GRASSMANNIAN FIXED POINT RATIOS

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Abstract. We provide estimates for the fixed point ratios in the permutation representations of a finite classical group over a field of order q on k-subspaces of its natural n-dimensional module. For sufficiently large n each element must either have a negligible ratio or act linearly with a large eigenspace. We obtain an asymptotic estimate in the latter case, which in most cases is q^{-dk} where d is the codimension of the large eigenspace. We also obtain more specific results that are applicable to the monodromy conjecture of Guralnick and Thompson.

Introduction

If G is a finite group, $g \in G$, and M is a maximal subgroup of G, then the fixed point ratio of g in the associated primitive permutation action is denoted by $f(g, M)_G$, or simply $f(g, M)$. Our concern here is with the fixed point ratios $f(g, M)_G$ when G is a nearly simple finite classical group and $M \cap F^*(G)$ stabilizes a nontrivial proper subspace of the natural module for $F^*(G)$, the generalized Fitting subgroup of G. [Following, [11], we say that a group G is *nearly simple* if there is a nonabelian simple group S such that $G/Z(G)$ is isomorphic to a subgroup of $\mathrm{Aut}(S)$ containing $\mathrm{Inn}(S)$.] Apart from a few small exceptions, these permutation actions include for each classical group the permutation actions of lowest degree in which $F^*(G)$ acts non-trivially.

In [8], Guralnick and Thompson conjectured that for every non-negative integer g there is a finite set $\mathcal{E}(g)$ of finite simple groups such that if S is a nonabelian composition factor of $\mathrm{Gal}(N/\mathbb{C}(t))$ where N is the normal closure of a finite, genus g extension L of $\mathbb{C}(t)$ and t is transcendental over $\mathbb{C}$ then either S is an alternating group or $S \in \mathcal{E}(g)$. They also suggested that it would be desirable to know a minimal set $\mathcal{E}^*(0)$ explicitly. The Riemann-Hurwitz formula connects the genus of L to the permutation action of $\mathrm{Gal}(N/\mathbb{C}(t))$, and the work of several authors shows that to establish the conjecture it suffices to consider primitive permutation actions of nearly simple groups. (See [8] for a more detailed discussion.)

Several recent papers (see [3], [4], [5], [6], [10], [13], [14], [15], and [16]) have investigated fixed point ratios in nearly simple finite groups. The collective effect of such analyses (and the Classification of Finite Simple Groups) is that the Guralnick-Thompson Conjecture has been verified for all primitive permutation actions of all nearly simple finite groups apart from the ones considered here, and the non-classical members of $\mathcal{E}^*(0)$ have been determined. In a forthcoming paper [7], we use the results here to complete the proof of the Guralnick-Thompson Conjecture.

Permutation actions of sufficiently large degree in which all fixed point ratios are sufficiently small must have large genus. Liebeck and Saxl showed in [13] that,

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apart from a few exceptions, if $F^*(G)$ is a group of Lie type written over $\mathbf{F}_q$, $g \in G^\sharp$, and M is a maximal subgroup of G not containing $F^*(G)$ then $f(g, M) < 4/3q$.

We refine that here by producing element-wise bounds. We show that for every $\varepsilon > 0$ there is an integer n such that if G has Lie rank at least n, then $f(g, M) < 1/q + \varepsilon$ for every $g \in G \setminus Z(G)$ and every maximal subgroup M under consideration here. More generally, we show that if G is a classical group of large Lie rank defined over $\mathbf{F}_q$ then the only elements with non-negligible fixed point ratios are elements that act linearly on the natural module with a large eigenspace. We further show that the fixed point ratios for such elements are essentially q^{-dk} where d is the codimension of the eigenspace and k is the smaller of the dimension or codimension of the subspace stabilized by the maximal subgroup.

We show specifically that if G is a classical group with natural module V of dimension n over $\mathbf{F}_q$, M is a maximal subgroup of G such that $M \cap F^*(G)$ stabilizes a subspace of V of dimension or codimension k, where $k \leq n/2$, then either $f(g, M) \leq (4000 + 150n)q^{-(n-6)/2}$ or g acts linearly on V with an eigenspace of codimension d, with $dk < n/2$, and $f(g, M) \leq q^{-dk} + 2000q^{-(n-6)/2}$.

Contents

1. Preliminaries

1.1. **Nearly simple groups of classical type.** For the most part, we follow standard conventions, but the reader should note that the parameter q in the definition of unitary groups is the size of the field over which the associated vector space is written: our q is the square of the q used elsewhere, for example, [12]. Furthermore, we consider odd-dimensional orthogonal spaces in characteristic 2 in order to qualify $O^\epsilon(n,q)$ as a subspace stabilizing subgroup of $Sp(n,q) \cong O(n+1,q)$.

Group elements, and more generally linear transformations, will act on the right, but other functions such as polynomials, will typically appear on the left, as in $f(x)$.

Throughout this paper, $\mathbf{F}$ will be a finite field of order q and characteristic p, and V will be an n-dimensional vector space over $\mathbf{F}$. Say that V is a classical vector space provided V admits a 1 or 2-form κ satisfying one of the following conditions.

linear: κ is identically 0;
symplectic: $\kappa = \mathbf{f}$ is a non-degenerate, alternating, bilinear 2-form;
unitary: $\kappa = \mathbf{f}$ is a non-degenerate, hermitian 2-form;
orthogonal: $\kappa = \mathbf{q}$ is a quadratic 1-form, and either the associated symmetric bilinear form $\mathbf{f}$ is non-degenerate or rad $\mathbf{f}$ is nonsingular with respect to $\mathbf{q}$.

Following [12], we say that a symplectic, orthogonal, or unitary vector subspace W of V is *non-degenerate* if rad $\mathbf{f} \cap W = 0$. If V is orthogonal, then we say that W is *nonsingular* if $\mathbf{q}(v) \neq 0$ for every non-zero vector v in rad $\mathbf{f} \cap W$. In this terminology, if κ is not identically 0, then V is nonsingular in all cases considered. The only non-trivial, non-degenerate case here is that of an odd-dimensional vector space in even characteristic. In that case, rad $\mathbf{f}$ has dimension 1. A subspace W of V is *totally singular* if κ vanishes identically on W. The subspace W is *totally isotropic* if $\mathbf{f}$ vanishes identically on W.

Let $\mathcal{L}$ be the set of all subspaces of V, and let $\Delta(V)$ $[\Gamma(V)]$ be the group of all linear [linear or semilinear] transformations of V that preserve κ up to scalar multiplication. The action of $\Gamma(V)$ on V induces a permutation action of $\Gamma(V)$ on $\mathcal{L}$.

If κ is not trivial, set $A(V) = \Gamma(V)$. If κ is trivial, let ι be a 'transpose-inverse' automorphism of $\Gamma(V)$. Set $A(V) = \Gamma(V)\langle\iota\rangle$, and let $V^* = \mathrm{Hom}_{\mathbf{F}}(V, \mathbf{F})$ be the dual of V. Then $\Gamma(V)$ acts on V^* by the rule $\phi^g(v) = \phi(v^{g^{-1}})$, and there is a basis $\mathcal{B}$ of V such that ι acts on $V \oplus V^*$ by interchanging the members of $\mathcal{B}$ with the corresponding members of the dual basis for V^*. Under this action, each member g of $A(V) \setminus \Gamma(V)$ naturally takes each subspace U of V to a subspace U^g of V^*. Setting $W^\vee = \{v \in V : v \in \ker w \text{ for all } w \in W\}$, we have an action of g on $\mathcal{L}$ via the rule $g : U \mapsto (U^g)^\vee$. This extends the action of $\Gamma(V)$ on $\mathcal{L}$ to an action of $A(V)$ on $\mathcal{L}$.

Assume throughout this paper that $F^*(I(V))/Z(I(V))$ is simple and that G is a subgroup of $A(V)$ such that $F^*(I(V)) \subseteq G$. Assume also that M is a maximal subgroup of G such that $M \cap \Gamma(V)$ fixes some member of $\mathcal{L}$ that is not invariant under $\Gamma(V)$, in other words, a subspace different from 0, rad V, ndV. For convenience, call such a subgroup a *subspace stabilizing* maximal subgroup of G.

We identify six types of subspace stabilizing maximal subgroups:

Linear: V is a linear space, U is a subspace of V, and M stabilizes U.
Non-degenerate: V is a non-degenerate symplectic, orthogonal, or unitary space, U is a non-degenerate subspace, and M stabilizes U.
Totally singular: V is a non-degenerate symplectic, orthogonal, or unitary space, κ vanishes identically on U, and M stabilizes U.
Nonsingular point: V is a non-degenerate orthogonal space in even characteristic, U is a 1-subspace on which $\mathbf{q}$ is not trivial, and M stabilizes U.

Hyperplane: V is a nonsingular odd-dimensional orthogonal space in even characteristic, U is a complement to rad $\mathbf{f}$, and M stabilizes U.

Graph: V is a linear space, U and U' are distinct subspaces of V such that $\dim U + \dim U' = \dim V$, and $\{U, U'\}$ is an M-orbit on $\mathcal{L}$.

The following lemma shows that this list is complete in the sense that every subspace stabilizing maximal subgroup arises in one of the situations described.

Lemma 1.1. *If $M \cap \Gamma(V)$ stabilizes a subspace of V that is not $\Gamma(V)$-invariant, then one of the following holds.*

1. *V is a linear space, $G \subseteq \Gamma(V)$, and M stabilizes a subspace of V.*
2. *V admits a non-degenerate form, and M stabilizes a non-degenerate subspace of V.*
3. *V admits a non-degenerate form, and M stabilizes a totally singular subspace of V.*
4. *V is an orthogonal space, and q is even. Furthermore, M stabilizes a subspace U of V such that one of the following holds:*
   (a) *n is even, $\dim U = 1$, and U is nonsingular.*
   (b) *n is odd, and U is a complement to rad V.*
   (c) *n is odd, rad $V \subseteq U$, and either rad $U =$ rad V or U is totally singular.*
5. *V is a linear space, $G \not\subseteq \Gamma(V)$, and M stabilizes a subspace of V of dimension $n/2$.*
6. *V is a linear space, and M has an orbit $\{U, U'\}$ where $\dim U' =$ codim U' and either $U \cap U' = 0$ or $U \subset U'$.*

Proof. Assume first that $G \subseteq \Gamma(V)$. If κ is trivial, then (1) holds, so we may assume that κ is not trvial. Suppose V is non-degenerate (all cases except odd-dimensional orthogonal spaces in even characteristic). Since $G_U \subseteq G_{U \cap U^\perp}$ we may assume that either $U \cap U^\perp = 0$ or $U \subseteq U^\perp$. In the former case, U is non-degenerate, and (2) holds. If $U \subseteq U^\perp$, then U is totally isotropic (with respect to $\mathbf{f}$), and either U is totally singular (with respect to $\mathbf{q}$), or q is even, $\dim U = 1$ and U is nonsingular. Thus (3) or (4a) must hold.

If V is degenerate then V is an odd-dimensional orthogonal space and q is even. Furthermore, rad V has dimension 1. If $V = U + \text{rad } V$, then (4b) must hold. Otherwise, since M stabilizes $U + \text{rad } V$, we may assume that rad $V \subseteq U$, whence rad $V \subseteq$ rad U. Since $G_U \subseteq G_{\text{rad } U}$, and G_U is maximal in G, it follows that either rad $U =$ rad V or U is totally singular. This shows that (4c) holds.

Now assume that $G \not\subseteq \Gamma(V)$, so that κ is trivial. Set $G_0 = G \cap \Gamma(V)$ and $M_0 = M \cap G_0$. Then $[G : G_0] = 2$. Let U be a non-trivial M_0-invariant subspace of minimal dimension k, and let $h \in M \setminus M_0$. Then U^h is a k-subspace of V^* and $(U^h)^\vee$ is an $n-k$-subspace of V. Set $U' = (U^h)^\vee$. If $U' = U$, then $k = n/2$, and (5) holds. Otherwise, since U' is M_0-invariant, we must have either $U \cap U' = U$ or $U \subseteq U'$, so (6) holds.

Remarks 1. (1) In the Linear case we can insure that $\dim U \leq n/2$ by replacing V with V^* if necessary.

