A THIRD LOOK AT WEIGHT DIAGRAMS

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Abstract. In this paper, which is a sequel of [PSV], we develop a completely elementary approach to calculations in Chevalley groups $G = G(\Phi, R)$ of types $\Phi = \mathrm{E}_6$ and E_7 over a commutative ring R using only the weight diagrams (alias, crystal graphs) of their minimal modules. After an elementary construction of a crystal base we explicitly describe action of root subgroups and of the extended Weyl group, multilinear invariants, equations defining the orbit of the highest weight vector and Freudenthal tranvections. As an illustration of our methods we give the first complete a priori proof of the central step in the method of decomposition of unipotents (see [VPS], [V2], [VP], [SV], [VPe]) for these cases. Namely we prove that any singular column v is stabilised by a non-trivial Freudenthal transvection of a certain type ('fake root unipotent') and that there are in fact enough of those to generate the whole elementary group of type Φ over R as the v ranges over the columns of a matrix $g \in G$. It is known that this result immediately implies the main structure theorems for G (description of normal subgroups, standard commutator formulae and the like). The results of the present paper provide complete proofs for the algebraic part of [V2] in the cases of E_6 and E_7, complete proofs for the geometric part of the above paper are given in [V7].

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Introduction

Let Φ be a reduced irreducible root system, $W = W(\Phi)$ be the corresponding Weyl group. We fix an order on Φ and denote by $\Pi = \{\alpha_1, \dots, \alpha_l\}$, Φ^+ and Φ^- the corresponding sets of fundamental, positive and negative roots respectively. Then the Weyl group is generated by the set S of fundamental reflections $s_1 = w_{\alpha_1}, \dots, s_l = w_{\alpha_l}$. As usual $P(\Phi)$ denotes the lattice of integral weights of Φ and $P(\Phi)_{++}$ is the cone of dominant integral weights. Recall that any weight $\omega \in P(\Phi)_{++}$ is a non-negative integral linear combination of the fundamental weights $\overline{\omega}_1, \dots, \overline{\omega}_l$.

Further, let $G = G(\Phi, R)$ be the *simply connected* Chevalley group of type Φ over a commutative ring R with 1. One can find all the relevant notions in [A], [B], [C], [Hé], [H], [M], [St1], [S] (see [V2], [V4], [VP] for many additional references). Fix a dominant weight $\omega \in P(\Phi)_{++}$ and let $V = V(\omega)$ be the Weyl module of G with the highest weight ω. The corresponding representation $G \longrightarrow \mathrm{GL}(V)$ will be denoted by π. By $\Lambda(\pi)$ we denote the set of weights of the representation π *with multiplicities*. An *admissible base* v^λ, $\lambda \in \Lambda(\pi)$, of V consists of weight vectors and has the property that the action of the root unipotents $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$, is described by matrices whose entries are polynomials in ξ with integral coefficients.

The *weight diagram* of π is a marked graph, whose nodes correspond to the weights of π (usually with multiplicities) and two nodes λ and μ are joined by a bond marked i if their difference $\lambda - \mu = \alpha_i$ is the i-th fundamental root. Eventually, one should indicate the positive direction as well. Sometimes this is done by drawing arrows instead of lines. We draw the diagrams in such a way that a larger weight stands to the left of and/or higher than a smaller one, with the landscape orientation usually being primary. Another convention is that we omit at least one of the two equal labels at the opposite sides of a parallelogramm.

Below we reproduce two typical weight diagrams, the one of the representation of the Chevalley group $G(\mathrm{E}_6, R)$ with the highest weight $\overline{\omega}_1$,

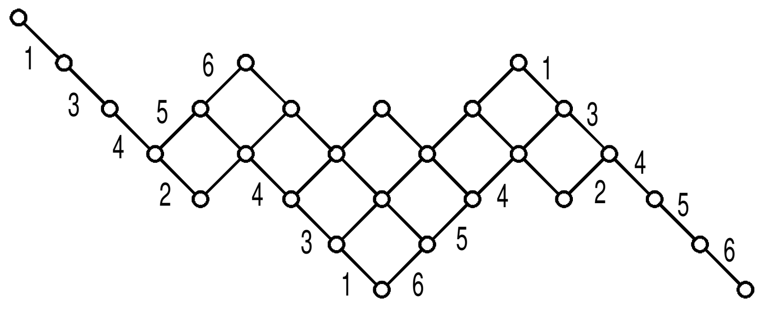

Figure 1. $(\mathrm{E}_6, \overline{\omega}_1)$

and the one of the representation of the Chevalley group $G(\mathrm{E}_7, R)$ with the highest weight $\overline{\omega}_7$.

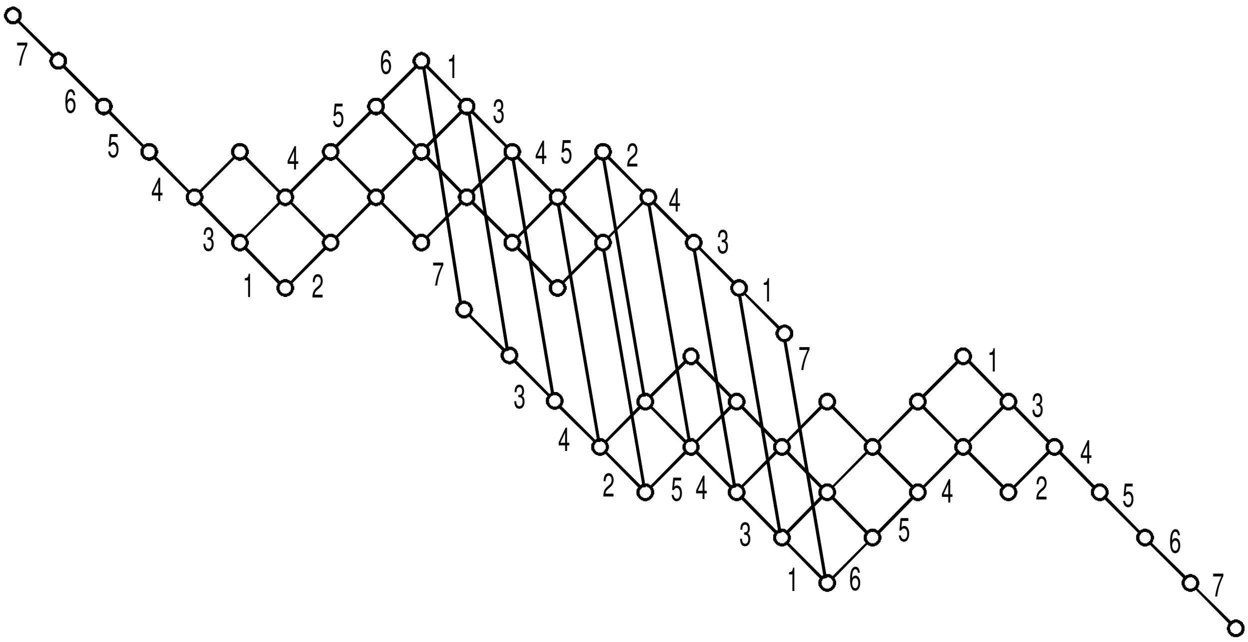

FIGURE 2. $(\mathrm{E}_7, \overline{\varpi}_7)$

Here, as always, the numbering of the fundamental roots follows that of [B1]. The weight diagrams are especially useful when – as in the above cases – all weights (apart, probably, from the zero weight) have multiplicity one, in particular, for basic representations [M]. Recall, that a representation is called *basic*, if its non-zero weights form one orbit with respect to the action of the Weyl group. All basic representations, apart from a unique representation for each type, are *microweight* [B2], i.e. they do not have zero weight, so that the weights in fact form a single Weyl orbit, as in the above cases $(\mathrm{E}_6, \overline{\varpi}_1)$ and $(\mathrm{E}_7, \overline{\varpi}_7)$. Diagrams of all basic and adjoint representations are collected in [PSV].

These diagrams arise in a number of contexts, ranging from representation theory of semisimple Lie algebras and algebraic groups to invariant theory, algebraic geometry, algebraic K-theory, differential geometry and combinatorics. One can find detailed discussion of the diagrams and some of their uses, as well as many additional references, in [Ho], [PR], [PSV], [Sch], [V2], [V5], [VP]. Now, *a posteriori*, weight diagrams are a special case of *crystal graphs* of M.Kashiwara[1] [K1] – [K3]. That the weight diagrams as described here and depicted in [PSV] do indeed coincide with the crystal graphs is obvious for microweights (the microweight representations do not melt – any temperature is like temperature zero; see also [Li2] for a more general result describing crystal graphs for almost all fundamental

[1]Which are intimately related to to the canonical bases of Lusztig [L1] – [L3]. An explicit description of crystal bases is ultimately provided by Littelmann's path theory [Li1], [Li3]. One can find a very accessible introduction to this circle of ideas in [J].

weights). For the adjoint representations it is checked in [Ma]. For the classical cases it follows also from the explicit construction of crystal graphs in [KN].

The first appearance of the weight diagrams in print, which we could trace[2], is [CIK]. There the weight diagrams were considered purely combinatorially, as the adjacency diagrams for the cosets of the Weyl group $W = W(\Phi)$ modulo a parabolic subgroup W_J, $J \subseteq \Pi$. For example the nodes of the above diagram may be interpreted as the cosets $W/W_J = W(\mathrm{E}_6)/W(\mathrm{D}_5)$. Two cosets w_1W_J and w_2W_J are joined by a bond marked i if $w_2W_J = s_iw_1W_J$ for the i-th fundamental reflection s_i. In the case of microweight representations these diagrams are – up to labels – the Hasse diagrams of the (reversed) induced Bruhat order. This is the most common interpretation of the diagrams, see, for example, [BE], [CC2], [Hi1], [Hi2], [PR], [P1], [P2], [Sch], [St], [V2], [V5] and references there.

Weight diagrams are also sometimes used as a shorthand form of the *weight graphs*. This graph has the same nodes as the corresponding weight diagram, whereas its bonds correspond to all positive roots, rather than just the fundamental ones. In other words, two weights λ, μ are joined by a bond marked $\alpha \in \Phi^+$ if $\lambda - \mu = \alpha$. The weight graphs of types $(\mathrm{E}_6, \overline{\omega}_1)$ and $(\mathrm{E}_7, \overline{\omega}_7)$ have special names, they are called the *Schläfli graph* and the *Gosset graph* respectively, see [BCN] and references there. Weight graphs have very strong regularity properties and have been extensively studied in combinatorics, especially in the context of finite geometries and sphere packings. Historically the graphs of types $(\mathrm{E}_6, \overline{\omega}_1)$ and $(\mathrm{E}_7, \overline{\omega}_7)$ first appeared in the theory of algebraic surfaces. The Schläfli graph describes the configuration of the 27 lines on the surface obtained from the projective plane $\mathbb{P}^2$ by blowing up 6 points, whereas the Gosset graph describes the configuration of the 56 nonsingular rational curves with negative self-intersection on the surface obtained from $\mathbb{P}^2$ by blowing up 7 points, see [Hr], [Mn] for details and [V5] for further references and an explicit identification of the curves with the nodes of Figures 1 and 2.

The *second look* at the weight diagrams was started by the paper of M.R.Stein [St2]. The usual techniques based on the calculations with canonical forms (Bruhat decomposition, etc.) does not work for groups over rings. This is why one has to find a substitute for matrix calculations which works also for exceptional groups. H.Matsumoto [M] developed techniques which allow to calculate with one column or one row of a matrix representing an element of a Chevalley group G in a basic representation (V, π). In particular, he has shown that one may normalize an admissible base v^λ of V in such a way that the action of $x_\alpha(\xi)$ is described by

[2]E.B.Vinberg told us that weight diagrams were used by E.B.Dynkin and his students in Moscow in mid-fifties. To describe their shape Dynkin even coined a special word *veretenoobraznost'*, meaning approximatively "the property of having form similar to that of a spindle" – what would nowadays be called 'rank symmetry' and 'rank unimodality'. But they never made their way to the published works of Dynkin's school, as far as I can see.

very nice formulae. In the case of a microweight representation all unipotents are quadratic and the formulae become especially simple:

$$x_\alpha(\xi)v^\lambda = \begin{cases} v^\lambda \pm \xi v^{\lambda+\alpha}, & \text{if } \lambda+\alpha \in \Lambda(\pi), \\ v^\lambda, & \text{otherwise} \end{cases}$$

(which is a special case of the formulae, expressing the action in a canonical base, see the footnote in § 2). In the presence of zero weight the formulae are slightly more complicated.

In [St2] the weight diagrams were used to visualize these calculations. Namely, we may conceive a vector $a = (a_\lambda) \in V$ as a marked graph as follows: put a_λ to the node of the diagram corresponding to λ. Then the diagram shows how $x_\alpha(\xi)$ acts on the components of a. A positive/negative fundamental root unipotent $x_{\pm\alpha_i}(\xi)$ acts along the bonds marked i in the positive/negative direction. For an arbitrary root α the action of $x_\alpha(\xi)$ is described by directed paths with the labels, corresponding to the expansion of α into a linear combination of the fundamental roots, see [St2], [V2], [PSV], [VP].

Now the matrix of an element $\pi(g)$, $g \in G$, with respect to the base v^λ, $\lambda \in \Lambda(\Phi)$, is defined in the usual way. Its columns and rows are indexed by the weights $\lambda, \mu \in \Lambda(\Phi)$ and the μ-th column consists of the coefficients in the expansion of $\pi(g)v^\mu$ with respect to v^λ. Thus the columns of the matrix may be interpreted as vectors from V. By the same token, the rows of the matrix are vectors from the dual module V^*. It is essential that the columns and the rows of this matrix are not linearly ordered, but *partially* ordered by the corresponding weight diagram or its dual, respectively. Using the weight diagrams one can fairly efficiently calculate with such matrices.

However in [M], [St2] and subsequent publication almost all calculations were performed *up to signs*. As M.R.Stein himself puts it: “It should be noted that in describing the elementary transformations no attempt to fix signs has been made” [St2]. In the present paper we make yet another step. Namely, we show that in fact a weight diagram encodes also the information about the signs. Consider the action constants $c_{\lambda\alpha}$ defined by $x_\alpha(1)v^\lambda = v^\lambda + c_{\lambda\alpha}v^{\lambda+\alpha}$. We show that the signs of $c_{\lambda\alpha}$'s are easily determined by looking at the weight diagram.

Let us illustrate this in the above example of $(\mathrm{E}_6, \overline{\omega}_1)$, see § 2 for the precise statements. It turns out, that one can normalize an admissible base in such a way that for a fundamental or a negative fundamental root α all $c_{\lambda\alpha}$ are $+1$ (Theorem 1). Now contemplating Figure 1 one notices that, for instance, the six paths with labels $\{1,3\}$ corresponding to the root $\alpha_1+\alpha_3$ are of two different kinds: when read in the positive direction three of them have labels $(3,1)$, whereas three others have labels $(1,3)$. This means precisely that (for the standard choice of structure constants for E_6, see [C], [GS], [V3] and references there) three of the action constants $c_{\lambda,\alpha_1+\alpha_3}$ are $+1$, whereas the other three are -1. The same applies to all roots: one may compute the sign of the action constants from the order of labels in the paths corresponding to a given root (Theorem 2).

In other words, a weight diagram (Φ, ω) (together with the structure constants of the corresponding Lie algebra, or, what is the same, together with the weight diagram (Φ, ad) of the adjoint representation of the corresponding type) contains all information necessary to *completely* describe the action of G on the module with the highest weight ω. Of course, a weight diagram is much easier to memorize and to use, than a table of the action constants. With the help of the weight diagrams one gets an explicit control over the action of the root unipotents $x_\alpha(\xi)$ in the minimal representation of the exceptional groups. In fact, one can think of the above diagrams as a mnemotechnical device, which encodes in a compact form most of the information you might be willing to recover about the groups of type E_6 or E_7 if you pretend to calculate in them on a beach and have forgotten your Bourbaki at home.

In this paper we illustrate some of the possible applications of this idea. First of all, what we said above means that in fact a weight diagram describes the action of the *extended* Weyl group $\widetilde{W} = \widetilde{W}(\Phi)$ (also called the *Tits-Demazure group* [MPS]), not just of the Weyl group itself. This is extremely important, since the extended Weyl group controls signs in most of the calculations related to G. In all usual cases G has multilinear invariants on V which have very few $\widetilde{W}$-orbits of monomials (say, just one, two or three). This means that the weight diagrams allow to *explicitly* control the equations on the entries of matrices representing elements of G. After an explicit construction of the cubic form for E_6 and some remarks concerning E_7 we take a special case of this problem and explicitly determine the equations defining the orbit of the highest weight vector (Theorem 3).

