{1}{(1-T)^n}$, and L_1 has the basis

$G(\mathrm{I}) = \{ \prod_{q=1}^{s} u_q^{p^{r_q}} | u_q \in G(\mathrm{I}_q),\ 1\le q\le s\}$, thus

$P(L_1,T) = \frac{1}{(1-T)^n} \prod_{q=1}^{s} \Big(\sum_{u_q\in G(I_q)} T^{p^{r_q}\deg u_q}\Big)$, since

$\deg \prod_{q=1}^{s} u_q^{p^{r_q}} = \sum_{q=1}^{s} p^{r_q} \deg u_q.$

For all $j \ge 2$, by Theorem 2.5, L_j has the basis

$\{ \prod_{t>q} u_t^{p^{r_t}} f(\sigma; u_q)^{p^{r_q}} | 1\le q\le s,\ |\sigma| = j-1,\ \ \sigma = (i_1,\ldots,i_{j-1}),$
$i_{j-1} < m(u_q),\ u_t \in G(\mathrm{I}_t)\}.$

Since $\deg u_t^{p^{r_t}} f(\sigma;u_q)^{p^{r_q}} = p^{r_t}\deg u_t + p^{r_q}\deg u_q + (j-1)p^{r_q}$, it follows:

$$P(L_j,T) = \sum_{q=1}^{s} \sum_{u_q\in G(I_q)} \tfrac{1}{(1-T)^n} \begin{pmatrix} m(u_q)-1 \\ j-1 \end{pmatrix} T^{p^{r_q}(j-1)+p^{r_q}\deg u_q} \times$$
$$\times \prod_{t>q} \Big(\sum_{u_t\in G(\mathrm{I}_t)} T^{p^{r_t}\deg u_t}\Big).$$

Then we obtain:

$$P(R/I,T)=\tfrac{1}{(1-T)^n}-\tfrac{1}{(1-T)^n}\prod_{q=1}^{s}\Big(\sum_{u_q\in G(I_q)}T^{p^{r_q}\deg u_q}\Big)+$$

$$+\sum_{q=1}^{s}\sum_{u_q\in G(I_q)}T^{p^{r_q}\deg u_q}\prod_{t>q}\Big(\sum_{u_t\in G(\mathrm{I}_t)}T^{p^{r_t}\deg u_t}\Big)\times$$

$$\times\sum_{j\geq 2}(-1)^j\begin{pmatrix} m(u_q)-1 \\ j-1\end{pmatrix}\tfrac{T^{p^{r_q}(j-1)}}{(1-T)^n}.$$

But $\sum\limits_{j\geq 2}(-1)^j\begin{pmatrix} m(u_q)-1 \\ j-1\end{pmatrix}\tfrac{T^{p^{r_q}(j-1)}}{(1-T)^n}=\tfrac{1}{(1-T)^n}-\tfrac{1}{(1-T)^{n-p^{r_q}(m(u_q)-1)}}$.
Replacing this sum we obtain the desired formula. ■

Corollary 3.10. *In the hypothesis of Corollary 2.6,*

$$P\,(R/I,T)=\tfrac{1}{(1-T)^n}+\tfrac{1}{(1-T)^n}\sum_{c=2}^{s}\prod_{t\geq c}\begin{pmatrix} a_t+q_t-1 \\ q_t-1\end{pmatrix}T^{a_tp^{r_t}}-$$

$$-\sum_{c=1}^{s}\prod_{t>c}\begin{pmatrix} a_t+q_t-1 \\ q_t-1\end{pmatrix}T^{a_tp^{r_t}}\cdot\tfrac{T^{a_cp^{r_c}}}{(1-T)^{n+p^{r_c}}}\sum_{l=0}^{q_c-1}(1-T)^{(l+1)p^{r_c}}\begin{pmatrix} a_c+l \\ l\end{pmatrix}.$$

Proof. In the hypothesis of Corollary 2.6, $\deg u_t=a_t$ for all $u_t\in G(\mathrm{I}_t)$ and $|G(\mathrm{I}_t)|=\begin{pmatrix} a_t+q_t-1 \\ q_t-1\end{pmatrix}$. ■

Corollary 3.11. *In the hypothesis of Corollary 2.7,*

$$P\,(R/I,T)=\tfrac{1}{(1-T)^n}+\tfrac{1}{(1-T)^n}\sum_{c=2}^{s}\prod_{t\geq c}\begin{pmatrix} a_t+q_t-1 \\ q_t-1\end{pmatrix}T^{p^{t-1}a_t}-$$

$$-\sum_{c=1}^{s}\prod_{t>c}\begin{pmatrix} a_t+q_t-1 \\ q_t-1\end{pmatrix}T^{p^{t-1}a_t}\tfrac{T^{p^{c-1}a_c}}{(1-T)^{n+p^{c-1}}}\sum_{l=0}^{q_c-1}(1-T)^{(l+1)p^{c-1}}\begin{pmatrix} a_c+l \\ l\end{pmatrix}.$$

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