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CLASSIFICATION OF 2-REFLECTIVE HYPERBOLIC LATTICES OF RANK 4

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Abstract. A hyperbolic lattice is called 2-reflective if its automorphism group contains a finite index subgroup generated by 2-reflections. We find all 2-reflective hyperbolic lattices of rank 4. (For all other ranks this was done by V. V. Nikulin.)

In this work we classify 2-reflective hyperbolic lattices of rank 4 (see the definitions below). The classification was done by the author in 1981 and mentioned in [N3] but the proof was not published up to the present. A classification of 2-reflective hyperbolic lattices is the key step in the classification of algebraic $K3$-surfaces whose automorphism group is finite [N1], [N2].

A *quadratic lattice* L is a free abelian group equipped with a non-degenerate integral symmetric bilinear form, which is called the *scalar product*. The lattice L is called *Euclidean*, if the scalar product in it is positive definite, and *hyperbolic*, if the scalar product is of signature $(n, 1)$.

One can think of L as of a lattice in the (pseudo)Euclidean vector space $V = L \otimes \mathbb{R}$. The automorphism group $\mathrm{O}(L)$ of L is a lattice in the (pseudo)orthogonal group $\mathrm{O}(V)$.

In the hyperbolic case, one of two connected components of the hyperboloid

$$(x, x) = -1 \tag{1}$$

can be considered as a model of n-dimensional Lobachevsky space L^n in such a way that the group of motions of L^n is the subgroup $\mathrm{O}'(V)$ of index 2 in $\mathrm{O}(V)$ consisting of transformations leaving invariant each connected component of the hyperboloid (1). The Lobachevsky planes in this model are non-empty intersections of L^n with subspaces of V. The points at infinity of L^n correspond to isotropic one-dimensional subspaces of V. The group

$$\mathrm{O}'(L) = \mathrm{O}(L) \cap \mathrm{O}'(V)$$

is a discrete group of motions of L^n with a finite covolume.

A primitive vector e of a quadratic lattice L is called a *root* or, more exactly, a *k-root*, if $(e, e) = k > 0$ and

$$2(e, x) \in k\mathbb{Z} \quad \forall x \in L. \tag{2}$$

Any root e gives rise to an orthogonal reflection

$$R_e : x \mapsto x - \frac{2(e,x)}{(e,e)} e,$$

which leaves L invariant, i.e. belongs to $\mathrm{O}(L)$. In the hyperbolic case, R_e defines a reflection in the the hyperplane

$$H_e = \{x \in L^n : (e,x) = 0\}$$

of the space L^n. The reflection defined by a k-root, is called a *k-reflection*. Obviously, a root is defined by the corresponding reflection up to sign.

We are especially interested in 2-reflections. Note that any vector $e \in L$ with $(e,e) = 2$ automatically satisfies (2), i.e. is a root.

Denote by $\mathrm{O}_r(L)$ (resp. by $\mathrm{O}_r^{(2)}(L)$) the (normal) subgroup of $\mathrm{O}(L)$ generated by all reflections (resp. by all 2-reflections) in $\mathrm{O}(L)$. A hyperbolic lattice L is called *reflective* (resp. *2-reflective*) if $\mathrm{O}_r(L)$ (resp. $\mathrm{O}_r^{(2)}(L)$) has a finite index in $\mathrm{O}(L)$.

If L' is a sublattice of L, then $\mathrm{O}_r^{(2)}(L') \subset \mathrm{O}_r^{(2)}(L)$, while $\mathrm{O}(L')$ and $\mathrm{O}(L)$ are commensurable. It follows that if L' is 2-reflective, then L is also 2-reflective.

A quadratic lattice L is called *even*, if $(x,x) \in 2\mathbb{Z}$ for any $x \in L$, and *odd* otherwise. Any odd lattice L contains a unique even sublattice L^{even} of index 2, namely,

$$L^{even} = \{x \in L : (x,x) \in 2\mathbb{Z}\}.$$

Since $\mathrm{O}(L) \subset \mathrm{O}(L^{even})$ and any 2-root of L^{even} is a 2-root of L, we have

$$\mathrm{O}_r^{(2)}(L) = \mathrm{O}_r^{(2)}(L^{even}).$$

It follows that an odd hyperbolic lattice L is 2-reflective if and only if L^{even} is 2-reflective. This allows us to cosider only even lattices.

Let L be a hyperbolic lattice. Then the groups $\mathrm{O}_r(L)$ and $\mathrm{O}_r^{(2)}(L)$ are discrete reflection groups in L^n. Denote by P and $P^{(2)}$ their fundamental polyhedra (in general infinite). The lattice L is reflective (resp. 2-reflective) if and only if $\operatorname{vol} P < \infty$ (resp. $\operatorname{vol} P^{(2)} < \infty$) or, equivalently, if P (resp. $P^{(2)}$) is finite [V2]. There is an algorithm allowing for any concrete L to find recursively all the faces of P or $P^{(2)}$ and to determine if there are only finitely many of them [V2], [V4], [B].

V. V. Nikulin [N1], [N2] classified all 2-reflective hyperbolic lattices of rank $\neq 4$. R. Scharlau [SW] published a tentative list of all reflective isotropic hyperbolic lattices L of rank 4 with a maximal group $\mathrm{O}_r(L)$, but no proof has appeared as yet.

In order to formulate the result, let us introduce some notation.

[C] – the quadratic lattice, the scalar product in which is given by a (symmetric) matrix C in some basis,

$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ – the standard 2-dimensional hyperbolic lattice,

A_n – the (Euclidean) root lattice of type A_n,

$L \oplus M$ – the orthogonal direct sum of lattices L and M,

$[k]L$ – the quadratic lattice obtained from L by multiplying all scalar products by $k \in \mathbb{Z}$.

Theorem. *1) Any isotropic 2-reflective even hyperbolic lattice of rank 4 is isomorphic to one of the following 12 lattices:*

$$
\begin{aligned}
&[-8] \oplus [2] \oplus [2] \oplus [2], \\
&[k]U \oplus [2] \oplus [2], \quad k = 1, 2, 3, 4, \\
&[k]U \oplus A_2, \quad k = 1, 2, 3, 6, \\
&\begin{bmatrix} 0 & 3 \\ 3 & 2 \end{bmatrix} \oplus A_2, \\
&[-4] \oplus [4] \oplus A_2, \\
&[-4] \oplus A_3.
\end{aligned}
$$

2) Any anisotropic 2-reflective even hyperbolic lattice of rank 4 is isomorphic to one of the following 2 lattices:

$$
\begin{bmatrix} -2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}, \quad
\begin{bmatrix} -12 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.
$$

All the listed lattices are really 2-reflective and non-isomorphic to one another. For more information about them see the body of the paper. We only note here that the latter two lattices are the even sublattices of index 2 in the 2-reflective odd lattices

$$[-7] \oplus [1] \oplus [1] \oplus [1], \qquad [-15] \oplus [1] \oplus [1] \oplus [1],$$

respectively.

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1. The key consideration

A vertex of a 3-dimentional convex polyhedron is called *simple*, if it belongs to exactly 3 faces.

A convex polyhedron of finite volume in L^3 is the convex hull of finitely many ordinary points and points at infinity.

Let P be a Coxeter polyhedron of finite volume in L^3. Then any ordinary vertex of P is simple, while any vertex at infinity either is simple, or belongs to 4 faces, each two opposite ones of them being parallel and each two adjacent ones perpendicular.

Denote by v, e and f the numbers of vertices, edges, and faces of P, respectively. The Euler equation says that

$$v - e + f = 2. \tag{3}$$

Denote by v_3 and v_4 the numbers of simple and non-simple vertices of P, and, for any face F, by $v_3(F)$ and $v_4(F)$ the numbers of simple and non-simple vertices of P belonging to F. Obviously,

$$v = v_3 + v_4, \tag{4}$$

$$2e = 3v_3 + 4v_4. \tag{5}$$

It follows from (3)-(5) that

$$f - \frac{1}{2}v_3 - v_4 = 2,$$

whence

$$f > \frac{1}{2}v_3 + v_4 = \sum_F (\frac{1}{6}v_3(F) + \frac{1}{4}v_4(F)).$$

Therefore, there is a face F such that

$$\frac{1}{6}v_3(F) + \frac{1}{4}v_4(F) < 1.$$

Clearly, the number of sides of such F cannot exceed 5. More precisely, analyzing all the possibilities, we obtain

Proposition. Any Coxeter polyhedron of finite volume in L^3 has either a 3-face, or a 4-face with $\leqslant 3$ non-simple vertices, or else a 5-face with $\leqslant 1$ non-simple vertex.

(Here "k-face" means a face with k sides.)

We apply this to the fundamental polyhedron $P^{(2)}$ of the group $\mathrm{O}_r^{(2)}(L)$ for a 2-reflective hyperbolic lattice L of rank 4.

Let a face F of $P^{(2)}$ be of one of the types listed in Proposition. Let $F_1, \dots, F_k$ ($k = 3, 4, 5$) be the adjacent faces. Denote by $u_1, \dots, u_k, u$ the 2-roots corresponding to $F_1, \dots, F_k, F$, respectively. We shall see that there are only few possibilitiess for the Gram matrix

$$G = G(u_1, \dots, u_k, u)$$

of the vectors $u_1, \dots, u_k, u$ and, thereby, for the lattice L' generated by them.

Obviously, L' has finite index in L, so L is between L' and the dual lattice $(L')^*$. We have

$$[(L')^* : L'] = |d(L)|, \quad [L : L']^2 | d(L),$$

where $d(L)$ is the discriminant of L. So, for a given L' there are only finitely many (in fact, few) possibilities for L.

In such a way we shall obtain a finite (in fact, relatively short) list of lattices that may be 2-reflective.

2. Invariants of a quadratic lattice

In this section we summarize some results on the classification of indefinite quadratic lattices. For further details and proofs we refer the reader to [C] and [CS].

Let A be a principal ideal ring. A *quadratic A-module* is a free A-module of finite rank equipped with a nondegenerate symmetric bilinear form with values in A, which is called the *scalar product*. In particular, a quadratic $\mathbb{Z}$-module is called a quadratic lattice.

The *discriminant* $d(L)$ of a quadratic A-module L is the determinant of the Gram matrix of a basis of L. It is defined up to multiplication by an element of the group $(A^*)^2$ (where A^* stands for the group of units of A) and can be considered as an element of the semigroup $A/(A^*)^2$. A quadratic A-module L is called *unimodular*, if $d(A) \in A^*$.

In the case, when $2 \notin A^*$, a quadratic A-module L is called *even* if $(x, x) \in 2A$ for any $x \in L$, and *odd* otherwise.