In [V2, §§ 10–14] we had to find a non-trivial element of root type stabilizing a given vector $v \in V$. In the calculation reproduced there we had to quote explicit knowledge of the action constants as well as of the equations defining the orbit of the highest weight vector. Here we show that in fact this is not necessary, one can check that the unipotent elements constructed there stabilize a given (singular) vector simply by looking at the order of labels in certain paths and that these unipotents actually generate the whole elementary group as the vector ranges over the columns of a matrix $g \in G$ (Theorems 4 and 5). Thus the present paper may be regarded as an updated version of the algebraic half of [V2]. We do not try to include here complete proofs for the geometric part, since this would more than double the length of the paper.

To avoid some further technical complications related to the presence of zero weight and to present the ideas in their simplest form, in this paper we focus on the microweight representations, especially on those of types $(\mathrm{E}_6, \overline{\omega}_1)$ and $(\mathrm{E}_7, \overline{\omega}_7)$. In particular for types E_6 and E_7 the contents of the present paper suffices to supply complete proofs for what was left open in [VPS], [V2]. In [V7] I revise also the geometric part of [V2] giving a complete proof of Theorem 1 of [V2] for these cases. On the other hand, to analyse the case of E_8 one is compelled to work in the adjoint module and this is, probably, the correct approach for all types. Classical cases are described in detail in [V2] and especially in my joint works with A.V.Stepanov [SV] and E.Ya.Perelman [VPe], whereas [V6] treats the adjoint case for simply-laces systems. A systematic treatment with the emphasis on the adjoint and the short-root modules will appear in my forthcoming joint papers with E.B.Plotkin, "Structure of Chevalley groups over commutative rings", see [V2], [VP] for a description of the whole project (in fact the five sections of the present paper are toy versions of [VP] and subsequent papers).

It is assumed that the reader actually looks at Figures 1 and 2 – and, possibly, at other figures from [V2], [PSV] – while going through §§ 2, 3 and 5. But there is much more hidden in them, than what we could possibly mention here. For example, the number of all paths from the left end to the right end is exactly the multiplicity of the highest self-intersection of the cycle of codimension 1 in the Chow ring $A^*(G/P_i)$, where $i = 1$ or 6 for E_6 and 7 for E_7, see [Hi1], [P1]. Many other similar observations from various sources are collected in [PSV], [V5]. The pictures certainly deserve a further look ("... und weise die Gedanken oder Träume nicht ab, die dir dabei etwa kommen").

§ 1. Preliminaries

This section contains some background material, related to basic representations, structure constants of Lie algebras and realization of microweight representations in the unipotent radicals of parabolic subgroups.

1°. Basic representations. We keep notation from the introduction. In particular, π is a basic representation of a Chevalley group G on a Weyl module V, ω is the highest weight of this representation. Recall that by $\Lambda(\pi)$ we denote the set of weights of π *with multiplicities*. All non-zero weights have multiplicity one, and we denote the set of *non-zero* weights of π by $\Lambda^*(\pi)$. In turn, the multiplicity of the zero weight equals the number of the fundamental roots which are weights of this representation. In other words, $m = \mathrm{mult}(0) = |\Delta(\pi)|$, where $\Delta(\pi) = \Lambda^*(\pi) \cap \Pi$. Then $\Lambda(\pi)$ has m 'distinct' zero weights $\widehat{\alpha}$, one for each $\alpha \in \Delta(\pi)$.

The above definition of a basic representation is equivalent to the following one: if the difference $\alpha = \lambda - \mu$ of two weights $\lambda, \mu \in \Lambda^*(\pi)$ is a (fundamental) root, then $\mu = w_\alpha(\lambda)$, see [M]. This means that for any non-zero weight λ of a basic representation $w_\alpha(\lambda)$ takes one of the following three values $\lambda - \alpha$, λ, or $\lambda + \alpha$.

Lemma 1. *Let $\alpha \in \Phi$ and $\lambda, \lambda + \alpha \in \Lambda^*(\pi)$. Then $(\alpha, \alpha) = -2(\lambda, \alpha)$.*

Proof. By definition of a basic representation $w_\alpha(\lambda) = \lambda + \alpha$. It remains to compare this with the definition of a reflection with respect to a root.

Lemma 2. *Let $\alpha, \beta, \alpha + \beta \in \Phi$ be such that $\lambda, \lambda + \alpha, \lambda + \beta, \lambda + \alpha + \beta \in \Lambda(\pi)$. Then at least one of the following two assertion holds:*

(1) *one of the weights $\lambda, \lambda + \alpha, \lambda + \beta, \lambda + \alpha + \beta$ is zero,*

(2) $(\alpha, \beta) = 0$.

Proof. Suppose that the weights $\lambda, \lambda+\alpha, \lambda+\beta, \lambda+\alpha+\beta$ are all non-zero. Then by the preceding lemma

$$(\alpha+\beta,\alpha+\beta) = -2(\lambda,\alpha+\beta) = -2(\lambda,\alpha) - 2(\lambda,\beta) = (\alpha,\alpha) + (\beta,\beta),$$

which means exactly that $(\alpha,\beta)=0$.

In particular, if the weights $\lambda, \lambda+\alpha, \lambda+\beta, \lambda+\alpha+\beta$ are all non-zero, then α and β must be orthogonal short roots, whose sum is a long root. For simply-laced root systems this is impossible. This does not occur for the *fundamental* roots either: indeed, two orthogonal fundamental roots generate a subsystem, isomorphic to $\mathrm{A}_1 + \mathrm{A}_1$ and not to B_2.

Lemma 3. *Assume that Φ is simply laced, π is a microweight representation and $\alpha, \beta, \alpha+\beta \in \Phi$. If for a given weight $\lambda \in \Lambda(\pi)$ one has $\lambda+\alpha+\beta \in \Lambda(\pi)$, then $\lambda+\alpha \in \Lambda(\pi)$ or $\lambda+\beta \in \Lambda(\pi)$, but not both.*

Proof. Since Φ is simply laced and $\alpha+\beta \in \Phi$, the roots α and β cannot be orthogonal. Thus, by the preceding lemma, $\lambda+\alpha$ and $\lambda+\beta$ cannot be both weights of π. On the other hand, if neither of them is a weight, then $w_\alpha(\lambda) = \lambda-\alpha, \lambda$ and $w_\beta(\lambda) = \lambda-\beta, \lambda$. However by assumption $w_{\alpha+\beta}(\lambda) = \lambda+\alpha+\beta$ and now the equality $w_{\alpha+\beta} = w_\alpha w_\beta w_\alpha$ leads to a contradiction.

In the sequel we use two notions of distance between two weights $\lambda, \mu \in \Lambda(\pi)$. As usual we define the distance as the length of a shortest path between λ and μ. The distance between λ and μ in the weight graph is denoted by $d(\lambda,\mu)$. In other words, $d(\lambda,\mu)=1$ if $\lambda-\mu$ is a root, in this case the weights λ and μ are called *adjacent*. Similarly, $d(\lambda,\mu)=2$ if $\lambda-\mu$ is not a root, but there exists a weight ν such that both $\lambda-\nu$ and $\nu-\mu$ are roots. For reasons which become apparent in 3^o two weights at distance 2 are called *orthogonal*. In the case of $(\mathrm{E}_6, \overline{\omega}_1)$ one has $d(\lambda,\mu) \le 2$ so that any two distinct weights are either adjacent, or orthogonal. But in the case of $(\mathrm{E}_7, \overline{\omega}_7)$ for any λ there exists exactly one μ such that $d(\lambda,\mu)=3$. This μ will be denoted by λ^* and called the *opposite* of λ.

We will use also the distance between two weights in the weight diagram itself, which will be denoted by $h(\lambda,\mu)$. First, let λ and μ be two comparable weights, $\lambda \ge \mu$. Then $\lambda-\mu$ is a linear combination of the fundamental roots with nonnegative coefficients and $h(\lambda,\mu) = \mathrm{ht}(\lambda-\mu)$. In general $h(\lambda,\mu) = h(\nu,\lambda) + h(\nu,\mu)$, where ν is the least upper bound of λ and μ. For a microweight representation $h(\lambda,\mu)$ can be described also as the length of the shortest element $w \in W$ such that $w(\lambda)=\mu$ (that such a w exists is exactly what it means for a representation to be microweight). The maximal values of $h(\lambda,\mu)$ for $(\mathrm{E}_6, \overline{\omega}_1)$ and $(\mathrm{E}_7, \overline{\omega}_7)$ are 16 and 27 respectively.

2^o. Structure constants. Let L be the complex semisimple Lie algebra of type Φ with the Lie bracket $[\ \ ,\ \ ]$, H be a Cartan subalgebra of L. Consider the root space decomposition $L = H \oplus \sum L_\alpha$, $\alpha \in \Phi$, of L with respect to H. Choose a *Chevalley system* $e_\alpha \in L_\alpha \setminus \{0\}$, $\alpha \in \Phi$, see [B2], [Hè]. Recall that together with the fundamental coroots h_β, $\beta \in \Pi$, the elements e_α form a Chevalley base. In particular, for all roots $\alpha, \beta \in \Phi$ such that $\alpha + \beta \neq 0$ the structure constants $N_{\alpha\beta}$, where $[e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta}$, are integers.

The constants $N_{\alpha\beta}$ arise also as structure constants in the Chevalley commutator formula. For two elements $x, y \in G$ we denote by $[x, y]$ their commutator $xyx^{-1}y^{-1}$. Let now $\alpha, \beta \in \Phi$, $\alpha+\beta \neq 0$, $\xi, \eta \in R$. Then the Chevalley commutator formula asserts, that

$$[x_\alpha(\xi), x_\beta(\eta)] = \prod x_{i\alpha+j\beta}(N_{\alpha\beta ij}\xi^i\eta^j),$$

where the product on the right hand side is taken over all roots of the form $i\alpha + j\beta \in \Phi$, $i, j \in \mathbb{N}$, in any given order. One has $N_{\alpha\beta ij} = 0, \pm 1, \pm 2, \pm 3$ and the primes $p = 2, 3$ which actually appear in the formula are called *very bad* for G (or for Φ). The constants $N_{\alpha\beta ij}$ are called the *structure constants of the Chevalley group*, and $N_{\alpha\beta 11} = N_{\alpha\beta}$. For a simply laced root system the product on the right hand side has at most one factor, so that $N_{\alpha\beta}$ are the only structure constants.

For the simply laced case one has $N_{\alpha\beta} = 0, \pm 1$ and the only problem is to determine the *signs* of the structure constants. Below we describe the choice of signs which will be used throughout the paper. Let $\alpha \in \Phi^+$ be a positive root. Then $\alpha = \sum m_i\alpha_i$, $\alpha_i \in \Pi$, where m_i are non-negative integers. Their sum $\sum m_i$ is called the *height* of the root α and is denoted by $\mathrm{ht}(\alpha)$. The sequence $m_1m_2\dots m_l$ is called the *string Dynkin form* of α (as opposed to the usual Dynkin form, where m_i's are put into the corresponding vertices of the Dynkin diagram).

Let us choose the *height lexicographic* ordering of positive roots which is *regular* (i.e. a root of smaller height always precedes a root of larger height) and *lexicographic* at the roots of a given height. It is a total ordering of Φ^+ and we write $\alpha \prec \beta$ if α precedes β with respect to this ordering. By definition this means that either $\mathrm{ht}(\alpha) < \mathrm{ht}(\beta)$ or $\mathrm{ht}(\alpha) = \mathrm{ht}(\beta)$ and the integer represented by the string Dynkin form of α is *bigger* than the integer represented by the string Dynkin form of β.

Recall that a pair (α, β) of positive roots is called *special* if $\alpha+\beta \in \Phi$ and $\alpha \prec \beta$ with respect to the ordering described above. A pair (α, β) is called *extraspecial* if it is special and for any special pair (γ, δ) such that $\alpha + \beta = \gamma + \delta$ one has $\alpha \prec \gamma$. Then the values of the structure constants $N_{\alpha\beta}$ may be taken arbitrarily at the extraspecial pairs and all the other structure constants may be uniquely determined using only the standard properties of the structure constants, see [C, p. 58 – 60].

In this paper we always assume that the signs of $N_{\alpha\beta}$ at all extraspecial pairs is taken to be "+". In other words, $N_{\alpha_i\beta}$ is positive, if $\alpha_i + \beta \in \Phi^+$ and there is no $j < i$ such that $\alpha_i + \beta = \alpha_j + \gamma$ for some $\gamma \in \Phi^+$. Tables of the structure constants under this assumption may be found in [GS], [V3] and [VP].

In the sequel we make a very heavy use of some well-known properties of the structure constants, see [C]. The following obvious property will be used without any specific reference:

$$N_{\alpha\beta} = N_{-\beta,-\alpha} = -N_{-\alpha,-\beta} = -N_{\beta\alpha}.$$

The other two are somewhat more complicated and to save space we state them only for simply laced systems since we only use them in this case:

$$N_{\alpha\beta} = N_{\beta\gamma} = N_{\gamma\alpha}, \quad \text{if} \quad \alpha+\beta+\gamma = 0, \tag{1}$$

and

$$N_{\alpha\beta}N_{\gamma\delta} + N_{\beta\gamma}N_{\alpha\delta} + N_{\gamma\alpha}N_{\beta\delta} = 0, \quad \text{if} \quad \alpha+\beta+\gamma+\delta = 0. \tag{2}$$

Observe that in the last formula only two of the summands can be non-zero. Indeed, $N_{\alpha\beta} \neq 0$ and $N_{\beta\gamma} \neq 0$ implies that $\alpha+\beta, \beta+\gamma \in \Phi$. Thus β forms angle $2\pi/3$ with both α and γ. Suppose $N_{\alpha\gamma} \neq 0$. Then the angle between α and γ also equals $2\pi/3$ and thus α, β, γ lie in one plane. But then $\delta = 0$, a contradiction. Thus (2) is equivalent to a piece of the 2-cocycle equation:

$$N_{\beta\gamma}N_{\alpha,\beta+\gamma} = N_{\alpha+\beta,\gamma}N_{\alpha\beta}.$$

This last formula is extremely important for the rest of the paper.

3°. Internal Chevalley modules. Let $T = T(\Phi, R)$ be a split maximal torus in G. In the sequel the elementary root unipotents $x_\alpha(\xi)$, etc. are always defined with respect to this torus. For a root $X_\alpha = \{x_\alpha(\xi),\ \xi \in R\}$ denotes the corresponding (elementary) root subgroup. Since G is simply connected, T is generated by the elements $h_\alpha(\varepsilon)$, $\alpha \in \Phi$, $\varepsilon \in R^*$, where, as always, $h_\alpha(\varepsilon) = w_\alpha(\varepsilon)w_\alpha(1)^{-1}$ and $w_\alpha(\varepsilon) = x_\alpha(\varepsilon)x_{-\alpha}(-\varepsilon^{-1})x_\alpha(\varepsilon)$. As usual, we set

$$U(\Phi, R) = \langle x_\alpha(\xi),\ \alpha \in \Phi^+,\ \xi \in R\rangle,$$

(here $\langle X\rangle$ denotes the subgroup of G generated by the subset X.) Let $B = B(\Phi, R) = TU$ be the standard Borel subgroup of G.

Further, let $J \subseteq \Pi$ be a subset of fundamental roots and Φ_J be the subsystem of Φ generated by J. We denote by P_J the corresponding standard parabolic subgroup, by L_J its standard Levi component and by U_J its unipotent radical, respectively. Then $P_J = L_JU_J$ is a semidirect product, where L_J acts on U_J via conjugation. When $R = K$ is a field, P_J is generated by B and X_α, $\alpha \in \Phi_J$, whereas $L_J = T\langle X_\alpha,\ \alpha \in \Phi_J\rangle$ and $U_J = \langle X_\alpha,\ \alpha \in \Phi^+ \setminus \Phi_J\rangle$. We consider also the opposite parabolic subgroup P_J^-, which has the same Levi subgroup L_J and the opposite unipotent radical $U_J^- = \langle X_{-\alpha},\ \alpha \in \Phi^+ \setminus \Phi_J\rangle$.