(2) In the Non-degenerate case, we can arrange to have $\dim U \leq n/2$ because U and $U^\perp$ have the same stabilizer.

(3) In case (3), properties of non-degenerate bilinear forms imply that $\dim U \leq n/2$,

(4) If 4c holds then $V/\mathrm{rad}\ V$ is a non-degenerate symplectic space, and $U/\mathrm{rad}\ V$ is either non-degenerate or totally singular. The action of G on the cosets of M can therefore be studied in the context of symplectic groups.

(5) Case (5) can be subsumed into case (1).

1.2. **Statement of results.** The following hypothesis applies throughout.

Hypothesis $\mathcal{G}$. V is a vector space of dimension n over $\mathbf{F}_q$; $F^*(I(V))/Z(I(V))$ is simple, and $F^*(I(V)) \subseteq G \subseteq A(V)$; $g \in G$; U is a subspace of V, of dimension or codimension k where $k \leq n/2$; M is a maximal subgroup of G, and U is $M \cap \Gamma(V)$-invariant. One of the following is true:

1. V is a linear space.
2. V is a non-degenerate symplectic, orthogonal, or unitary space, and U is non-degenerate.
3. V is a non-degenerate symplectic, orthogonal, or unitary space, and U is totally singular.
4. V is a non-degenerate orthogonal space, q is even, and U is a nonsingular 1-subspace of V.
5. V is a nonsingular orthogonal space, q is even, and U is a non-degenerate hyperplane of V.

Main Theorem 1. *For every $\varepsilon > 0$ there is an integer N, depending on ε and q, such that if $\mathcal{G}$ holds for $n > N$ then either*

$$f(g, M) < \varepsilon$$

or g is a linear transformation with an eigenspace of codimension d, and

$$|f(g, M) - q^{-\nu dk}| < \varepsilon$$

where ν is the length of the M-orbit containing U. More specifically, the conclusion holds if $\varepsilon = (4000 + 150N)q^{-(N-6)/2}$ and $n \geq N$.

We will establish this theorem by proving more detailed results for each of six hypotheses, the linear case being subdivided according to whether U is M-invariant.

Theorem 1. *Assume $\mathcal{G}(1)$ and that U is M-invariant. Then one of the following is true*

1. $f(g, M) \leq 12q^{-n/2}$.
2. *g is a linear transformation on V, g has an eigenspace of codimension $d \leq n/2k$, and $q^{-dk} - q^{-n} \leq f(g, M) \leq q^{-dk} + 11q^{-n/2}$.*
3. *$n = 4$, $k = n/2$, g is a 'graph' automorphism, and $f(g, M) < q^{-1}$.*

Theorem 2. *Assume $\mathcal{G}(2)$. Then either*

1. $$f(g, M) \leq 340q^{-n/2}, \text{ or}$$
2. *g is a linear transformation on V, g has an eigenspace of codimension $d < n/2k$, and*
$$q^{-dk} - 3q^{-(n-1)/2} < f(g, M) \leq q^{-dk} + 220q^{-(n-1)/2}.$$

Theorem 3. *Assume $\mathcal{G}(3)$. Then either*

1. $$f(g, M) \leq (4000 + 150n)q^{-(n-6)/2}, \text{ or}$$

2. *g is a linear transformation on V, g has an eigenspace of codimension $d < n/2k$ in V, and*

$$q^{-dk} - 3q^{(n-2)/2} < f(g,M) \le q^{-dk} + 2000q^{-(n-6)/2}.$$

Theorem 4. *Assume $\mathcal{G}(4)$. Then either*

1.
$$f(g,M) \le 4q^{-n/2} \text{ or}$$

2. *g is a linear transformation on V, g has an eigenspace of codimension $d < n/2$, and*

$$q^{-d} - 2q^{-n/2} \le f(g,M) \le q^{-d} + 3q^{-n/2}.$$

Theorem 5. *Assume $\mathcal{G}(5)$. Then either*

1.
$$f(g,M) \le 4q^{-(n+1)/2}, \text{ or}$$

2. *g is a linear transformation on V, g has an eigenspace of codimension $d < n/2$, and*

$$q^{-d} - 2q^{-(n-1)/2} \le f(g,M) \le q^{-d} + 5q^{-(n-1)/2}.$$

Theorem 6. *Assume $\mathcal{G}(1)$ and that U is not M-invariant. Then either*

1.
$$f(g,M) \le 12q^{-(n-1)/2}, \text{ or}$$

2. *g is a linear transformation on V, g has an eigenspace of codimension d where $dk < n/2$, and*

$$q^{-2dk} - 2q^{-n/2} \le f(g,M) \le q^{-2dk} + 35q^{-n/2}.$$

1.3. **Overview of the proof.** In Section 2 we present the basic counts and estimates for numbers of subspaces of various types.

Section 3 provides computational results used in bounding coefficients in section 2 and elsewhere.

In Section 4 we show that the conclusions to Theorems 1, 2, 3, and 4 hold when $k = 1$ and g is a linear transformation. We obtain more detailed estimates in many special cases, especially when $q > 2$.

The heart of our result is that if g is a linear transformation and g has an eigenspace of dimension d where $dk < n/2$, then $f(g,M)$ is approximately q^{-dk}, with the significant contribution coming from the subspaces of the largest eigenspace. That the contribution is on the order of q^{-dk} follows easily from the results in Section 2. In Section 5 we show that this contribution is $q^{-dk} + \varepsilon$ where ε is at most on the order of $q^{-(n-2)/2}$.

In Section 6 we show that the conclusion of Theorem 1 holds when $k \ge 2$ and g is a linear transformation, and we apply this in Section 7 to show that the conclusion of Theorem 2 holds under the same conditions.

To show that the conclusion to Theorem 3 holds for linear transformations, we delineate six cases in Section 8. In Section 9 we handle the cases that correspond to semisimple elements that do not have large eigenspaces, and in Section 10 we handle the remaining cases.

Section 11 contains the technical calculations that bound the exponents involved in various sums.

In Section 12 we show that the conclusion to Theorem 5 holds when g is a linear transformation.

In Section 13 we show that the conclusions to Theorems 1, 2, 3, 4, and 5 hold when g is a semilinear transformation on V.

In Section 14 we first show that the conclusion of Theorem 1 holds in the only remaining case, that is when g is a graph automorphism and $k = n/2$. We then prove Theorem 6 in all cases.

Using a transitivity result, the summary section verifies that the numerical results in Theorems 1-6 are indeed implied by the more specific results obtained in the previous sections.

The final section contains a list of the notation used in this paper.

1.4. **Notation.** If W is a linear space, let $\mathcal{L}(W,k)$ be the collection of all k-subspaces of W. If W is a symplectic, orthogonal, or unitary space, let $\mathcal{S}(W,k)$ be the set of all totally singular k-subspaces of W. If U is non-degenerate, let $\mathcal{N}(W,U)$ be the set of all non-degenerate subspaces of W that are isometric to U.

When m is even there are two isometry types of non-degenerate orthogonal m-spaces over $\mathbf{F}_q$. Say that a $2m$-space has type $+$ if it contains a totally singular subspace of dimension m, and type $-$ if it does not. When q and n are both odd there are also two distinct isometry types of non-degenerate orthogonal n-spaces over $\mathbf{F}_q$, but the isometry groups are isomorphic. We shall abuse notation by saying that an odd-dimensional non-degenerate orthogonal space has type $\circ$. Note that $|\mathcal{N}(W,U)|$ depends only on $\dim U$ when U has odd dimension.

For an orthogonal space W in even characteristic, let $\mathcal{P}(W)$ be the set of 1-subspaces on which the quadratic form $\mathbf{q}$ does not vanish. For an odd-dimensional orthogonal space W in even characteristic and a complement X to rad W, let $\mathcal{H}(W,X)$ be the collection of subspaces of W isometric to X. Note that every element of $\mathcal{H}(W)$ is a complement to rad W.

Under suitable conditions the forms on W naturally induce forms on quotient spaces. If W admits an alternating, symmetric, or hermitian bilinear form $\mathbf{f}$, and $X \subseteq W^\perp \cap W$, the radical of W, then $\tilde{\mathbf{f}}(w_1 + X, w_2 + X) = \mathbf{f}(w_1, w_2)$ defines a form on W/X of the same kind. Suppose $\mathbf{f}$ is symmetric, $X \subseteq W^\perp$, and that the quadratic form $\mathbf{q}$ satisfies $\mathbf{f}(v_1, v_2) = \mathbf{q}(v_1) + \mathbf{q}(v_2) - \mathbf{q}(v_1 + v_2)$. If $\mathbf{q}$ vanishes on X, then $\tilde{\mathbf{q}}(w + X) = \mathbf{q}(w)$ defines a quadratic form on W/X compatible with $\tilde{\mathbf{f}}$.

If V is symplectic or unitary, or if V has characteristic not equal to 2, then set $T(W) = \text{rad }(W) = W^\perp \cap W$, for all subspaces W of V. If V is orthogonal and q is even, let $T(W)$ be the largest totally singular subspace of rad (W). Such a subspace exists and has codimension at most 1 in rad (W) because the quadratic form $\mathbf{q}$ is a semilinear transformation from rad (W) to $\mathbf{F}$. We will refer to $T(W)$ as the *totally singular radical* of W. Define the type of an arbitrary orthogonal space W to be the type of $W/T(W)$. Note that if Y is a subspace of rad W and $Y \not\subseteq T(W)$, then W/Y is a symplectic space.

Let $\mathcal{S}_i(W,k)$ denote the collection of totally singular k-subspaces of W that meet rad W in a subspace of dimension i. In particular, $\mathcal{S}_0(W,k)$ is the set of totally singular k-subspaces of W that are disjoint from rad W.

2. Numbers of subspaces

In this section we provide estimates for the number of subspaces of a given type contained in an arbitrary subspace of V. Throughout this section, V is an n-dimensional vector space over $\mathbf{F}_q$.

Lemma 2.1. *If $\mathcal{L}(V,k)$ is the collection of k-subspaces of V, $0 \le k \le n$, then*

$$q^{k(n-k)} \le |\mathcal{L}(V,k)| < C_1 q^{k(n-k)}$$

where

$$C_1 = \begin{cases} 3.5, & q = 2 \\ \dfrac{q^2}{q^2-q-1}, & q > 2. \end{cases}$$

Lemma 2.2. *Assume that V admits a non-degenerate symplectic, unitary, or orthogonal structure and $n \ge 3$. If U is a non-degenerate k-subspace of V and $\mathcal{N}(V,U)$ is the collection of subspaces of V that are isometric to U, then*

$$c_2 q^{k(n-k)} \le |\mathcal{N}(V,U)| \le C_2 q^{k(n-k)}.$$

where

$$c_2 = \begin{cases} 1/8, & q = 2 \\ \dfrac{(q-1)^2}{2q^2}, & q > 2; \end{cases}$$

$$C_2 = \begin{cases} 15/8, & q = 2 \\ \dfrac{q^4}{q^4-q^2-1}, & q > 2. \end{cases}$$

Lemma 2.3. *Assume that V admits a non-degenerate symplectic, unitary, or orthogonal structure. Let $\mathcal{S}(V,k)$ be the collection of totally isotropic k-subspaces of V. Assume that $k \le n/2$, and if V is of type $\mathbf{O}^-$ assume that $k < n/2$. Then*

$$c_3 q^{k(n-k)-\psi(k)} \le |\mathcal{S}(V,k)| \le C_3 q^{k(n-k)-\psi(k)}$$

where

$$\psi(k) = \begin{cases} k(k-1)/2, & V \text{ symplectic} \\ k(k+1)/2, & V \text{ orthogonal} \\ k^2/2, & V \text{ unitary}; \end{cases}$$

$$c_3 = \begin{cases} 11/24, & q = 2 \\ \dfrac{q^4-q^2-1}{q^4+q^3}, & q > 2; \text{ and} \end{cases}$$

$$C_3 = \begin{cases} 5, & q = 2 \\ 2\dfrac{q^6-q^4-q^2+1}{q^6-q^5-q^4}, & q > 2. \end{cases}$$

Remark . If U is a k-space, then $\psi(k)$ is $\dim \Lambda^2(U)$, $\dim S^2(U)$, $\frac{1}{2}\dim U \otimes U$, respectively, in the symplectic, orthogonal, and unitary cases.

