SUBGROUPS OF SL_n OVER A SEMILOCAL RING

Nikolai Vavilov

Department of Mathematics and Mechanics
University of Sanct-Petersburg
Petrodvorets, 198904, Russia

Let R be a commutative semilocal ring, $G = \mathrm{SL}(n, R)$ be the special linear group of degree $n \geq 3$ over R and $T = \mathrm{SD}(n, R)$ be its subgroup of diagonal matrices. In the present paper we describe subgroups of G containing T under some mild restrictions on the residue fields of R. More precisely we prove the following result (we refer to Sections 2^o and 3^o for an explanation of all definitions and notation used in this statement).

Theorem 1. *Let $n \geq 3$ be a natural number and R be a semilocal ring, all of whose residue fields contain at least $3n+2$ elements. Then the standard description of subgroups of $\Gamma = \mathrm{SL}(n, R)$ containing the diagonal subgroup $T = \mathrm{SD}(n, R)$ holds. In other words, for any such subgroup H there exists a unique D-net of ideals σ such that $\Gamma(\sigma) \leq H \leq N_\Gamma(\sigma)$.*

This result generalises my earlier results, published in [V3], where R was supposed to be a field. However the proof here does not repeat the proofs presented in [V3]. To the contrary, it is based on a reduction to the results of [V3]. From a general viewpoint Theorem 1 is a description of overgroups of a split maximal torus in a Chevalley group of type A_l. It is worth to stress the fact that unlike the case of *extended* Chevalley groups description of such overgroups in ordinary Chevalley groups was known only for fields.

The paper is organised as follows. In 1^o we discuss previously known related results and in 2^o and 3^o we introduce the necessary notation and

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state some lemmas used in the proof. In 4^o we reduce the proof of Theorem 1 to the special case of subradical subgroups, embodied in Theorem 2 and in 5^o and 6^o finish the proof of Theorem 2. Finally, in 7^o we mention some further related problems.

1^o. **Introduction.** A brief historical survey is in order to put Theorem 1 in context. We refer the reader to the surveys [V6], [V7], [V9] and [Z] for a much broader picture of the field and many further related references. First of all, observe that for the general linear group analogous results are substantially easier. Subgroups of the general linear group $\mathrm{GL}(n,R)$ over a semilocal ring R containing the diagonal subgroup $\mathrm{D}(n,R)$ were described some two decades ago in a series of papers by Z.I.Borewicz and the present author [B3], [BV], [V1], [V2]. The ring R was not even assumed to be commutative. The methods of the above papers based on the use of pseudoreflections turned out to be extremely productive and were generalized in several directions. In particular, they carry over also to other extended classical groups over rings, like the general symplectic group $\mathrm{GSp}(2l,R)$ or the special orthogonal group $\mathrm{SO}(n,R)$, see [DV], [V6], [V7], [V9] or [HV] for a detailed survey. At least for the case of skew fields these methods found definitive explanations in [V4] and [V10].

The case of the special linear group SL_n turned out to be much harder even over fields. For finite fields K, $|K|\geq 13$, $\mathrm{char}\,K\neq 2$, it was solved by G.M.Seitz [S]. For infinite fields one should distinguish between the cases $n\geq 3$ and $n=2$, the group SL_2 behaving like a symplectic group rather than a linear one. The case $n\geq 3$ was treated by the author, first in [V3, I] for infinite fields and then in [V3, II – IV] for all fields K such that $|K|\geq 7$. While the methods of [V3, I] still relied on the presence of a non-trivial 'homology' (i.e. a scalar multiple of a pseudoreflection $d\neq e$) in $\mathrm{SL}(n,K)$, they introduced calculations with the conjugates of the elements $d_{rs}(\varepsilon)=e+(\varepsilon-1)e_{rr}+(\varepsilon^{-1}-1)e_{ss}$, which are, from the point of view of algebraic groups, nothing else than semisimple root elements in SL_n. These methods too were later generalised to arbitrary Chevalley groups, see [V7] – [V9] and references there.