We are interested in the realisations of microweight modules for $G = L_J$ as factors of the unipotent radical U_J of some parabolic subgroup P_J of a larger Chevalley group. The consecutive factors $U_J(h)/U_J(h+1)$ of the lower central series of U_J have an obvious structure of R-modules and may be considered as L_J-modules via conjugation. The decomposition of those into indecomposable summands is very well understood, see [ABS]. However we do not need to reproduce results of [ABS] in full generality, since the only case we need is that of maximal parabolic subgroup, where a shape is completely determined by its level and everything becomes essentially trivial.

Fix an r, $1 \leq r \leq l$, and let $J = J_r = \Pi \setminus \{\alpha_r\}$. The corresponding maximal parabolic subgroup, its Levi component and unipotent radical will be denoted simply by P_r, L_r and U_r, respectively. Denote by $\Sigma_r(h)$ the set of roots $\alpha \in \Phi$ of α_r-level h:

$$\Sigma_r(h) = \big\{\alpha = \sum m_i\alpha_i \mid m_r = h\big\}.$$

The union of all $\Sigma_r(h)$, $h \geq 1$, will be denoted by Σ_r. Then (at least under condition that the *very bad primes* of G are invertible in R, which automatically holds for the simple laced case when there are no very bad primes), a factor

$$U_r(h)/U_r(h+1) \cong \prod X_\alpha, \qquad \alpha \in \Sigma_r(h),$$

is an irreducible L_r-module with the highest weight ω, where ω is the element of $\Sigma_r(h)$ of the largest height.

Since our focus is on the (commutator subgroup of the) Levi component L_r, rather then on the ambient group itself, we slightly change the notation: in the sequel Φ will refer to what was called Φ_J before, whereas the root system of the ambient group will be denoted by Δ. When r is fixed, we write simply Σ instead of Σ_r and $\Sigma(h)$ instead of $\Sigma_r(h)$. Thus, $\Delta = \Phi \cup \Sigma \cup (-\Sigma)$.

We will be interested essentially in the following two cases. First, let $(\Delta, \Phi) = (\mathrm{E}_7, \mathrm{E}_6)$, $r = 7$. In this case the r-level of the maximal root equals 1, so that $\Sigma = \Sigma(1)$ and the unipotent radical U_7 is *abelian*. The representation of $G(\mathrm{E}_6, R)$ on U_7 is the 27-dimensional representation with the highest weight $\overline{\omega}_1$, see Figure 1. Observe, that [ABS] works with the dual representation in U_J^-, rather than with that in U_J. In our case the representation of $G(\mathrm{E}_6, R)$ on U_7^- is the 27-dimensional representation contragredient to the one above and its highest weight equals $\overline{\omega}_6$.

Second, let $(\Delta, \Phi) = (\mathrm{E}_8, \mathrm{E}_7)$, $r = 8$. In this case the r-level of the maximal root equals 2, so that $\Sigma = \Sigma(1) \cup \Sigma(2)$, where $\Sigma(2)$ consists of the maximal root only, and the unipotent radical U_7 is *extraspecial*. The representation of $G(\mathrm{E}_7, R)$ on $U_7(1)/U_7(2)$ is the 56-dimensional representation with the highest weight $\overline{\omega}_1$, see Figure 2.

In this realization weights of V correspond to roots of $\Sigma(1)$ and the distance between two weights in the weight graph is expressed as follows: $d(\lambda, \mu) = 1, 2, 3$ depending on whether $\widehat{(\lambda, \mu)} = \pi/3, \pi/2$, or $2\pi/3$. We freely switch between these two languages and substantially use these realisations of the representations in the sequel.

§ 2. Signs of the action constants

In this section we describe how to read off the signs of the action constants from a weight diagram. All details are given for the microweght modules for E_6 and E_7. In particular, the results of this section may be considered as an elementary definition of the simply connected groups of these types.

1°. Action of the fundamental root unipotents. The microweight modules for E_6 and E_7 have the following nice property[3]. This result remains valid for all microweight representations, even when they cannot be realised via an internal Chevalley module. A different proof, based on the geometric results of [PR], is given in [V6].

Theorem 1. *Assume that $\Phi = \mathrm{E}_6$ or E_7 and π is a microweight representation. Then there exists an admissible base v^λ of V in which $c_{\lambda\alpha} = +1$ whenever α is a fundamental or a negative fundamental root.*

Proof. Keep the notation from the preceding section. In particular, if $\Phi = \mathrm{E}_l$, $l = 6, 7$, then $\Delta = \mathrm{E}_{l+1}$ and $\Sigma = \Sigma_{l+1}(1)$ is the set of roots $\alpha \in \Delta$ such that in the expansion $\alpha = \sum m_i\alpha_i$ of α into a linear combination of the fundamental roots one has $m_{l+1} = 1$. Let first $\omega = \overline{\omega}_1$ for E_6 or $\omega = \overline{\omega}_7$ for E_7. Then V may be interpreted as the first factor $U_{l+1}(1)/U_{l+1}(2)$ of the lower central series of the unipotent radical U_{l+1} of the parabolic subgroup P_{l+1} in $G(\Delta, R)$. The weights of V may be interpreted as the roots of Σ.

We set $v^\gamma = x_\gamma(1)$. Recall that the elementary unipotents of the group G act by conjugation. Since the case when $\alpha + \gamma$ is not a root is trivial, we may assume that $\alpha + \gamma \in \Sigma$ and thus

$$x_\alpha(1)v^\gamma = x_\gamma(1)x_{\alpha+\gamma}(N_{\alpha\gamma}).$$

In other words, in this case $c_{\gamma\alpha} = N_{\alpha\gamma}$. Observe that the following fact shows that it suffices to consider the case when α is fundamental.

Lemma 4. *For all $\alpha \in \Phi$, $\lambda \in \Lambda(\pi)$ such that $\lambda + \alpha \in \Lambda(\pi)$ one has $c_{\lambda+\alpha,-\alpha} = c_{\lambda,\alpha}$.*

Indeed, applying (1) to $\alpha+\gamma-(\alpha+\gamma) = 0$, we get $N_{\alpha\gamma} = N_{-\gamma-\alpha,\alpha} = N_{-\alpha,\gamma+\alpha}$. In particular if all $c_{\gamma\alpha} = +1$ when α is fundamental, then the same holds when α is negative fundamental.

Now we argue by induction in the height of γ. If $\mathrm{ht}(\gamma) = 1$, then $\gamma = \alpha_{l+1}$ and if α is a fundamental root one has $N_{\alpha\gamma} = 1$ by our convention about the

[3]Since in the microweight modules there is an essentially unique choice of an admissible base, a nice admissible base *must* be a *canonical base* of G.Lusztig [L1] – [L3] and a *crystal base* of M.Kashiwara [K1] – [K3]. As was observed by R.Carter, this means that Theorem 1 is in fact a special case of the positivity property of canonical bases [L1].

structure constants. Let now $\mathrm{ht}(\gamma) \geq 2$ and assume that our assertion holds for all roots $\delta \in \Sigma$ of smaller height. If α is the smallest fundamental root such that $(\gamma+\alpha)-\alpha \in \Sigma$, then again $N_{\alpha\gamma} = 1$ by our choice of the structure constants. Thus it remains to consider the case when there exists a fundamental root $\beta \in \Pi$ such that $\gamma+\alpha-\beta \in \Sigma$, which precedes α. Since $\alpha-\beta \notin \Delta$, one must have $\gamma-\beta \in \Delta$, and thus $\gamma-\beta \in \Sigma$, $\mathrm{ht}(\gamma-\beta) < \mathrm{ht}(\gamma)$. Take such a β minimal with respect to the order $\prec$ and apply (2) to $\alpha+\gamma-\beta-(\alpha+\gamma-\beta)$. Since $N_{-\beta,\alpha} = 0$, we get

$$N_{\alpha,\gamma}N_{-\beta,-\alpha-\gamma+\beta} + N_{\gamma,-\beta}N_{\alpha,-\alpha-\gamma+\beta} = 0.$$

Now $N_{-\beta,-\alpha-\gamma+\beta} = -N_{\beta,\alpha+\gamma-\beta} = -1$, by minimality of β and thus (applying (1) twice) we get

$$N_{\alpha,\gamma} = N_{\gamma,-\beta}N_{\alpha,-\alpha-\gamma+\beta} = N_{\beta,\gamma-\beta}N_{\alpha,\gamma-\beta}.$$

But both factors on the right hand side are $+1$ by the induction hypothesis.

It remains to consider the case when $\omega = \overline{\omega}_6$ for E_6. Here V may be interpreted as the unipotent radical of the parabolic subgroup of E_7 opposite to P_7. In other words, this time the weights of V are $-\gamma$, $\gamma \in \Sigma$. If we set $v^{-\gamma} = x_{-\gamma}(1)$ as above, then the case of $\omega = \overline{\omega}_1$ implies that $N_{\alpha,-\gamma} = -N_{-\alpha,\gamma} = -1$ for all fundamental roots, which is not quite what we wanted. However, since the poset Σ is *ranked* by height, the solution is to switch the signs of the base vectors of odd rank. In other words, the base $v^{-\gamma} = x_{-\gamma}((-1)^{\mathrm{ht}(\gamma)})$ satisfies conclusion of the theorem.

In the sequel we fix a base v^λ as above. The theorem was stated for E_6 and E_7, but such bases exist also in the classical cases. For the fundamental modules of a group of type A_l these are exactly the standard bases $v_{i_1} \wedge v_{v_2} \wedge \ldots \wedge v_{i_m}$, $1 \leq i_1 < i_2 < \ldots < i_m \leq l+1$. The natural representations of other classical groups are usually considered with respect to the bases such that for a fundamental root one of the action constants is $+1$ whereas another one is -1. Clearly one can switch the signs of some base vectors to make all of the action constants to be $+1$, only that such a base would be less convenient than the usual one. The spin case is addressed in the second part of [VP].

2°. Action of arbitrary root unipotents. Let now $\alpha \in \Phi$ be an arbitrary root and $\lambda \in \Lambda(\pi)$ be a weight such that $\lambda+\alpha \in \Lambda(\pi)$. We want to calculate $c_{\lambda\alpha}$. Lemma 4 shows that $c_{\lambda\alpha} = c_{\lambda+\alpha,-\alpha}$, so that we may from the very start assume that $\alpha \in \Phi^+$. First we give an inductive formula which reduces the problem to the roots of smaller height.

Let β be the smallest fundamental root such that $\alpha-\beta \in \Phi$. Then by Lemma 3 either $\lambda+\beta$ is a root, or $\lambda+(\alpha-\beta)$ is a root, but not both. This allows to reduce calculation of $c_{\lambda\alpha}$ to that of $c_{\mu,\alpha-\beta}$.

Lemma 5. *Let $\alpha \in \Phi^+ \setminus \Pi$, $\lambda \in \Lambda(\pi)$, and let β be the smallest fundamental root such that $\alpha-\beta \in \Phi$. Then*

$$c_{\lambda\alpha} = \begin{cases} c_{\lambda,\alpha-\beta}, & \text{if } \lambda+\beta \notin \Lambda(\pi), \\ -c_{\lambda+\beta,\alpha-\beta}, & \text{if } \lambda+\beta \in \Lambda(\pi). \end{cases}$$

Proof. Our choice of β guarantees that $x_\alpha(1) = [x_\beta(1), x_{\alpha-\beta}(1)]$. This means that

$$x_\alpha(1)v^\lambda = x_\beta(1)x_{\alpha-\beta}(1)x_\beta(-1)x_{\alpha-\beta}(-1)v^\lambda.$$

Calculating the expression on the right hand side, we get

$$c_{\lambda\alpha} = \begin{cases} c_{\lambda+\alpha-\beta,\beta}c_{\lambda,\alpha-\beta}, & \text{if } \lambda+\beta \notin \Lambda(\pi), \\ -c_{\lambda\beta}c_{\lambda+\beta,\alpha-\beta}, & \text{if } \lambda+\beta \in \Lambda(\pi). \end{cases}$$

It remains to apply Theorem 1.

Now it is completely clear how to calculate $c_{\lambda\alpha}$ inductively: one has to iterate Lemma 5. To formalize this idea we introduce the following notion. Define the *canonical string* of a root $\alpha \in \Phi^+$ as follows. The canonical string of a fundamental root α_i is i. If $\mathrm{ht}(\alpha) \geq 2$ and α_i is the smallest fundamental root such that $\alpha - \alpha_i \in \Phi$, then the canonical string of α is the string join of i and the canonical string of $\alpha - \alpha_i$ (in other words, i must be appended to the canonical string of $\alpha - \alpha_i$ on the left). For example, the canonical strings of the maximal roots of E_6 and E_7 are 24315423456 and 13425431654234567 respectively.

Now to calculate $c_{\lambda\alpha}$ we proceed as follows. Let $i_1 \dots i_h$, where $h = \mathrm{ht}(\alpha)$, be the canonical string of α. We search for a path in the *negative* direction starting at $\lambda_1 = \lambda + \alpha$ which has labels $i_1, \dots, i_h$ in the same order. Such a path does not necessarily exist. If there is a bond labeled i_1 hanging on λ_1 in the negative direction, we set $\lambda_2 = \lambda_1 - \alpha_{i_1}$, otherwise we say that i_1 is *nasty* for λ and set $\lambda_2 = \lambda_1$. Now if there is a bond labeled i_2 hanging on λ_2 in the negative direction, we set $\lambda_3 = \lambda_2 - \alpha_{i_2}$, otherwise we say that i_2 is *nasty* for λ and set $\lambda_3 = \lambda_2$, etc. We proceed like this until we get to the end of the canonical string. Let $n = n(\alpha, \lambda)$ be the number of labels in the canonical string of α nasty for λ. Then Lemma 5 immediately implies the following recipe.

Theorem 2. *Let v^λ be an admissible base satisfying conclusion of Theorem 1. Then for all $\alpha \in \Phi^+$ and all $\lambda \in \Lambda(\pi)$ such that $\lambda + \alpha \in \Lambda(\pi)$ one has*

$$c_{\lambda\alpha} = (-1)^{n(\alpha,\lambda)}.$$

With this rule it is a matter of less than one minute to calculate the explicit action of $x_\alpha(\xi)$ on V in each case. It is further simplified by the following facts, which one immediately notices contemplating Figure 1 and Figure 2 (of course, one could give an independent *a priori* proof along the lines of the above proof of Theorem 1).

Proposition 1. *The action constants $c_{\lambda\alpha}$ enjoy the following properties:*

(1) *If α is the maximal root, then $c_{\lambda\alpha} = 1$ for all $\lambda, \lambda + \alpha \in \Lambda(\pi)$.*
(2) *For any root α the parity of the number of negative structure constants $c_{\lambda\alpha}$ is opposite to the parity of* $\mathrm{ht}(\alpha)$. *In other words, $\prod c_{\lambda\alpha} = (-1)^{\mathrm{ht}(\alpha)-1}$, where the product is taken over all $\lambda \in \Lambda(\pi)$ such that $\lambda + \alpha \in \Lambda(\pi)$.*

(3) *For any root $\alpha \in \Phi^+$ the last non-trivial action constant $c_{\lambda\alpha}$ is always $+1$. In other words, if $\mu + \alpha \notin \Lambda(\pi)$, for all $\mu < \lambda$, but $\lambda + \alpha \in \Lambda(\pi)$, then $c_{\lambda\alpha} = +1$.*

Several recent papers explicitly tabulate the action of $x_\alpha(\xi)$ in these representations, see [Te], [DMV] and references there.

3°. Action of the extended Weyl group. Extended Weyl group $\widetilde{W} = \widetilde{W}(\Phi)$ of type Φ is the subgroup of G generated by $w_\alpha(1)$. If char $R \neq 2$, it is an extension of the ordinary Weyl group $W = W(\Phi)$ by an elementary abelian group of order 2^l, where $l = \mathrm{rk}(\Phi)$, see [T]. Let $N = N(\Phi, R)$ be the normalizer of T in G in the sense of algebraic groups. In other words, N is generated by T and $w_\alpha(\varepsilon)$, $\alpha \in \Phi$, $\varepsilon \in R^*$. Then $\widetilde{W}(\Phi)$ is isomorphic to $N(\Phi, \mathbb{Z})$.

Since $w_\alpha(\varepsilon)$ are defined in terms of $x_\alpha(\xi)$, it is clear from the above, that a weight diagram controls the action of $w_\alpha(\varepsilon)$. This means that a weight diagram describes action of the corresponding extended Weyl group, not just of the ordinary Weyl group. Since this action is responsible for the signs of monomials in equations among the matrix entries of a matrix $\pi(g)$, $g \in G$, in this subsection we make this description explicit. In fact, the following two lemmas describe the action of N.