We shall establish more precise counts in terms of the function β. Set

$$\beta(n,x) = (1-x^{-1})(1-x^{-2})\dots(1-x^{-n})$$

$$\beta_{\mathbf{L}}(n) = \beta(n,q)$$

$$\beta_{\mathbf{S}}(n) = \beta(n/2,q^2)$$

$$\beta_{\mathbf{O}^\epsilon}(n) = \frac{1}{1+\epsilon q^{-n/2}}\beta_{\mathbf{S}}(n),\ n \text{ even},\ (n,\epsilon) \neq (0,-)$$
$$\beta_{\mathbf{O}^\circ}(n) = \beta_{\mathbf{S}}(n-1),\ n \text{ odd}$$
$$\beta_{\mathbf{U}}(n) = \beta(n, -q^{1/2})$$

Remark . The trivial 0-dimensional orthogonal space has type $+$ and $\beta_{\mathbf{O}^+}(0) = 1/2$.

2.4. 1. *The sequence* $\langle\beta_{\mathbf{L}}(m)\rangle$ *is decreasing, and* $\dfrac{q^2-q-1}{q^2} < \beta_{\mathbf{L}}(m) < 1$ *for all* $m > 0$*. If* $q = 2$*, then* $\beta_{\mathbf{L}}(m) > 2/7$ *for all* m*. In every case* $1 < \dfrac{1}{\beta_{\mathbf{L}}(m)} < C_1$*, where* C_1 *is defined in Lemma 2.1.*

2. *If* $\alpha(m) = \beta_{\mathbf{S}}(2m)$*,* $\beta_{\mathbf{O}^\circ}(2m+1)$*, or* $\beta_{\mathbf{O}^-}(2m)$*, then the sequence* $\langle\alpha(m)\rangle$ *is decreasing and* $\dfrac{q^4-q^2-1}{q^4} < \alpha(m)$ *for all* m*.*
3. *The sequence* $\langle\beta_{\mathbf{O}^+}(2m)\rangle$ *is increasing and* $\beta_{\mathbf{O}^+}(2m) < \dfrac{q^6-q^4-q^2+1}{q^6}$ *for all* m*.*
4. $\lim \beta_{\mathbf{S}}(2m) = \lim \beta_{\mathbf{O}^\circ}(2m+1) = \lim \beta_{\mathbf{O}^-}(2m) = \lim \beta_{\mathbf{O}^+}(2m)$.
5. *The sequence* $\langle\beta_{\mathbf{U}}(2m+1)\rangle$ *is decreasing and* $\dfrac{(\sqrt{q}+1)(q-1)}{q\sqrt{q}} < \beta_{\mathbf{U}}(2m+1)$ *for all* m*.*
6. $\beta_{\mathbf{U}}(2m) < \dfrac{(\sqrt{q}+1)(q-1)(q\sqrt{q}+1)}{q^3}$ *for all* m*, and the sequence* $\langle\beta_{\mathbf{U}}(2m)\rangle$ *is increasing.*
7. $\lim \beta_{\mathbf{U}}(2m) = \lim \beta_{\mathbf{U}}(2m+1)$.

Proof. It is clear that $\beta_{\mathbf{L}}(1) < 1$

The sequence $\beta_{\mathbf{L}}(m)$ decreases because $\beta_{\mathbf{L}}(m+1) = (1-q^{-(m+1)})\beta_{\mathbf{L}}(m)$. It follows from [1] that $\beta_{\mathbf{L}}(m) > (q^2-q-1)/q^2$ for all m. To complete the proof of (1), it suffices to show that if $q = 2$ then $\beta_{\mathbf{L}}(m) > 2/7$ for all m. Setting $x_i = 1 - q^{-i/2}$ and $y_i = 1 + q^{-i/2}$ we have $\beta_{\mathbf{L}}(m) = \prod_{i=1}^m x_i y_i$. It is clear that $\langle x_i\rangle$ increases to 1 and $\langle y_i\rangle$ decreases to 1. By (3.2), if $i > 2$, then $x_i y_i < 1 < x_{i+1} y_i$. Applying (3.1) we have $\beta_{\mathbf{L}}(m) > \lim \beta_{\mathbf{L}}(i) > \beta_{\mathbf{L}}(r-1)(1-q^{-r/2})$ whenever $r > 3$. If $q = 2$, then $\beta_{\mathbf{L}}(13)(1-q^{-7}) > 2/7$ by straightforward numerical calculation.

Since $\beta_{\mathbf{S}}(2(m+1)) = (1-q^{-2(m+1)})\beta_{\mathbf{S}}(2m)$, we have that the sequence $\beta_{\mathbf{S}}(2m)$ is decreasing. By [1], $\beta_{\mathbf{S}}(2m) > (q^4-q^2-1)/q^4$ for all m. This establishes (2) for $\alpha(m) = \beta_{\mathbf{S}}(2m)$. Since $\beta_{\mathbf{O}^\circ}(2m+1) = \beta_{\mathbf{S}}(2m)$ the same conclusion holds for $\alpha(m) = \beta_{\mathbf{O}^\circ}(2m+1)$. This also implies that $\lim \beta_{\mathbf{S}}(2m)$ exists.

We have $\beta_{\mathbf{O}^-}(2m) = (1-q^{-(m-1)})(1+q^{-m})\beta_{\mathbf{O}^-}(2m-2)$, from which it follows that $\beta_{\mathbf{O}^-}(2m)$ is a decreasing sequence. Because $\beta_{\mathbf{O}^-}(2m) = \beta_{\mathbf{S}}(2m)/(1-q^{-m})$ this sequence must converge to $\lim_m \beta_{\mathbf{S}}(2m)$. It follows that, for each m, $\beta_{\mathbf{O}^-}(2m) > \lim_m \beta_{\mathbf{O}^-}(2m) = \lim_m \beta_{\mathbf{S}}(2m) \geq (q^4-q^2-1)/q^4$. This completes the proof of (2).

We have $\beta_{\mathbf{O}^+}(2m) = (1+q^{-(m-1)})(1-q^{-m})\beta_{\mathbf{O}^+}(2m-2)$. It follows from (3.2) that $\beta_{\mathbf{O}^+}(2m)$ is an increasing sequence. Since $\beta_{\mathbf{O}^+}(2m) = \beta_{\mathbf{S}}(2m)/(1+q^{-m})$, we also have $\lim_m \beta_{\mathbf{O}^+}(2m) = \lim_m \beta_{\mathbf{S}}(2m)$. Therefore $\beta_{\mathbf{O}^+}(2m) \leq \lim_m \beta_{\mathbf{S}}(2m)$ for all m. Because $\beta_{\mathbf{S}}(2m)$ is decreasing we have $\lim_m \beta_{\mathbf{S}}(2m) < \beta_{\mathbf{S}}(4) = (1-q^{-2})(1-q^{-4}) < \dfrac{q^6-q^4-q^2+1}{q^6}$. This establishes (3).

If m is even, then $\beta_{\mathbf{U}}(m+2) = (1+q^{-(m+1)/2})(1-q^{-(m+2)/2})\beta_{\mathbf{U}}(m)$ so that $\beta_{\mathbf{U}}(m+2) > \beta_{\mathbf{U}}(m)$ by (3.2). If m is odd, then $\beta_{\mathbf{U}}(m+2) = (1-q^{-(m+1)/2})(1+q^{-(m+2)/2})\beta_{\mathbf{U}}(m) < \beta_{\mathbf{U}}(m)$. Therefore $\beta_{\mathbf{U}}(2m)$ increases and $\beta_{\mathbf{U}}(2m+1)$ decreases. It follows from (3.1) that $\lim_m \beta_{\mathbf{U}}(m)$ exists. Therefore $\beta_{\mathbf{U}}(2k) < \lim_m \beta_{\mathbf{U}}(m) < \beta_{\mathbf{U}}(2l+1)$ for all choices of k and l. Thus $\beta_{\mathbf{U}}(2m+1) > \beta_{\mathbf{U}}(2) = \dfrac{(\sqrt{q}+1)(q-1)}{q\sqrt{q}}$ and $\beta_{\mathbf{U}}(2m) < \beta_{\mathbf{U}}(3) = \dfrac{(\sqrt{q}+1)(q-1)(q\sqrt{q}+1)}{q^3}$ for all m. This completes the proof.

Proof of Lemma 2.1: We first express $|\mathcal{L}(V,k)|$ explicitly in terms of $\beta_{\mathbf{L}}$.

2.5. *Let V be a linear space of dimension n over $\mathbf{F}$.*
1. $|GL(V)| = q^{n^2}\beta_{\mathbf{L}}(n)$.
2. *If $0 \le k \le n$, then $|\mathcal{L}(V,k)| = \gamma_1(n,k)q^{k(n-k)}$ where*

$$\gamma_1(n,k) = \frac{\beta_{\mathbf{L}}(n)}{\beta_{\mathbf{L}}(n-k)\beta_{\mathbf{L}}(k)}.$$

Proof. : Set $G = GL(V)$. If U is a k-subspace of V, then $|\mathcal{L}(V,k)| = [G : G_U]$ where G_U is the G-stabilizer of U. By counting ordered bases for V, we have $|G| = \prod_{i=1}^{n}(q^n - q^i) = q^{n^2}\beta_{\mathbf{L}}(n)$.

There is a surjective map $\phi : G_U \to GL(U) \times GL(V/U)$ and $\ker\phi$ acts regularly on the set $\mathcal{C}(V,U)$ of complements to U in V. We have $|GL(U)| = q^{k^2}\beta_{\mathbf{L}}(k)$ and $|GL(V/U)| = q^{(n-k)^2}\beta_{\mathbf{L}}(n-k)$. Since $|\mathcal{C}(V,U)| = q^{k(n-k)}$, we have

$$|\mathcal{L}(V,k)| = \frac{q^{n^2}\beta_{\mathbf{L}}(n)}{q^{k(n-k)}q^{k^2}\beta_{\mathbf{L}}(k)q^{(n-k)^2}\beta_{\mathbf{L}}(n-k)} = \frac{\beta_{\mathbf{L}}(n)}{\beta_{\mathbf{L}}(k)\beta_{\mathbf{L}}(n-k)}q^{k(n-k)}.$$

□

2.6. *Let $\gamma_1(n,k)$ be as in (2.5). Then* $\gamma_1(n,k) = \displaystyle\prod_{i=0}^{k-1}\frac{1-q^{-(n-i)}}{1-q^{-(k-i)}}$ *and*

$$1 \le \gamma_1(n,k) \le \frac{1}{\beta_{\mathbf{L}}(k)}.$$

Proof. It follows from the definitions that $\beta_{\mathbf{L}}(n) = \beta_{\mathbf{L}}(n-k)(1-q^{-(n-k+1)})(1-q^{-(n-k+2)})\dots(1-q^{-n})$. The first assertion is a straightforward consequence. Since $k \le n$, we have $1-q^{-(n-i)} \ge 1-q^{-(k-i)}$ for all i. Thus all terms in the product $\displaystyle\prod_{i=0}^{k-1}\frac{1-q^{-(n-i)}}{1-q^{-(k-i)}}$ are larger than 1, and we have $1 \le \gamma_1(n,k)$. It is clear that $\beta_{\mathbf{L}}(n) \le \beta_{\mathbf{L}}(n-k)$. Therefore $\gamma_1(n,k) = \dfrac{\beta_{\mathbf{L}}(n)}{\beta_{\mathbf{L}}(n-k)\beta_{\mathbf{L}}(k)} \le \dfrac{1}{\beta_{\mathbf{L}}(k)}$, as required. □

The conclusion of Lemma 2.1 follows from (2.6) and (2.4).(1).

Proof of Lemma 2.2: Assume that V is a non-degenerate symplectic, orthogonal, or unitary space and U is a non-singular k-space. We first record the precise value of $|\mathcal{N}(V,U)|$. If V is orthogonal, assume that V has type ϵ, U has type δ, and $U^\perp$ have type η. Note that η is determined by ϵ and δ.

2.7. 1. *If V is symplectic [or unitary] then*

(a) $|I(V)| = \beta_{\mathbf{X}}(n)q^{\varphi(n)}$ *where* $\varphi(n) = n^2 - \psi(n) = n(n-\mathrm{a})/2$.