For the group SL_n itself in the matrix language the basic trick consisted in considering a conjugate $y(\varepsilon)=gd_{rs}(\varepsilon)g^{-1}$ of the semisimple root element by an element g from the subgroup $T\leq H\leq G$. Then for any pair $i\neq j$ the condition $y(\varepsilon)_{ij}=0$ is an equation in ε of degree ≤ 2 one of whose roots equals 1 and we are almost done since one usually can solve *linear* equations over fields (maybe after varying g, since we want our root to be distinct from 0 and 1). When $|K|$ or n (or both) are large enough the proof goes smoothly, but the remaining small cases required substantial extra-work.

As it turned out, the case of SL_2 over an infinite field is incredibly tough and was actually the last one to be treated. Partial results were obtained by the present author and E.V.Dybkova (see, for example, [VD] and [V6]) under additional assumptions $|K^*/K^{*2}| > |K^*|$ or $-1 \in K^{*2}$, the latter case depending on a calculation of O.King in [K1]. Both conditions later appeared in a proof of a slightly weaker result by Li Shang Zhi in [L]. The final result was proven by O.King in [K2] by a tremendous calculation including 32 factors from H and T to produce a non-trivial transvection in H (in characteristic $p \geq 7$, whereas characteristics 2,3 and 5 required separate arguments).

The results of [V3] have been generalised to the group $\mathrm{SL}(n, R)$ over a commutative uniserial ring R in the Ph. D. Thesis of I.Hamdan [H1], [H2]. Recall that a ring R is called uniserial, if its ideals are linearly ordered by inclusion. I.Hamdan discovered that this condition allows to apply the basic computational trick described above and obtain a matrix with a zero entry in a subgroup of G contaning T by a single commutation. The basic reason why it works is exceedingly simple: for uniserial rings condition $\sum g_{ih} g'_{hj} = 0$, $i \neq j$, implies that at least two of the products $g_{ir} g'_{rj}$ and $g_{is} g'_{sj}$ generate the same ideal and thus differ by an invertible factor. But it required quite some work to take care for all details since now one cannot choose r and s arbitrarily. Observe that this result easily implies the standard description for all finite rings with large residue fields.

For some mysterious reason at that time it has not been noticed that another easy modification of the approach of [V3] gives the corresponding result for all commutative semilocal rings. In fact the present paper hardly contains any new idea as compared with [B3], [BV], [V1], [V3] and [BLS], it only combines the calculations already present in those papers in a slightly different fashion. The proof of Theorem 1 works as follows. First we observe that modulo the results for fields the general case reduces to the study of subradical intermediate subgroups. Even for a subradical subgroup H we cannot guarantee that any of the equations $y(\varepsilon)_{ij} = 0$ has a nontrivial invertible root, for an arbitrary element $g \in H$. But we can do it if g itself is a 'homology' – this is exactly the new twist, as compared with [V3], which allows to push the results of [V3] from fields to semilocal rings.

2^o. **Basic notation.** An excellent background reference for linear groups over rings is [HO]. The following standard notation will be used throughout the paper. Let R be a commutative ring, R^* be its multiplicative group, $J = J(R)$ be its Jacobson radical. Further, let $G = \mathrm{GL}(n, R)$ and $\Gamma = \mathrm{SL}(n, R)$ be the general linear and the special linear group of degree n over R respectively. For a matrix $g \in G$ we denote by g_{ij} its entry in the position (i, j), so that $g = (g_{ij})$, $1 \leq i, j \leq n$. As usual $g^{-1} = (g'_{ij})$ denotes the inverse of g, e denotes the identity matrix and e_{ij} is a standard matrix unit, i.e. the matrix whose entry in the position (i, j) is 1 and all the remaining entries are zeros. Thus $g = \sum g_{ij} e_{ij}$. As usual we denote by $\mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n)$

the diagonal matrix with diagonal entries $\varepsilon_1, \dots, \varepsilon_n$.