Lemma 6. *Assume that $\alpha \in \Phi$, $\lambda \in \Lambda(\pi)$, $\varepsilon \in R^*$. Then*

$$w_\alpha(\epsilon)v^\lambda = \begin{cases} v^\lambda, & \text{if } \lambda \pm \alpha \notin \Lambda(\pi), \\ c_{\lambda\alpha}\epsilon v^{\lambda+\alpha}, & \text{if } \lambda + \alpha \in \Lambda(\pi), \\ -c_{\lambda,-\alpha}\epsilon^{-1}v^{\lambda-\alpha}, & \text{if } \lambda - \alpha \in \Lambda(\pi). \end{cases}$$

Proof. The above formulae immediately follow from the definition of $w_\alpha(\varepsilon) = x_\alpha(\varepsilon)x_{-\alpha}(-\varepsilon^{-1})x_\alpha(\varepsilon)$, the definition of $c_{\lambda\alpha}$ and Lemma 4.

Now Theorem 2 gives a rule how to read off the action constants $c_{\lambda\alpha}$ – and thus also the coefficients in the formulae appearing in Lemma 6 – from the weight diagram. In particular, this means that the weight diagram encodes the whole information about the action of $\widetilde{W}$. In many cases it is more convenient though to read off the action of $\widetilde{W}$ directly from Theorem 1. Namely if α is a fundamental root, then $c_{\lambda\alpha} = 1$ whenever $\lambda + \alpha \in \Lambda(\pi)$, and $c_{\lambda,-\alpha} = 1$, whenever $\lambda - \alpha \in \Lambda(\pi)$, so that the formula in Lemma 6 simplifies to

$$w_\alpha(1)v^\lambda = \begin{cases} v^\lambda, & \text{if } \lambda \pm \alpha \notin \Lambda(\pi), \\ v^{\lambda+\alpha}, & \text{if } \lambda + \alpha \in \Lambda(\pi), \\ -v^{\lambda-\alpha}, & \text{if } \lambda - \alpha \in \Lambda(\pi). \end{cases}$$

Now to find wv^λ for an element $w \in \widetilde{W}$ one has simply to pick up a decomposition of w as a product of the fundamental generators $w_\alpha(1)$, $\alpha \in \Pi$, and apply the above rule to each factor.

The above lemma describes also the action of $h_\alpha(\varepsilon)$'s.

Lemma 7. *Assume that $\alpha \in \Phi$, $\lambda \in \Lambda(\pi)$, $\varepsilon \in R^*$. Then*

$$h_\alpha(\epsilon)v^\lambda = \begin{cases} v^\lambda, & \textit{if } \lambda \pm \alpha \notin \Lambda(\pi), \\ \epsilon^{-1}v^\lambda, & \textit{if } \lambda + \alpha \in \Lambda(\pi), \\ \epsilon v^\lambda, & \textit{if } \lambda - \alpha \in \Lambda(\pi). \end{cases}$$

Proof. The above formulae immediately follow from the definition of $h_\alpha(\varepsilon) = w_\alpha(\varepsilon)w_\alpha(1)^{-1}$, Lemma 6 and Lemma 4.

§ 3. Defining equations

In this section we show that the results of the preceding section allow to fix signs also in the equations defining a matrix $\pi(g)$, $g \in G$. It is well known that the simply connected groups of types E_6 and E_7 can be described as the isometry groups of appropriate multilinear invariants on their minimal modules. Using Theorems 1 and 2 one can explicitly determine the signs of monomials of the cubic form on the free module of rank 27 and the quartic form on the free module of rank 56, invariant under the action of these groups. In fact in the rest of the paper we use not the cubic/quartic invariants themselves, but rather their first/second partial derivatives defining the orbit of the highest weight vector. Here we explicitly derive these quadratic equations in a slightly different form (Theorem 3). This result is crucial both for the definition of fake roots unipotents in § 4 and for the proof of Theorems 4 and 5 in § 5. Some of the basic references for the present section are [A1], [A2], [Br], [CC1], [CC2], [CW], [C2], [FF], [G], [Ha], [LS1], [LS2], [Se]. Some further details and many references to earlier publications may be found in [V2] and [V5].

1°. The cubic form for E_6**.** Let $V = V(\varpi_1)$ be the 27-dimensional module of the Chevalley group $G = G_{\mathrm{sc}}(\Phi, R)$ of type E_6. Then there exists a three-linear form $F : V \times V \times V \longrightarrow R$ such that G can be identified with the full isometry group of the form F, i.e. with the group of all $g \in \mathrm{GL}(V)$ such that $F(gu, gv, gw) = F(u, v, w)$ for all $u, v, w \in V$. The similarities of the form F, i.e. transformations g such that $F(gu, gv, gw) = \lambda F(u, v, w)$ for a scalar $\lambda \in R^*$, form the *extended* Chevalley group $\overline{G} = \overline{G}(\mathrm{E}_6, R)$.

This form was discovered by L.E.Dickson in 1901 (!), used by E.Cartan in the geometric study of real Lie groups and further studied by C.Chevalley and R.D.Schafer in 1950–51. An especially elementary and elegant construction of this form was proposed by H.Freudenthal in 1952. In fact, the construction gives not the three-linear form F itself, but rather the corresponding *cubic* form Q. Clearly one can identify V with the 27-dimensional R-module $M(3, R)^3$. Now for an element $(a, b, c) \in M(3, R)^3$ one defines a cubic form in 27 variables by

$$Q((a, b, c)) = \det(a) + \det(b) + \det(c) - \mathrm{tr}(abc).$$

It can be proven (see [A1], [A2]) that over a field the isometry group of the cubic form Q coincides with the isometry group of its complete polarisation F. Actually M.Aschbacher uses a different construction of the form, not in terms of $3\mathrm{A}_2$, as above, but in terms of A_5 (the essence of this construction is expressed by the partition 27=6+15+6), but the resulting forms are equivalent. This is in fact a characteristic free result[4] which extends to all commutative rings (see [V2], § 6).

Another interpretation of the cubic form Q is as the norm form of the exceptional 27-dimensional Jordan algebra, see [FF]. As such it was studied by H.Freudenthal, T.Springer, F.Veldkamp, N.Jacobson and others. This interpretation is intimately related with the realisation of the Chevalley group of type E_6 as the *structure group* of the *split* exceptional Jordan algebra (for fields see [S2], where it is phrased in a slightly different, but essentially equivalent language of J-systems, for rings one should consider quadratic Jordan algebras instead, if one is not inclined to sacrifice the case $2 \notin R^*$).

When $R = K$ is a field, there are exactly 3 orbits of the Chevalley group G on the one-dimensional subspaces of V or, what is the same, there are exactly 4 orbits of the *extended* Chevalley group $\overline{G}$ on the vectors $v \in V$, see [Mr], [A1], [CC1]. The generic $\overline{G}$-orbit consists of the vectors $v \in V$ such that $F(v, v, v) \neq 0$, in [CC1] such vectors are called *black*. Over a field such that $K^* \neq {K^*}^3$ this orbit may split into several G-orbits. The orbit of the submaximal dimension consists of the vectors v such that $F(v, v, v) = 0$, but there exists a vector u such that $F(v, v, u) \neq 0$. Such vectors are called *grey*. Grey vectors always form one G-orbit. The next orbit consists of the vectors $v \neq 0$ such that $F(v, v, u) = 0$ for all $u \in V$. Such vectors are called *white*. Finally there is the zero vector $v = 0$. Vectors v such that $F(v, v, u) = 0$ for all $u \in V$ are also often called *singular*, see [A1]. In the above terminology a vector is singular if it is either zero or white.

The condition that a vector v is singular may be also expressed in terms of the cubic form Q or its partial polarisation T. In terms of T one must have $T(u, v) = 0$. In other words, v is singular if and only if it is zero of the 27 partial derivatives of the form Q, see [CC2]. These partial derivatives are quadratic polynomials, which are explicitly listed in [CC2] and (in a slightly different language) in Theorem 3 below.

Next we recall the construction of the cubic form from [V2], where we can now give a very simple rule for the signs. It is known that the form has one Weyl orbit

[4]If the form Q were non-degenerate, this would fail in characteristics 2 and 3. But the form Q is highly degenerate – it *must* be, for the semisimple part of the isometry group of a nondegenerate cubic form is *finite*. In fact M.Aschbacher considers 3-forms, which are triples consisting of the cubic form Q, its partial polarisation T, linear in the first argument and quadratic the second one, and its complete polarisation F, see [A1] – [A3]. To get a fully satisfactory theory over rings one has to generalise the notion of a 3-form along the lines suggested by A.Bak's theory of quadratic forms over form rings, see [HOM].

of monomials, and that this orbit corresponds to triads of weights. Namely, a triple (λ, μ, ν) of distinct weights is called a *triad* if λ, μ, ν are pair-wise orthogonal (or, what is the same, their differences $\lambda - \mu$, $\lambda - \nu$, $\mu - \nu$ are not weights). In the terminology of [A1] a triple of weights (λ, μ, ν) is a triad exactly when v^λ, v^μ, v^ν generate a *special plane*. Let Θ be the set of triads, $|\Theta| = 27 \cdot 10$. Then the three-linear form F takes the following values: $F(v^\lambda, v^\mu, v^\nu) = \pm 1$ if $(\lambda, \mu, \nu) \in \Theta$ and $F(v^\lambda, v^\mu, v^\nu) = 0$ otherwise. The signs are determined by the condition that the form F is invariant under the action of $\widetilde{W}$. A model triad in the realisation from § 1 is

$$(\lambda_0, \mu_0, \nu_0) = (\begin{smallmatrix}000001\\0\end{smallmatrix}, \begin{smallmatrix}012221\\1\end{smallmatrix}, \begin{smallmatrix}234321\\2\end{smallmatrix})$$

and we set $F(\lambda_0, \mu_0, \nu_0) = 1$ for this particular one. For any other triad (λ, μ, ν) the sum

$$\lambda + \mu + \nu = \lambda_0 + \mu_0 + \nu_0 = \begin{smallmatrix}246543\\3\end{smallmatrix}$$

is orthogonal to all fundamental roots $\alpha_1, \dots, \alpha_6$. Thus for any fundamental reflection $w_\alpha \in W(\mathrm{E}_6)$ one has the following alternative: either $w_\alpha(\lambda, \mu, \nu) = (\lambda, \mu, \nu)$, or exactly two of the weights λ, μ, ν are moved by w_α, necessarily in opposite directions (say, $w_\alpha(\lambda) = \lambda + \alpha$, $w_\alpha(\mu) = \mu = \alpha$, $w_\alpha(\nu) = \nu$). This means that the sign of $f(v^\lambda, v^\mu, v^\nu)$ can be calculated in terms of the distance of the triad (λ, μ, ν) from the model triad $(\lambda_0, \mu_0, \nu_0)$. Namely, let

$$h(\lambda, \mu, \nu) = 1/2(h(\lambda, \lambda_0) + h(\mu, \mu_0) + h(\nu, \nu_0))$$

be half of the sum of distances between respective terms of the triads. Then one sets $F(v^\lambda, v^\mu, v^\nu) = (-1)^{h(\lambda,\mu,\nu)}$. The cubic form can be defined similarly, only that to avoid the coefficient 6 which would cause problems in characteristics 2 and 3, now one has to sum only over the set Θ_0 of unordered triads $\{\lambda, \mu, \nu\}$. Clearly, $|\Theta_0| = |\Theta|/6 = 45$. Then for a vector $x = \sum x_\lambda v^\lambda$ one has

$$Q(x) = \sum (-1)^{h(\lambda,\mu,\nu)} x_\lambda x_\mu x_\nu,$$

where the sum is taken over $\{l, \mu, \nu\} \in \Theta_0$.

2°. The quartic form for E_7**.** There is a similar, but more complicated description of the simply connected group of type E_7 acting on the 56-dimensional module $V = V(\overline{\varpi}_7)$. In this case one needs two invariants to define the group, one of degree 2, another of degree 4. First of all, the module V is self-dual and supports a unimodular symplectic form h. Further, there exists a four-linear form $F : V \times V \times V \times V \longrightarrow R$ such that G can be identified with the full isometry group of the pair h, F, i.e. with the group of all $g \in \mathrm{GL}(V)$ such that $h(gu, gv) = h(u, v)$ and $F(gu, gv, gx, gy) = F(u, v, x, y)$ for all $u, v, x, y \in V$. The similarities of the pair form the *extended* Chevalley group $\overline{G} = \overline{G}(\mathrm{E}_7, R)$.

It is obvious, how to construct h. The construction of the quartic invariant is somewhat more complicated and classically one constructed not the four-linear form F itself, but rather the corresponding quartic form[5]. That the group G preserves a quartic form in 56 variables was first observed by E.Cartan, at least in characteristic 0, but his explicit construction of the form seems to be an error. A very elegant construction of such an invariant over a field K of characteristic not 2 was given by H.Freudenthal. Namely, he identifies V with the space $A(8,K)^2$, where $A(8,K)$ is the space of anti-symmetric 8×8 matrices, and considers the following symplectic product and quartic form:

$$h((a_1,b_1),(a_2,b_2)) = \frac{1}{2}(\mathrm{tr}(a_1b_2^t) - \mathrm{tr}(a_2b_1^t)),$$

$$Q((a,b)) = \mathrm{pf}(a) + \mathrm{pf}(b) - \frac{1}{4}\,\mathrm{tr}((ab)^2) + \frac{1}{16}\,\mathrm{tr}(ab)^2.$$

Then in all characteristics distinct from 2 one can identify the isometry group of this pair as the simply-connected Chevalley group G of type E_7 over K, see [A2], [C2]. The constructions of the form by M.Aschbacher and B.Cooperstein are slightly different. In fact [A2] constructs the form in terms of A_6 (the essence of this construction is expressed by the partition $56 = 7+21+21+7$), whereas the construction in [C2] is closer to that of Freudenthal as it is phrased in terms of A_7 (where 56=28+28). The isometry group of Q alone is spanned by G and a diagonal element of order 2, see [C2]. In characteristics $p\geq 5$ everything works smoothly, whereas characteristic 3 needs some extra care.

But the thing breaks down in characteristic 2. Not only the above construction does not work, but apparently in characteristic 2 there is no non-trivial G-invariant symmetric four-linear form on V (see [A2]). This is because in characteristic 2 the form

$$f(u,v,x,y) = h(u,v)h(x,y) + h(u,x)h(v,y) + h(u,y)h(v,x)$$

obtained by squaring the symplectic form becomes symmetric, which it is not in characteristics ≥ 3. In fact M.Aschbacher [A2] constructs a 4-linear G-invariant form F in characteristic 2, symmetric with respect to *even* permutations.

There are other constructions of the form Q, notably that of R.Brown [Br], which works in characteristics $\neq 2,3$. Let V be a space which supports a non-degenerate inner product. Then to define a three-linear form on V is essentially the same as to give V an algebra structure. By the same token to define a four-linear form on V is essentially the same as to give V a structure of a *ternary* algebra. In fact, there is a remarkable ternary algebra of dimension 56 constructed in terms of the exceptional Jordan algebra $\mathbb{J}$, see [Br], [FF] and references there. The algebra consists of 2×2 matrices over $\mathbb{J}$ with scalar diagonal entries, $56 = 1+27+27+1$, which is exactly the way how we draw Figure 2.

[5]The first appearance of the quartic form is again in a 1901 paper of L.E.Dickson, in the context of the 28 bitangents and thus of the Weyl group $W(\mathrm{E}_7)$. Apparently the explicit connection with the group of type E_7 was missing. Otherwise Chevalley groups would have been discovered half a century earlier!

The orbits of $G = G(\mathrm{E}_7, K)$ on the 56-dimensional module were classified in [Ha] in the absolute case, in [LS] for finite fields and in [C2] in general. These orbits are essentially described in terms of the four-linear form. Again characteristics 2 and 3 require some additional attention and one needs the notion of a 4-form (see [A2], [C3]), which is a quartic form together with its polarisations, to account for all details. So let $\operatorname{char} K \neq 2, 3$. Then a vector $u \in V$ is called *singular* if $F(u, u, x, y) = 0$ for all $x, y \in V$, *brilliant* if $F(u, u, u, x) = 0$ for all $x \in V$ and *lumenescent* if $F(u, u, u, u) = 0$. Otherwise, i.e. if $F(u, u, u, u) \neq 0$, the vector u is called *dark*. The orbits of G on V are as follows: 0, non-zero singular vectors, non-singular brilliant vectors, luminescent vectors which are non-brilliant, and finally one or several orbits of dark vectors, parametrised by K^*/K^{*2} (these orbits are fused by the action of the *extended* Chevalley group $\overline{G}$ of type E_7).