We extend to quadratic modules the notations $[C], U, A_n, L \oplus M, [k]L$ introduced above for quadratic lattices.

Since $(\mathbb{Z}^*)^2 = \{1\}$, the discriminant $d(L)$ of a quadratic lattice L is just an integer. The invariant factors of the Gram matrix of a basis of L are called the *invariant factors* of L. Their product equals $|d(L)|$. They can be interpreted as the invariant factors of L as a subgroup (of finite index) in the group

$$L^* = \{x \in L \otimes \mathbb{Q} : (x, y) \in \mathbb{Z} \quad \forall y \in L\},$$

which is a free abclian group of the same rank.

Any quadratic lattice L gives rise to a quadratic real vector space $L_\infty = L \otimes \mathbb{R}$ and, for any prime p, a quadratic $\mathbb{Z}_p$-module $L_p = L \otimes \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the ring of p-adic integers. The signature of L_∞ is called the *signature* of L. Obviously, if quadratic lattices L and M are isomorphic, they have the same signature and $L_p \simeq M_p$ for any prime p. The converse is also true under the following restrictions:

(I) L is indefinite;

(II) for any prime p there are two invariant factors of L divided by the same power of p.

The structure of quadratic $\mathbb{Z}_p$-modules is described as follows. Any such module L admits a *Jordan decomposition*

$$L = L^{(0)} \oplus [p]L^{(1)} \oplus [p^2]L^{(2)} \oplus \dots,$$

where $L^{(0)}, L^{(1)}, L^{(2)}, \dots$ are unimodular quadratic $\mathbb{Z}_p$-modules. The latter modules are uniquely defined by L up to isomorphism, unless $p = 2$. For $p = 2$ it is true, however, that the rank and the parity (the property of being even or odd) of each $L^{(k)}$ are uniquely defined by L.

Following [CS], define the *sign* $s(L)$ of a unimodular quadratic $\mathbb{Z}_p$-module L as follows. For $p \neq 2$, set $s(L) = 1$ if $d(L) \in (\mathbb{Z}_p^*)^2$, and $s(L) = -1$ otherwise. For $p = 2$, set $s(L) = 1$ if $d(L) \equiv \pm 1 \pmod 8$, and $s(L) = -1$ otherwise.

For $p \neq 2$, any unimodular quadratic $\mathbb{Z}_p$-module is isomorphic to

$$[1] \oplus \dots \oplus [1] \oplus [u] \quad (u \in \mathbb{Z}_p^*)$$

and, hence, is uniquely defined by its rank and sign.

To any quadratic $\mathbb{Z}_p$-module L ($p \neq 2$) we associate the *symbol*, which is a formal product of formal powers of those p^k, for which $L^{(k)} \neq 0$, with exponents $\operatorname{rk} L^{(k)}$ and with bars in the cases when $s(L^{(k)}) = -1$. For example, the symbol of

$$L = [1] \oplus [1] \oplus [3]([1] \oplus [1] \oplus [-1]) \oplus [27]$$

is $1^2\bar{3}^3 27^1$.

It follows from the above that two quadratic $\mathbb{Z}_p$-modules ($p \neq 2$) are isomorphic if and only if they have the same symbol.

For $p = 2$ the situation is more complicated. Following the approach of [CS], one can uniquely and consistently define the *oddity* $o(L)$ of a unimodular quadratic $\mathbb{Z}_2$-module L as a residue modulo 8, satisfying the following conditions:

- $o([u]) \equiv u \pmod 8 \quad (u \in \mathbb{Z}_2^*)$;
- $o(L) = 0$, if L is even;
- $o(L \oplus M) = o(L) + o(M)$.

Any even unimodular quadratic $\mathbb{Z}_2$-module is isomorphic to

$$U \oplus \dots \oplus U \oplus K,$$

where $K = U$ or A_2 and, hence, is uniquely defined by its rank (which is always even) and sign.

Any odd unimodular quadratic $\mathbb{Z}_2$-module L decomposes into a direct sum of an even quadratic $\mathbb{Z}_2$-module and one or two (depending on the parity of $\operatorname{rk} L$) quadratic $\mathbb{Z}_2$-modules of rank 1. It is uniquely defined by its rank, sign and oddity. Clearly,

$$o(L) \equiv \operatorname{rk} L \pmod 2.$$

To any quadratic $\mathbb{Z}_2$-module L and a Jordan decomposition

$$L = L^{(0)} \oplus [2]L^{(1)} \oplus [4]L^{(2)} \oplus \dots \tag{6}$$

we associate the *pre-symbol*, which is a formal product of formal powers of those 2^k, for which $L^{(k)} \neq 0$, with exponents $\operatorname{rk} L^{(k)}$, with bars in the cases when $s(L^{(k)}) = -1$, and with subscripts $o(L^{(k)})$ in the cases when $L^{(k)}$ is odd. For example, the pre-symbol of

$$L = A_2 \oplus [4](U \oplus [3]) \oplus [-8]$$

is $\bar{1}^2\bar{4}^3_3 8^1_{-1}$.

In general the pre-symbol depends on the choice of a Jordan decomposition of L. To explain the situation, we use some notions introduced in [CS].

A *compartment* is a maximal sequence of consecutive (non-trivial) odd modules $L^{(k)}$, $L^{(k+1)}, \dots, L^{(l)}$ in the decomposition (6). The *oddity* of a compartment is the sum of the oddities of all its elements. The *symbol* of L with respect to the decomposition (6) is obtained from the pre-symbol by putting in parentheses the factors corresponding to each compartment and replacing their subscripts by one subscript related to the whole compartment and equalling its oddity. (In practice we do not put in parentheses compartments consisting of a single element.) For instance, in the above example the symbol is $\bar{1}^2(\bar{4}^3 8^1)_2$).

A pair of consecutive factors of a symbol is called *flexible* in the following cases:

- they correspond to consecutive modules $L^{(k)}, L^{(k+1)}$, at least one of which is odd;
- they correspond to odd modules $L^{(k-1)}, L^{(k+1)}$ with $L^{(k)} = 0$.

For example, in the above symbol $\bar{1}^2(\bar{4}^3 8^1)_2$ the second pair of factors is flexible, while the first one is not.

An *elementary transformation* of a symbol is the change of signs of the factors of a flexible pair combined with adding 4 to the oddities of all (one or two) compartments containing them. Two symbols are called *equivalent*, if they can be obtained from one another by a number of elementary transformations. For example,

$$\begin{aligned}
&\bar{1}^2(\bar{4}^3 8^1)_2 \sim \bar{1}^2(4^3 \bar{8}^1)_{-2} \nsim 1^2(4^3 8^1)_{-2}, \\
&\bar{1}^2_2 4^1_{-1} \bar{8}^2 \overline{16}^2 \sim 1^2_{-2} \bar{4}^1_3 \bar{8}^2 \overline{16}^2 \sim 1^2_{-2} 4^1_{-1} 8^2 \overline{16}^2 \nsim 1^2_{-2} 4^1_{-1} \bar{8}^2 16^2.
\end{aligned}$$

Two quadratic $\mathbb{Z}_2$-modules are isomorphic if and only if their symbols are equivalent.

Let now L be a quadratic lattice and $L_p = L \otimes \mathbb{Z}_p$ for any prime p. Then $\operatorname{rk} L_p^{(k)}$ equals the number of the invariant factors of L divided exactly by p^k. Further, if $d(L) = p^m u, (p, u) = 1$, then for $p \neq 2$

$$\prod_k s(L_p^{(k)}) = \begin{cases} 1, & \text{if} \quad u \bmod p \quad \text{is a square in} \quad \mathbb{Z}/p\mathbb{Z}, \\ -1, & \text{otherwise.} \end{cases}$$

In an analogous way,

$$\prod_k s(L_2^{(k)}) = \begin{cases} 1, & \text{if} \quad u \equiv \pm 1 \pmod 8, \\ -1, & \text{otherwise.} \end{cases}$$

In particular, if $p \neq 2$ does not divide $d(L)$, then L_p is uniquely defined by $\operatorname{rk} L$ and $d(L)$. In an analogous way, if $d(L)$ is odd and L is even, then L_2 is uniquely defined by $\operatorname{rk} L$ and $d(L)$.

The symbol of L_p is called the *p-symbol* of L.

Resuming, we can say that under the restrictions (I) and (II) two even quadratic lattices L and M of the same signature are isomorphic if and only if

- $d(L) = d(M)$;
- for any odd prime divisor p of $d(L)$ the p-symbols of L and M coincide;
- if $d(L)$ is even, the 2-symbols of L and M are equivalent.

In conclusion note the following useful isomorphism:

$$U \oplus [1] \simeq [-1] \oplus [1] \oplus [1]. \tag{7}$$

3. Triangular faces with 3 non-simple vertices

Let F be a 3-face of $P^{(2)}$, all whose vertices are non-simple, and u_1, u_2, u_3, u be defined as in §2. Then u_1, u_2, u_3 are orthogonal to u, and the Gram matrix $G = G(u_1, u_2, u_3, u)$ has the form

$$G = \begin{pmatrix} 2 & -2 & -2 & 0 \\ -2 & 2 & -2 & 0 \\ -2 & -2 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

After some simplifications we obtain:

$$L' \simeq [4]U \oplus [2] \oplus [2],$$

the invariant factors of L' being $2, 2, 4, 4$. It follows that $[L : L'] \leqslant 8$. Finding all even extensions of index 2 of L', then all their even extensions of index 2 and so on, and taking into account the isomorphism (7), we obtain the following possibilities for L:

No.	L	Invariant factors	Symbols
1	$[4]U \oplus [2] \oplus [2]$	$2, 2, 4, 4$	$2_2^2 4^2$
2	$[2]U \oplus [2] \oplus [2]$	$2, 2, 2, 2$	2_2^4
3	$U \oplus [2] \oplus [2]$	$1, 1, 2, 2$	$1^2 2_2^2$

We shall continue this list in the subsequent sections. Let us agree to denote the lattice No. k from the list by $L(k)$.

4. Triangular faces with 2 non-simple vertices

Let F be a 3-face of $P^{(2)}$ with 2 non-simple vertices. Then u_1, u_2, u_3 are still orthogonal to u, and the matrix G has one of the following forms:

$$G_1 = \begin{pmatrix} 2 & -2 & -1 & 0 \\ -2 & 2 & -2 & 0 \\ -1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

In the first case

$$L' \simeq [3]U \oplus [2] \oplus [2],$$

with invariant factors $1, 1, 6, 6$. There are no even extensions of L' of index 2, and all the extensions of index 3 are isomorphic to $L(3)$. So we obtain nothing new, except for L' itself.