By $t_{ij}(\xi) = e + \xi e_{ij}$ for $\xi \in R$ and $1 \le i \ne j \le n$ we denote an *elementary transvection*. If $\varepsilon \in R^*$ is invertible, we denote by $d_i(\varepsilon) = e + (\varepsilon - 1)e_{ii}$ an 'elementary' *pseudoreflection*. But the pseudoreflections do not belong to Γ so they have to be replaced in the calculations by the two following types of diagonal matrices with determinant 1. First, we consider products of two pseudoreflections $d_{ij}(\varepsilon) = d_i(\varepsilon)d_j(\varepsilon^{-1}) = e + (\varepsilon - 1)e_{ii} + (\varepsilon^{-1} - 1)e_{jj}$. These elements are exactly the semisimple root elements of $\mathrm{SL}(n, R)$ considered as a Chevalley group. Second, we consider scalar multiples of pseudoreflections, traditionally referred to as 'homologies': $d^i(\varepsilon) = \varepsilon d_i(\varepsilon^{-n}) = \varepsilon(e + (\varepsilon^{-n} - 1)$. In other words, $d^i(\varepsilon) = \mathrm{diag}(\varepsilon, \dots, \varepsilon, \varepsilon^{1-n}, \varepsilon, \dots, \varepsilon)$, where ε^{1-n} stands in the i-th position.

We use the following subgroups of G: the group $B = \mathrm{B}(n, R)$ of upper triangular matrices, the group $B^- = \mathrm{B}^-(n, R)$ of lower triangular matrices, the group $D = \mathrm{D}(n, R)$ of diagonal matrices, the monomial subgroup $N = \mathrm{N}(n, R)$. We denote by $U = \mathrm{U}(n, R)$ and $U^- = \mathrm{U}^-(n, R)$ the unipotent radicals of B and B^- respectively, i.e. the groups of upper or lower unitrianglular matrices. The names of the corresponding subgroups in Γ are prefixed by 'S'. Thus $\mathrm{SD}(n, R) = \mathrm{D}(n, R) \cap \Gamma$ is the group of diagonal matrices with determinant 1, $\mathrm{SB}(n, R) = \mathrm{B}(n, R) \cap \Gamma$, is the group of upper triangular matrices with determinant 1, etc.

For an ideal $I \trianglelefteq R$ one denotes by $\Gamma_I = \mathrm{SL}(n, R, I)$ the principal congruence subgroup of Γ, which consists of all matrices from Γ congruent to 1 modulo I. In the present paper we will use mainly the congruence subgroup Γ_J whose level is the Jacobson radical J of R. Another group which is crucial in our proof is the group $\mathrm{SB}(n, R, J)$, consisting of all matrices $x = (x_{ij}) \in \Gamma$ such that $x_{ij} \in J$ for all $i > j$. Since $R^* + J = R^*$ one has $\mathrm{SB}(n, R, J) = \mathrm{B}(n, R)\Gamma_J$, or, what is the same, $\mathrm{SB}(n, R, J) = \mathrm{B}(n, R)\,\mathrm{U}^-(n, R)$.

3^o. **Net subgroups.** The following definitions were introduced by Z.I.Borewicz in [B1] – [B3], [BV]. Let, as above, R be a commutative ring with 1. For a natural n we consider a matrix σ_{ij}, $1 \le i, j \le n$, of order n whose entries are ideals σ_{ij} of R. Such a matrix is called a **net** of ideals in R of order n if one has $\sigma_{ir}\sigma_{rj} \subseteq \sigma_{ij}$ for all i, j, r. Since we are only interested in overgroups of the diagonal subgroup we need to consider only ***D*-nets** satisfying the additional condition $\sigma_{ii} = R$ for all i.

For any net σ we denote by $M(\sigma)$ the subring of the full matrix ring $\mathrm{M}(n, R)$ consisting of all matrices $x = (x_{ij})$ such that $x_{ij} \in \sigma_{ij}$ for all i, j. For a D-net one has $e \in \mathrm{M}(\sigma)$. The largest subgroup of $G = \mathrm{GL}(n, R)$ contained in the multiplicative system $e + M(\sigma)$ is called the **net subgroup** of G, corresponding to the net σ, and Theorem 1 of [B3] implies that over a commutative ring one has $G(\sigma) = G \cap (e + M(\sigma))$. We set $\Gamma(\sigma) = G(\sigma) \cap \Gamma$

and call $\Gamma(\sigma)$ the **net subgroup** of Γ corresponding to the net σ.