In the next subsection we give an *ad hoc* definition of a singular vector, for use in §§ 4 and 5 and do not try to verify that our singular vectors actually coincide with the singular vectors as defined by B.Cooperstein in [C2]. This is because although a construction of the form Q similar to the one given in the preceding subsection for E_6 is possible, it is already much more complicated, and taking care of all details requires serious effort. Instead we explain where the problem lies.

In fact, suspecting that E_7 stands in the same relation to E_6 as E_6 itself does to D_5, one is immediately tempted to define the quartic form on V as follows. Take a base vector v^λ. Then the vectors v^μ, $d(\lambda, \mu) = 2$, generate a 27-dimensional module U which carries the cubic E_6-form. One defines *tetrades* as quadruples $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ of pair-wise orthogonal weights. Let Θ and Θ_0 be the sets of ordered and unordered tetrades respectively. Clearly, $|\Theta| = 56 \cdot 27 \cdot 10$, whereas $|\Theta_0| = |\Theta|/24 = 630$. Then tentatively one defines the quartic form Q_{tent} as $Q_{\text{tent}}(x) = \sum \pm x_{\lambda_1} x_{\lambda_2} x_{\lambda_3} x_{\lambda_4}$, where the sum is taken over $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \in \Theta_0$ and the signs are defined by the condition that the resulting form is invariant under the action of $\widetilde{W}$. One should be slightly more cautious here, than in the case of E_6, because now w_α can move all 4 weights of a tetrade, two in positive, two in negative direction, in which case the sign is not changed, but the expression of the sign in terms of $h(\lambda_i, \mu_i)$ still works. This is essentially the same, as define a four-linear form F_{tent} by setting $F_{\text{tent}}(v^{\lambda_1}, v^{\lambda_2}, v^{\lambda_3}, v^{\lambda_4}) = (-1)^{h(\lambda_1, \lambda_2, \lambda_3, \lambda_4)}$, for a tetrade $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \Theta$ and $F_{\text{tent}}(v^{\lambda_1}, v^{\lambda_2}, v^{\lambda_3}, v^{\lambda_4}) = 0$ otherwise. By construction the form is invariant under the action of $\widetilde{W}$ and it remains only to check that it is preserved by a root subgroup X_α for some root $\alpha \in \Phi$. Unfortunately this is not the case. In fact, for any tetrade $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and any elementary root unipotent $g = x_\alpha(\xi)$ one has

$$F_{\text{tent}}(gv^{\lambda_1}, gv^{\lambda_2}, gv^{\lambda_3}, gv^{\lambda_4}) = F_{\text{tent}}(v^{\lambda_1}, v^{\lambda_2}, v^{\lambda_3}, v^{\lambda_4}).$$

Unfortunately there are quadruples of weights for which the right-hand side is zero, whereas the left-hand side is not. Take, for example, four weights $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that $\lambda_1 + \alpha, \lambda_2 + \alpha, \lambda_3 + \alpha, \lambda_4 - \alpha$ are weights and the 8 above weights form a cube (i.e. the corresponding weight diagram is the tensor product of three copies of (A_1, ϖ_1), see [C2], [C3], [PSV]). One of the weights $\lambda_1, \lambda_2, \lambda_3$ is adjacent to the other two, say $d(\lambda_1, \lambda_2) = d(\lambda_1, \lambda_3) = 1$, so that $F_{\mathrm{tent}}(v^{\lambda_1}, v^{\lambda_2}, v^{\lambda_3}, v^{\lambda_4}) = 0$. On the other hand, expanding $F_{\mathrm{tent}}(gv^{\lambda_1}, gv^{\lambda_2}, gv^{\lambda_3}, gv^{\lambda_4})$ by linearity we get 8 summands of which exactly one, namely $F_{\mathrm{tent}}(v^{\lambda_1+\alpha}, v^{\lambda_2}, v^{\lambda_3}, v^{\lambda_4})$, corresponds to a tetrade and is equal to ± 1. Thus the form F_{tent} is not preserved by X_α.

Well, in itself this is not tragic since one hopes to repair the situation throwing in another Weyl orbit of monomials. But this is where the real trouble starts. In the above example throwing in another orbit would give you *two* further summands, which are non-zero, which would leave you either 0, or twice something. This means that you could not define the value of the form on the tetrades to be ± 1, but should have started with ± 2 instead. It is here that one starts feeling extremely unhappy. But in fact in characteristic $\neq 2$ the above construction is *essentially* correct, in the sense that it tells you what the *relevant* part of the quartic form is, which accounts for the reduction to E_6. For fix a base vector v^λ. Then $F(v^\lambda, *, *, *)$ consists of two parts, the form F_{tent} defined above and another part thrown in to make the form G-invariant, having the form $F(v^\lambda, v^{\lambda^*}, *, *)$. But the second part reveals you nothing new as compared with the fact that the group preserves a symplectic form.

In [V7] I give this last observation a precise technical meaning by describing the group of type E_7 as being essentially the stabiliser of the ideal in the polynomial ring in 56 variables, generated by 126 partial derivatives of F_{tent}. These 126 quadratic forms are determined in Theorem 3 below and in the remarks immediately following its statement I explain where one sees all these 126 forms on Figure 2. This identification works over $\mathbb{Z}$ and as opposed to the description via a quartic invariant, characteristic 2 plays no special role whatesoever. The proof in [V7] almost word for word follows the general scheme of 'matrix preserver problems' approach of W.C.Waterhouse (see [W] as an excellent introduction to his method). The only place where one has to work slightly harder is a certain Lie algebra calculation necessary to ensure the fact that the stabiliser of the above ideal is *smooth*.

3°. Orbit of the highest weight vector. It is well known that in any representation of G the orbit Gv^+ of the highest weight vector v^+ is an intersection of quadrics [Li]. In this subsection we explicitly describe the equations defining the orbit of v^+ for the microweight representations of types E_6 and E_7. Of course, for these cases the corresponding equations have been determined some 40 years ago by H.Freudenthal and J.Tits, (see also [Se], [LS1], [LS2], [LW], [CC2]), but again we would like to show how one can read the equations directly from the weight diagram.

Let $\omega = \varpi_1$ for E_6 or $\omega = \varpi_7$ for E_7, the case $(\Phi, \omega) = (\mathrm{E}_6, \varpi_6)$ follows by duality. In the next theorem we use the same interpretation of the modules, as in § 1, 2. In particular, $\Phi = \mathrm{E}_l$, $l = 6, 7$, $\Delta = E_{l+1}$, and $\Sigma = \Sigma_{l+1}(1)$. The group $G = G(\Phi, R)$ acts on $V = U_{l+1}(1)/U_{l+1}(2) \cong \prod X_\alpha$, $\alpha \in \Sigma$, by conjugation. Since we are interested only in the equations satisfied by an element of the orbit Gv^+

of the highest weight vector v^+, we may assume that $R = K$ is a(n algebraically closed) field[6].

In both cases one may take $v^+ = v^\rho = x_\rho(1)$ as a highest weight vector, where

$$\rho = {234321 \atop 2} \qquad \text{or} \qquad \rho = {2465431 \atop 3}$$

is the maximal root of E_7 or the unique submaximal root of E_8, respectively. Recall that to a vector $a = (a_\alpha) \in V$ there corresponds the product $x = \prod x_\alpha(a_\alpha) \in \prod X_\alpha$, $\alpha \in \Sigma$. In the case of E_7 this product is considered modulo $U_8(2) = X_{\rho+\alpha_8}$, the root subgroup, corresponding to the maximal root of E_8. The coefficients a_α will be called the *coordinates* of x. The action $a \mapsto ga$ of $G = G(\Delta, K)$ on V is expressed by conjugation $x \mapsto gxg^{-1}$. Thus in these terms the orbit Gv^+ may be described as the set of *root unipotents* contained in $V \cong X_\gamma$, $\gamma \in \Sigma$.

When I reported about this work in a *Darstellungstheorie* seminar in Bielefeld, C.M.Ringel asked whether the fact that a longest regular sequence of such quadratic equations defining the orbit of the highest weight vector has exactly 10 terms has something to do with the fact that there are 10 small squares in Figure 1. It does indeed, and the answer is called the theory of standard monomials. In fact, there are 27 rectangles in this figure, and each of them corresponds to an equation. Only that the orbit of the highest weight vector is not a complete intersection and thus these equations are not independent. More generally, there is an equation for any pair of *incomparable* weights (clearly rectangles in Figure 1 correspond to such pairs). For the microweight representations this is exactly the message of [Se] (see also [LS1], [LS2] for more general results and [PSV] for further related observations).

Theorem 3. *Let K be a field, and let G be the simply connected Chevalley group of type $\Phi = \mathrm{E}_6$ or E_7 over K. Let $V = V(\overline{\omega}_1)$ for $\Phi = \mathrm{E}_6$ and $V = V(\overline{\omega}_7)$ for $\Phi = \mathrm{E}_7$. If a vector $a = (a_\alpha) \in V$, $\alpha \in \Sigma$, is in the orbit of the highest weight vector then for every pair of roots $\alpha, \beta \in \Sigma$ forming angle of $\pi/2$ one has*

$$a_\alpha a_\beta = \sum N_{\alpha,-\gamma} N_{\beta,-\delta} a_\gamma a_\delta, \tag{3}$$

where the sum is taken over all unordered pairs $\{\gamma, \delta\}$, $\gamma, \delta \in \Sigma$, such that $\gamma + \delta = \alpha + \beta$ and $\{\gamma, \delta\} \neq \{\alpha, \beta\}$.

Now, let R be an arbitrary commutative ring. In view of the preceding subsections it is natural to call a vector $a = (a_\alpha) \in V$ *singular* if it satisfies Equations (3)

[6]Essentially we ask when a column may be completed to a matrix from G. The obvious necessary conditions are that this column should be unimodular and satisfy the equations satisfied by the vectors in the orbit of v^+. Over fields these conditions are also sufficient, but over rings there are further obstructions related to the non-triviality of lower K-functors and their analogues, which we do not discuss here. In general over a ring a unimodular column cannot be even completed to an invertible matrix, unless all finitely generated projective modules are free.

listed in this theorem. A subspace of V is called *singular*, if it consists of singular vectors. Any column of a matrix $g = (g_{\lambda\mu}) \in G$ of the Chevalley group G of type E_l, $l = 6, 7$, in the minimal representation is singular. This fact will be crucial in § 5. Another application of these equations is in the K-theory of exceptional groups. In [Pl1] – [Pl3] E.B.Plotkin introduced a new stability condition, phrased in terms of these equations, which is a vast generalisation of Vaserstein's stability condition for orthogonal groups, depending on one quadratic equation. For the case of E_6 all 27 equations are explicitly listed in [CC2]. For the case of E_7 the 28 independent equations are listed in [Pl3], *up to signs*.

Now we explain, how to find Equations (3) on Figures 1 and 2 and to determine the corresponding signs. First, let $\Phi = \mathrm{E}_6$ and $\Lambda = \Lambda(\overline{\omega}_1)$. For each weight $\lambda \in \Lambda$ there are 10 orthogonal ones (distance 2). This gives us 270 equations, but in this way each one of them is counted 10 times. Thus there are 27 equations. Indeed, the equations naturally correspond to the weights $\nu \in \Lambda$ as follows. For any two orthogonal weights ν, λ there is a unique weight μ such that (λ, μ, ν) form a triad. Thus for any $\nu \in \Lambda$ we define

$$\Omega(\nu) = \{\lambda \in \Lambda \mid d(\lambda, \nu) = 2\}.$$

Then $|\Omega(\nu)| = 10$ and for every $\lambda \in \Omega(\nu)$ there is a unique $\mu \in \Omega(\nu)$ such that $d(\lambda, \mu) = 2$ and this μ will be denoted by $\lambda^\sharp$. The equations have the form $\sum \pm a_\lambda a_{\lambda^\sharp}$, where the sum is taken over all unordered pairs $\{\lambda, \lambda^\sharp\}$, $\lambda \in \Omega(\nu)$, and the signs are explicitly determined by Theorem 3.

Now, let $\Phi = \mathrm{E}_7$ and $\Lambda = \Lambda(\overline{\omega}_7)$. For each weight $\lambda \in \Lambda$ there are 27 weights at distance 2 from λ. This gives us $27 \cdot 56$ equations, but in this way each one of them is counted 12 times. Thus there are 126 equations. Indeed, the equations naturally correspond to the roots $\alpha \in \Phi$ as follows. For any root $\alpha \in \Phi$ we define

$$\Omega(\alpha) = \{\lambda \in \Lambda \mid \lambda - \alpha \in \Lambda\}.$$

Then $|\Omega(\alpha)| = 12$ and for every $\lambda \in \Omega(\alpha)$ there is a unique $\mu \in \Omega(\alpha)$ such that $d(\lambda, \mu) = 2$ and this μ will be denoted by $\lambda^\sharp$. The equations have the form $\sum \pm a_\lambda a_{\lambda^\sharp}$, where as above the sum is taken over all unordered pairs $\{\lambda, \lambda^\sharp\}$, $\lambda \in \Omega(\alpha)$, and the signs are explicitly determined by Theorem 3.

For the sake of geometers, observe, that $\Omega = \Omega(\nu)$ and $\Omega = \Omega(\alpha)$ correspond to certain objects of the building of G, namely the objects of type 6 in E_6 and of type 1 in E_7. In fact, consider the subspace U generated by v^λ, $\lambda \in \Omega$. Then U is one of the summands in the D_5 branching for E_6 or the D_6 branching for E_7. In other words, the stabilizer P of U is a parabolic subgroup with the Levi factor of type D_5 or D_6, respectively. In the case E_6 there are two conjugacy classes of parabolic subgroups with the Levi factor D_5, but P is a P_6 parabolic. In other words, P is conjugate to P_6 in E_6 or P_1 in E_7, respectively, by an element of the Weyl group. There are 27 weights of the form $w\overline{\omega}_6$, $w \in W(\mathrm{E}_6)$), which gives you exactly 27 possibilitites for P in the case of E_6, the Weyl involution establishes a natural bijection with the weights of V. On the other hand, in the case of E_7 the fundamental weight $\overline{\varpi}_1$ is the maximal root which gives you 126 conjugates of P by the elements of $W(\mathrm{E}_7)$

The signs are explicitly determined by Theorem 3 and now we explain how one can read off the signs from the diagram. Let, as above, $\Omega = \Omega(\nu)$ or $\Omega = \Omega(\alpha)$ and fix a pair $\lambda, \lambda^\sharp \in \Omega$. Set the sign of the monomial $a_\lambda a_{\lambda^\sharp}$ to be '+'. We want to determine the signs of the remaining monomials $a_\mu a_{\mu^\sharp}$, $\mu, \mu^\sharp \in \Omega$, $\mu \neq \lambda, \lambda^\sharp$. In this case $\mu - \lambda$ is a root and switching λ and $\lambda^\sharp$, if necessary, we may even assume that $\mu - \lambda \in \Phi^+$.

First of all, if $\mu - \lambda$ is a fundamental root, Theorem 1 immediately implies that the sign of $a_\mu a_{\mu^\sharp}$ is '$-$'. Iterating this simple rule, we can determine signs of some further monomials, in some cases of all monomials. Thus, in the notation of § 5 we get for E_6 equations

$$a_{\gamma_1} a_{\sigma_1} - a_{\gamma_2} a_{\sigma_2} + a_{\gamma_3} a_{\sigma_3} - a_{\gamma_4} a_{\sigma_4} + a_{\gamma_5} a_{\sigma_5} = 0,$$

where $\sigma = \delta, \varepsilon$.

In general, Theorem 3 tells us that $a_\mu a_{\mu^\sharp}$ appears with the sign $-N_{\lambda,-\mu} N_{\lambda^\sharp,\mu^\sharp}$. By (1) and the interpretation of $c_{\lambda\alpha}$ in § 2 this sign equals $c = -c_{\lambda,\mu-\lambda} c_{\lambda^\sharp,\lambda-\mu} = -c_{\lambda,\mu-\lambda} c_{\mu^\sharp,\mu-\lambda}$. Now Theorem 2 gives us an algorithm to compute this sign by looking at the weight diagram. For example, let $\mu - \lambda = \alpha_i + \alpha_j$ be a sum of two fundamental roots. Then the sign c depends on whether the order of labels in the $(\mu - \lambda)$-paths starting at λ and $\mu^\sharp$ is the same, in which case $c = -1$, or inverted, in which case $c = +1$. One can proceed similarly for other cases.