In the second case $L' \simeq L(2)$, so we obtain nothing new.

Thus, we add only one lattice to the list:

No.	L	Invariant factors	Symbols
4	$[3]U \oplus [2] \oplus [2]$	$1, 1, 6, 6$	$1^2 2_2^2, 1^2 3^2$

5. Triangular faces with 1 non-simple vertex

Let F be a 3-face of $P^{(2)}$ with 1 non-simple vertex. Then at least two of the vectors u_1, u_2, u_3 are orthogonal to u, and the matrix G has one of the following forms:

$$G_1 = \begin{pmatrix} 2 & -2 & -1 & 0 \\ -2 & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 2 & -2 & -1 & 0 \\ -2 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

$$G_3 = \begin{pmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad G_4 = \begin{pmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

In the first case, replasing u_1 by $u_1 + 2u_2 + 2u_3 + u_4$, we obtain:

$$L' \simeq [-4] \oplus A_3,$$

with invariant factors 1, 1, 4, 4.

In the second case L' has invariant factors 1, 1, 2, 8. It contains a primitive sublattice of type A_2. Since A_2 is unimodular over $\mathbb{Z}_2$, we have the following isomorphism over $\mathbb{Z}_2$:

$$L_2' \simeq A_2 \oplus [2] \oplus [-24].$$

It follows that the symbol of L_2' is $\bar{1}^2 2_1^1 \bar{8}_{-3}^1 \sim 1^2 2_1^1 8_1^1$ and, hence,

$$L' \simeq U \oplus [2] \oplus [8].$$

In the both preceding cases L' admits only one even extension of index 2. It is generated by $\frac{1}{2}(u_1 + u_2)$ and is isomorphic to $L(3)$.

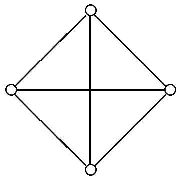
In the two remaining cases L' itself is isomorphic to $L(3)$.

Thus, we add two lattices to the list:

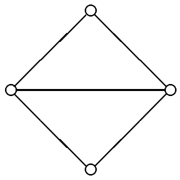
No.	L	Invariant factors	Symbols
5	$[-4] \oplus A_3$	$1, 1, 4, 4$	$\bar{1}^1 \bar{4}_2^2$
6	$U \oplus [2] \oplus [8]$	$1, 1, 2, 8$	$1^2 2_1^1 8_1^1$

6. Triangular faces without non-simple vertices

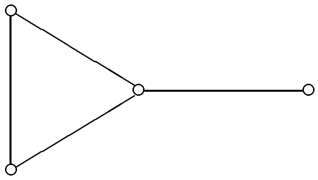
Let F be a 3-face of $P^{(2)}$, all whose vertices are simple. Then all scalar products of the vectors u_1, u_2, u_3, u are -1 or 0. The Coxeter diagram Σ for the vectors u_1, u_2, u_3, u and the Gram matrix can have only one of the following forms:

Σ_1 

$$G_1 = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{pmatrix},$$

Σ_2 

$$G_2 = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & -1 \\ -1 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix},$$

Σ_3 

$$G_3 = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

In each of these cases Σ defines a Coxeter polyhedron of finite volume, which must coincide with $P^{(2)}$ [V4]. Hence the lattice L' and any its extension are 2-reflective.

It is easy to see that the lattices defined by the matrices G_1, G_2, G_3, are isomorphic to $[k]U \oplus A_2, \quad k = 3, 2, 1$, respectively. All the even extensions of the first two of them are isomorphic to the third one, while the latter does not admit any extensions.

Thus, we add three new lattices to our list (but we already know that they are 2-reflective):

No.	L	Invariant factors	Symbols
7	$[3]U \oplus A_2$	$1, 3, 3, 3$	$\bar{1}^1 3^3$
8	$[2]U \oplus A_2$	$1, 1, 2, 6$	$\bar{1}^2 2^2, 1^3 \bar{3}^1$
9	$U \oplus A_2$	$1, 1, 1, 3$	$1^3 \bar{3}^1$

7. Quadrilateral faces with 3 non-simple vertices

Let F be a 4-face of $P^{(2)}$ with 3 non-simple vertices, and u_1, u_2, u_3, u_4, u be defined as in §2. Then u_1, u_2, u_3, u_4 are orthogonal to u, and the Gram matrix $G = G(u_1, u_2, u_3, u_4, u)$ has the following form:

$$G = \begin{pmatrix} 2 & -2 & -x & -\varepsilon & 0 \\ -2 & 2 & -2 & -y & 0 \\ -x & -2 & 2 & -2 & 0 \\ -\varepsilon & -y & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

where $\varepsilon = 0$ or 1, while x and y are some integers $\geqslant 3$.

Since $\operatorname{rk} G = 4$, we have $\det G = 2(x+2)(y+2)[(x-2)(y-2) - 4(\varepsilon+2)] = 0$, whence we obtain the following possibilities for ε, x, y (up to permutation of x and y):

ε	1	1	1	0	0
x	3	4	5	3	4
y	14	8	6	10	6

It is easy to see that in all the cases but the last one $L' \simeq L(3)$; in the last case $L' \simeq L(2)$. Thus, we obtain nothing new.

8. Quadrilateral faces with 2 opposite non-simple vertices

Let F be a 4-face of $P^{(2)}$ with 2 opposite non-simple vertices. Then u_1, u_2, u_3, u_4 are still orthogonal to u, and the matrix G has the following form:

$$G = \begin{pmatrix} 2 & -2 & -x & -\varepsilon & 0 \\ -2 & 2 & -\eta & -y & 0 \\ -x & -\eta & 2 & -2 & 0 \\ -\varepsilon & -y & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

where $\varepsilon, \eta = 0$ or 1, while x and y are some integers $\geqslant 3$.

We have $\det G = 2[(x+2)(y+2) - (2-\varepsilon)(2-\eta)][(x-2)(y-2) - (\varepsilon+2)(\eta+2)]] = 0$, whence we obtain the following possibilities for ε, η, x, y (up to permutations of ε and η, x and y):

ε	1	1	0	0	0	0
η	1	1	1	1	0	0
x	3	5	3	4	3	4
y	11	5	8	5	6	4

In the third and fourth cases $L' \simeq L(3)$.

In all the other cases $u_2 + u_4 = k(u_1 + u_3) \quad (k = 1, 2, 3)$, so u_1, u_2, u_3, u constitute a basis of L'.

In the first case the invariant factors of L' are 1, 1, 2, 32. Replacing u_1 with $u_1 + u_2$ and u_3 with $-u_3$, we obtain

$$L' \simeq \begin{bmatrix} 0 & 0 & 4 \\ 0 & 2 & 1 \\ 4 & 1 & 2 \end{bmatrix} \oplus [2].$$

There is only one extension of index 2 of L'. It is generated by $\frac{1}{2}(u_1+u_2)$ and is isomorphic to $L(6)$.

In the second case

$$L' \simeq [-24] \oplus A_2 \oplus [2],$$

with invariant factors 1, 1, 6, 24. There are two extensions of index 3 of L', which are isomorphic to $L(6)$, and there is only one extension of index 2. The latter is isomorphic to

$$[-6] \oplus A_2 \oplus [2] \simeq L(4).$$

In the fifth case

$$L' \simeq \begin{bmatrix} 0 & 3 \\ 3 & 2 \end{bmatrix} \oplus [2] \oplus [2],$$

with invariant factors 1, 1, 2, 18. There are no even extensions of index 2 of L', and there is only one extension of index 3, which is isomorphic to $L(3)$.

In the sixth case

$$L' \simeq [-8] \oplus [2] \oplus [2] \oplus [2].$$

There is only one even extension of index 2 of L'. It is isomorphic to $L(2)$.

Thus, we add four new lattices to our list:

No.	L	Invariant factors	Symbols
10	$\begin{bmatrix} 0 & 0 & 4 \\ 0 & 2 & 1 \\ 4 & 1 & 2 \end{bmatrix} \oplus [2]$	$1, 1, 2, 32$	$\bar{1}^2 2_1^1 \overline{32}_{-3}^1$
11	$[-24] \oplus A_2 \oplus [2]$	$1, 1, 6, 24$	$\bar{1}^2 2_1^1 \bar{8}_{-3}^1, 1^2 \bar{3}^2$
12	$\begin{bmatrix} 0 & 3 \\ 3 & 2 \end{bmatrix} \oplus [2] \oplus [2]$	$1, 1, 2, 18$	$1^2 2_2^2, \bar{1}^3 9^1$
13	$[-8] \oplus [2] \oplus [2] \oplus [2]$	$2, 2, 2, 8$	$2_3^3 8_{-1}^1$

9. Quadrilateral faces with 2 adjacent non-simple vertices

Let F be a 4-face of $P^{(2)}$ with 2 adjacent non-simple vertices. Then at least 3 of the vectors u_1, u_2, u_3, u_4 are orthogonal to u, and the matrix G has the form

$$G = \begin{pmatrix} 2 & -2 & -x & -\varepsilon & 0 \\ -2 & 2 & -2 & -y & 0 \\ -x & -2 & 2 & -\eta & 0 \\ -\varepsilon & -y & -\eta & 2 & -\theta \\ 0 & 0 & 0 & -\theta & 2 \end{pmatrix},$$

where $\varepsilon, \eta, \theta = 0$ or 1, $x \geqslant 3, y \geqslant 2$, and if $\theta = 0$, then $y \geqslant 3$.

We have

$$\det G = 2(x+2)[(x-2)y^2 - (4-\theta)(x+2) - 4(\varepsilon+\eta)y - 4\varepsilon\eta] = 0,$$

whence

$$(x-2)y^2 - (4-\theta)(x+2) - 4(\varepsilon+\eta)y - 4\varepsilon\eta = 0. \tag{8}$$

Let $\theta = 0$. Set $(x-2)(y-2) = k$ and rewrite the equation (7) in the form

$$[k - 4(\varepsilon + \eta)]y = 2[2(4 + \varepsilon\eta) - k].$$

Since $y > 0$, k is strictly between $4(\varepsilon + \eta)$ and $2(4 + \varepsilon\eta)$. For each pair of values of ε, η this condition allows only few values of k. Analysing all of them, one can easily see that the equation (7) has no solutions in this case.

Let now $\theta = 1$. Set $(x-2)(y-\sqrt{3}) = k$ and rewrite the equation (7) in the form

$$[k - 4(\varepsilon + \eta)]y = 12 + 4\varepsilon\eta - k\sqrt{3}.$$

We see that k is between $4(\varepsilon + \eta)$ and $\frac{12+4\varepsilon\eta}{\sqrt{3}}$. Analysing all the possibilities one can show that the equation (7) in this case has only the following solutions:

ε	1	0	0	0	0
η	1	1	0	1	0
x	34	22	14	6	4
y	2	2	2	3	3

In the first case

$$L' \simeq U \oplus [2] \oplus [18].$$

There are no even extensions of index 2, and there is only one extension of index 3, which is isomorphic to $L(3)$.