We shall also need two further subgroups of Γ, associated with a net. Subgroup $E(\sigma)$ generated by all elementary transvections contained in $\Gamma(\sigma)$, or, in other words, by all matrices, $t_{ij}(\xi)$, where $\xi \in \sigma_{ij}$ and $i \neq j$, is called the **elementary net subgroup** corresponding to the net σ. Further, the normaliser of $\Gamma(\sigma)$ in $\Gamma = \mathrm{SL}(n, R)$ will be denoted by $N_\Gamma(\sigma)$. We need the following three elementary facts about these subgroups. We do not try to formulate the following lemmas in maximal possible generality, and state them in the form we use in the proof of Theorem 1.

First, let H be a subgroup of $\Gamma = \mathrm{SL}(n, R)$ containing $T = \mathrm{SD}(n, R)$. In the following lemma we construct the maximal net of ideals such that $E(\sigma) \leq H$. All the assertions are obvious from the commutator relations among the elementary transvections $t_{ij}(\xi)$ and the diagonal elements $d_{ij}(\varepsilon)$.

Lemma 1. *Let R be a semilocal ring all of whose residue fields contain at least 3 elements and $n \geq 3$. Then for any subgroup H, $T \leq H \leq \Gamma$, there exists a unique largest D-net σ such that $E(\sigma) \leq H$. Namely, σ is definied as follows: $\sigma_{ii} = R$ and σ_{ij} for $i \neq j$ consists of all $\xi \in R$ such that $t_{ij}(\xi) \in H$.*

The net constructed in this lemma will be called the D-net **associated** with the subgroup H. In fact our next lemma (a special case of Theorem 1 of [B1]) implies that for the associated net one has $\Gamma(\sigma) \leq H$.

Lemma 2. *For any D-net σ over a semilocal ring one has $\Gamma(\sigma) = E(\sigma) \cdot T$.*

The group $\Gamma(\sigma)$ is defined by linear congruences in its entries. The next lemma gives a similar characterisation of the matrices from $N_\Gamma(\sigma)$.

Lemma 3. *Let R be a semilocal ring all of whose residue fields contain at least 4 elements and $n \geq 3$. Then for any D-net σ the necessary and sufficient condition for a matrix $x = (x_{ij}) \in \Gamma$ to belong to $N_\Gamma(\sigma)$ is embodied in the following system of inclusions: $x_{ir}\sigma_{rs}x'_{sj} \subseteq \sigma_{ij}$, for all i, j, r, s.*

For the general linear group this is exactly the characterisation given in Theorem 3 of [BV], for subgroups in SL_n is proven similarly. The only aspect which is not quite trivial here is the necessity of inclusions $x_{ir}x'_{rj} \in \sigma_{ij}$, but these last inclusions follow by considering inclusions $xd_{rs}(\varepsilon)x^{-1} \in \Gamma(\sigma)$ and $xd_{rs}(\eta)x^{-1} \in \Gamma(\sigma)$ for two invertible elements $\varepsilon, \eta \in R^*$ such that $\varepsilon - \eta \in R^*$.

4^o. **Reduction modulo the radical.** In view of the Lemmas 1–3 the following result is deduced from Theorem of [V3, IV], by the same argument as in 3^o of [V1] with $d_i(\varepsilon)$ replaced everywhere by $d_{ij}(\varepsilon)$

Proposition 1. *Let $R = K_1 \oplus \ldots \oplus K_s$, where $K_1, \ldots, K_s$ are fields containing at least 7 elements each. Then the description of subgroups of $G = \mathrm{SL}(n, R)$, $n \geq 3$, containing $T = \mathrm{SD}(n, R)$ is standard. In other* *words for any such subgroup H there exists a unique D-net σ such that $\Gamma(\sigma) \leq H \leq N_\Gamma(\sigma)$.*

Let σ be a net of ideals in R. Set $\sigma^* = (\sigma^*_{ij})$, where $\sigma^*_{ij} = \sigma_{ij} + J$. Clearly σ^* forms a net of ideals in R. The following result is an immediate corollary of Lemma 6 of [BV].

Lemma 4. *For any net σ over a commutative ring R one has $\Gamma(\sigma^*) = \Gamma(\sigma)\Gamma_J$.*

Now the same argument as in 5^o of [V1] using an our Lemma 4 instead of Lemma 6 of [BV], shows that Proposition 1 implies the following result reducing the proof of Theorem 1 to the case of subradical subgroups (see, for example, [H2], 3^o).