4°. Proof of Theorem 3. We subdivide the proof in several lemmas. Some of them are being taken over and generalised in [V6] as they play a key role in the 'decomposition of unipotents' for the adjoint case.

Lemma 8. *Let $a \in Gv^+$ be a vector in the orbit of the highest weight. Assume that for some $\beta \in \Sigma$ one has $a_\beta \neq 0$ and $a_\gamma = 0$ for all $\gamma \in \Sigma$, $(\widehat{\beta, \gamma}) = \pi/3$. Then $a \in Kv^+$, or, in other words, $a_\alpha = 0$ for all $\alpha \in \Sigma$, $\alpha \neq \beta$.*

Proof. Let $w \in \widetilde{W}$ be such that $w\beta = \rho$. Set $b = wa$. Then $b_{w(\alpha)} = 0$ if and only if $a_\alpha = 0$. Thus it suffices to prove the lemma for $\beta = \rho$.

Let Q be the stabilizer of the line Kv^+ in G. Then Q is a maximal parabolic subgroup of G with the Levi factor of type D_5 or E_6, respectively. In particular, the unipotent radical U_Q of Q – and thus also the unipotent radical U_Q^- of the opposite parabolic subgroup with the same Levi factor – is *abelian*. Clearly $\Omega = U_Q^- Q$ contains the *big cell* $B^- B$ and hence is a dense open subset of G. Thus, $X = \Omega v^+ = U_Q^- K v^+$ is a dense open subset of Gv^+. In the above interpretation X consists exactly of those root unipotents $x = \prod x_\gamma(a_\gamma)$, $\gamma \in \Sigma$, $a_\gamma \in K$, for which $a_\rho \neq 0$. As we have just seen, if an $x \in X$ is conjugate to $x_\rho(a_\rho)$, it must be conjugate to $x_\rho(a_\rho)$ already by an element of U_Q^-.

Now U_Q^- is a product of 16 or 27 commuting root subgroups, respectively, which correspond to roots $\gamma - \beta$, where $\gamma \in \Sigma$, $(\widehat{\beta, \gamma}) = \pi/3$. This means that for any $u \in U_Q^-$, $u \neq e$ at least one of the coefficients b_γ, $\gamma \in \Sigma$, $(\widehat{\beta, \gamma}) = \pi/3$, in the product $b = uv^+$ is distinct from zero. By assumption $a = u\xi v^+$ for some $u \in U_Q^-$, $\xi \in K$ and $a_\gamma = 0$ for all $\gamma \in \Sigma$, $(\widehat{\beta, \gamma}) = \pi/3$. This means that $a \in Kv^+$ which proves the lemma.

In the following lemma we take a product of the form $u = \prod x_{\gamma-\beta}(c_\gamma)$ over all $\gamma \in \Sigma$, $(\widehat{\beta, \gamma}) = \pi/3$. Since u is conjugate to an element of U_Q the order of the factors may be taken to be arbitrary.

Lemma 9. *Take a root $\beta \in \Sigma$ such that $a_\beta \neq 0$. Set $u = \prod x_{\gamma-\beta}(-N_{\gamma-\beta,\beta} a_\gamma a_\beta^{-1})$, where the product is taken over all $\gamma \in \Sigma$, $(\widehat{\beta, \gamma}) = \pi/3$ and let $b = ua$. Then for any $\alpha \in \Sigma$ orthogonal to β one has*

$$b_\alpha = a_\alpha - \sum N_{\alpha,-\gamma} N_{\beta,-\delta} a_\gamma a_\beta^{-1} a_\delta,$$

where the sum is taken over all unordered pairs $\{\gamma, \delta\}$, $\gamma, \delta \in \Sigma$, such that $\gamma + \delta = \alpha + \beta$.

Proof. By the choice of u one has $y = uxu^{-1} = \prod x_\gamma(b_\gamma)$, $\gamma \in \Sigma$, where $b_\beta = a_\beta$, whereas $b_\gamma = 0$ if $\gamma - \beta$ is a root, or, in other words, if g forms angle $\pi/3$ with ρ. A straightforward calculation shows that

$$b_\alpha = a_\alpha - \sum N_{\gamma-\beta,\alpha-\gamma+\beta} N_{\gamma-\beta,\beta} a_\gamma a_\beta^{-1} a_{\alpha-\gamma+\beta},$$

where the sum is taken over certain roots $\gamma \in \Sigma$ forming angle $\pi/3$ with β. Namely, γ must be such that $\alpha - \gamma + \beta$ is itself a root (then it automatically belongs to Σ). Moreover, if $\delta = \alpha - \gamma + \beta$ itself forms angle $\pi/3$ with β, only one of the roots γ and δ appears in the sum. Indeed, since conjugation by $x_\gamma(*)$ makes $b_\gamma = 0$, subsequent conjugation by $x_\delta(*)$ does not alter b_α.

As we know, $N_{\gamma-\beta,\delta} = N_{\beta-\gamma,\alpha}$. Thus, the above expression may be rewritten in the form

$$b_\alpha = a_\alpha - \sum N_{-\sigma,\alpha} N_{\sigma,\beta} a_{\alpha-\sigma} a_\beta^{-1} a_{\beta+\sigma} = 0,$$

$\sigma \in \Phi$ is such that $\beta + \sigma$, $\alpha - \sigma$ are roots and, if, moreover, $\alpha - \sigma = \beta + \tau$ for some root $\tau \in \Phi$, only one of σ, τ appears in the above sum. Using (1) once more, we get $N_{-\sigma,\alpha} = N_{\alpha,-\gamma}$ and $N_{\sigma,\beta} = N_{\beta,-\delta}$, which proves the lemma.

Next we prove that the equations in Theorem 3 are in fact consistent.

Lemma 10. *Equation (3) depends only on $\alpha + \beta$.*

Proof. Obviously Equation (3) is symmetric with respect to α and β. Let $\gamma, \delta \in \Sigma$ be another pair of orthogonal weights such that $\gamma + \delta = \alpha + \beta$. Then (up to renaming γ and δ) one has $\gamma=\alpha-\sigma$, $\delta=\beta+\sigma$ for some root $\sigma\in\Phi$. Equation (3) written in terms of γ, δ instead of α, β has the same summands $a_\varepsilon a_\eta$, where $\varepsilon+\eta=\alpha+\beta=\gamma+\delta$, and the only problem is to check that the signs also coincide. By (1) one has $N_{-\sigma,\alpha}N_{\sigma,\beta}=N_{-\sigma,\delta}N_{\sigma,\gamma}$, so there is no problem with the summand, corresponding to the pair (γ,δ).

Now, let (ε,η) be any other pair appearing in the right hand side of (3), say $\varepsilon=\alpha-\tau$, $\eta=\beta+\tau$. Then $\varepsilon=\gamma-(\tau-\sigma)$, $\eta=\delta+(\tau-\sigma)$. By (1) one has

$$N_{\alpha,-\varepsilon}N_{\alpha,-\gamma}=N_{-\varepsilon,-\tau}N_{-\gamma,-\sigma}\qquad\text{and}\qquad N_{\beta,-\eta}N_{\beta,-\delta}=N_{-\eta,\tau}N_{-\delta,\sigma}$$

and now applying (2) twice, to $\gamma-\varepsilon+\sigma-\tau=0$ and to $\delta-\eta+\tau-\sigma=0$, respectively, we get

$$N_{\alpha,-\varepsilon}N_{\alpha,-\gamma}N_{\beta,-\eta}N_{\beta,-\delta}=N_{\sigma,-\tau}N_{-\sigma,\tau}N_{\gamma,-\varepsilon}N_{\delta,-\eta}.$$

It follows that Equation (3) depends only on the sum $\alpha+\beta$.

Now we are in a position to establish Equation (3). Indeed, let $a=(a_\alpha)$, $\alpha\in\Sigma$ be in the orbit of v^+. Let $\{\alpha,\beta\}$, $\alpha,\beta\in\Sigma$, $(\widehat{\alpha,\beta})=\pi/2$, be any pair of orthogonal weights. First, let $a_\beta\neq 0$. Take the same u as in Lemma 9 and let $b=ua$. By the choice of u one has $b_\gamma=0$ for all $\gamma\in\Sigma$ forming angle $\pi/3$ with β. Then by Lemma 8 also $b_\alpha=0$. Now Lemma 9 gives us Equation 4 for the pair $\{\alpha,\beta\}$. It remains to prove that the same holds also when $a_\beta=0$. If $a_\gamma=0$ for all $\gamma\in\Sigma$ forming part of another orthogonal pair $\{\gamma,\delta\}$ such that $\gamma+\delta=\alpha+\beta$, there is nothing to prove, since in this case Equation (3) holds automatically. On the other hand if $a_\gamma\neq 0$ for such a γ we can apply the same argument as above to the pair $\{\gamma,\delta\}$ instead of the pair $\{\alpha,\beta\}$. By Lemma 10 we still get the required equation. This finishes the proof of Theorem 3.

§ 4. Elements of root type

In this section we introduce a class of elements which generalizes the class of root unipotents and preserves most of their properties. The property which is relevant for our purposes is that they satisfy an analogue of the Whitehead–Vaserstein lemma.

1°. Elements of root type versus root unipotents. Recall that a long root unipotent x is a conjugate of an elementary long root unipotent $x_\delta(\xi)$, where $\delta\in\Phi$ is the maximal root and $\xi\in R$. When $R=K$ is a field, long root unipotents can be defined by polynomial equations. But when R is a ring, the class of long root unipotents as defined above is usually too narrow to be useful.

To explain, where the problem lies, look at the root unipotents in $\mathrm{SL}(n,K)$. In the natural representation they correspond to transvections, which are defined by the condition $\mathrm{rk}(x-e)\leq 1$, which amounts to a set of polynomial equations: triviality of all minors of degree 2. Namely, let $y=x-e$. Then $\mathrm{rk}(y)\leq 2$ amounts to $y_{ih}y_{jl} = y_{il}y_{jh}$ for all $i, j, h, l = 1, \dots, n$. (The other condition in the usual definition of a transvection over a field $\mathrm{tr}(x - e) = 0$ follows from the fact that $x \in \mathrm{SL}(n, K)$). Over a field any solution of these equations is a conjugate of the elementary transvection $t_{1n}(\xi) = e + \xi e_{1n}$. This is not necessarily the case over a ring. Let $n = 3$ and take a Dedekind ring R other than a discrete valuation ring and let $\zeta, \theta \in R$ generate a non-principal ideal I. Then the matrix

$$t_{12}(\zeta)t_{13}(\theta) = \begin{pmatrix} 1 & \zeta & \theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

satisfies the above equations, but it is not conjugate to a matrix of the form

$$t_{13}(\xi) = \begin{pmatrix} 1 & 0 & \xi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(For if it were, this would mean that I is generated by ξ, contrary to the assumption.)

A class of elements which obviously satisfy the above equations are the elements of the form $e + \alpha\xi\beta = (\delta_{ij} + \alpha_i\xi\beta_j)$, $1 \leq i, j \leq n$, where $\xi \in R$, $\alpha = (\alpha_1, \dots, \alpha_n)^t \in R^n$ is a column of height n, $\beta = (\beta_1, \dots, \beta_n) \in {}^nR$ is a row of length n, and $\beta\alpha = \beta_1\alpha_1 + \dots + \beta_n\alpha_n = 0$. In the theory linear groups over rings such elements are usually called *transvections*. Obviously the element $t_{12}(\zeta)t_{13}(\theta)$ considered above is a transvection (take $\xi = 1$, $\alpha = (1, 0, 0)^t$ and $\beta = (0, \zeta, \theta)$). Transvections have extremely nice properties. First of all, a conjugate of a transvection by an element of $G = \mathrm{SL}(n, K)$ is again a transvection.

In general it is not obvious even how to define the “elements of root type” which are analogues of transvections for other Chevalley groups. Below we simply list elements of root type in some of the easiest cases, namely the minimal representations of the classical groups, E_6 and E_7. In [VPe] we define “transvections” in polyvector representations of SL_n. In § 1 of [VPe] one can find a much more detailed discussion of the ring theoretic aspects of various definitions of a transvection.

For the classical groups these are the *Eichler–Siegel–Dickson transvections* (the *ESD-transvections* for short). In these cases V carries a non-degenerate bilinear form $(\ ,\ ) : V \times V \longrightarrow R$, which identifies V^* with V. Let $\widetilde{u}$ be the row corresponding to u under this isomorphism. Thus in these cases we can define transvections in terms of two columns, instead of a column and a row. In the symplectic case u and v will be arbitrary orthogonal columns. In the orthogonal case V supports also a quadratic form $Q : V \longrightarrow R$. In this case a column u is called *singular*, if $Q(u) = 0$. In the orthogonal case u and v will be singular orthogonal columns. A typical ESD-transvections has the form

$$T_{uv}(\xi) = e + u\xi\widetilde{v} - v\xi\widetilde{u}.$$

In fact the above conditions can be relaxed, and a more general type of transformations can be defined, without assumption that v is singular.

Now, let $\Phi = \mathrm{E}_6$, $V = V(\overline{\varpi}_1)$. In this case V is not selfdual, elements of V will be represented by columns and elements of V^* by rows. There is a natural pairing $(\ ,\ ) : V^* \times V \longrightarrow R$ given by multiplication. There are also vector products $V \times V \longrightarrow V^*$ and $V^* \times V^* \longrightarrow V$ defined by the trilinear form F. Thus for two columns u, v the row $u \times v$ is defined by the equality $(u \times v)w = F(u, v, w)$. Assume that u is a singular column, v is a singular row and $vu = 0$ (these conditions can be relaxed, but we do not discuss this possibility here). Then for any $\xi \in R$ one defines a Freudenthal transvection $T_{uv}(\xi)$ by its action on $x \in V$ as follows

$$T_{uv}(\xi)x = x + (v, x)u + v \times (u \times x).$$

These transformations indeed preserve the trilinear form F. This is checked in [S1] for a more general type of transformations, the Freudenthal–Springer transvections. See also [V2] for many further references.

Finally, let $\Phi = \mathrm{E}_7$, $V = V(\overline{\varpi}_7)$. In this case, as in the classical cases, $V \cong V^*$ so that we can define the transvections by two columns $u, v \in V$, rather than by a column and a row. Assume for simplicity that u and v generate a singular subspace (this condition can be relaxed but this leads to more complicated formulae). Then the element of root type $T_{uv}(\xi)$ in G can be defined as

$$T_{uv}(\xi)x = x + (u, x)\xi v - (v, x)\xi u + \xi\{uvx\},$$

where $\{uvx\}$ is the ternary product defined by the four-linear form F and the symplectic form h as follows $h(\{uvx\}, y) = F(u, v, x, y)$. Again it can be checked that these transformations preserve both the symplectic product and the form F and thus sit inside E_7.

2°. Fake root unipotents. For most purposes it is sufficient to work with the conjugates of the elements of root type contained in U. First we take a field $R = K$ and look at the equations satisfied by the long root unipotents contained in U. The following lemma shows that everything happens in the unipotent radicals of the maximal parabolic subgroups.

Lemma 11. *Any long root unipotent $x \in U$ is contained in the unipotent radical U_r of a maximal parabolic subgroup P_r, $1 \leq r \leq l$.*

Proof. Let $x = gx_\delta(\xi)g^{-1}$ for some $g \in G$. Let $g = uwdv$, where $u, v \in U$, $w \in W$, $d \in T$, be the Bruhat decomposition of g. Since X_δ is normalised by the Borel subgroup, one has $x = uwx_\delta(\zeta)w^{-1}u^{-1} = ux_{w(\delta)}(\zeta)u^{-1}$ for some $\zeta \in R$. The case $w(\delta) < 0$ is obviously impossible. Let $\alpha = w(\delta) > 0$ and let $\alpha = \sum m_i\alpha_i$ be the expansion of α as a linear combination of the fundamental roots. If $m_r \neq 0$, then $x_\alpha(\zeta) \in U_r$ and thus also $ux_\alpha(\zeta)u^{-1} \in U_r$.

In the preceding subsections we considered an explicit form of the equations which $x \in U_r$ has to satisfy to be a root unipotent in the special case when U_r

is *abelian*. All of them are quadratic equations of the form $\sum \pm u_\alpha u_\beta = 0$, where the angle between α and β equals $\pi/2$ or $2\pi/3$. Let now Σ be a set of roots such that all angles $(\widehat{\alpha,\beta})$ between $\alpha, \beta \in \Sigma$ are equal to $\pi/3$. Then all of the above equations are void. Thus we get the following fact.