In the second case $L' \simeq L(3)$.

In the third case $L' \simeq L(6)$.

In the fourth case $L' \simeq L(3)$.

In the fifth case $L' \simeq L(4)$.

Thus, we add one new lattice to our list:

No.	L	Invariant factors	Symbols
14	$U \oplus [2] \oplus [18]$	$1, 1, 2, 18$	$1^2 2_2^2, 1^3 \bar{9}^1$

10. Quadrilateral and pentagonal faces with at most 1 non-simple vertex

Let F be a 4- or 5-face of $P^{(2)}$ with $\leqslant 1$ non-simple vertex, and let F_1, F_2, F_3, F_4 be adjacent faces. (So, it F is pentagonal, we omit one of the adjacent faces.)

The Gram matrix $G = G(u_1, u_2, u_3, u_4, u)$ has the form

$$G = \begin{pmatrix} 2 & -\varepsilon_{12} & -x & -z & -\varepsilon_1 \\ -\varepsilon_{12} & 2 & -\varepsilon_{23} & -y & -\varepsilon_2 \\ -x & -\varepsilon_{23} & 2 & -\varepsilon_{34} & -\varepsilon_3 \\ -z & -y & -\varepsilon_{34} & 2 & -\varepsilon_4 \\ -\varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 & -\varepsilon_4 & 2 \end{pmatrix},$$

where

$$\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 = 0 \quad \text{or} \quad 1, \tag{9}$$

$$\varepsilon_{12}, \varepsilon_{23}, \varepsilon_{34} = 0, 1 \quad \text{or} \quad 2, \tag{10}$$

$$x, y \geqslant 2, \quad z \geqslant 0.$$

Due to our assumption about non-simple vertices

$$\text{at most one of} \quad \varepsilon_{ij}\text{'s equals} \quad 2, \tag{11}$$

and

$$\text{if} \quad \varepsilon_{ij} = 2, \quad \text{then} \quad \varepsilon_i = \varepsilon_j = 0. \tag{12}$$

We may assume that x and y do not equal 2 simultaneously, because otherwise F_1, F_2, F_3, F_4 have a common vertex at infinity and $P^{(2)}$ has triangular faces. We also may assume that $x \geqslant y$.

If F is a 4-face and $\varepsilon_{12} = \varepsilon_{23} = \varepsilon_{34} = z = 0$, then again F_1, F_2, F_3, F_4 have a common vertex at infinity [A], and $P^{(2)}$ is a quadrilateral pyramid. We do not need to consider this case, so we may assume that $z \geqslant 1$. (This automatically holds, if F is a 5-face.)

If F is a 5-face, we can dispose of the choice of the omitted adjacent face in such a way that z take the minimal possible value. Then, in particular, $z \leqslant x, y$. (This automatically holds, if F is a 4-face.)

As a result, we obtain the following restrictions for x, y, z:

$$x \geqslant y \geqslant z, \quad x \geqslant 3, \quad y \geqslant 2, \quad z \geqslant 1. \tag{13}$$

In addition, since F_2, F_4 and F have no common vertex at infinity,

$$y = 2 \quad \text{implies} \quad \varepsilon_2 + \varepsilon_4 > 0. \tag{14}$$

Moreover, if $y = 2$ and $\varepsilon_{23} = \varepsilon_{34} = 0$, then F_2, F_4 and F_3 have a common vertex at infinity, which implies that the face F_3 is triangular. Therefore, we may assume that

$$y = 2 \quad \text{implies} \quad \varepsilon_{23} + \varepsilon_{34} > 0, \tag{15}$$

Let us now turn to the condition $\det G = 0$. We have

$$\begin{aligned}
&\tfrac{1}{2} \det G = x^2 y^2 - xyz(2\varepsilon_{23} + \varepsilon_2\varepsilon_3) + \\
&+ x^2(\varepsilon_2\varepsilon_4 y + \varepsilon_2^2 + \varepsilon_4^2 - 4) + y^2(\varepsilon_1\varepsilon_3 x + \varepsilon_1^2 + \varepsilon_3^2 - 4) + z^2(\varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_{23}^2 + \varepsilon_2\varepsilon_3\varepsilon_{23} - 4) - \\
&- xy(\varepsilon_1\varepsilon_4\varepsilon_{23} + \varepsilon_1\varepsilon_2\varepsilon_{34} + \varepsilon_3\varepsilon_4\varepsilon_{12} + 2\varepsilon_{12}\varepsilon_{34}) - \\
&- xz(\varepsilon_2\varepsilon_4\varepsilon_{23} + 2\varepsilon_3\varepsilon_4 + (4 - \varepsilon_2^2)\varepsilon_{34}) - yz(\varepsilon_1\varepsilon_3\varepsilon_{23} + 2\varepsilon_1\varepsilon_2 + (4 - \varepsilon_3^2)\varepsilon_{12}) - \\
&- x((4 - \varepsilon_4^2)\varepsilon_{12}\varepsilon_{23} + 2\varepsilon_1\varepsilon_2\varepsilon_{23} + 2\varepsilon_2\varepsilon_3\varepsilon_{12} + 2\varepsilon_1\varepsilon_4\varepsilon_{34} + \varepsilon_2\varepsilon_4\varepsilon_{12}\varepsilon_{34} + 4\varepsilon_1\varepsilon_3) - \\
&- y((4 - \varepsilon_1^2)\varepsilon_{23}\varepsilon_{34} + 2\varepsilon_3\varepsilon_4\varepsilon_{23} + 2\varepsilon_2\varepsilon_3\varepsilon_{34} + 2\varepsilon_1\varepsilon_4\varepsilon_{12} + \varepsilon_1\varepsilon_3\varepsilon_{12}\varepsilon_{34} + 4\varepsilon_2\varepsilon_4) - \\
&- z((4 - \varepsilon_{23}^2)\varepsilon_1\varepsilon_4 + 2\varepsilon_{12}\varepsilon_{23}\varepsilon_{34} + 2\varepsilon_1\varepsilon_3\varepsilon_{34} + 2\varepsilon_2\varepsilon_4\varepsilon_{12} + \\
&+ \varepsilon_3\varepsilon_4\varepsilon_{12}\varepsilon_{23} + \varepsilon_2\varepsilon_3\varepsilon_{12}\varepsilon_{34} + \varepsilon_1\varepsilon_2\varepsilon_{23}\varepsilon_{34}) \\
&+ 16 - 4(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 + \varepsilon_{12}^2 + \varepsilon_{23}^2 + \varepsilon_{34}^2) - 4(\varepsilon_1\varepsilon_2\varepsilon_{12} + \varepsilon_3\varepsilon_4\varepsilon_{34} + \varepsilon_2\varepsilon_3\varepsilon_{23}) - \\
&- \varepsilon_1\varepsilon_4\varepsilon_{12}\varepsilon_{23}\varepsilon_{34} + (\varepsilon_1\varepsilon_{23} - \varepsilon_3\varepsilon_{12})^2 + (\varepsilon_2\varepsilon_{34} - \varepsilon_4\varepsilon_{23})^2 + \\
&+ \varepsilon_1^2\varepsilon_{34}^2 + \varepsilon_4^2\varepsilon_{12}^2 + \varepsilon_{12}^2\varepsilon_{34}^2 + \varepsilon_1\varepsilon_2\varepsilon_{12}\varepsilon_{34}^2 + \varepsilon_3\varepsilon_4\varepsilon_{34}\varepsilon_{12}^2.
\end{aligned}$$

Consider the case $y = 2$ (and $z = 1$ or 2). Then $\varepsilon_2 + \varepsilon_4 > 0$. If $\varepsilon_2 = \varepsilon_4 = 1$, one can show that

$$\frac{1}{2}\det G \geqslant 4x^2 - 45x - 101,$$

whence $x \leqslant 13$. If $\varepsilon_2 + \varepsilon_4 = 1$, one can show that

$$\frac{1}{2}\det G \geqslant x^2 - 33x - 69,$$

whence $x \leqslant 34$.

Let now $y \geqslant 3$. Then

$$\begin{aligned}
\tfrac{1}{2}\det G \ &\geqslant x^2y^2 - 4xyz - 4x^2 - 4y^2 - 4z^2 - 6xy - 8xz - 8yz + \\
&+\varepsilon_2\varepsilon_4(x^2y - x\varepsilon_{12}\varepsilon_{34} - 4y - 2z\varepsilon_{12}) + \varepsilon_1\varepsilon_3(xy^2 - 4x - y\varepsilon_{12}\varepsilon_{34} - 2z\varepsilon_{34}) + \\
&+\varepsilon_2\varepsilon_{34}z(x - \varepsilon_1\varepsilon_{23}) + \varepsilon_3\varepsilon_{12}z(y - \varepsilon_4\varepsilon_{23}) - \\
&-x(4\varepsilon_{12}\varepsilon_{23} + 2\varepsilon_1\varepsilon_2\varepsilon_{23} + 2\varepsilon_2\varepsilon_3\varepsilon_{12} + 2\varepsilon_1\varepsilon_4\varepsilon_{34}) - \\
&-y(4\varepsilon_{23}\varepsilon_{34} + 2\varepsilon_3\varepsilon_4\varepsilon_{23} + 2\varepsilon_2\varepsilon_3\varepsilon_{34} + 2\varepsilon_1\varepsilon_4\varepsilon_{12}) - \\
&-z(4\varepsilon_1\varepsilon_4 + 2\varepsilon_{12}\varepsilon_{23}\varepsilon_{34} + \varepsilon_2\varepsilon_3\varepsilon_{12}\varepsilon_{34}) + \\
&+16 + (y^2 - 4)\varepsilon_1 + (x^2 + z^2 - 4)\varepsilon_2 + (y^2 + z^2 - 4)\varepsilon_3 + (x^2 - 4)\varepsilon_4 - \\
&-4\varepsilon_{12}^2 - (4 - z^2)\varepsilon_{23}^2 - 4\varepsilon_{34}^2 - 4\varepsilon_1\varepsilon_2\varepsilon_{12} - 4\varepsilon_3\varepsilon_4\varepsilon_{34} - 4\varepsilon_2\varepsilon_3\varepsilon_{23} - \varepsilon_1\varepsilon_4\varepsilon_{12}\varepsilon_{23}\varepsilon_{34}.
\end{aligned}$$