Proposition 2. *Let R be a commutative semilocal ring all of whose residue fields contain at least 7 elements. Then for the description of subgroups in $\Gamma = \mathrm{SL}(n, R)$, $n \geq 3$, containing $T = \mathrm{SD}(n, R)$ to be standard it is necessary and sufficient that the description of subgroups in $\mathrm{SB}(n, R, J)$ containing $T = \mathrm{SD}(n, R)$ is standard.*

Thus to prove Theorem 1 it remains only to prove the following result.

Theorem 2. *Let $n \geq 3$ be a natural number and R be a commutative semilocal ring, all of whose residue fields contain at least $3n+2$ elements. Then the description of $\mathrm{SB}(n, R, J)$ containing $T = \mathrm{SD}(n, R)$ is standard, in other words, for any such subgroup H there exists a unique D-net σ such that $H = \Gamma(\sigma)$.*

The rest of the paper is devoted to the proof of Theorem 2. In fact an analogue of Theorem 2 holds for not necessarily semilocal commutative rings provided they have enough units (as is readily seen from the proof below), but not to accumulate technical conditions on units in the statements of auxiliary results, we prefer to restrict ourselves to semilocal rings, where all these conditions are conveniently expressed in terms of cardinalities of residue fields.

5^o. **Lemmas on extraction of transvections.** First observe that an analogue of Theorem 2 for the general linear group holds even under weaker restrictions on R. The following result is a special case of Theorem 3 of [V1].

Lemma 5. *Let R be a commutative semilocal ring, all of whose residue fields contain at least 5 elements. Then the description of $\mathrm{B}(n, R, J)$ containing $D = \mathrm{D}(n, R)$ is standard, in other words, for any such subgroup H there exists a unique D-net σ such that $H = G(\sigma)$.*

In the following lemmas we consider a subgroup H of $\Gamma = \mathrm{SL}(n, R)$ containing $T = \mathrm{SD}(n, R)$. For such a subgroup we denote by σ the associated

D-net constructed in Lemma 1. In the sequel we always assume that $n \geq 3$, some of the lemmas fail to be true without this assumption. The following result is a corollary of Theorems 1,5,6 of [B2] (or more general results pertaining to arbitrary Chevalley groups, see references in [V7]).

Lemma 6. *Let R be a commutative semilocal ring, all of whose residue fields contain at least 5 elements. Further, let H be a subgroup of $\Gamma = \mathrm{SL}(n, R)$ containing $T = \mathrm{SD}(n, R)$. Then for an arbitrary upper triangular matrix $b \in H$ one has $b \in G(\sigma)$.*

In other words, this lemma asserts that $t_{ij}(\beta_{ij}) \in H$ for all $i < j$. The following result allows to argue by induction in degree. It immediately follows from the two preceding Lemmas by the same arguments as Lemma 4 in [V3, I] (see also a more general lemma 2 of [H2]).

Lemma 7. *Let R be a commutative semilocal ring, all of whose residue fields contain at least 5 elements. Further, let H be a subgroup of $\mathrm{SB}(n, R, J)$ containing $\mathrm{SD}(n, R)$. Then if for a matrix $x = (x_{ij}) \in H$ for a fixed r one has $x_{rj} = 0$ for all $j \neq r$, then $x \in G(\sigma)$.*

As the preceding lemma, this one may be rephrased as the statement that $t_{ij}(x_{ij}) \in H$ for all $i \neq j$. We intend to show that the conclusion of Lemma 7 should hold already for a matrix x having one zero entry. To avoid technical complications we impose stronger restrictions on the residue fields of R, to guarantee that Γ contains non-trivial 'homologies'. The following lemma is essentially Lemma 5 of [H2], but we slightly relax condition on R.