(b) $|\mathcal{N}(V,U)| = \gamma_2(n,k)q^{k(n-k)}$ *where*

$$\gamma_2(n,k) = \frac{\beta_{\mathbf{X}}(n)}{\beta_{\mathbf{X}}(n-k)\beta_{\mathbf{X}}(k)},\ \mathbf{X} = \mathbf{S}\ [\mathit{or}\ \mathbf{U}].$$

2. *If V is orthogonal then*

(a) $|I(V)| = 2\beta_{\mathbf{O}^\epsilon}(n)q^{\varphi(n)}$ *where* $\varphi(n) = n^2 - \psi(n) = n(n-1)/2$.

(b) $|\mathcal{N}(V,U)| = \gamma_2(n,k,\epsilon,\delta)q^{k(n-k)}$ *where*

$$\gamma_2(n,k,\epsilon,\delta) = \frac{\beta_{\mathbf{O}^\epsilon}(n)}{2\beta_{\mathbf{O}^\eta}(n-k)\beta_{\mathbf{O}^\delta}(k)}.$$

Proof. It follows from Witt's Theorem that $I(V)$ is transitive on subspaces isometric to U. The $I(V)$-stabilizer of U is isomorphic to $I(U) \times I(U^\perp)$. Therefore

$$|\mathcal{N}(V,U)| = \frac{|I(V)|}{|I(U)||I(U^\perp)|}.$$

If V is symplectic then n is even, say $n = 2m$, and we have $|I(V)| = q^{m^2}\prod_{i=1}^{m}(q^{2i} - 1) = q^{m^2+m(m+1)}\beta_{\mathbf{S}}(2m)$ by counting standard (ordered) symplectic bases for V. This implies that

$$|I(V)| = q^{n(n+1)/2}\beta_{\mathbf{S}}(n).$$

If V is orthogonal of type ϵ, $\epsilon = +$ or $-$, then $n = 2m$ for some m, and the count of standard bases shows that $|I(V)| = 2q^{m(m-1)}(q^m - \epsilon 1)\prod_{i=1}^{m-1}(q^{2i} - 1) = 2q^{m(m-1)+m+m(m-1)}\beta_{\mathbf{O}^\epsilon}(2m)$. This implies that

$$|I(V)| = 2q^{n(n-1)/2}\beta_{\mathbf{O}^\epsilon}(n).$$

If V is orthogonal of type $\circ$, then $n = 2m+1$ for some m, and q is odd. Counting standard bases, we have $|I(V)| = 2q^{m^2}\prod_{i=1}^{m}(q^{2i} - 1) = 2q^{m^2+m(m+1)}\beta_{\mathbf{O}^-}(2m+1)$. Therefore

$$|I(V)| = 2q^{n(n-1)/2}\beta_{\mathbf{O}^\circ}(n).$$

If V is unitary, then $q = {q_0}^2$ for some integer q_0, and $|I(V)| = {q_0}^{n(n-1)/2}\prod_{i=1}^{n}({q_0}^i - (-1)^i) = {q_0}^{n(n-1)/2+n(n+1)/2}\beta_{\mathbf{U}}(n)$ by the standard basis count. This implies that

$$|I(V)| = q^{n^2/2}\beta_{\mathbf{U}}(n).$$

Observing that the quadratic function $\varphi(x) = x(x-\mathrm{a})/2$ satisfies $\varphi(n) - \varphi(n-k) - \varphi(k) = k(n-k)$, it follows easily that if V is symplectic [or unitary], then

$$|\mathcal{N}(V,U)| = \frac{|I(V)|}{|I(U)||I(U^\perp)|} = \frac{\beta_{\mathbf{X}}(n)}{\beta_{\mathbf{X}}(k)\beta_{\mathbf{X}}(n-k)}q^{k(n-k)},$$

where $\mathbf{X} = \mathbf{S}$ [or $\mathbf{U}$]. If V is orthogonal of type ϵ, U has type δ, and $U^\perp$ has type η, then

$$|\mathcal{N}(V,U)| = \frac{|I(V)|}{|I(U)||I(U^\perp)|} = \frac{\beta_{\mathbf{O}^\epsilon}(n)}{2\beta_{\mathbf{O}^\delta}(k)\beta_{\mathbf{O}^\eta}(n-k)}q^{k(n-k)}.$$

□

To complete the proof of Lemma 2.2, it suffices to show that $\gamma_2(n,k)$ lies between c_2 and C_2 whenever $0 \le k \le n$. Since $\gamma_2(n,0) = \gamma_2(n,n) = 1$ and $c_2 < 1 < C_2$ we may assume that $0 < k < n$.

2.8. *Let γ_2 be defined as in (2.7), and let $0 < k < n$.*

1. *If V is symplectic, then*
$$1 < \gamma_2(n,k) < \frac{q^4}{q^4 - q^2 - 1}.$$
2. *If V is orthogonal, then*
$$\frac{(q-1)^2}{2q^2} \le \gamma_2(n,k,\epsilon,\delta) \le \frac{1}{2}\frac{q^4-1}{q^2(q-1)^2}.$$
3. *If V is unitary, then*
$$\frac{\sqrt{q}-1}{\sqrt{q}} \le \gamma_2(n,k) \le \frac{q-\sqrt{q}+1}{q-1} \le 1.$$

Proof. Because $\gamma_2(n,k) = \gamma_2(n,n-k)$ we may assume that $k \le n/2$.

It follows from the definitions that $\beta_{\mathbf{S}}(n) = \beta_{\mathbf{S}}(n-k)(1-q^{-(n-k+2)})(1-q^{-(n-k+4)})\dots(1-q^{-n})$. Therefore
$$\gamma_2(n,k) = \frac{(1-q^{-(n-k+2)})(1-q^{-(n-k+4)})\dots(1-q^{-n})}{\beta_{\mathbf{S}}(k)}$$
$$= \frac{(1-q^{-(n-k+2)})(1-q^{-(n-k+4)})\dots(1-q^{-n})}{(1-q^{-k})(1-q^{-(k-2)})\dots(1-q^{-2})}.$$
Since $k < n-k+2$ this implies that $1 < \gamma_2(n,k)$ and that $\gamma_2(n,k) < 1/\beta_{\mathbf{S}}(k)$. By [1] we have $1/\beta_{\mathbf{S}}(k) < q^4/(q^4-q^2-1)$. This establishes (1).

If V is orthogonal and $n = 2$, then $k = 1$ and it is easy to see that (2) holds. We may therefore assume that $n \ge 3$. It follows from (2.4) that if k is even, then $\beta_{\mathbf{O}^+}(2) \le \beta_{\mathbf{O}^+}(k) \le \lim \beta_{\mathbf{O}^+}(2m)$ and $\lim \beta_{\mathbf{O}^-}(2m) \le \beta_{\mathbf{O}^-}(k) \le \beta_{\mathbf{O}^-}(2)$. If k is odd, then $\lim \beta_{\mathbf{O}^\circ}(2m+1) \le \beta_{\mathbf{O}^\circ}(k) \le \beta_{\mathbf{O}^\circ}(1)$. Since $\beta_{\mathbf{O}^\circ}(1) = 1 < 1 + q^{-1} = \beta_{\mathbf{O}^-}(2)$ and $\lim \beta_{\mathbf{O}^+}(2m) = \lim \beta_{\mathbf{O}^-}(2m) = \lim \beta_{\mathbf{O}^\circ}(2m+1)$ we have $\beta_{\mathbf{O}^+}(2) \le \beta_{\mathbf{O}^\delta}(k) \le \beta_{\mathbf{O}^-}(2)$ and $\beta_{\mathbf{O}^+}(2) \le \beta_{\mathbf{O}^\eta}(n-k) \le \beta_{\mathbf{O}^-}(2)$. Therefore $(\beta_{\mathbf{O}^+}(2))^2 \le \beta_{\mathbf{O}^\delta}(k)\beta_{\mathbf{O}^\eta}(n-k) \le (\beta_{\mathbf{O}^-}(2))^2$. Since $\beta_{\mathbf{O}^\circ}(3) < \beta_{\mathbf{O}^-}(4)$, and $n \ge 3$, it follows that $\beta_{\mathbf{O}^+}(4) \le \beta_{\mathbf{O}^\epsilon}(n) \le \beta_{\mathbf{O}^-}(4)$. Therefore $\dfrac{\beta_{\mathbf{O}^+}(4)}{2(\beta_{\mathbf{O}^-}(2))^2} \le \gamma_2(n,k,\epsilon,\delta) \le \dfrac{\beta_{\mathbf{O}^-}(4)}{2(\beta_{\mathbf{O}^+}(2))^2}$. Since $\dfrac{\beta_{\mathbf{O}^+}(4)}{2(\beta_{\mathbf{O}^-}(2))^2} = \dfrac{(q-1)^2}{2q^2}$ and $\dfrac{\beta_{\mathbf{O}^-}(4)}{2(\beta_{\mathbf{O}^+}(2))^2} = \dfrac{1}{2}\dfrac{q^4-1}{q^2(q-1)^2}$ the second assertion follows.

To prove the last statement, note that by (2.4), $\beta_{\mathbf{U}}(2) \le \beta_{\mathbf{U}}(k) \le \lim \beta_{\mathbf{U}}(m)$ when k is even and $\lim \beta_{\mathbf{U}}(m) \le \beta_{\mathbf{U}}(k) \le \beta_{\mathbf{U}}(1)$ when k is odd. Furthermore, $\beta_{\mathbf{U}}(n) \le \beta_{\mathbf{U}}(3)$ because $n \ge 2$. Therefore $\dfrac{\beta_{\mathbf{U}}(2)}{(\beta_{\mathbf{U}}(1))^2} \le \gamma_2(n,k) \le \dfrac{\beta_{\mathbf{U}}(3)}{(\beta_{\mathbf{U}}(2))^2}$ when V is unitary. By computation, $\dfrac{\beta_{\mathbf{U}}(2)}{\beta_{\mathbf{U}}(1)^2} = \dfrac{\sqrt{q}-1}{\sqrt{q}}$ and $\dfrac{\beta_{\mathbf{U}}(3)}{\beta_{\mathbf{U}}(2)^2} = \dfrac{q-\sqrt{q}+1}{q-1} = 1 - \dfrac{\sqrt{q}-2}{q-1} \le 1$, and the assertion for the unitary case follows. □

It remains to verify that the ranges given in (2.8) lie within the interval $[c_2, C_2]$. For $q = 2$, we have

$$1 < \gamma_2(n,k) < \frac{16}{11}, \ V \text{ symplectic, and}$$

$$\frac{1}{8} \leq \gamma_2(n,k,\epsilon,\delta) \leq \frac{15}{8}, \ V \text{ orthogonal,}$$

as required.

If $q > 2$, then $\dfrac{(q-1)^2}{2q^2} < \dfrac{\sqrt{q}-1}{\sqrt{q}} < 1$, so $\dfrac{(q-1)^2}{2q^2} < \gamma_2(n,k)$. Furthermore $1 < \dfrac{q^4}{q^4-q^2-1}$ and $\dfrac{q^4-1}{2q^2(q-1)^2} < \dfrac{q^4}{q^4-q^2-1}$. Therefore $\gamma_2(n,k) \leq \dfrac{q^4}{q^4-q^2-1}$. This completes the proof of Lemma 2.2.

Proof of Lemma 2.3: Let V again be a non-degenerate symplectic, unitary, or orthogonal space. If V is symplectic [or unitary], let $\mathbf{X} = \mathbf{S}[$ or $\mathbf{U}]$. If V is orthogonal of type ϵ, let $\mathbf{X} = \mathbf{O}^\epsilon$. Assume that $k \leq n/2$, and if V is orthogonal of type $-$ assume that $k < n/2$.

2.9. $|\mathcal{S}(V,k)| = \gamma_3(n,k) q^{k(n-k)-\psi(k)}$ *where* $\gamma_3(n,k) = \dfrac{\beta_{\mathbf{X}}(n)}{\beta_{\mathbf{X}}(n-2k)\beta_{\mathbf{L}}(k)}$.