All the expressions in parentheses are non-negative, so

$$\begin{aligned}
\tfrac{1}{2}\det G \geqslant\ & x^2y^2 - 4xyz - 4x^2 - 4y^2 - 4z^2 - 6xy - 8xz - 8yz - 10x - 10y - 8z + \\
& +16 + 5\varepsilon_1 + 6\varepsilon_2 + 6\varepsilon_3 + 5\varepsilon_4 - 4\varepsilon_{12}^2 - 3\varepsilon_{23}^2 - 4\varepsilon_{34}^2 - \\
& -4\varepsilon_1\varepsilon_2\varepsilon_{12} - 4\varepsilon_3\varepsilon_4\varepsilon_{34} - 4\varepsilon_2\varepsilon_3\varepsilon_{23} - \varepsilon_1\varepsilon_4\varepsilon_{12}\varepsilon_{23}\varepsilon_{34} \geqslant \\
\geqslant\ & x^2y^2 - 4xyz - 4x^2 - 4y^2 - 4z^2 - 6xy - 8xz - 8yz - 10x - 10y - 8z - 7 \\
\geqslant\ & x^2y^2 - 4xy^2 - 4x^2 - 16y^2 - 14xy - 10x - 18y - 7.
\end{aligned}$$

It follows that

$$x^2(y^2 - 4) - 2x(2y^2 + 7y + 5) - (16y^2 + 18y + 7) \leqslant 0.$$

Giving different values to y, one can get from this upper bounds for x. Taking into account our preceding consideration for $y = 2$, we obtain the following table:

y	2	3	4	5	6	7	8
$x \leqslant$	34	19	12	10	9	9	8

For $y \geqslant 9$ there are no admissible values of x.

This leaves less than 40000 possibilities for G. The determinants of all these matrices were calculated by A. V. Alekseevsky with help of a computer. As a result all matrices G satisfying (8)–(14) with $\det G = 0$ were found. Some of them differ only by the enumeration of the faces F_1, F_2, F_3, F_4. Taking this into account, 75 essentially different matrices were obtained. They are listed below. In all the cases when the corresponding lattice L' is not isomorphic to one of the lattices already found, we provide it with a number and indicate a simple form of it, the invariant factors, and symbols.

$$G = \begin{pmatrix} 2 & 0 & -3 & -1 & 0 \\ 0 & 2 & -1 & -2 & -1 \\ -3 & -1 & 2 & 0 & 0 \\ -1 & -2 & 0 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{pmatrix}$$

$L' = L(15) \simeq U \oplus \begin{bmatrix} 2 & 1 \\ 1 & 10 \end{bmatrix}$
Invariant factors : $1, 1, 1, 19$
Symbols : $1^3\overline{19}^1$

$$G = \begin{pmatrix} 2 & 0 & -5 & -1 & 0 \\ 0 & 2 & 0 & -2 & -1 \\ -5 & 0 & 2 & -1 & 0 \\ -1 & -2 & -1 & 2 & 0 \\ 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

$L' = L(16) \simeq U \oplus \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$
Invariant factors : $1, 1, 1, 7$
Symbols : $\bar{1}^3 7$

$$G = \begin{pmatrix} 2 & 0 & -5 & -1 & -1 \\ 0 & 2 & -1 & -2 & -1 \\ -5 & -1 & 2 & 0 & -1 \\ -1 & -2 & 0 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix}$$

$L' = L(17) \simeq U \oplus \begin{bmatrix} 2 & 1 \\ 1 & 14 \end{bmatrix}$
Invariant factors : $1, 1, 1, 27$
Symbols : $1^3\overline{27}^1$

$$G = \begin{pmatrix} 2 & 0 & -6 & -1 & 0 \\ 0 & 2 & 0 & -2 & 0 \\ -6 & 0 & 2 & -1 & 0 \\ -1 & -2 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$L' = L(18) \simeq U \oplus [2] \oplus [4]$
Invariant factors : $1, 1, 2, 4$
Symbols : $1^2(2^1 4^1)_2$

$$G = \begin{pmatrix} 2 & -1 & -6 & -1 & 0 \\ -1 & 2 & -1 & -2 & -1 \\ -6 & -1 & 2 & -1 & 0 \\ -1 & -2 & -1 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{pmatrix}$$

$L' = L(19) \simeq [-8] \oplus A_3$
Invariant factors : $1, 1, 4, 8$
Symbols : $\bar{1}^2(\bar{4}^1 8^1)_2$

$$G = \begin{pmatrix} 2 & 0 & -7 & -1 & -1 \\ 0 & 2 & -1 & -2 & -1 \\ -7 & -1 & 2 & -1 & -1 \\ -1 & -2 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix}$$

$L' \simeq L(9)$

$$G = \begin{pmatrix} 2 & 0 & -9 & -1 & -1 \\ 0 & 2 & 0 & -2 & -1 \\ -9 & 0 & 2 & -1 & -1 \\ -1 & -2 & -1 & 2 & 0 \\ -1 & -1 & -1 & 0 & 2 \end{pmatrix}$$

$L' = L(20) \simeq U \oplus \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}$
Invariant factors : $1, 1, 1, 11$
Symbols : $1^3\overline{11}^1$

$$G = \begin{pmatrix} 2 & 0 & -10 & -1 & -1 \\ 0 & 2 & 0 & -2 & 0 \\ -10 & 0 & 2 & -1 & -1 \\ -1 & -2 & -1 & 2 & -1 \\ -1 & 0 & -1 & -1 & 2 \end{pmatrix}$$

$L' = L(21) \simeq U \oplus [2] \oplus [6]$
Invariant factors : $1, 1, 2, 6$
Symbols : $1^2\bar{2}_4^2, 1^3\bar{3}^1$

$$G = \begin{pmatrix} 2 & -1 & -10 & -1 & -1 \\ -1 & 2 & -1 & -2 & -1 \\ -10 & -1 & 2 & -1 & -1 \\ -1 & -2 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix}$$

$L' = L(22) \simeq [4]U \oplus A_2$
Invariant factors : $1, 1, 4, 12$
Symbols : $\bar{1}^2 4^2, 1^3\bar{3}^1$

$$G = \begin{pmatrix} 2 & 0 & -16 & -1 & -1 \\ 0 & 2 & -1 & -2 & 0 \\ -16 & -1 & 2 & -1 & -1 \\ -1 & -2 & -1 & 2 & -1 \\ -1 & 0 & -1 & -1 & 2 \end{pmatrix}$$

$L' \simeq L(9)$

$$G = \begin{pmatrix} 2 & -1 & -17 & -1 & 0 \\ -1 & 2 & -1 & -2 & 0 \\ -17 & -1 & 2 & -1 & 0 \\ -1 & -2 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$L' \simeq L(15)$

$$G = \begin{pmatrix} 2 & -1 & -25 & -1 & -1 \\ -1 & 2 & -1 & -2 & 0 \\ -25 & -1 & 2 & -1 & -1 \\ -1 & -2 & -1 & 2 & -1 \\ -1 & 0 & -1 & -1 & 2 \end{pmatrix}$$

$L' \simeq L(17)$

$$G = \begin{pmatrix} 2 & 0 & -3 & -1 & 0 \\ 0 & 2 & -1 & -3 & 0 \\ -3 & -1 & 2 & 0 & 0 \\ -1 & -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$L' = L(23) \simeq U \oplus [4] \oplus [6]$
Invariant factors : $1, 1, 2, 12$
Symbols : $1^2(\bar{2}^1 4^1)_4$

$$G = \begin{pmatrix} 2 & 0 & -3 & -1 & 0 \\ 0 & 2 & -1 & -3 & 0 \\ -3 & -1 & 2 & 0 & -1 \\ -1 & -3 & 0 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

$L' = L(24) \simeq \begin{bmatrix} -2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$
Invariant factors : $1, 1, 2, 14$
Symbols : $1^2\bar{2}^2, 1^3\bar{7}^1$

$$G = \begin{pmatrix} 2 & 0 & -3 & -1 & 0 \\ 0 & 2 & -1 & -3 & -1 \\ -3 & -1 & 2 & 0 & -1 \\ -1 & -3 & 0 & 2 & 0 \\ 0 & -1 & -1 & 0 & 2 \end{pmatrix}$$

$L' = L(25) \simeq \begin{bmatrix} 0 & 3 \\ 3 & 2 \end{bmatrix} \oplus A_2$
Invariant factors : $1, 1, 3, 9$
Symbols : $1^2\bar{3}^1 9^1$

$$G = \begin{pmatrix} 2 & 0 & -3 & -1 & -1 \\ 0 & 2 & -1 & -3 & -1 \\ -3 & -1 & 2 & 0 & -1 \\ -1 & -3 & 0 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix}$$

$L' = L(26) \simeq [-4] \oplus [4] \oplus A_2$
Invariant factors : $1, 1, 4, 12$
Symbols : $\bar{1}^2 4_0^2, 1^3\bar{3}^1$

$$G = \begin{pmatrix} 2 & -1 & -4 & -1 & 0 \\ -1 & 2 & -1 & -3 & -1 \\ -4 & -1 & 2 & -1 & 0 \\ -1 & -3 & -1 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{pmatrix}$$

$L' = L(27) \simeq \begin{bmatrix} -12 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$
Invariant factors : $1, 1, 2, 30$
Symbols : $\bar{1}^2\bar{2}^2, 1^3 3^1, 1^3\bar{5}^1$

$$G = \begin{pmatrix} 2 & 0 & -4 & -1 & 0 \\ 0 & 2 & -1 & -3 & -1 \\ -4 & -1 & 2 & -1 & -1 \\ -1 & -3 & -1 & 2 & 0 \\ 0 & -1 & -1 & 0 & 2 \end{pmatrix}$$

$L' \simeq L(9)$

$$G = \begin{pmatrix} 2 & 0 & -4 & -1 & -1 \\ 0 & 2 & -1 & -3 & 0 \\ -4 & -1 & 2 & -1 & -1 \\ -1 & -3 & -1 & 2 & -1 \\ -1 & 0 & -1 & -1 & 2 \end{pmatrix}$$

$L' \simeq L(9)$

$$G = \begin{pmatrix} 2 & 0 & -4 & -1 & -1 \\ 0 & 2 & -1 & -3 & -1 \\ -4 & -1 & 2 & -1 & -1 \\ -1 & -3 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix}$$

$L' \simeq L(9)$

$$G = \begin{pmatrix} 2 & -1 & -5 & -1 & -1 \\ -1 & 2 & -1 & -3 & 0 \\ -5 & -1 & 2 & -1 & -1 \\ -1 & -3 & -1 & 2 & 0 \\ -1 & 0 & -1 & 0 & 2 \end{pmatrix}$$