Lemma 8. *Let R be a commutative semilocal ring, all of whose residue fields contain at least $n + 2$ elements. Further, let H be a subgroup of $\mathrm{SB}(n, R, J)$ containing $\mathrm{SD}(n, R)$. Then if for a matrix $x = (x_{ij}) \in H$ for fixed p, r one has $x_{pr} = 0$ then $x_{ir}x'_{rj} \in \sigma_{ij}$ for all i, j.*

Proof. The proof works almost exactly as the proof of Lemma 5 in [V3, I] and is even somewhat easier. Namely, take an $\varepsilon \in R^*$ such that $\varepsilon^n - 1 \in R^*$ (our condition on R and the Chinese remainder theorem imply that such an ε exists) and consider the matrix $y = xd^r(\varepsilon^{-1})x^{-1} \in H$. A direct calculation shows that $y_{ij} = \varepsilon^{-1}\left(\delta_{ij} + x_{ir}(\varepsilon^n - 1)x'_{rj}\right)$. Since $x_{pr} = 0$, the p-th row of y coincides with the p-th row of the scalar matrix $\varepsilon^{-1}e$ and thus by the preceding lemma $y \in \Gamma(\sigma)$ (observe that $n + 2 \geq 5$). This means that for all $i \neq j$ one has $y_{ij} \in \sigma_{ij}$ and, we get the conclusion of the lemma by the invertibility of $\varepsilon^n - 1$.

The two preceding lemmas imply the following result (essentially Lemma 6 of [H2], again slightly relaxing condition on R).

Proposition 3. *Let R be a commutative semilocal ring, all of whose residue fields contain at least $n + 2$ elements. Further, let H be a subgroup of*

$\mathrm{SB}(n, R, J)$ *containing* $\mathrm{SD}(n, R)$. *Then if for a matrix* $x = (x_{ij}) \in H$ *for fixed* $p \neq r$ *one has* $x_{pr} = 0$ *then* $x \in \Gamma(\sigma)$.

Proof. By the preceeding lemma $x_{ir} x'_{rr} \in \sigma_{ir}$ and, since x'_{rr} is invertible, $x_{ir} \in \sigma_{ir}$, ot, in other words, $t_{ir}(x_{ir}) \in \Gamma(\sigma) \leq H$ for all $i \neq r$. Setting $y = \prod t_{ir}(-x_{ir} x_{rr}^{-1}) x \in H$, we get $y_{ir} = 0$ for all $i \neq r$. Applying to the matrix y Lemma 7 we conclude that $y \in \Gamma(\sigma)$ and thus also $x \in \Gamma(\sigma)$.

We are going to use this Proposition in the following form.

Corollary. *Let* R *be a commutative semilocal ring, all of whose residue fields contain at least* $n+2$ *elements. Further, let* H *be a subgroup of* $\mathrm{SB}(n, R, J)$ *containing* $\mathrm{SD}(n, R)$. *Then if for a matrix* $x = (x_{ij}) \in H$ *for fixed* $p \neq r$ *one has* $x_{pr} \in \sigma_{pr}$, *then* $x \in \Gamma(\sigma)$.

Proof. In the matrix $y = tx$, where $t = t_{pr}(-x_{pr} x_{rr}^{-1}) \in \Gamma(\sigma)$ one has $y_{pr} = 0$. Thus by Proposition 3 one has $y \in \Gamma(\sigma)$ and finally, $x = t^{-1} y \in \Gamma(\sigma)$.

6^o. **Proof of Theorem 2.** Let, as above H be a subgroup of $\mathrm{SB}(n, R, J)$ containing $T = \mathrm{SD}(n, R)$. By Lemmas 1 and 2 for the net σ associated with H one has $\Gamma(\sigma) \leq H$ and we have to prove that conversely $H \leq \Gamma(\sigma)$. Take an arbitrary matrix $x \in H$ an arbitrary $\varepsilon \in R^*$ such that $\varepsilon^n - 1 \in R^*$ and arbitrary $r \neq s$. Next we consider the homology $y(\varepsilon) = x d^h(\varepsilon) x^{-1}$ and finally form the matrix $z = z(\varepsilon, \theta) = y(\varepsilon) d_{rs}(\theta) y(\varepsilon^{-1})$ for a $\theta \in R^*$. A straightforward calculation shows that

$$y(\varepsilon)_{ij} = \varepsilon \delta_{ij} + \varepsilon x_{ih} (\varepsilon^{-n} - 1) x'_{hj}$$

and

$$z_{ij} = \delta_{ij} + y(\varepsilon)_{ir} (\theta - 1) y(\varepsilon^{-1})_{rj} + y(\varepsilon)_{is} (\theta^{-1} - 1) y(\varepsilon^{-1})_{sj}.$$