Lemma 12. *Let K be a field and Σ be a set of roots such that $(\widehat{\alpha,\beta}) = \pi/3$ for all $\alpha, \beta \in \Sigma$. Then for any choice of $u_\alpha \in K$ the product $\prod x_\alpha(u_\alpha)$, $\alpha \in \Sigma$, is a root unipotent.*

In the paper [C1] this lemma is expressed by saying that $\prod X_\alpha$, $\alpha \in \Sigma$, forms a $(m-1)$-dimensional subspace in the geometry of root subgroups, where $m = |\Sigma|$. Let now R be a ring. Then as the above example of SL_3 shows, the product $\prod x_\alpha(u_\alpha)$, $\alpha \in \Sigma$, $u_\alpha \in R$, is not a root unipotent anymore, but it can be checked that these elements and their conjugates behave as root unipotents for all practical matters. This motivates the following definition.

Let Σ be a set of roots such that $(\widehat{\alpha,\beta}) = \pi/3$ for all $\alpha, \beta \in \Sigma$, $|\Sigma| = m$. Then the elements of the form $\prod x_\alpha(u_\alpha)$, $\alpha \in \Sigma$, $u_\alpha \in R$, are called *elementary fake root unipotents* of shape A_m. Their conjugates are called *fake root unipotents* of shape A_m.

These are the only fake root unipotents which we encounter in § 5. However in the case $\Phi = \mathrm{E}_8$ (and in general in the study of the adjoint representations) one needs a more general type of fake root unipotents, the ones of shape D_m. Namely, let Σ be a set of roots such that for all $\alpha \in \Sigma$ one has $(\widehat{\alpha,\beta}) = \pi/3$ for all $\beta \in \Sigma$ except exactly one of them, denoted by α^*, whereas $(\widehat{\alpha,\alpha^*}) = \pi/2$. First, let K be a field. As we have seen above, to be a root unipotent a product $\prod x_\alpha(u_\alpha)$, $\alpha \in \Sigma$, $u_\alpha \in K$, has to satisfy a single quadratic equation, of the form $\sum \pm u_\alpha u_{\alpha^*} = 0$. Now, let R be a ring. A product $\prod x_\alpha(u_\alpha)$, $\alpha \in \Sigma$, $u_\alpha \in R$, satisfying the above equation is called a fake root unipotent of shape D_{m+1}, where $|\Sigma| = 2m$.

3°. Whitehead–Vaserstein lemma. In this subsection we state without proof an analogue of the Whitehead–Vaserstein lemma. This lemma is *not* used in the proof of the main results of the next section. It is needed rather to show how these results imply the main structure theorems for Chevalley groups over commutative rings. A similar result is valid for elements of root type, but to define these elements and to prove the addition and commutation formulae for them would require a much more detailed analysis of the geometry of minimal modules, than what we are willing to offer in the present paper.

Here we will need the result only in the following case. Let P be one of the following parabolic subgroups in G: P_1 or P_6 for $\Phi = \mathrm{E}_6$ and P_7 for $\Phi = \mathrm{E}_7$. First, let $\Phi = \mathrm{E}_6$. After an appropriate renumbering of the admissible base the matrices representing the elements of P_1 and P_6 on the minimal module $V = V(\overline{\omega}_1)$ have the following block forms, respectively:

$$\begin{pmatrix} \boxed{1} & * & * \\ 0 & \boxed{16} & * \\ 0 & 0 & \boxed{10} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \boxed{10} & * & * \\ 0 & \boxed{16} & * \\ 0 & 0 & \boxed{1} \end{pmatrix},$$

where the boxed numbers are the degrees of the diagonal blocks. Clearly, these degrees are just the dimensions of the irreducible summands of the restriction of V to the Levi factor of P, which in both cases has type D_5 (these are the trivial, the vector and the half-spin modules of the group $\mathrm{Spin}(10, R)$ – with P_1 and P_6 giving contragredient half-spins). The conjugacy classes represented by these two subgroups are fused by the external automorphism of $G(\mathrm{E}_6, R)$ and in the sequel we will speak only of P_1.

In the case $\Phi = \mathrm{E}_7$ the group P_7 on the module $V = V(\overline{\omega}_7)$ is represented by matrices of the form

$$\begin{pmatrix} \boxed{1} & * & * & * \\ 0 & \boxed{27} & * & * \\ 0 & 0 & \boxed{27} & * \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$

Here again the degrees of the diagonal blocks are the dimensions of the irreducible summands of the restriction of V to the Levi factor of type E_6 (two trivial ones and two contragredient modules of dimension 27).

There are $|W(\mathrm{E}_6)/W(\mathrm{D}_5)| = 27$ subgroups wPw^{-1} conjugate in $G = G(\mathrm{E}_6, R)$ to $P = P_1$ by an element w of the extended Weyl group $\widetilde{W}(\mathrm{E}_6)$. By the same token there are $|W(\mathrm{E}_7)/W(\mathrm{E}_6)| = 56$ subgroups wPw^{-1} conjugate in $G = G(\mathrm{E}_7, R)$ to $P = P_7$ by an element w of the extended Weyl group $\widetilde{W}(\mathrm{E}_7)$. Here is the version of the Whitehead–Vaserstein lemma, we need.

Proposition 2. *Let $G = G(\Phi, R)$, where $\Phi = \mathrm{E}_l$, $l = 6, 7$. Further, let $P = P_1, P_7$ depending on whether $l = 6$ or $l = 7$. Assume that $z \in G(\Phi, R)$ is a fake root unipotent of shape A_m contained in a proper parabolic subgroup wPw^{-1}, $w \in \widetilde{W}(\Phi)$. Then $z \in E(\Phi, R)$.*

In this form the proposition suffices to deduce from Theorem 4 normality of $E(\Phi, R)$ in $G(\Phi, R)$. But stronger statements concerning the commutators of the relative elementary subgroups/congruence subgroups require explicit expressions of z as a product of elementary transvections. There are indeed very pretty polynomial formulae which express z as a product of elements of the unipotent radicals

U_P, U_P^- of P and the opposite parabolic. See [V1] for a uniform purely algebraic proof of a slightly weaker result for all parabolic subgroups in groups of all types. In [BV], § 6 one can find a very detailed geometric treatment of the classical cases. In fact the maximal parabolic subgroups are only one type of reducible subgroups, the ones stabilizing totally isotropic submodules. For classical types other than A_l there are also stabilizers of non-degenerate submodules and in fact [BV] contains also other Whitehead-type lemmas, like the Kopeiko–Taddei lemma, which allow to conclude that some further ESD-transvections, not contained in proper parabolics, still belong to the elementary subgroup. The general case is considered geometrically in a forthcoming paper by the author.

§ 5. Decomposition of unipotents

In this section we show that the results of the three preceding sections suffice to complete the key step in the proof of the main structure theorems (normality of the elementary subgroup, standard commutator formulae, description of normal subgroups, centrality of K_2, and the like) for the types E_6 and E_7 sketched in [VPS], [V2], [VP].

1°. Statement of results. The central step in the above proof was the possibility to find enough root type unipotents stabilizing a column v of a matrix $\pi(g)$, representing an element $g \in G$ in a minimal representation π. The entries of these unipotents had to be polynomials in the components of v with integral coefficients. For the classical types this is fairly easy, see [V2], §§ 10 – 12, or [SV], which is specifically devoted to the case of GL_n and other split classical groups. However our original proof for the exceptional groups was purely computational and referred to the explicit tables of structure constants, [GS]. We show that already Theorem 2 suffices to give an elementary proof of the following result for the types E_6 and E_7.

Theorem 4. *Let $\Phi = \mathrm{E}_6$ or E_7 and fix a $g \in G = G(\Phi, R)$. Then the elementary subgroup $E(\Phi, R)$ is generated by the elementary fake root unipotents z such that $zv = v$, as v runs over the columns $g_{*\mu}$ of the matrix $g = (g_{\lambda\mu})$, $\lambda, \mu \in \Lambda(\pi)$.*

To illustrate the significance of this result we observe that it immediately implies normality of the elementary subgroup $E(\Phi, R)$ in the Chevalley group $G(\Phi, R)$, and more. Indeed, take an element $g \in G(\Phi, R)$. By Theorem 4 the elementary subgroup $E(\Phi, R)$ is generated by the elementary fake root unipotents z such that $zg'_{*\mu} = g'_{*\mu}$ for the μ-th column of the matrix g^{-1}. For such a unipotent z the μ-th column of the conjugate gzg^{-1} equals the μ-th column of the identity matrix v_μ. Let P the same parabolic subgroup, as in § 4, i.e. $P = P_1$ for $\Phi = \mathrm{E}_6$ and $P = P_7$ for $\Phi = \mathrm{E}_7$. The above means precisely that gzg^{-1} is contained in a conjugate of P by an element of $\widetilde{W}$ and thus Proposition 2 implies that $gzg^{-1} \in E(\Phi, R)$ for every such fake root unipotent z. Since they generate the elementary subgroup, it follows that $gE(\Phi, R)g^{-1} \leq E(\Phi, R)$. For the explanation of how one can deduce other main structure theorems for Chevalley groups from Theorem 4 see [VPS], [V2] or [SV].

Below we present the first complete proof of Theorem 4 in the case of exceptional groups. In fact [VPS] contained only the statement, whereas in [V2] only a sketch of the proof for E_6 was given, dropping the explicit technical check that the signs of the action actually agrees with the signs of the equations. Our present proof is based on the following result.

Theorem 5. *Let (V,π) be a microweight module of a Chevalley group $G = G(\Phi, R)$ of type E_6 or E_7. Then for any singular vector $u = (u_\lambda)$, $\lambda \in \Lambda(\pi)$, there exists a non-trivial elementary fake root unipotent $z \in \mathrm{U}(\Phi, R)$ of shape A_m such that $zv = v$. Moreover, one may choose such a z so that its coordinates are equal to $\pm u_\lambda$, $\lambda \in \Lambda(\pi)$.*

As we shall see from the explicit construction of z below, $m = 5$ for $\Phi = \mathrm{E}_6$ and $m = 7$ for $\Phi = \mathrm{E}_7$. This means that in both cases we use fake root elements corresponding to a maximal "singular subspace" in the geometry of root subgroups, [C1]. Namely, let $\beta_1, \dots, \beta_m$ be a maximal set of roots forming mutual angles $\pi/3$. In both cases one may find a required z of the form $z = \prod x_{\beta_i}(\pm u_{\lambda_i})$, $i = 1, \dots, m$.

In other words, the proofs for E_6 and E_7 are based on the existence of embeddings $\mathrm{A}_5 \subseteq \mathrm{E}_6$ and $\mathrm{A}_7 \subseteq \mathrm{E}_7$. As a matter of comparison observe that the proofs for classical groups are based on the embeddings $\mathrm{A}_2 \subseteq \mathrm{A}_l$, $\mathrm{C}_2 \subseteq \mathrm{C}_l$, $\mathrm{B}_2 \subseteq \mathrm{B}_l$ or $\mathrm{D}_3 \subseteq \mathrm{B}_l$ and $\mathrm{D}_3 \subseteq \mathrm{D}_l$ respectively (see [V2] or [SV]). Thus for the classical groups one can find a required unipotent z in a very small subgroup. Another feature of the proof for the classical cases is that no equations are imposed on the columns, there are enough unipotents in a classical group to stabilize *any* column.

The proofs for E_6 and E_7 are more demanding: the unipotents come from a fairly large subgroup and we can stabilize only *singular* columns. But (as in the simply laced classical cases) the coordinates of the unipotent z coincide with the components of u up to sign. The situation for other exceptional groups is still worse: more complicated patterns are required to stabilize a column, usually related to geometries of type D_m. The coordinates of z are polynomials of degree two or three, which involve up to six components of u. Even to check that the unipotent z which we construct in this way is actually a fake root unipotent one has to use equations.

2°. An A_5-proof of Theorem 5 for E_6. Let us recall that in E_6 the maximal number of roots every two of which form the angle $\pi/3$ is five. Now we fix such a set which is maximal with respect to the chosen order of positive roots. We take the following roots

$$\beta_1 = \substack{12321\\2}, \quad \beta_2 = \substack{12321\\1}, \quad \beta_3 = \substack{12221\\1}, \quad \beta_4 = \substack{12211\\1}, \quad \beta_5 = \substack{12210\\1}\,.$$

(Note that the notation we use in the present paper dramatically differs from the notation of [V2], § 13). Since all of their differences are roots too they form such a set. Now the products

$$z = x_{\beta_1}(z_1)x_{\beta_2}(z_2)x_{\beta_3}(z_3)x_{\beta_4}(z_4)x_{\beta_5}(z_5)$$

are elements of long root type for all values of $z_1, \dots, z_5 \in R$. The action of the elements $x_{\beta_1}(z_1), \dots, x_{\beta_5}(z_5)$ on the 27-dimensional module V with the highest weight $\overline{\omega}_1$ may be described as follows. Recall that we may interpret the weights of V as the roots of $\Sigma = \Sigma_7 = \Sigma_7(1)$ in E_7. Consider the following three series of weights:

$$\gamma_1 = \begin{smallmatrix}001111\\1\end{smallmatrix}, \quad \gamma_2 = \begin{smallmatrix}001111\\0\end{smallmatrix}, \quad \gamma_3 = \begin{smallmatrix}000111\\0\end{smallmatrix}, \quad \gamma_4 = \begin{smallmatrix}000011\\0\end{smallmatrix}, \quad \gamma_5 = \begin{smallmatrix}000001\\0\end{smallmatrix},$$

$$\delta_1 = \begin{smallmatrix}011111\\0\end{smallmatrix}, \quad \delta_2 = \begin{smallmatrix}011111\\1\end{smallmatrix}, \quad \delta_3 = \begin{smallmatrix}012111\\1\end{smallmatrix}, \quad \delta_4 = \begin{smallmatrix}012211\\1\end{smallmatrix}, \quad \delta_5 = \begin{smallmatrix}012221\\1\end{smallmatrix},$$

$$\varepsilon_1 = \begin{smallmatrix}111111\\0\end{smallmatrix}, \quad \varepsilon_2 = \begin{smallmatrix}111111\\1\end{smallmatrix}, \quad \varepsilon_3 = \begin{smallmatrix}112111\\1\end{smallmatrix}, \quad \varepsilon_4 = \begin{smallmatrix}112211\\1\end{smallmatrix}, \quad \varepsilon_5 = \begin{smallmatrix}112221\\1\end{smallmatrix}.$$

Then a straightforward calculation shows that

(1) $\beta_i + \gamma_j \in \Sigma$ if and only if $i \neq j$,
(2) $\gamma_{ij} = \beta_i + \gamma_j = \beta_j + \gamma_i$,
(3) $\beta_i + \rho_j \in \Sigma$, $\rho = \delta, \varepsilon$, if and only if $i = j$,
(4) $\rho = \beta_i + \rho_i = \beta_j + \rho_j$, $\rho = \delta, \varepsilon$,
(5) $\beta_i + \alpha \notin \Sigma$, if $\alpha \neq \gamma_j, \delta_i, \varepsilon_i$,
(6) The 27 weights γ_i, δ_i, ε_i, $\gamma_{ij} = \beta_i + \gamma_j$, $i < j$, $\delta = \beta_1 + \delta_1$ and $\varepsilon = \beta_1 + \varepsilon_1$ are all distinct.

This completely describes the action of z on an *arbitary* vector $u = (u_\lambda) \in V$ up to signs. Indeed, $x_{\beta_i}(z_i)$ multiplies u_{γ_j} by z_i and adds it to or subtracts it from $u_{\gamma_{ij}}$. Moreover, $x_{\beta_i}(z_i)$ multiplies u_{ρ_i}, $\rho = \delta, \varepsilon$, by z_i and adds it to or subtracts them from u_ρ.

Now we take a *singular* vector $u = (u_\lambda) \in V$ and try to choose the coefficients $z_1, \dots, z_5$ in the expression for z so that z stabilizes u. There is little doubt that if we want z to stabilize u we have to set $z_1 = \pm\xi u_{\gamma_1}, \dots, z_5 = \pm\xi u_{\gamma_5}$, and the real problem is to choose the signs in such a way that everything cancels. Indeed, in this case $x_{\beta_i}(z_i)$ and $x_{\beta_j}(z_j)$ add $\pm\xi u_{\gamma_i} u_{\gamma_j}$ and $\pm\xi u_{\gamma_j} u_{\gamma_i}$ respectively to $u_{\gamma_{ij}}$.