$L' = L(28) \simeq U \oplus \begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix}$
Invariant factors : $1, 1, 1, 35$
Symbols : $1^3\bar{5}^1, 1^3 7^1$

$$G = \begin{pmatrix} 2 & -1 & -6 & -1 & 0 \\ -1 & 2 & -1 & -3 & 0 \\ -6 & -1 & 2 & -1 & 0 \\ -1 & -3 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$L' = L(29) \simeq U \oplus [4] \oplus [10]$
Invariant factors : $1, 1, 2, 20$
Symbols : $1^2(\bar{2}^1 4^1)_{-2}, 1^3\bar{5}^1$

$$G = \begin{pmatrix} 2 & -1 & -6 & -1 & -1 \\ -1 & 2 & -1 & -3 & 0 \\ -6 & -1 & 2 & -1 & -1 \\ -1 & -3 & -1 & 2 & -1 \\ -1 & 0 & -1 & -1 & 2 \end{pmatrix}$$

$L' \simeq L(26)$

$$G = \begin{pmatrix} 2 & -1 & -4 & -1 & 0 \\ -1 & 2 & -1 & -4 & 0 \\ -4 & -1 & 2 & -1 & 0 \\ -1 & -4 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$L' = L(30) \simeq [-12] \oplus [2] \oplus A_2$
Invariant factors : $1, 1, 6, 12$
Symbols : $\bar{1}^2(2^1\bar{4}^1)_{-2}, 1^2 3^2$

$$G = \begin{pmatrix} 2 & -1 & -4 & -1 & 0 \\ -1 & 2 & -1 & -4 & 0 \\ -4 & -1 & 2 & -1 & -1 \\ -1 & -4 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

$L' = L(31) \simeq U \oplus \begin{bmatrix} 6 & 3 \\ 3 & 14 \end{bmatrix}$
Invariant factors : $1, 1, 1, 75$
Symbols : $1^3\bar{3}^1, 1^3\overline{25}^1$

$$G = \begin{pmatrix} 2 & -1 & -4 & -1 & -1 \\ -1 & 2 & -1 & -4 & -1 \\ -4 & -1 & 2 & -1 & -1 \\ -1 & -4 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix}$$

$L' = L(32) \simeq [6]U \oplus A_2$
Invariant factors : $1, 3, 6, 6$
Symbols : $\bar{1}^2 2^2, \bar{1}^1 3^3$

$$G = \begin{pmatrix} 2 & -1 & -9 & -2 & -1 \\ -1 & 2 & 0 & -2 & -1 \\ -9 & 0 & 2 & -1 & -1 \\ -2 & -2 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix}$$

$L' \simeq L(9)$

$$G = \begin{pmatrix} 2 & 0 & -12 & -2 & -1 \\ 0 & 2 & -1 & -2 & 0 \\ -12 & -1 & 2 & 0 & 0 \\ -2 & -2 & 0 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$L' \simeq L(16)$

$$G = \begin{pmatrix} 2 & -1 & -13 & -2 & -1 \\ -1 & 2 & -1 & -2 & -1 \\ -13 & -1 & 2 & -1 & -1 \\ -2 & -2 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & 0 & -14 & -2 & 0 \\ 0 & 2 & 0 & -2 & -1 \\ -14 & 0 & 2 & -2 & 0 \\ -2 & -2 & -2 & 2 & 0 \\ 0 & -1 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(6)$$

$$G = \begin{pmatrix} 2 & 0 & -22 & -2 & 0 \\ 0 & 2 & -1 & -2 & -1 \\ -22 & -1 & 2 & -2 & 0 \\ -2 & -2 & -2 & 2 & 0 \\ 0 & -1 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(3)$$

$$G = \begin{pmatrix} 2 & -1 & -22 & -2 & -1 \\ -1 & 2 & -1 & -2 & -1 \\ -22 & -1 & 2 & 0 & -1 \\ -2 & -2 & 0 & 2 & 0 \\ -1 & -1 & -1 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(21)$$

$$G = \begin{pmatrix} 2 & -1 & -34 & -2 & 0 \\ -1 & 2 & -1 & -2 & -1 \\ -34 & -1 & 2 & -2 & 0 \\ -2 & -2 & -2 & 2 & 0 \\ 0 & -1 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(14)$$

$$G = \begin{pmatrix} 2 & -1 & -34 & -2 & -1 \\ -1 & 2 & -1 & -2 & -1 \\ -34 & -1 & 2 & -1 & -1 \\ -2 & -2 & -1 & 2 & 0 \\ -1 & -1 & -1 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & 0 & -4 & -2 & 0 \\ 0 & 2 & 0 & -3 & -1 \\ -4 & 0 & 2 & -2 & 0 \\ -2 & -3 & -2 & 2 & 0 \\ 0 & -1 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(4)$$

$$G = \begin{pmatrix} 2 & -1 & -5 & -2 & -1 \\ -1 & 2 & 0 & -3 & -1 \\ -5 & 0 & 2 & -1 & 0 \\ -2 & -3 & -1 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & -1 & -5 & -2 & -1 \\ -1 & 2 & 0 & -3 & -1 \\ -5 & 0 & 2 & -1 & -1 \\ -2 & -3 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(8)$$

$$G = \begin{pmatrix} 2 & 0 & -6 & -2 & 0 \\ 0 & 2 & -2 & -3 & 0 \\ -6 & -2 & 2 & 0 & 0 \\ -2 & -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(12)$$

$$G = \begin{pmatrix} 2 & 0 & -6 & -2 & 0 \\ 0 & 2 & -1 & -3 & -1 \\ -6 & -1 & 2 & -2 & 0 \\ -2 & -3 & -2 & 2 & 0 \\ 0 & -1 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(3)$$

$$G = \begin{pmatrix} 2 & -1 & -7 & -2 & -1 \\ -1 & 2 & -1 & -3 & -1 \\ -7 & -1 & 2 & -1 & -1 \\ -2 & -3 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & 0 & -8 & -2 & 0 \\ 0 & 2 & -2 & -3 & 0 \\ -8 & -2 & 2 & -1 & 0 \\ -2 & -3 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(3)$$

$$G = \begin{pmatrix} 2 & -1 & -8 & -2 & -1 \\ -1 & 2 & -1 & -3 & -1 \\ -8 & -1 & 2 & -1 & -1 \\ -2 & -3 & -1 & 2 & 0 \\ -1 & -1 & -1 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & -1 & -11 & -2 & 0 \\ -1 & 2 & -2 & -3 & 0 \\ -11 & -2 & 2 & -1 & 0 \\ -2 & -3 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(10)$$

$$G = \begin{pmatrix} 2 & 0 & -4 & -2 & 0 \\ 0 & 2 & -2 & -4 & 0 \\ -4 & -2 & 2 & 0 & 0 \\ -2 & -4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(13)$$

$$G = \begin{pmatrix} 2 & 0 & -4 & -2 & -1 \\ 0 & 2 & -2 & -4 & 0 \\ -4 & -2 & 2 & 0 & 0 \\ -2 & -4 & 0 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(10)$$

$$G = \begin{pmatrix} 2 & 0 & -5 & -2 & 0 \\ 0 & 2 & -2 & -4 & 0 \\ -5 & -2 & 2 & -1 & 0 \\ -2 & -4 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(3)$$

$$G = \begin{pmatrix} 2 & 0 & -5 & -2 & -1 \\ 0 & 2 & -2 & -4 & 0 \\ -5 & -2 & 2 & -1 & 0 \\ -2 & -4 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(3)$$

$$G = \begin{pmatrix} 2 & -1 & -5 & -2 & -1 \\ -1 & 2 & -1 & -4 & -1 \\ -5 & -1 & 2 & -1 & 0 \\ -2 & -4 & -1 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & -1 & -5 & -2 & -1 \\ -1 & 2 & -1 & -4 & -1 \\ -5 & -1 & 2 & -1 & -1 \\ -2 & -4 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & -1 & -5 & -2 & 0 \\ -1 & 2 & -2 & -5 & 0 \\ -5 & -2 & 2 & -1 & 0 \\ -2 & -5 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(11)$$

$$G = \begin{pmatrix} 2 & -1 & -5 & -2 & -1 \\ -1 & 2 & -2 & -5 & 0 \\ -5 & -2 & 2 & -1 & 0 \\ -2 & -5 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$L' = L(33) \simeq [-36] \oplus A_3$
Invariant factors : $1, 1, 4, 36$
Symbols : $\bar{1}^2\bar{4}^2_2, 1^3\bar{9}^1$

$$G = \begin{pmatrix} 2 & 0 & -6 & -3 & -1 \\ 0 & 2 & 0 & -3 & -1 \\ -6 & 0 & 2 & -2 & 0 \\ -3 & -3 & -2 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(3)$$

$$G = \begin{pmatrix} 2 & -1 & -6 & -3 & -1 \\ -1 & 2 & 0 & -3 & -1 \\ -6 & 0 & 2 & -1 & 0 \\ -3 & -3 & -1 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(25)$$

$$G = \begin{pmatrix} 2 & -1 & -6 & -3 & -1 \\ -1 & 2 & 0 & -3 & -1 \\ -6 & 0 & 2 & -1 & -1 \\ -3 & -3 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & -1 & -7 & -3 & 0 \\ -1 & 2 & -1 & -3 & 0 \\ -7 & -1 & 2 & 0 & 0 \\ -3 & -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(23)$$

$$G = \begin{pmatrix} 2 & -1 & -7 & -3 & -1 \\ -1 & 2 & -1 & -3 & 0 \\ -7 & -1 & 2 & 0 & -1 \\ -3 & -3 & 0 & 2 & -1 \\ -1 & 0 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(24)$$

$$G = \begin{pmatrix} 2 & -1 & -7 & -3 & -1 \\ -1 & 2 & -1 & -3 & -1 \\ -7 & -1 & 2 & 0 & -1 \\ -3 & -3 & 0 & 2 & 0 \\ -1 & -1 & -1 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(25)$$

$$G = \begin{pmatrix} 2 & 0 & -9 & -3 & -1 \\ 0 & 2 & -1 & -3 & -1 \\ -9 & -1 & 2 & -2 & 0 \\ -3 & -3 & -2 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{pmatrix} \qquad \begin{array}{l} L' = L(34) \simeq [-16] \oplus A_3 \\ \text{Invariant factors} : 1, 1, 4, 16 \\ \text{Symbols} : \bar{1}^2 \bar{4}^1_3 16^1_{-1} \end{array}$$