Proof. Set $G = I(V)$, and let U be a totally singular k-subspace of V. Since G acts transitively on the set of all totally singular k-spaces, we have $|\mathcal{S}(V,k)| = [G : G_U]$. By (2.7), $|I(V)| = \mu\beta_{\mathbf{X}}(n)q^{\varphi(n)}$ where $\varphi(n) = n^2 - \psi(n)$ and $\mu = 1$ in the symplectic and unitary cases and $\mu = 2$ in the orthogonal case.

To compute $|G_U|$, consider the series $1 \subseteq C_G(U^\perp) \subseteq C_H(U^\perp/U) \subseteq H \subseteq G_U$ where $H = C_G(U)$. We have $G_U/H \cong GL(U)$, and $H/C_H(U^\perp/U) \cong I(U^\perp/U)$. Also, $C_H(U^\perp/U)/C_G(U^\perp)$ acts regularly on the set of $q^{k(n-2k)}$ complements to U in $U^\perp$. Therefore $|G_U| = |GL(U)|\ |I(U^\perp/U)|\ q^{k(n-2k)}\ |C_G(U^\perp)|$.

To compute $|C_G(U^\perp)|$, write $U^\perp = U \perp W$ where W is non-degenerate. Then $C_G(U^\perp)$ acts regularly on the collection of totally singular complements to U in $W^\perp$ because each such complement has a unique basis dual to a given basis for U. Therefore $|C_G(U^\perp)| = q^{\varphi(k)}$.

It follows that

$$[G : G_U] = \frac{|I(V)|}{q^{k(n-2k)}q^{\varphi(k)}\ |I(U^\perp/U)|\ |GL(U)|}.$$

We observe that, in the orthogonal case, $U^\perp/U$ has the same type as V. This is true because $U^\perp/U$ has a totally singular subspace of dimension $n/2 - k = \dim(U^\perp/U)/2$ if and only if V has a totally singular subspace of dimension $n/2$. Thus $|I(U^\perp/U)| = \mu q^{\varphi(n-2k)}\beta_{\mathbf{X}}(n-2k)$, μ as above. Therefore

$$|\mathcal{S}(V,k)| = \frac{\beta_{\mathbf{X}}(n)}{\beta_{\mathbf{X}}(n-2k)\beta_{\mathbf{L}}(k)} \frac{q^{\varphi(n)}}{q^{k(n-2k)}q^{\varphi(k)}q^{\phi(n-2k)}q^{k^2}}$$

because $|GL(U)| = \beta_{\mathbf{L}}(k)q^{k^2}$. This shows that $|\mathcal{S}(V,k)| = \gamma_3(n,k)q^h$ where $h = \varphi(n) - \varphi(n-2k) - \varphi(k) - k(n-2k) - k^2$. Since $\varphi(x) = x^2/2 - ax/2$ it follows that $\varphi(x+y) = \varphi(x) + \varphi(y) + xy$. Therefore $h = \varphi(2k) + 2k(n-2k) - \varphi(k) - k(n-2k) - k^2 = \varphi(k) + k^2 + k(n-2k) - k^2 = k(n-k) - k^2 + \varphi(k) = k(n-k) - \psi(k)$, as required. $\square$

Since $\gamma_3(n,0)=1$, and $c_3<1<C_3$ we may assume for the rest of the proof of Lemma 2.3 that $k\geq 1$.

2.10. 1. $\dfrac{q^4-q^2-1}{q^4}<\dfrac{\beta_{\mathbf{S}}(n)}{\beta_{\mathbf{S}}(n-2k)}<1.$

2. $\dfrac{q^4-q^2-1}{q^4}<\dfrac{\beta_{\mathbf{O}^\circ}(n)}{\beta_{\mathbf{O}^\circ}(n-2k)}<1.$

3. *If* $k\leq n/2-1$, *then* $\dfrac{q^4-q^2-1}{q^4+q^3}<\dfrac{\beta_{\mathbf{O}^-}(n)}{\beta_{\mathbf{O}^-}(n-2k)}<1.$

4. $1<\dfrac{\beta_{\mathbf{O}^+}(n)}{\beta_{\mathbf{O}^+}(n-2k)}<2\dfrac{q^6-q^4-q^2+1}{q^6}.$

5. *If* n *is odd, then* $\dfrac{q-1}{q}<\dfrac{\beta_{\mathbf{U}}(n)}{\beta_{\mathbf{U}}(n-2k)}<1.$

6. *If* n *is even, then* $1<\dfrac{\beta_{\mathbf{U}}(n)}{\beta_{\mathbf{U}}(n-2k)}<\dfrac{(q^{1/2}+1)(q-1)(q^{3/2}+1)}{q^3}.$

Proof. If $\alpha(m)$ is a decreasing sequence such that $\alpha(m)>L$ for all m, and $a\leq n-2k<n$, then $\dfrac{L}{\alpha(a)}<\dfrac{\alpha(n)}{\alpha(n-2k)}<1$. Applying this to $\beta_{\mathbf{S}}(n)$, $\beta_{\mathbf{O}^\circ}(2m+1)$, $\beta_{\mathbf{O}^-}(2m)$, and $\beta_{\mathbf{U}}(2m+1)$ with $a=0,1,2,1$, respectively, and the bounds from (2.4), we have:

$$\frac{q^4-q^2-1}{q^4\beta_{\mathbf{S}}(0)}<\frac{\beta_{\mathbf{S}}(n)}{\beta_{\mathbf{S}}(n-2k)}<1;$$

$$\frac{q^4-q^2-1}{q^4\beta_{\mathbf{O}^\circ}(1)}<\frac{\beta_{\mathbf{O}^\circ}(n)}{\beta_{\mathbf{O}^\circ}(n-2k)}<1;$$

$$\frac{q^4-q^2-1}{q^4\beta_{\mathbf{O}^-}(2)}<\frac{\beta_{\mathbf{O}^-}(n)}{\beta_{\mathbf{O}^-}(n-2k)}<1;$$

and

$$\frac{(\sqrt{q}+1)(q-1)}{q\sqrt{q}\beta_{\mathbf{U}}(1)}<\frac{\beta_{\mathbf{U}}(n)}{\beta_{\mathbf{U}}(n-2k)}<1,\ n \text{ odd}.$$

The inequalities (1), (2), (3), and (5) now follow because $\beta_{\mathbf{S}}(0)=\beta_{\mathbf{O}^\circ}(1)=1$, $\beta_{\mathbf{O}^-}(2)=1+q^{-1}$, and $\beta_{\mathbf{U}}(1)=(\sqrt{q}+1)/\sqrt{q}$.

For an increasing sequence $\alpha(m)$ with $\alpha(m)<L$ for all m, we have $1<\dfrac{\alpha(n)}{\alpha(n-2k)}<\dfrac{L}{\alpha(0)}$ whenever $0\leq n-2k<n$. Using the bounds in (2.4) we have

$$1<\frac{\beta_{\mathbf{O}^+}(n)}{\beta_{\mathbf{O}^+}(n-2k)}<\frac{q^6-q^4-q^2+1}{q^6\beta_{\mathbf{O}^+}(0)}$$

and

$$1<\frac{\beta_{\mathbf{U}}(n)}{\beta_{\mathbf{U}}(n-2k)}<\frac{(\sqrt{q}+1)(q-1)(q\sqrt{q}+1)}{q^3\beta_{\mathbf{U}}(0)}.$$

The assertion now follows because $\beta_{\mathbf{O}^+}(0)=1/2$ and $\beta_{\mathbf{U}}(0)=1$.

Combining (2.10) with (2.4)(1) and the definition of $\gamma_3(n,k)$, we have the following.

2.11. 1. *If* V *is symplectic or orthogonal of type* $\circ$, *then*

$$\frac{q^4-q^2-1}{q^4}<\gamma_3(n,k)<C_1.$$

2. *If V is orthogonal of type $-$, then*

$$\frac{q^4-q^2-1}{q^4+q^3} < \gamma_3(n,k) < C_1.$$

3. *If V is orthogonal of type $+$, then*

$$1 < \gamma_3(n,k) < 2C_1\frac{q^6-q^4-q^2+1}{q^6}.$$

4. *If V is unitary and n is odd, then*

$$\frac{q-1}{q} < \gamma_3(n,k) < C_1.$$

5. *If V is unitary and n is even, then*

$$1 < \gamma_3(n,k) < C_1\frac{(q^{1/2}+1)(q-1)(q^{3/2}+1)}{q^3}.$$

Since $\dfrac{q^4-q^2-1}{q^4+q^3} < \dfrac{q-1}{q} < 1$ and $\dfrac{q^4-q^2-1}{q^4+q^3} < \dfrac{q^4-q^2-1}{q^4}$, it follows that $\gamma_3(n,k) > \dfrac{q^4-q^2-1}{q^4+q^3}$ in all cases. In particular, if $q=2$, then $\gamma_3 > 11/24$. This establishes the lower bound of Lemma 2.3.

Since $2\dfrac{q^6-q^4-q^2+1}{q^6} > 1$ and $2\dfrac{q^6-q^4-q^2+1}{q^6} > \dfrac{(q^{1/2}+1)(q-1)(q^{3/2}+1)}{q^3}$, it follows that $\gamma_3(n,k) \le 2C_1\dfrac{q^6-q^4-q^2+1}{q^6}$.

If $q=2$, then $\gamma_3(n,k) \le 7\cdot\dfrac{45}{64} < 5$. If $q>2$, then $\gamma_3(n,k) \le 2\dfrac{q^6-q^4-q^2+1}{q^6-q^5-q^4}$. This completes the proof of Lemma 2.3.

The next result shows that the formulas for the numbers of non-degenerate and totally singular subspaces of a non-degenerate space carry over for subspaces of a space with a nonsingular radical.

Lemma 2.12. *Let W be an orthogonal space of dimension m over $\mathbf{F}_q$ where m is odd and q is even. Assume that rad W is nonsingular of dimension 1.*

1. *If $k \le m/2$ then $|\mathcal{S}(W,k)| = \dfrac{\beta_{\mathbf{O}^\circ}(m)}{\beta_{\mathbf{O}^\circ}(m-2k)\beta_{\mathbf{L}}(k)}q^{k(m-k)-\psi(k)}$.*
2. *If U is a non-degenerate k-subspace of W of type δ, then*

$$|\mathcal{N}(W,U)| = \frac{\beta_{\mathbf{O}^\circ}(m)}{2\beta_{\mathbf{O}^\circ}(m-k)\beta_{\mathbf{O}^\delta}(k)}q^{k(m-k)}.$$

Proof. Consider the function from $\mathcal{L}(W)$ to $\mathcal{L}(W/\text{rad } W)$ defined by $\phi(X) = (X + \text{rad } W)/\text{rad } W$. Because rad W contains no singular points, we have $X \cap \text{rad } W = 0$ and $\phi(X) \in \mathcal{S}(W/\text{rad } W, k)$ for all $X \in \mathcal{S}(W,k)$. Furthermore, if $Z/\text{rad } W \in \mathcal{S}(W/\text{rad } W, k)$ then the totally isotropic elements of Z form a complement Z' to rad W in Z, and Z' is consequently the unique element of $\mathcal{S}(W,k)$ with $\phi(Z') = Z$. This shows that ϕ induces a bijection between $\mathcal{S}(W,k)$ and $\mathcal{S}(W/\text{rad } W, k)$.

Therefore,

$$\begin{aligned}|\mathcal{S}(W,k)| &= |\mathcal{S}(W/\text{rad } W,k)| \\ &= \frac{\beta_{\mathbf{S}}(m-1)}{\beta_{\mathbf{S}}(m-2k-1)\beta_{\mathbf{L}}(k)} q^{k(m-1-k)-k(k-1)/2} \\ &= \frac{\beta_{\mathbf{O}^\circ}(m)}{\beta_{\mathbf{O}^\circ}(m-2k)\beta_{\mathbf{L}}(k)} q^{k(m-k)-k(k+1)/2}.\end{aligned}$$

This establishes (1).