In particular,

$$z_{rs} = x_{rh} x'_{hs} \theta^{-1} (1 - \theta)(a - \theta b),$$

where

$$a = a(\varepsilon) = (\varepsilon^{-n} - 1)(1 + x_{sh} (\varepsilon^n - 1) x'_{hs}),$$

$$b = b(\varepsilon) = (\varepsilon^n - 1)(1 + x_{rh} (\varepsilon^{-n} - 1) x'_{hr}).$$

It is easy to see that $a, b \in R^*$ and thus there exists a unique $\theta = \theta(\varepsilon) \in R^*$ such that $a - \theta b = 0$, namely $\theta = b^{-1} a$. If $h \neq r, s$, then $a \equiv \varepsilon^{-n} - 1 (\operatorname{mod} J)$ and $b \equiv \varepsilon^n - 1 (\operatorname{mod} J)$ and thus $\theta \equiv -\varepsilon^{-n} (\operatorname{mod} J)$. We denote the matrix $z(\varepsilon, \theta)$ for this particular choice of θ simply by $z(\varepsilon)$. By the very construction of θ one has $z(\varepsilon)_{rs} = 0$ and thus by Proposition 3 we have the inclusion $z_{hs} \in \sigma_{hs}$, or, in other words,

$$z(\varepsilon)_{hs} = y(\varepsilon)_{hr} (\theta - 1) y(\varepsilon^{-1})_{rs} + y(\varepsilon)_{hs} (\theta^{-1} - 1) y(\varepsilon^{-1})_{ss} \in \sigma_{hs}$$

We want to deduce from here that $y(\varepsilon)_{hs} \in \sigma_{hs}$. An obvious necessary condition for this to be possible is $\theta^{-1} - 1 \in R^*$ and since $\theta \equiv -\varepsilon^{-n} (\mathrm{mod}\, J)$, we must have $\varepsilon^n + 1 \in R^*$, which imposes at most n further restrictions on the class of ε in any residue field. Now take another $\eta \in R^*$ for which $\eta^{2n} - 1 \in R^*$ and write the same inclusion for the matrix $z(\eta)$, namely $z(\eta)_{hs} \in \sigma_{hs}$. Considering $z(\varepsilon)_{hs}, z(\eta)_{hs} \in \sigma_{hs}$ as a system of two linear congruences in $x_{hh}x'_{hr}x_{rh}x'_{hs}$ and $x_{hh}x'_{hs}$, and calculating the determinant of this system, we see that this determinant is congruent to $(1-\varepsilon^{-2n})(1-\eta^{-2n})(\varepsilon^n - \eta^n)$ modulo J. If we assume additionally that $\varepsilon^n - \eta^n \in R^*$, then this system of congruences implies that $x_{hh}x'_{hs} \in \sigma_{hs}$ and, since x itself belongs to $B(n,R,J)$, that $x'_{hs} \in \sigma_{hs}$. Observe that for every residue field our last restriction on η prohibits at most n further values of the class of η on top of not more than $2n+1$ values excluded above, which gives at most $3n+1$ prohibited values altogether and thus we can certainly find in R two invertible elements ε and η such that $\varepsilon^{2n} - 1$, η^{2n} and $\eta^n - \varepsilon^n$ are all invertible. Applying Corollary of the Proposition 3 to the matrix x^{-1} we conclude that $x^{-1} \in \Gamma(\sigma)$ and thus, finally, $x \in \Gamma(\sigma)$. This completes the proof of Theorem 2 and thus in view of Proposition 2 also of Theorem 1.

7^o. **Concluding remarks.** I cannot think of any counterexamples to the following conjecture.

Problem A. *The standard description of subgroups in* $\mathrm{SL}(n,R)$*,* $n \geq 3$*, containing* $\mathrm{SD}(n,R)$*, holds for a semilocal ring* R *whenever* $|R/\mathfrak{m}| \geq 7$ *for all residue fields of* R*.*

Similarly Theorem 2 should be valid for all semilocal rings all of whose residue fields contain at least 5 elements. However it seems that a proof of these statements is not easy. Already a generalisation of Lemma 8 to all semilocal rings subject to the last condition is not immediate. In the absence of reduction from [V3, II] (specific to the field case) all attempts to directly push the approach of [V3, III] which requires the presence of two zeros in one row or column lead to formidable calculations. Probably such a result would require an entirely different computational trick.