$$G = \begin{pmatrix} 2 & -1 & -10 & -3 & -1 \\ -1 & 2 & -1 & -3 & -1 \\ -10 & -1 & 2 & -1 & -1 \\ -3 & -3 & -1 & 2 & 0 \\ -1 & -1 & -1 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(8)$$

$$G = \begin{pmatrix} 2 & -1 & -4 & -3 & -1 \\ -1 & 2 & 0 & -4 & 0 \\ -4 & 0 & 2 & -1 & -1 \\ -3 & -4 & -1 & 2 & -1 \\ -1 & 0 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & -1 & -6 & -3 & -1 \\ -1 & 2 & -1 & -4 & 0 \\ -6 & -1 & 2 & -1 & -1 \\ -3 & -4 & -1 & 2 & -1 \\ -1 & 0 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & -1 & -6 & -3 & -1 \\ -1 & 2 & -1 & -4 & -1 \\ -6 & -1 & 2 & -1 & 0 \\ -3 & -4 & -1 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & -1 & -6 & -3 & -1 \\ -1 & 2 & -1 & -4 & -1 \\ -6 & -1 & 2 & -1 & -1 \\ -3 & -4 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(8)$$

$$G = \begin{pmatrix} 2 & -2 & -4 & -4 & 0 \\ -2 & 2 & 0 & -4 & 0 \\ -4 & 0 & 2 & 0 & -1 \\ -4 & -4 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(3)$$

$$G = \begin{pmatrix} 2 & -1 & -5 & -4 & -1 \\ -1 & 2 & -1 & -4 & -1 \\ -5 & -1 & 2 & 0 & 0 \\ -4 & -4 & 0 & 2 & -1 \\ -1 & -1 & 0 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & -1 & -5 & -4 & -1 \\ -1 & 2 & 0 & -4 & -1 \\ -5 & 0 & 2 & -1 & -1 \\ -4 & -4 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & 0 & -6 & -4 & 0 \\ 0 & 2 & -1 & -4 & -1 \\ -6 & -1 & 2 & -2 & 0 \\ -4 & -4 & -2 & 2 & 0 \\ 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

$L' = L(35) \simeq U \oplus [2] \oplus [50]$
Invariant factors : $1, 1, 2, 50$
Symbols : $1^2 2_2^2, \bar{1}^3 \overline{25}^1$

$$G = \begin{pmatrix} 2 & 0 & -6 & -4 & -1 \\ 0 & 2 & -2 & -4 & 0 \\ -6 & -2 & 2 & 0 & 0 \\ -4 & -4 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(3)$$

$$G = \begin{pmatrix} 2 & -1 & -7 & -4 & -1 \\ -1 & 2 & -1 & -4 & -1 \\ -7 & -1 & 2 & -1 & 0 \\ -4 & -4 & -1 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & -1 & -7 & -4 & -1 \\ -1 & 2 & -1 & -4 & -1 \\ -7 & -1 & 2 & -1 & -1 \\ -4 & -4 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(7)$$

$$G = \begin{pmatrix} 2 & -1 & -5 & -5 & 0 \\ -1 & 2 & -1 & -5 & -1 \\ -5 & -1 & 2 & 0 & -1 \\ -5 & -5 & 0 & 2 & -1 \\ 0 & -1 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(8)$$

$$G = \begin{pmatrix} 2 & -1 & -6 & -5 & 0 \\ -1 & 2 & -1 & -5 & -1 \\ -6 & -1 & 2 & -1 & -1 \\ -5 & -5 & -1 & 2 & -1 \\ 0 & -1 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & -1 & -6 & -5 & -1 \\ -1 & 2 & -1 & -5 & 0 \\ -6 & -1 & 2 & -1 & -1 \\ -5 & -5 & -1 & 2 & -1 \\ -1 & 0 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(8)$$

$$G = \begin{pmatrix} 2 & -1 & -7 & -7 & 0 \\ -1 & 2 & -2 & -7 & 0 \\ -7 & -2 & 2 & -1 & 0 \\ -7 & -7 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(10)$$

$$G = \begin{pmatrix} 2 & -1 & -7 & -7 & -1 \\ -1 & 2 & -2 & -7 & 0 \\ -7 & -2 & 2 & -1 & 0 \\ -7 & -7 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(34)$$

As the result, we have obtained 21 new lattices $L(15)$–$L(35)$. All their even extensions of prime indices are described as follows.

Each of the lattices $L(19), L(26), L(33), L(34)$ admits one even extension of index 2, which is isomorphic to $L(18), L(8), L(14), L(5)$, respectively. The lattice $L(22)$ admits 3 even extensions of index 2, two of which are isomorphic to $L(8)$ and one to $L(21)$. The lattice $L(32)$ admits two even extensions of index 2, which are isomorphic to L(7).

Each of the lattices $L(17), L(25), L(33)$ admits one extension of index 3, which is isomorphic to $L(9), L(9), L(5)$, respectively. The lattice $L(32)$ admits 4 extensions of index 3, which are isomorphic to $L(8)$.

Each of the lattices $L(31), L(35)$ admits one extension of index 5, which is isomorphic to $L(9), L(3)$, respectively.

Thus, there are no new lattices among these extensions, so all 2-reflective lattices of signature (3, 1) are among the lattices $L(1)$–$L(35)$.

11. Non-2-reflectivity of some isotropic lattices

In order to prove that some of the lattices $L(1)$–$L(35)$ are not 2-reflective, we apply a method going back to [V3]. Note that this method can be applied only to isotropic lattices.

A Euclidean lattice M is called 2-*reflective* if its 2-roots span the space $M \otimes \mathbb{R}$.

Let L be a hyperbolic lattice of rank $n+1$ and $v \in L$ a primitive isotropic vector. Then

$$L_v = (\mathbb{Z}v)^{\perp}/\mathbb{Z}v$$

is a Euclidean lattice of rank $n-1$.

Lemma. If L is 2-reflective, then L_v is also 2-reflective.

Proof. Let L be 2-reflective, and let q be the point at infinity of the Lobachevsky space L^n defined by v. Since $v \in L$, q is a vertex of a fundamental polyhedron $P^{(2)}$ of the group $\mathrm{O}_r^{(2)}(L)$ (see, e.g., [V3]). The 2-roots of L corresponding to the walls of $P^{(2)}$ passing through q are orthogonal to v, and their images in the Euclidean space $(\mathbb{R}v)^{\perp}/\mathbb{R}v$ are 2-roots of the lattice L_v. Since the intersection of $P^{(2)}$ with the horosphere centered at q is compact, the above vectors span the space $(\mathbb{R}v)^{\perp}/\mathbb{R}v = L_v \otimes \mathbb{R}$. □

Note that there are only two 2-reflective Euclidean lattices of rank 2: $[2] \oplus [2]$ and A_2. It immediately follows from this that the lattices $L(6)$, $L(14)$, $L(15)$, $L(16)$, $L(17)$, $L(18)$, $L(20)$, $L(21)$, $L(23)$, $L(28)$, $L(29)$, $L(31)$, $L(35)$ are not 2-reflective. Hence, their sublattices $L(10)$, $L(11)$, $L(19)$, $L(22)$, $L(33)$ are not 2-reflective, either.

Comparing the symbols, one can get

$$L(12) \simeq U \oplus \begin{bmatrix} 4 & 2 \\ 2 & 10 \end{bmatrix},$$

which shows that $L(12)$ is not 2-reflective.

Finally, the lattice $L(34)$ has the Gram matrix

$$\begin{pmatrix} -16 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

with respect to some basis $\{v_0, v_1, v_2, v_3\}$. The vector

$$v = v_0 + 2(v_1 - v_3)$$

is isotropic, and

$$(\mathbb{Z}v)^{\perp} = \langle v, v_2, v_1 + v_3 \rangle.$$

It follows that

$$(\mathbb{Z}v)^{\perp}/\mathbb{Z}v \simeq \langle v_2, v_1+v_3\rangle = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}.$$

This shows that the lattice $L(34)$ is not 2-reflective.

12. Investigation of the remaining lattices

In order to determine if a hyperbolic lattice L is 2-reflective, it is useful to consider all reflections in $\mathrm{O}(L)$. In the notation of the introduction, we have the following

Lemma $\mathrm{O}_r(L) = \mathrm{O}_r^{(2)}(L) \rtimes \Delta$, where Δ is the subgroup of $\mathrm{O}_r(L)$ generated by those reflections in the walls of P which are not 2-reflections.

Proof. Decompose the set S of all reflections in the walls of P into two parts: the set S_2 of all 2-reflections and the set S' of all other reflections in S. Since any reflection in $\mathrm{O}_r(L)$ is conjugate to some reflection of S, $\mathrm{O}_r^{(2)}(L)$ is the normal subgroup of $\mathrm{O}_r(L)$ generated by S_2.

Note that the exponent of the Coxeter relation between any $s \in S_2$ and $s' \in S'$ is even, because otherwise s and s' would be conjugate, which is impossible. It follows that the defining relations for the quotient $\mathrm{O}_r(L)/\mathrm{O}_r^{(2)}(L)$ are exactly the same as those for Δ, which means that Δ is mapped isomorphically onto $\mathrm{O}_r(L)/\mathrm{O}_r^{(2)}(L)$. This implies the assertion of the lemma. (Cf. [V2], Lemma 1.)

It follows that the lattice L is 2-reflective if and only if it is reflective and the group Δ is finite.

To find P is usually easier than to find $P^{(2)}$, and to determine if the group Δ is finite does not represent any problem due to the well-known classification of finite Coxeter groups.

Let us investigate the remaining lattices $L(1)$–$L(5)$, $L(13)$, $L(24)$–$L(27)$, $L(30)$, $L(32)$ by this method. In all the cases the lattice L will turn to be reflective. In order to find P, we shall use the algorithm described in [V2], [V4].

The following remark is useful for calculations: a lattice L may contain k-roots only if k divides the doubled last invariant factor of L [V5, Proposition 24].

Consider the lattice $L(1)$. In some co-ordinates u, v, x_1, x_2 the scalar square in it is given by the quadratic form

$$2(-4uv + x_1^2 + x_2^2).$$

There may be k-roots for $k = 2, 4, 8$. Their co-ordinates are defined by the conditions

$$x_1^2 + x_2^2 = 4uv + \frac{k}{2}, \quad 2x_1 \equiv 2x_2 \equiv 0 \pmod{\frac{k}{2}}.$$

Taking $(1,0,0,0)$ for the base point of the algorithm, we obtain the first 4 simple roots as follows:

No.	u	v	x_1	x_2	k
1	0	0	-1	1	4
2	0	0	0	-1	2
3	1	0	2	0	8
4	-1	1	0	0	8

Their Coxeter diagram

define a tetrahedron with one vertex at infinity. We can conclude that the lattice is reflective, and P coincides with the above tetrahedron.