With the same choice of an admissible base, as in Theorem 1 we fix the signs of z_i as follows:

$$z = x_{\beta_1}(\xi u_{\gamma_1})x_{\beta_2}(\xi u_{\gamma_2})x_{\beta_3}(\xi u_{\gamma_3})x_{\beta_4}(\xi u_{\gamma_4})x_{\beta_5}(-\xi u_{\gamma_5}).$$

The canonical strings of the roots β_i, $i = 1, \dots, 5$ are as follows:

$$24315423456, \quad 4315423456, \quad 315423456, \quad 31423456, \quad 3142345.$$

A straightforward calculation[7] based on Theorem 2 shows that with this choice of signs one has

$$x_{\beta_i}(z_i)v^{\gamma_j} - v^{\gamma_j} = \begin{cases} \xi u_{\gamma_i} v^{\gamma_{ij}}, & \text{if} \quad i < j, \\ -\xi u_{\gamma_i} v^{\gamma_{ij}}, & \text{if} \quad i > j, \end{cases} \tag{4}$$

where $1 \le i \ne j \le 5$. In other words, the contributions of $x_{\beta_i}(z_i)$ and $x_{\beta_j}(z_j)$ to $u_{\gamma_{ij}}$ always appear with opposite signs and, thus, cancel. Notice that in terms of the structure constants this formula asserts that $N_{\beta_i,\gamma_j} = -N_{\beta_j,\gamma_i}$ if $1 \le i \ne j \le 4$, but $N_{\beta_5,\gamma_j} = N_{\beta_j,\gamma_5}$ when $i = 5$ (this is exactly why we had to set $z_5 = -\xi u_{\gamma_5}$ to make the above contributions cancel).

It remains to check that z does not change u_δ and u_ε. In reality multiplication by z adds

$$\xi(N_{\beta_1,\rho_1} u_{\gamma_1} u_{\rho_1} + \dots + N_{\beta_4,\rho_4} u_{\gamma_4} u_{\rho_4} - N_{\beta_5,\rho_5} u_{\gamma_5} u_{\rho_5})$$

to u_ρ where $\rho = \delta, \varepsilon$, and for an arbitrary column there is no reason to expect that these sums are zeros. However we assume that the column is singular.

In other words, we claim, that, always for $\rho = \delta$ or ε, one has

$$u_{\gamma_5} u_{\rho_5} = \sum N_{\beta_5,\rho_5} N_{\beta_i,\rho_i} u_{\gamma_i} u_{\rho_i}, \qquad 1 \le i \le 4.$$

It is almost the same as the equation in the statement of Theorem 3, only that the coefficients in the right hand side should be $N_{\gamma_5,-\gamma_i} N_{\rho_5,-\rho_i}$, instead of $\mathrm{N}_{\beta_5,\rho_5} \mathrm{N}_{\beta_i,\rho_i}$. It remains to check that the coefficients actually coincide.

But this almost immediately follows from the usual properties of the structure constants and (4) above. In fact, applying (2) to $\beta_5 + \rho_5 - \beta_i - \rho_i = 0$ we get

$$N_{\beta_5,\rho_5} N_{\beta_i,\rho_i} = N_{\beta_5,\rho_5} N_{-\rho_i,-\beta_i} = -N_{\rho_5,-\rho_i} N_{\beta_5,-\beta_i}$$

(since $N_{-\rho_i,-\beta_5} N_{\rho_5,-\beta_i} = 0$). Thus, it remains only to check that $N_{\beta_5,-\beta_i} = -N_{\gamma_5,-\gamma_i}$. Applying (2) to $\beta_5 - \beta_i + \gamma_i - \gamma_5 = 0$ and using (4) we get

$$N_{\beta_5,-\beta_i} N_{\gamma_i,-\gamma_5} = -N_{\gamma_i,\beta_5} N_{-\beta_i,-\gamma_5} = -N_{\beta_5,\gamma_i} N_{\beta_i,\gamma_5} = -1$$

(since $N_{-\beta_i,g_i} N_{\beta_5,-\gamma_5} = 0$). This finishes the proof of Theorem 5 for E_6.

3°. An A_7-proof of Theorem 5 for E_7. Let us recall that in E_7 the maximal number of roots every two of which form the angle $\pi/3$ is seven. We fix such a set

[7] This calculation and the similar calculation for $\Phi = \mathrm{E}_7$ were performed by hand and then checked using Mathematica 2.0 for DEC RISC.

which is maximal with respect to the chosen order of positive roots. We take the following roots

$$\beta_1 = \begin{smallmatrix}234321\\2\end{smallmatrix}, \quad \beta_2 = \begin{smallmatrix}134321\\2\end{smallmatrix}, \quad \beta_3 = \begin{smallmatrix}124321\\2\end{smallmatrix}, \quad \beta_4 = \begin{smallmatrix}123321\\2\end{smallmatrix},$$

$$\beta_5 = \begin{smallmatrix}123221\\2\end{smallmatrix}, \quad \beta_6 = \begin{smallmatrix}123211\\2\end{smallmatrix}, \quad \beta_7 = \begin{smallmatrix}123210\\2\end{smallmatrix}.$$

Since all of their differences are roots too they form such a set. As above the products

$$z = x_{\beta_1}(z_1)x_{\beta_2}(z_2)\dots x_{\beta_7}(z_7)$$

are elements of long root type for all values of $z_1, \dots, z_7 \in R$. The action of the elements $x_{\beta_1}(z_1), \dots, x_{\beta_7}(z_7)$ on the 56-dimensional module V with the highest weight $\overline{\omega}_7$ may be described as follows. Recall that we may interpret the weights of V as the roots of $\Sigma = \Sigma_8(1)$ in E_8. Consider the following series of weights:

$$\gamma_1 = \begin{smallmatrix}1111111\\0\end{smallmatrix}, \quad \gamma_2 = \begin{smallmatrix}0111111\\0\end{smallmatrix}, \quad \gamma_3 = \begin{smallmatrix}0011111\\0\end{smallmatrix}, \quad \gamma_4 = \begin{smallmatrix}0001111\\0\end{smallmatrix},$$

$$\gamma_5 = \begin{smallmatrix}0000111\\0\end{smallmatrix}, \quad \gamma_6 = \begin{smallmatrix}0000011\\0\end{smallmatrix}, \quad \gamma_7 = \begin{smallmatrix}0000001\\0\end{smallmatrix}.$$

Let

$$\rho = \begin{smallmatrix}2465432\\3\end{smallmatrix}$$

be the maximal root of E_8. Then a straightforward calculation shows that

(1) $\beta_i + \gamma_j \in \Sigma$ if and only if $i \neq j$,
(2) $\gamma_{ij} = \beta_i + \gamma_j = \beta_j + \gamma_i$,
(3) The 56 weights γ_i, γ_{ij}, $i < j$, $\gamma_{ij}^* = \rho - \gamma_{ij}$, $i < j$, $\gamma_i^* = \rho - \gamma_i$, are all distinct.
(4) $\beta_i + \gamma_{jh}^* \in \Sigma$, if and only if $i = j$ or $i = h$, in fact $\beta_i + \gamma_{ij}^* = \gamma_j^*$,
(5) $\beta_i + \alpha \notin \Sigma$, if $\alpha \neq \gamma_j, \gamma_{ij}^*$.

The four above series of roots (which correspond to the branching of the restriction of the 56-dimensional module for E_7 to A_6) may be described as follows: γ_i are precisely the roots $\gamma \in \Sigma$ such that $m_2(\gamma) = 0$; γ_{ij}^* are precisely the roots such that $m_2(\gamma) = 1$; γ_{ij} are precisely the roots such that $m_2(\gamma) = 2$ and, finally, γ_i^* are precisely the roots $\gamma \in \Sigma$ such that $m_2(\gamma) = 3$;

This completely describes the action of z on an *arbitary* vector $u = (u_\lambda) \in V$ up to signs. Indeed, $x_{\beta_i}(z_i)$ multiplies u_{γ_j} by z_i and adds it to or subtracts it from $u_{\gamma_{ij}}$. Moreover, $x_{\beta_i}(z_i)$ multiplies $u_{\gamma_{ij}^*}$, by z_i and adds it to or subtracts them from $u_{\gamma_j^*}$.

Now we take a *singular* vector $u=(u_\lambda)\in V$ and try to choose the coefficients $z_1,\dots,z_7$ in the expression for z so that z stabilizes u. As above we have to set $z_1=\pm\xi u_{\gamma_1},\ \dots,\ z_7=\pm\xi u_{\gamma_7}$. Indeed, in this case $x_{\beta_i}(z_i)$ and $x_{\beta_j}(z_j)$ add $\pm\xi u_{\gamma_i}u_{\gamma_j}$ and $\pm\xi u_{\gamma_j}u_{\gamma_i}$ respectively to $u_{\gamma_{ij}}$.

With the same choice of an admissible base, as in Theorem 1 we fix the signs of z_i as follows:

$$z=x_{\beta_1}(\xi u_{\gamma_1})x_{\beta_2}(\xi u_{\gamma_2})x_{\beta_3}(\xi u_{\gamma_3})x_{\beta_4}(\xi u_{\gamma_4})x_{\beta_5}(\xi u_{\gamma_5})x_{\beta_6}(\xi u_{\gamma_6})x_{\beta_7}(-\xi u_{\gamma_7}).$$

The canonical strings of the roots β_i, $i=1,\dots,7$ are as follows:

13425431654234567, 3425431654234567, 425431654234567,

25431654234567, 2431654234567, 243154234567, 24315423456.

Again a straightforward calculation based on Theorem 2 shows that with this choice of signs an analogue of (4) holds for all $1\le i\ne j\le 7$ and thus the contributions of $x_{\beta_i}(z_i)$ and $x_{\beta_j}(z_j)$ to $u_{\gamma_{ij}}$ cancel.

It remains to check that z does not change $u_{\gamma_i^*}$. In reality multiplication by z adds to $u_{\gamma_i^*}$ the following sum

$$\xi(N_{\beta_1,\gamma_{i1}^*}u_{\gamma_1}u_{\gamma_1^*}+\dots+\widehat{N_{\beta_i,\gamma_{ii}^*}u_{\gamma_i}}u_{\gamma_{ii}^*}+\dots+N_{\beta_6,\gamma_{i6}^*}u_{\gamma_6}u_{\gamma_{i6}^*}-N_{\beta_7,\gamma_{i7}^*}u_{\gamma_7}u_{\gamma_{i7}^*}),$$

where the i-th summand is omitted. We have to check, that these sums are zeros. This could follow only from the equations defining a singular column.

First, let $1\le i\le 6$. Then the condition that the above sum equals zero coincides with the equation in Theopem 3 up to signs. In fact, we claim that for all $i\ne 7$ one has

$$u_{\gamma_7}u_{\gamma_{i7}^*}=\sum N_{\beta_7,\gamma_{i7}^*}N_{\beta_j,\gamma_{ij}^*}u_{\gamma_j}u_{\gamma_{ij}^*},\qquad 1\le j\le 6,\quad j\ne i. \tag{5}$$

It is almost the same as Equation (3) in Theorem 3, only that the coefficients in the right hand side should be $N_{\gamma_7,-\gamma_j}N_{\gamma_{i7}^*,-\gamma_{ij}^*}$, rather than $N_{\beta_7,\gamma_{i7}^*}N_{\beta_j,\gamma_{ij}^*}$. But exactly the same calculation as in the case of E_6 based on (2) and the equality $N_{\beta_7,\gamma_j}N_{\beta_j,\gamma_7}=1$ – which is a special case of (4) – shows that these two expressions do in fact coincide.

The case $i=7$ is treated similarly, only that now we must get the equation

$$u_{\gamma_6}u_{\gamma_{76}^*}=-\sum N_{\beta_6,\gamma_{76}^*}N_{\beta_j,\gamma_{7j}^*}u_{\gamma_j}u_{\gamma_{7j}^*},\qquad 1\le j\le 5.$$

This is indeed the same, as above, with the following modification: now it follows from (4) that $N_{\beta_6,\gamma_j}N_{\beta_j,\gamma_6}=-1$, $1\le j\le 5$, which accounts for the change of sign of the right hand side as compared with (5). This finishes the proof of Theorem 5 for E_7.

4°. Proof of Theorem 4. Let $H=H(g)$ be the group generated by the elementary fake root unipotents such that $zv=v$ for some column $v=g_{*\mu}$ of the matrix $g=(g_{\lambda\mu})$, $\lambda,\mu\in\Lambda(\pi)$. Let $g^{-1}=(g'_{\lambda\mu})$, $\lambda,\mu\in\Lambda(\pi)$, be the inverse matrix. The following fact is obvious from the definition.

Lemma 13. *Let Φ be a simply laced root system. Then $E(\Phi, R)$ is the smallest subgroup containing an elementary root subgroup X_α, for some $\alpha \in \Phi$, and normalized by the extended Weyl group $\widetilde{W}$.*

Now we prove that H satisfies the conditions of this lemma. Indeed, Theorem 5 immediately implies the following result. Here we return to the notation of the preceding subsections, in particular, $\Phi = E_l$, $l = 6, 7$, and β_1 has the same meaning as in the above proof of Theorem 5.

Lemma 14. *Let $g \in G(\Phi, R)$. Then the group $H = H(g)$ contains the root subgroup X_{β_1}.*

Proof. Let $v_\mu = g_{*\mu}$ be the μ-th column of the matrix $g = (g_{\lambda\mu})$, $\lambda, \mu \in \Lambda(\pi)$. Let, as usual $m = 5$ for $\Phi = \mathrm{E}_6$ and $m = 7$ for $\Phi = \mathrm{E}_7$. Then by the proof of Theorem 5 we know that for any $\zeta \in R$ the product

$$z_\mu = x_{\beta_1}(\zeta g_{\gamma_1\mu}) x_{\beta_2}(\zeta g_{\gamma_2\mu}) \dots x_{\beta_{m-1}}(\zeta g_{\gamma_{m-1}\mu}) x_{\beta_m}(-\zeta g_{\gamma_m\mu})$$

is an elementary fake root unipotent, stabilizing v_μ. Set $\zeta = g'_{\mu\gamma_1}\xi$, where, as usual, $g'_{\mu\gamma_1}$ is the entry of the inverse matrix $g^{-1} = (g'_{\lambda\mu})$, $\lambda, \mu \in \Lambda(\pi)$, in the position (μ, γ_1) and $\xi \in R$ is arbitrary. We claim that the product $\prod z_\mu$, $\mu \in \Lambda(\pi)$, equals $x_{\beta_1}(\xi)$. Indeed, belonging to a singular subspace in the geometry of root subgroups all X_{β_i}'s commute and by commutativity of R one has $\sum g'_{\mu\gamma_1} g_{\gamma_i\mu} = \sum g_{\gamma_i\mu} g'_{\mu\gamma_1} = \delta_{\gamma_1\gamma_i}$, $\mu \in \Lambda(\pi)$ (the product of a row of a matrix by the column of the inverse matrix). This shows that $x_{\beta_1}(\xi) \in H$ for all $\xi \in R$ and thus $X_{\beta_1} \le H$.

But since this is true for *any* matrix $g \in G$, the group $H = H(g)$ must be the whole elementary subgroup. This immediately follows from the next lemma.

Lemma 15. *Let $g \in G$, $w \in \widetilde{W}$. Then $H(g) = w^{-1}H(wg)w$.*

Proof. Take an arbitrary $z \in H(wg)$. Let, for example, $z(wg)_{*\mu} = (wg)_{*\mu}$ for some $\mu \in \Lambda(\pi)$. Then

$$(w^{-1}zw)g_{*\mu} = w^{-1}(z(wg)_{*\mu}) = w^{-1}(wg)_{*\mu} = g_{*\mu}.$$

This means that $w^{-1}zw \in H(g)$, so that, in particular, $w^{-1}H(wg)w \le H(g)$. But then also $H(g) = H(w^{-1}wg) \le w^{-1}H(wg)w$.

Now we are in a position to finish the proof of Theorem 4. Indeed, take an arbitrary $g \in G$ and an arbitrary $w \in \widetilde{W}$. By Lemmas 14 and 15 one has $X_{\beta_1} \le H(w^{-1}g) = w^{-1}H(g)w$ and thus $X_{w\beta_1} = wX_{\beta_1}w^{-1} \le H(g)$. This means that $H(g)$ contains all elementary root subgroups X_α, $\alpha \in \Phi$, and thus coincides with the elementary group $E(\Phi, R)$.

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