To prove (2), note that under the assumptions on U, we must have $U\cap\text{rad } W = 0$ so that $\phi : \mathcal{N}(W,U) \to \mathcal{N}(W/\text{rad } W, \tilde{U})$ where $\tilde{U}$ is the non-degenerate *symplectic* k-space $(U + \text{rad } W)/\text{rad } W$. If $Z/\text{rad } W \in \mathcal{N}(W/\text{rad } W, \tilde{U})$, then the pre-image of $Z/\text{rad } W$ under ϕ consists of the type δ complements to rad Z = rad W in Z. Considering the index of $O^\delta(k,q)$ in $O^\circ(k+1,q)$, we see that there are $q^k(1 + \delta q^{-k/2})/2$ type δ complements to rad Z in Z. Therefore

$$\begin{aligned}|\mathcal{N}(W,U)| &= |\mathcal{N}(W/\text{rad } W,\tilde{U})| \ \frac{q^k(1+\delta q^{-k/2})}{2} \\ &= \frac{\beta_{\mathbf{S}}(m-1)}{\beta_{\mathbf{S}}(m-k-1)\beta_{\mathbf{S}}(k)} \frac{1+\delta q^{-k/2}}{2} q^{k(m-k-1)+k} \\ &= \frac{\beta_{\mathbf{O}^\circ}(m)}{2\beta_{\mathbf{O}^\circ}(m-k)\beta_{\mathbf{O}^\delta}(k)} q^{k(m-k)}.\end{aligned}$$

We will need formulas for the numbers of non-degenerate and totally singular subspaces of an arbitrary space.

Lemma 2.13. *Let W be an arbitrary subspace of V, and set $T = T(W)$, the totally singular radical of W.*

1. *If U is a non-degenerate k-subspace of W then*
$$|\mathcal{N}(W,U)| = |T|^k \ |\mathcal{N}(W/T,U)|.$$
2. *If $\mathcal{S}_i(W,k)$ is the set of totally singular k-subspaces of W that meet T in a subspace of dimension i, and $\mathcal{S}_i(W,k) \neq \emptyset$, then*
$$|\mathcal{S}_i(W,k)| = (|T|q^{-i})^{k-i} \ |\mathcal{L}(T,i)| \ |\mathcal{S}(W/T,k-i)|.$$

Proof. (1) Consider the function $\phi : \mathcal{L}(W) \to \mathcal{L}(W/T)$ defined by $\phi(X) = (X + T)/T$. Then $\phi : \mathcal{N}(W,U) \to \mathcal{N}(W/T,U)$. For each $Z/T \in \mathcal{N}(W/T,U)$ the preimage of Z/T consists of the complements to T in Z. There are $|T|^k$ such complements, and the first assertion follows easily.

(2) Consider the function $\xi : \mathcal{S}_i(W,k) \to \mathcal{L}(T,i) \times \mathcal{S}_i(W/T,k-i)$, where $\xi(X) = (X \cap T, \phi(X))$. Given $Y \in \mathcal{L}(T,i)$ and $Z/T \in \mathcal{S}(W/T,k-i)$ we have $\xi(X) = (Y, Z/T)$ if and only if $X \cap T = Y$ and $X + T = Z$, which happens exactly when X/Y is a complement to T/Y in Z/Y. There are $q^{(t-i)(k-i)}$ such complements because T/Y has dimension $t - i$ and codimension $k - i$ in Z/Y.

For later estimates, we restate the previous result in a different form.

2.14. *Assume that W is an m-dimensional subspace of V and that the totally singular radical of W has dimension t.*

1. *If V is symplectic or unitary, and U is a non-degenerate k-subspace, then*
$$|\mathcal{N}(W,U)| = \gamma_2(m-t,k)q^{k(m-k)}.$$
2. *If W is orthogonal of type ϵ, and U is a non-degenerate k-subspace of type δ, then*
$$|\mathcal{N}(W,U)| = \gamma_2(m-t,k,\epsilon,\delta)q^{k(m-k)}.$$
3. *If V is symplectic, orthogonal, or unitary, and $\mathcal{S}_0(W,k)$ is nonempty then*
$$|\mathcal{S}_0(W,k)| = \gamma_3(m-t,k)q^{k(m-k)-\psi(k)}.$$
4. *If V is symplectic, orthogonal, or unitary, then*
$$|\mathcal{S}_i(W,k)| \leq C_1C_3q^{e(m,t,k,i)}$$
where $e(m,t,k,i) = (m-k)(k-i) + i(t-i) - \psi(k-i)$.

Proof. We have $\dim W/T(W) = m-t$. The first two statements follow from Lemma 2.13 and (2.7). The third statement follows from Lemma 2.13 and (2.9). By Lemma 2.13 and Lemmas 2.1 and 2.3, we have
$$|\mathcal{S}_i(W,k)| \leq C_1C_3q^{(t-i)(k-i)}q^{i(t-i)}q^{(k-i)(m-t-k+i)-\psi(k-i)}.$$
The fourth statement now follows because $e(m,t,k,i) = (t-i)(k-i) + i(t-i) + (k-i)(m-t-k+i) - \psi(k-i)$. □

2.15. *For an m-space W with $\dim T(W) = t$, set $\sigma^*(W,k) = \sum_i C_1C_3q^{e(m,t,k,i)}$ where $e(m,t,k,i)$ is defined in the previous statement, and every i in the sum satisfies $0 \leq i \leq k$ and $i \leq t$. If $\dim W' = \dim W$ and $\dim T(W') \leq \dim T(W)$ then $|\mathcal{S}(W',k)| \leq \sigma^*(W,k)$.*

Proof. It follows from (2.14) that $|\mathcal{S}(W',k)| \leq \sigma^*(W',k)$. Set $\mathcal{I}_x = \{i : 0 \leq i \leq k, i \leq x\}$, and suppose $\dim W' = s$. Then $s \leq t$, $\mathcal{I}_s \subseteq \mathcal{I}_t$, and $e(m,s,k,i) \leq e(m,t,k,i)$ for all i in $\mathcal{I}_s$. Therefore $\sigma^*(W',k) \leq \sigma^*(W,k)$, and the statement follows.

3. Numerical lemmas

This section contains the technical estimates that are used in bounding the coefficients.

3.1. *Assume that x_i and y_i are sequences of positive numbers such that $\lim x_i = \lim y_i = 1$, and for every positive integer i with $i \geq r$:*

1. $x_i < x_{i+1} < 1 < y_{i+1} < y_i$;
2. $x_iy_i < 1 < x_{i+1}y_i$.

If $z_m = \prod_{i=1}^m x_iy_i$, then $z = \lim z_m$ exists and $x_{m+1}z_m < z < z_m$ for every integer m with $m \geq r$.

Proof. The assumptions imply that the series $\log x_1 + \log y_1 + \log x_2 + \log y_2 + \ldots$ satisfies the hypotheses of the alternating series test of elementary calculus. We conclude from that test that the series converges to S where
$$\log(x_{m+1}z_m) < S < \log z_m$$
for every $m \geq r$. We have $S = \log z$, and the statement follows.

The following numerical result is useful in estimating γ_2.

3.2. 1. *If either* $0 < r \leq 1/2$ *and* $a \geq 0$ *or* $0 < r \leq 1/\sqrt{2}$ *and* $a \geq 2$, *then*

$$\frac{1}{1-r^{a+1}} \leq 1 + r^a.$$

2. *If* $b \geq 0$ *and* $0 < r \leq 1/2$, *then*

$$1 - r^b \leq \prod_{i=0}^{k}(1-(-1)^i r^{b+i}) \leq 1 \quad \text{and} \quad 1 \leq \prod_{i=0}^{k}(1+(-1)^i r^{b+i}) < 1 + r^b.$$

Proof. (1) We have $(1+r^a)(1-r^{a+1}) = 1 + r^a(1-r-r^{a+1})$. Either hypothesis implies that $1 - r - r^{a+1} \geq 0$. Therefore $(1+r^a)(1-r^{a+1}) \geq 1$, and (1) follows.

(2) We have $1 \leq (1+r^a)(1-r^{a+1})$ for every positive integer a by (1). It follows by induction on m that $\prod_{i=1}^{2m}(1-(-1)^i r^{b+i}) = \prod_{i=0}^{2m-1}(1+(-1)^i r^{b+i}) \geq 1$ for every positive integer m. This implies that $\prod_{i=1}^{k}(1-(-1)^i r^{b+i}) \geq 1$ for every positive integer k. Therefore $1 - r^b \leq \prod_{i=0}^{k}(1-(-1)^i r^{b+i})$ and $1 \leq \prod_{i=0}^{k}(1+(-1)^i r^{b+i})$.

Since $(1-r^a)(1+r^{a+1}) < (1-r^a)(1+r^a) = 1 - r^{2a} < 1$ we have $\prod_{i=1}^{2m}(1+(-1)^i r^{b+i}) = \prod_{i=0}^{2m-1}(1-(-1)^i r^{b+1+i}) < 1$ for every positive integer m. It follows that $\prod_{i=1}^{k}(1+(-1)^i r^{b+i}) \leq 1$. Therefore $\prod_{i=0}^{k}(1+(-1)^i r^{b+i}) \leq 1+r^b$ and $\prod_{i=0}^{k}(1-(-1)^i r^{b+1+i}) \leq 1$.

We also record a few other facts.

3.3. *If* $0 < r < 1$, $0 < s < 1$, *and* $0 < x$, *then*

1. $1 - r < \dfrac{1}{1+r}$.
2. $1 - r - s < (1-r)(1-s)$.
3. $1 - xr < (1-r)^x$.

3.4. *If* $0 < s \leq r/2$, $r \leq 1$, *and* a *and* b *are positive then*

1. $\dfrac{1}{1-s} \leq 1 + r$ *and*
2. $(1+ar)(1+br) \leq 1 + (a+b+ab)r$.

Proof. (1) $\dfrac{1}{1-s} = 1 + \dfrac{s}{1-s} \leq 1 + 2s \leq 1 + r$.

(2) $(1+ar)(1+br) = 1 + (a+b+abr)r \leq 1 + (a+b+ab)r$.

3.5. *If* $\alpha \leq 1/2$ *and* $r < 1/2$, *then*

1. $\dfrac{\alpha+r}{1-r} < \alpha + 3r$.
2. $\dfrac{\alpha-r}{1+r} < \alpha - 2r$.

Proof. (1) $\dfrac{\alpha+r}{1-r} = (\alpha+r)(1+\dfrac{r}{1-r}) = \alpha + r + \dfrac{\alpha}{1-r}r + \dfrac{r^2}{1-r} < \alpha + r + 2\alpha r + 2r^2 < \alpha + 3r$.

(2) $\dfrac{\alpha-r}{1+r} > (\alpha-r)(1-r) > \alpha - r - \alpha r > \alpha - 2r$.

4. Points

We consider in this section the actions of linear transformations on the collection of 1-dimensional subspaces.

We handle the linear case first. The following fact is well known and elementary:

4.1. *If V has dimension n then* $|\mathcal{L}(V,1)| = \dfrac{q^n-1}{q-1}$.

If g is a linear transformation on V, then every g-invariant point lies in some eigenspace of V. The following lemma delineates the distinction between elements with a large number of fixed points and elements with a small set of fixed points.

4.2. *If g is a linear transformation on V then either g has an eigenspace of dimension larger than $n/2$ or there are subspaces W_1, W_2, W_3 such that* $\dim W_i \leq n/2$, $\sum \dim W_i \leq n$, *and every g-invariant 1-subspace of V is contained in some W_i.*

Proof. Let E_i, $i \in I$, be the eigenspaces for g, and set $m_i = \dim E_i$. Assume without loss that $m_i \leq n/2$ for all i. For each $J \subseteq I$, set $m_J = \sum_{j\in J} m_j$. Choose J so that m_J is as large as possible subject to $m_J \leq n/2$, and set $W_1 = \sum_{j\in J} E_j$. If $J = I$, then set $W_2 = W_3 = 0$. Otherwise, let $l \in I \setminus J$, and set $W_2 = E_l$ and $W_3 = \sum_{i\notin J, i\neq l} E_i$. Then every eigenspace is contained in W_1, W_2, or W_3, so every g-invariant 1-subspace is also contained in some W_i. By choice of J, we have $\dim W_1 = m_J \leq n/2$, and $m_J + m_l \geq n/2$. Therefore $\dim W_3 = \sum_{i\notin J, i\neq l} m_i \leq n - m_J - m_l \leq n/2$. Since $\dim W_2 = m_l \leq n/2$, and $\dim W_1 + \dim W_2 + \dim W_3 = \sum m_i \leq n$ we are done.