Turning to a somewhat different direction, observe that Bui Xuan Hai [BXH1] and [BXH2] essentially generalized results of [V3] to the elementary group $\mathrm{E}(n,T)$ over a non-commutative skew-field T. In fact he proves that the standard description holds if either $n \geq 3$ and the centre of T is infinite or $n \geq 5$ and the centre of T contains at least 7 elements. Trying to closely follow the arguments from [V3] Bui Xuan Hai had to handle some non-trivial questions about the commutator subgroup $[T^*, T^*]$ of the multiplicative group of T, and the difficulties accumulate as n is small. On the other hand, I am not aware of any counterexamples to the following conjecture.

Problem B. *Standard description holds for any $n \geq 3$ and any non-commutative skew-field T, without exceptions.*

In fact, I suspect that this should hold even for $n = 2$ provided that the characteristic of T is distinct from 2 (if T is non-commutative then the centre and the commutator subgroup of T^* cannot be both small; if $\mathrm{Cent}(T^*)$ is large, one can hope to be able to imitate the arguments from [K2]; on the other hand, since $\mathrm{SD}(n, T)$ contains all pseudoreflections $d_i(\varepsilon)$, where $\varepsilon \in [T^*, T^*]$, when the commutator subgroup is large it should be possible to imitate the proof from [B3], [BV], [V1] along the lines of Theorem 1 of [VD, II]).

The following result should follow from Theorem 1 by essentially the same arguments as in [V5].

Problem C. *For all local rings subject to the conditions of Theorem 1 the subgroup* $\mathrm{SD}(n, R)$ *is pronormal in* $\mathrm{SL}(n, R)$.

Observe that pronormality may fail in general for a semilocal ring R, in which case $\mathrm{SD}(n, R)$ should satisfy a similar but weaker condition, see [V9] and references there.

A much more ambitious generalisation of Theorem 1 is embodied in the following problem.

Problem D. *Generalise Theorem 1 to all Chevalley groups over a semilocal ring R all of whose residue fields $R/\mathfrak{m}$ are large enough (say infinite).*

It seems that for classical types all the necessary tools are there. On the other hand for the exceptional groups, even for the easier cases of E_6 and E_7, one has either to work in their minimal representation, or to develop a new technique of calculations with something like Gauß decomposition. As far as I can see, both approaches encounter some serious technical obstacles.

The results of the present paper essentially solve Problem C from [V3, IV] and for the case of $\Phi = \mathrm{A}_l$ the problem posed in [V7] and repeated in § 11 in [V9]. I would like to note that meanwhile there has been a substantial progress on some further problems mentioned in [V7] (in the bibliography we only list the papers not appearing in [V9]). First of all, in some of the most important situations the problems concerning the description of overgroups of non-split tori were completely solved by V.P.Platonov [P] (local fields, global to local reduction) and V.A.Koibaev [Ko] (cyclic tori). Many concrete results for small degrees may be found in the papers of V.A.Koibaev, Z.I.Borewicz, and his research students, notably A.A.Bondarenko and Tran Ngoc Hoi (see the references in [V9] and [Bo1] – [Bo3]). In a different direction E.V.Dybkova recently obtained major progress in pushing results on overgroups of maximal tori to the generalised classical groups over a form ring (see [D1], [D2] and references there).

There is a further extremely interesting recent development not covered in [V9]. Namely A.V.Yakovlev and his research students A.Z.Simonian and A.A.Panine [Si], [Pa], [PY] generalised the results of [B3], [BV], [V1], [V2] to overgroups of certain subgroups in the automorphism groups of modular lattices. There is a strong analogy between their results and my results on the geometry of tori [V10], which is not completely understood. It would be interesting to push to their situation also the results of [V3] and the present paper.

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e-mail addresses: vavilov@pdmi.ras.ru
vavilov@mathematik.uni-bielefeld.de