The group Δ generated by the reflections in the walls of P which are not 2-reflections (the corresponding vertices of the Coxeter diagram are depicted by black circles) is a finite Coxeter group of type B_3. Hence, the lattice $L(1)$ is 2-reflective.

Since the lattices $L(2)$ and $L(3)$ contain $L(1)$, they are also 2-reflective.

In an analogous way, the scalar product in $L(4)$ is given by the quadratic form

$$2(-3uv + x_1^2 + x_2^2).$$

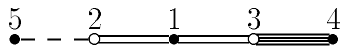
There may be k-roots for $k = 2, 4, 6, 12$. They are defined by the conditions

$$x_1^2 + x_2^2 = 3uv + \frac{k}{2}, \quad 3u \equiv 3v \equiv 2x_1 \equiv 2x_2 \equiv 0 \pmod{\frac{k}{2}}.$$

The first 5 simple roots are as follows:

No.	u	v	x_1	x_2	k
1	0	0	-1	1	4
2	0	0	0	-1	2
3	1	0	1	0	2
4	-1	1	0	0	6
5	2	2	3	3	12

Their Coxeter diagram



defines a triangular prism with one vertex at infinity, so the lattice is reflective, and P coincides with this prism. The group Δ is a finite Coxeter group of type $3A_1$. Hence, the lattice $L(4)$ is 2-reflective.

Consider the lattice $L(32)$. It is convenient to realize it as the sublattice defined by the equation

$$x_1 + x_2 + x_3 = 0 \tag{16}$$

in the lattice of rank 5 with the scalar square

$$-12uv + x_1^2 + x_2^2 + x_3^2.$$

There may be k-roots for $k = 2, 4, 6, 12$. They are defined by the conditions

$$x_1^2 + x_2^2 + x_3^2 = 12uv + k, \quad x_1 \equiv x_2 \equiv x_3 \pmod{\frac{k}{2}}.$$

The first 4 simple roots are as follows:

No.	u	v	x_1	x_2	x_3	k
1	0	0	-1	2	-1	6
2	0	0	0	-1	1	2
3	1	0	2	-1	-1	6
4	-1	1	0	0	0	12

Their Coxeter diagram

defines a tetrahedron with one vertex at infinity, so the lattice is reflective, and P coincides with this tetrahedron. (In particular, we see that there are no 4-reflections.) The group Δ is a finite Coxeter group of type B_3. Hence, the lattice $L(32)$ is 2-reflective.

In an analogous way, the lattice $L(25)$ is the sublattice defined by the equation (16) in the lattice of rank 5 with the scalar square

$$-6x_0u + 2x_0^2 + x_1^2 + x_2^2 + x_3^2.$$

There are k-roots for $k = 2, 6$. They are defined by the conditions

$$2x_0^2 + x_1^2 + x_2^2 + x_3^2 = 6x_0u + k, \quad x_0 \equiv 0 \pmod{\frac{k}{2}}, x_1 \equiv x_2 \equiv x_3 \pmod{\frac{k}{2}}.$$

The first 4 simple roots are as follows:

No.	u	x_0	x_1	x_2	x_3	k
1	0	0	-1	2	-1	6
2	0	0	0	-1	1	2
3	1	0	2	-1	-1	6
4	0	1	0	0	0	2

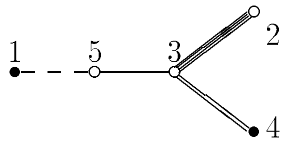
Their Coxeter diagram

defines a tetrahedron with 2 vertices at infinity, so the lattice is reflective, and P coincides with this tetrahedron. The group Δ is a finite Coxeter group of type A_2. Hence, the lattice $L(25)$ is 2-reflective.

The lattice $L(26)$ is the sublattice defined by the equation (16) in the lattice of rank 5 with the scalar square
$$-4x_0^2+4y^2+x_1^2+x_2^2+x_3^2.$$
There are k-roots for $k=2,4,6$. They are defined by the conditions
$$4y^2+x_1^2+x_2^2+x_3^2=4x_0^2+k,\quad 4x_0\equiv 4y\equiv 0 \pmod{\frac{k}{2}},\quad x_1\equiv x_2\equiv x_3 \pmod{\frac{k}{2}}.$$
Taking $(1,0,0,0,0)$ for the base point of the algorithm, we obtain the first 5 simple roots as follows:

No.	x_0	y	x_1	x_2	x_3	k
1	0	-1	0	0	0	4
2	0	0	-1	2	-1	6
3	0	0	0	-1	1	2
4	1	0	2	0	-2	4
5	1	1	1	0	-1	2

Their Coxeter diagram

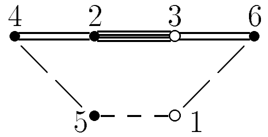


defines a triangular prism with one vertex at infinity, so the lattice is reflective, and P coincides with this prism. The group Δ is a finite Coxeter group of type $3A_1$. Hence, the lattice $L(26)$ is 2-reflective.

In an analogous way, the lattice $L(30)$ is the sublattice defined by the equation (16) in the lattice of rank 5 with the scalar square
$$-12x_0^2+2y^2+x_1^2+x_2^2+x_3^2.$$
There are k-roots for $k=2,4,6,12$. They are defined by the conditions
$$2y^2+x_1^2+x_2^2+x_3^2=12x_0^2+k,\quad 2y\equiv 0 \pmod{\frac{k}{2}},\quad x_1\equiv x_2\equiv x_3 \pmod{\frac{k}{2}}.$$
The first 6 simple roots are as follows:

No.	x_0	y	x_1	x_2	x_3	k
1	0	-1	0	0	0	2
2	0	0	-1	2	-1	6
3	0	0	0	-1	1	2
4	1	0	4	-2	-2	12
5	1	3	0	0	0	6
6	1	2	2	0	-2	4

Their Coxeter diagram



defines a bounded polyhedron (a tetrahedron with two truncated vertices), so the lattice is reflective. However, the group Δ is infinite. Hence, the lattice $L(30)$ is not 2-reflective.

The lattice $L(5)$ is the sublattice defined by the equation

$$x_1 + x_2 + x_3 + x_4 = 0$$

in the lattice of rank 5 with the scalar square

$$-4x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

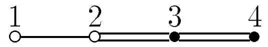
There are k-roots for $k = 2, 4, 8$. They are defined by the conditions

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 4x_0^2 + k, \quad x_1 \equiv x_2 \equiv x_3 \equiv x_4 \pmod{\frac{k}{2}}.$$

The first 4 simple roots are as follows:

No.	x_0	x_1	x_2	x_3	x_4	k
1	0	0	−1	1	0	2
2	0	0	0	−1	1	2
3	0	−1	1	1	−1	4
4	1	3	−1	−1	−1	8

Their Coxeter diagram



defines a tetrahedron with one vertex at infinity. The group Δ is a finite Coxeter group of type B_2. Hence, the lattice $L(5)$ is 2-reflective.

The scalar square in $L(13)$ is

$$2(-4x_0^2 + x_1^2 + x_2^2 + x_3^2).$$

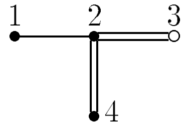
There may be k-roots for $k = 2, 4, 8, 16$. They are defined by the conditions

$$x_1^2 + x_2^2 + x_3^2 = 4x_0^2 + \frac{k}{2}, \quad 2x_1 \equiv 2x_2 \equiv 2x_3 \equiv 0 \pmod{\frac{k}{2}}.$$

The first 4 simple roots are as follows:

No.	x_0	x_1	x_2	x_3	k
1	0	-1	1	0	4
2	0	0	-1	1	4
3	0	0	0	-1	2
4	1	2	2	0	8

Their Coxeter diagram



defines a tetrahedron with one vertex at infinity. The group Δ is a finite Coxeter group of type B_3. Hence, the lattice $L(13)$ is 2-reflective.

In order to investigate the lattice $L(24)$, let us note that it admits an odd extension $\tilde{L}(24)$ of index 2 (with discriminant -7), so

$$L(24) = \tilde{L}(24)^{even}$$

The 7-symbol of $\tilde{L}(24)$ must coincide with that of $L(24)$, whence

$$\tilde{L}(24) \simeq [-7] \oplus [1] \oplus [1] \oplus [1].$$

The scalar square in $\tilde{L}(24)$ is given by the quadratic form

$$-7x_0^2 + x_1^2 + x_2^2 + x_3^2,$$

and the sublattice $L(24)$ is defined by the congruence

$$x_0 + x_1 + x_2 + x_3 \equiv 0 \pmod 2. \tag{17}$$

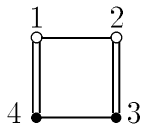
There are k-roots of $L(24)$ for $k = 2, 4$. They are defined by the conditions

$$x_1^2 + x_2^2 + x_3^2 = 7x_0^2 + k, \quad x_0 \equiv x_1 \equiv x_2 \equiv x_3 \pmod{\frac{k}{2}}.$$

The first 4 simple roots are as follows:

No.	x_0	x_1	x_2	x_3	k
1	0	-1	1	0	2
2	0	0	-1	1	2
3	0	0	0	-2	4
4	1	3	1	1	4

Their Coxeter diagram



defines a tetrahedron, so the lattice is reflective, and P coincides with this tetrahedron. The group Δ is a finite Coxeter group of type A_2. Hence, the lattice $L(24)$ is 2-reflective.

In an analogous way,

$$L(27) = \tilde{L}(27)^{even},$$

where

$$\tilde{L}(27) \simeq [-15] \oplus [1] \oplus [1] \oplus [1].$$

The scalar square in $\tilde{L}(27)$ is given by the quadratic form

$$-15x_0^2 + x_1^2 + x_2^2 + x_3^2,$$

and the sublattice $L(27)$ is defined by the congruence (17).

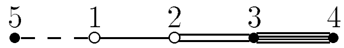
There are k-roots of $L(27)$ for $k = 2, 4, 10, 12$. They are defined by the conditions

$$x_1^2 + x_2^2 + x_3^2 = 15x_0^2 + k, \quad 15x_0 \equiv x_1 \equiv x_2 \equiv x_3 \pmod{\frac{k}{2}}.$$

The first 5 simple roots are as follows:

No.	x_0	x_1	x_2	x_3	k
1	0	-1	1	0	2
2	0	0	-1	1	2
3	0	0	0	-2	4
4	1	3	3	3	12
5	1	5	0	0	10

Their Coxeter diagram



defines a triangular prism, so the lattice is reflective, and P coincides with this prism. The group Δ is a finite Coxeter group of type $G_2 + A_1$. Hence, the lattice $L(27)$ is 2-reflective.

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