4.3. *Assume that W has codimension d in V where $d > 0$. Then* $q^{-d} - q^{-n} < \dfrac{|\mathcal{L}(W,1)|}{|\mathcal{L}(V,1)|} < q^{-d}$.

Proof. From (4.1), we have $|\mathcal{L}(W,1)| = (q^{n-d}-1)/(q-1)$ and $|\mathcal{L}(V,1)| = (q^n - 1)/(q-1)$, so that

$$\frac{|\mathcal{L}(W,1)|}{|\mathcal{L}(V,1)|} = \frac{q^{n-d}-1}{q^n-1} = q^{-d}\frac{1-q^{-(n-d)}}{1-q^{-n}}.$$

Since $1 - q^{-(n-d)} < (1-q^{-(n-d)})/(1-q^{-n}) < 1$ the result follows.

4.4. *If $a + d = b + c$ and $a < b < c < d$ then $q^a + q^d > q^b + q^c$.*

Proof. $\dfrac{d}{dx}(q^x + q^{a+d-x}) = (q^x - q^{a+d-x})\log q$, so $q^x + q^{a+d-x}$ decreases on the interval $(-\infty, (a+d)/2)$.

Proposition 4.5. *If g is a linear transformation on V, then either*

1. $|\mathcal{L}(V,1)_g| < 2q^{-n/2}|\mathcal{L}(V,1)|$ *or*
2. *g has an eigenspace of codimension d where $d < n/2$ and*

$$q^{-d} - q^{-n} < \frac{|\mathcal{L}(V,1)_g|}{|\mathcal{L}(V,1)|} < q^{-d} + q^{-n/2}.$$

Proof. Suppose first that g has an eigenspace W of codimension d where $d < n/2$, and let X be the sum of all other eigenspaces. Then $|\mathcal{L}(V,1)_g| \leq |\mathcal{L}(W,1)| + |\mathcal{L}(X,1)|$. We may assume that $d > 0$. By (4.3) we have $q^{-d} - q^{-n} < \dfrac{|\mathcal{L}(W,1)|}{|\mathcal{L}(V,1)|}$. Since codim $X \geq \dim W > n/2$, we also have $\dfrac{|\mathcal{L}(W,1)|}{|\mathcal{L}(V,1)|} + \dfrac{|\mathcal{L}(X,1)|}{|\mathcal{L}(V,1)|} < q^{-d} + q^{-n/2}$, and (2) holds.

If g has no eigenspace of codimension smaller than $n/2$ then by (4.2) there exist subspaces W_1, W_2, W_3, with $\dim W_i \le n/2$, $i = 1,2,3$, such that $|\mathcal{L}(V,1)_g| \le \sum |\mathcal{L}(W_i,1)|$ and $\sum \dim W_i \le n$.

Let W_i have dimension m_i, $i = 1,2,3$. Then $\sum |\mathcal{L}(W_i,1)| = \dfrac{1}{q-1}(q^{m_1} + q^{m_2} + q^{m_3} - 3)$ by (4.1). Since $m_i \le n/2$ we have, by (4.4), $q^{m_1} + q^{m_2} + q^{m_3} \le q^{n/2} + q^{m_2} + q^{m_1+m_3-n/2} \le q^{n/2} + q^{n/2} + q^{m_1+m_2+m_3-n} \le 2q^{n/2} + 1$. Therefore $\sum |\mathcal{L}(W_i,1)| \le \dfrac{2}{q-1}(q^{n/2}-1) = \dfrac{2}{q^{n/2}+1}|\mathcal{L}(V,1)| < 2q^{-n/2}|\mathcal{L}(V,1)|$ by (4.1).

Assume for the balance of this section that V admits a non-degenerate form. If V is symplectic, then the only relevant ratio is covered by Proposition 4.5 because $\mathcal{S}(V,1) = \mathcal{L}(V,1)$. Assume therefore that V is orthogonal or unitary. Since $F^*(I(V))$ is nearly simple, we have $n \ge 3$.

Proposition 4.6. *Let V be a non-degenerate orthogonal or unitary space over* $\mathbf{F}$*, and let $g \in \Delta(V)$.*

1. *If V is orthogonal and U is a non-degenerate 1-subspace of V then either*
   (a) $|\mathcal{N}(V,U)_g| < 12q^{-n/2}|\mathcal{N}(V,U)|$ *or*
   (b) *g has an eigenspace of codimension d, where $d < n/2$ and*
   $$q^{-d} - 2q^{-(n-1)/2} < \frac{|\mathcal{N}(V,U)_g|}{|\mathcal{N}(V,U)|} < q^{-d} + 7q^{-(n-1)/2}.$$
2. *If V is unitary and U is a non-degenerate 1-subspace of V then either*
   (a) $|\mathcal{N}(V,U)_g| < 8q^{-n/2}|\mathcal{N}(V,U)|$ *or*
   (b) *g has an eigenspace of codimension d, where $d < n/2$ and*
   $$q^{-d} - 2q^{-n/2} < \frac{|\mathcal{N}(V,U)_g|}{|\mathcal{N}(V,U)|} < q^{-d} + 7q^{-n/2}.$$
3. *If V is orthogonal and q is even, then either*
   (a) $|\mathcal{P}(V)_g| < 4q^{-n/2}|\mathcal{P}(V)|$ *or*
   (b) *g has an eigenspace of codimension d, where $d < n/2$ and*
   $$q^{-d} - 2q^{-n/2} < \frac{|\mathcal{P}(V)_g|}{|\mathcal{P}(V)|} < q^{-d} + 3q^{-n/2}.$$
4. *If V is orthogonal, then either*
   (a) $|\mathcal{S}(V,1)_g| < 4q^{-(n-2)/2}|\mathcal{S}(V,1)|$ *or*
   (b) *g has an eigenspace of codimension d, where $d < n/2$ and*
   $$q^{-d} - 2q^{-(n-2)/2} < \frac{|\mathcal{S}(V,1)_g|}{|\mathcal{S}(V,1)|} < q^{-d} + 7q^{-(n-2)/2}.$$
5. *If V is unitary, then either*
   (a) $|\mathcal{S}(V,1)_g| < 4q^{-(n-1)/2}|\mathcal{S}(V,1)|$ *or*
   (b) *g has an eigenspace of codimension d, where $d < n/2$ and*
   $$q^{-d} - 2q^{-(n-1)/2} < \frac{|\mathcal{S}(V,1)_g|}{|\mathcal{S}(V,1)|} < q^{-d} + 7q^{-(n-1)/2}.$$

Proof. Let U be a 1-subspace of V and, for each $\mathbf{F}_q$-space W, let $\mathcal{U}(W)$ be the set of subspaces of W isometric to U. For $W \subseteq V$, let $\mathcal{U}(W)_g$ denote the g-invariant members of $\mathcal{U}(W)$.

TABLE 1. a and b values

V	q	U	a	b
orthogonal	odd	non-degenerate	$\frac{1}{2}$	$q^{1/2}$
unitary		non-degenerate	$\frac{1}{1+q^{-1/2}}$	1
orthogonal	even	nonsingular	1	1
orthogonal		totally singular	$\frac{1}{q-1}$	q
unitary		totally singular	$\frac{q^{1/2}}{q-1}$	$q^{1/2}$

4.7. *Let a and b be as defined in Table 1 Then*

$$aq^{n-1}(1-bq^{-n/2}) \leq |\mathcal{U}(V)| \leq aq^{n-1}(1+bq^{-n/2}).$$

Proof. Assume first that V is a non-degenerate orthogonal space of type ϵ.

If U is non-degenerate then q is odd and $|I(U)| = 2$. We have $|\mathcal{U}(V)| = |\mathcal{N}(V,U)|$. If $U^{\perp}$ has type δ, then it follows from (2.7) that

$$|\mathcal{N}(V,U)| = \frac{|I(V)|}{2|I(U^{\perp})|} = \begin{cases} \frac{1}{2}q^{m-1}(q^m - \epsilon 1) = \frac{1}{2}q^{2m-1}(1-\epsilon q^{-m}), & \dim V = 2m; \\ \frac{1}{2}q^m(q^m + \delta 1) = \frac{1}{2}q^{2m}(1+\delta q^{-m}), & \dim V = 2m+1. \end{cases}$$

Therefore

$$\frac{1}{2}q^{n-1}(1-q^{-(n-1)/2}) \leq |\mathcal{N}(V,U)| \leq \frac{1}{2}q^{n-1}(1+q^{-(n-1)/2}).$$

If U is totally singular, then $\mathcal{U}(V) = \mathcal{S}(V,1)$. If n is even, say $n = 2m$, then $|\mathcal{S}(V,1)| = \dfrac{(q^m - \epsilon 1)(q^{m-1} + \epsilon 1)}{q-1}$. This implies that

$$q^{2m-1}\frac{1-q^{-(m-1)}}{q-1} \leq |\mathcal{S}(V,1)| \leq q^{2m-1}\frac{1+q^{-(m-1)}}{q-1}.$$

If n is odd, say $n = 2m+1$, then

$$|\mathcal{S}(V,1)| = \frac{q^{2m}-1}{q-1}.$$

In either case, $q^{n-1}\dfrac{1-q^{1-n/2}}{q-1} \leq |\mathcal{S}(V,1)| \leq q^{n-1}\dfrac{1+q^{1-n/2}}{q-1}$.

Suppose q is even and U is nonsingular. Then n is even and $\mathcal{U}(V) = \mathcal{P}(V)$. The nonsingular points of V are in bijective correspondence with the reflections in $I(V)$. For each such reflection τ, we have $C_{I(V)}(\tau) \cong Z_2 \times I(X)$ where X is a symplectic space of dimension $n-2$. This implies that $|\mathcal{P}(V)| = |I(V)|/(2|I(X)|)$. Therefore

4.8. $|\mathcal{P}(V)| = q^{n-1}(1-\epsilon q^{-n/2})$,

which we record for future reference. In either case,

$$q^{n-1}(1-q^{-n/2}) \le |\mathcal{P}(V)| \le q^{n-1}(1+q^{-n/2}).$$

Now assume that V is unitary.

If U is non-degenerate, then $|\mathcal{U}(V)| = |\mathcal{N}(V,U)| = \dfrac{|I(V)|}{|I(U)||I(U^\perp)|}$, by Witt's Theorem. We have $\dfrac{|I(V)|}{|I(U^\perp)|} = q^{n-1/2}(1-(-1)^n q^{-n/2})$ and $|I(U)| = q^{1/2}(1+q^{-1/2})$ by (2.7). Therefore $|\mathcal{N}(V,U)| = q^{n-1}\dfrac{1-(-1)^n q^{-n/2}}{1+q^{-1/2}}$, and

$$q^{n-1}\frac{1-q^{-n/2}}{1+q^{-1/2}} \le |\mathcal{N}(V,U)| \le q^{n-1}\frac{1+q^{-n/2}}{1+q^{-1/2}}.$$

The claim follows.

If U is totally singular, then

$$|\mathcal{U}(V)| = |\mathcal{S}(V,1)| = q^{n-1/2}\frac{(1-\epsilon q^{-n/2})(1+\epsilon q^{-(n-1)/2})}{q-1}$$

where $\epsilon = (-1)^n$ by (2.9). It follows that

$$q^{n-1/2}\frac{1-q^{-(n-1)/2}}{q-1} \le |\mathcal{S}(V,1)| \le q^{n-1/2}\frac{1+q^{-(n-1)/2}}{q-1},$$

as required. □

Recall that $T(W)$ is the largest totally singular subspace of rad W. Set $\mathcal{U}_0(W) = \{X \in \mathcal{U}(W) : X \cap T(W) = 0\}$.

4.9. *If W is a subspace of codimension d in V and $d < n/2$, then*

$$q^{-d} - 2bq^{-n/2} < \frac{|\mathcal{U}_0(W)|}{|\mathcal{U}(V)|} < q^{-d} + 3bq^{-n/2}.$$

Proof. It suffices to assume that $d \ge 1$ and $bq^{-n/2} < 1/2$. Let $m = \dim W/T(W)$ and $t = \dim W$. We claim that

$$aq^{t+m-1}(1-bq^{-m/2}) \le |\mathcal{U}_0(W)| \le aq^{t+m-1}(1+bq^{-m/2}).$$

To establish the claim, suppose first that $t = 0$. If W is non-degenerate then the claim follows from (4.7). Otherwise, q is even, W is orthogonal of odd dimension, and rad W has dimension 1. Furthermore the map $w \mapsto \mathbf{q}(w)$ is one-to-one on each coset of rad W. This implies that W contains q^{m-1} nonsingular points and $(q^{m-1}-1)/(q-1)$ totally singular points. The claim therefore holds when $t = 0$.

If $\bar{W} = W/T(W)$, then $aq^{m-1}(1-bq^{-m/2}) \le |\mathcal{U}(\bar{W})| \le aq^{m-1}(1+bq^{-m/2})$ by the previous paragraph. Consider the function $\phi : \mathcal{U}_0(W) \to \mathcal{U}(\bar{W})$ defined by $\phi(X) = (X+T(W))/T(W)$. For each $Z/T(W) \in \mathcal{U}(\bar{W})$, we have $\phi(X) = Z/T(W)$ if and only if X is a complement to $T(W)$ in Z. There are q^t such complements, and the claim follows.

Since $m = n-d-t$, we have $aq^{t+m-1} = aq^{n-1}q^{-d}$ and $aq^{t+m-1}\cdot bq^{-m/2} = aq^{n-1}\cdot bq^{-(n+d-t)/2}$. Therefore the claim implies that

$$aq^{n-1}(q^{-d} - bq^{-(n+d-t)/2}) \le |\mathcal{U}_0(W)| \le aq^{n-1}(q^{-d} + bq^{-(n+d-t)/2}).$$

Because $t \le \dim W^{\perp} = d$, it follows that

$$aq^{n-1}(q^{-d} - bq^{-n/2}) \le |\mathcal{U}_0(W)| \le aq^{n-1}(q^{-d} + bq^{-n/2}).$$

Setting $\alpha = q^{-d}$ and $r = bq^{-n/2}$, we have, by 4.7,

$$\frac{\alpha - r}{1+r} \le \frac{|\mathcal{U}_0(W)|}{|\mathcal{U}(V)|} \le \frac{\alpha + r}{1+r}.$$

Assertion (4.9) now follows from (3.5). □

4.10. *Assume that $bq^{-n/2} < 1/2$. Let $D_i = 1/(1-q^{-i})$. If $m = \dim X \le n/2$, then*

$$\frac{|\mathcal{U}(X)|}{|\mathcal{U}(V)|} < \begin{cases} 4D_1 q^{m-n}, & U \text{ non-degenerate}, V \text{ orthogonal} \\ 2D_{1/2} q^{m-n}, & U \text{ non-degenerate}, V \text{ unitary} \\ 2D_1 q^{m-n}, & U \text{ nonsingular}, V \text{ orthogonal} \\ 2bq^{m-n}, & U \text{ totally singular} \end{cases}$$

Proof. Set $r = bq^{-n/2}$. By (4.1), $|\mathcal{U}(X)| \le |\mathcal{L}(X)| = \dfrac{q^m - 1}{q-1} < D_1 q^{m-1}$. Since $|\mathcal{U}(V)| \ge aq^{n-1}(1-r)$ it follows that $\dfrac{|\mathcal{U}(X)|}{|\mathcal{U}(V)|} < \dfrac{D_1}{a(1-r)} q^{m-n} < \dfrac{2D_1}{a} q^{m-n}$ because $r < 1/2$. Since

$$\frac{D_1}{a} = \begin{cases} 2D_1, & U \text{ non-degenerate}, V \text{ orthogonal} \\ D_{1/2}, & U \text{ non-degenerate}, V \text{ unitary} \\ D_1, & U \text{ nonsingular}, V \text{ orthogonal} \\ b, & U \text{ totally singular} \end{cases}$$

the case assertions follow. □

We now turn to the proof of Proposition 4.6. Suppose g has no eigenspace of co-dimension smaller than $n/2$. Then by (4.2) there exist subspaces W_1, W_2, W_3, each of dimension at most $n/2$, such that $|\mathcal{U}(V)_g| \le \sum_i |\mathcal{U}(W_i)|$. If U is non-degenerate, then by (4.10) and (4.4) we have, similarly to the proof of Proposition 4.5,

$$\sum |\mathcal{N}(W_i, U)| < \begin{cases} 8D_1 q^{-n/2} |\mathcal{N}(V,U)|, & V \text{ orthogonal} \\ 4D_{1/2} q^{-n/2} |\mathcal{N}(V,U)|, & V \text{ unitary.} \end{cases}$$

If V is orthogonal and U is non-degenerate, then q is odd. Therefore $D_1 \le 3/2$, and $|\mathcal{N}(V,U)_g| < 12q^{-n/2}|\mathcal{N}(V,U)|$. If V is unitary, then $D_{1/2} \le 2$ because $q \ge 4$. We have $|\mathcal{N}(V,U)_g| < 8q^{-n/2}|\mathcal{N}(V,U)|$.

If V is non-degenerate orthogonal and q is even, then at most one eigenspace can contain nonsingular points, so $|\mathcal{P}(V)_g| < 2D_1 q^{-n/2}|\mathcal{P}(V)| \le 4q^{-n/2}|\mathcal{P}(V)|$.

For totally singular subspaces, we have

$$\sum |\mathcal{S}(W_i, 1)| < 4bq^{-n/2}|\mathcal{S}(V,1)|.$$

This establishes the (a) conclusions for the case that g has no eigenspace of dimension at least $n/2$.

Now suppose g has an eigenspace W of co-dimension d, where $d < n/2$. Then there is a space X, namely the sum of the other eigenspaces, with $\dim X \le n/2$, such that $|\mathcal{U}(W)| \le |\mathcal{U}(V)_g| \le |\mathcal{U}(W)| + |\mathcal{U}(X)|$.

When U is non-degenerate, $\mathcal{U}(W) = \mathcal{N}(W,U) = \mathcal{U}_0(W)$. If V is orthogonal, then $|\mathcal{U}(X)| < 4D_1 q^{-n/2} < 4q^{-(n-1)/2}$ by (4.10) because $q \geq 3$. If V is unitary, then $|\mathcal{U}(X)| \leq 2D_{1/2} \leq 4q^{-n/2}$ because $q \geq 4$. The (b) assertions for the non-degenerate cases now follow from (4.9).

When U is orthogonal, nonsingular in characteristic 2, $\mathcal{U}(V)_g = \mathcal{U}(W) = \mathcal{P}(W) = \mathcal{U}_0(W)$, and the (b) assertion follows from (4.9).

If U is totally singular, then $\mathcal{U}(W) = \mathcal{U}_0(W) \cup \mathcal{U}(T(W))$. Since $\dim T(W) \leq n/2$, the conclusions follow from (4.9) and (4.10) in this case.

5. Large eigenspaces and primary fixed points

We provide both upper and lower bounds here for the number of k-spaces contained in a large subspace W of V, where 'large' means $\dim W > \dim V/2$. When V is non-degenerate, we require the k-subspaces to be disjoint from rad W. Recall that $\mathcal{S}_0(W,k)$ is the collection of totally singular k-subspaces $U \subseteq W$ such that $U \cap \operatorname{rad} W = 0$.

Proposition 5.1. *Assume that W has codimension d in V. Then*

1.
$$q^{-dk} - q^{-n} < \frac{|\mathcal{L}(W,k)|}{|\mathcal{L}(V,k)|} \leq q^{-dk}.$$

2. *If V is non-degenerate, $2 \leq k \leq n/2$, and $0 < d < n/2$, then*
$$q^{-dk} - 3q^{-(n-1)/2} < \frac{|\mathcal{N}(W,U)|}{|\mathcal{N}(V,U)|} < q^{-dk} + 9q^{-(n-1)/2}.$$

3. *If V is non-degenerate, $2 \leq k \leq n/2$, and $0 < d < n/2$, then*
$$q^{-dk} - 3q^{-(n-2)/2} < \frac{|\mathcal{S}_0(W,k)|}{|\mathcal{S}(V,k)|} < q^{-dk} + 9q^{-(n-2)/2}.$$

Proof. It follows from (2.5) that
$$\frac{|\mathcal{L}(W,k)|}{|\mathcal{L}(V,k)|} = \frac{\gamma_1(n-d,k)}{\gamma_1(n,k)} q^{-dk} = \frac{\beta_{\mathbf{L}}(n-d)\beta_{\mathbf{L}}(n-k)}{\beta_{\mathbf{L}}(n-d-k)\beta_{\mathbf{L}}(n)} q^{-dk}.$$

It follows from (2.7) and (2.14) that if V is non-degenerate symplectic ($\mathbf{X} = \mathbf{S}$) or unitary ($\mathbf{X} = \mathbf{U}$) and $T(W)$ has dimension t, then
$$\frac{|\mathcal{N}(W,U)|}{|\mathcal{N}(V,U)|} = \frac{\gamma_2(n-d-t,k)}{\gamma_2(n,k)} q^{-dk} = \frac{\beta_{\mathbf{X}}(n-d-t)\beta_{\mathbf{X}}(n-k)}{\beta_{\mathbf{X}}(n-d-t-k)\beta_{\mathbf{X}}(n)} q^{-dk},$$

If V is non-degenerate orthogonal of type ϵ, $T(W)$ has dimension t, and W has type η then, for appropriate θ, ξ,
$$\frac{|\mathcal{N}(W,U)|}{|\mathcal{N}(V,U)|} = \frac{\gamma_2(n-d-t,k,\eta,\delta)}{\gamma_2(n,k,\epsilon,\delta)} q^{-dk} = \frac{\beta_{\mathbf{O}^\eta}(n-d-t)\beta_{\mathbf{O}^\xi}(n-k)}{\beta_{\mathbf{O}^\theta}(n-d-t-k)\beta_{\mathbf{O}^\epsilon}(n)} q^{-dk}.$$

By (2.9) and (2.14), if V is non-degenerate, then
$$\frac{|\mathcal{S}_0(W,k)|}{|\mathcal{S}(V,k)|} = \frac{\gamma_3(n-d-t,k)}{\gamma_3(n,k)} q^{-dk} = \frac{\beta_{\mathbf{X}'}(n-d-t)\beta_{\mathbf{X}}(n-2k)}{\beta_{\mathbf{X}'}(n-d-t-2k)\beta_{\mathbf{X}}(n)} q^{-dk},$$
where $\mathbf{X}$ and $\mathbf{X}'$ are $\mathbf{S}$, $\mathbf{O}^\epsilon$, or $\mathbf{U}$.

To estimate $\gamma_1(n-d,k)/\gamma_1(n,k)$ and $\gamma_i(n-d-t,k)/\gamma_i(n,k)$, $i = 2, 3$, we will use the following result.

5.2. *If $b < a$ then*

1.
$$(1-q^{-(b+1)})^{a-b} \le \frac{\beta_{\mathbf{L}}(a)}{\beta_{\mathbf{L}}(b)} \le (1-q^{-a})^{a-b}.$$
2.
$$(1-q^{-(b+2)})^{(a-b)/2} \le \frac{\beta_{\mathbf{S}}(a)}{\beta_{\mathbf{S}}(b)} \le (1-q^{-a})^{(a-b)/2}.$$
3.
$$\frac{1-q^{-b/2}}{1+q^{-a/2}}(1-q^{-(b+1)})^{(a-b)/2} \le \frac{\beta_{\mathbf{O}^{\eta}}(a)}{\beta_{\mathbf{O}^{\theta}}(b)} \le \frac{1+q^{-b/2}}{1-q^{-a/2}}(1-q^{-(a+1)})^{(a-b)/2}.$$
4.
$$1-q^{-(b+1)/2} \le \frac{\beta_{\mathbf{U}}(a)}{\beta_{\mathbf{U}}(b)} \le 1+q^{-(b+1)/2}.$$

Proof. Set $h = a-b$.

It follows from the definitions that $\frac{\beta_{\mathbf{L}}(a)}{\beta_{\mathbf{L}}(b)} = \prod_{i=1}^{h}(1-q^{-(b+i)})$. The first assertion follows easily from this because $1-q^{-(b+1)} \le 1-q^{-(b+i)} \le 1-q^{-a}$ for $1 \le i \le h$.

Similarly, we have $\frac{\beta_{\mathbf{S}}(a)}{\beta_{\mathbf{S}}(b)} = \prod_{i=1}^{h/2}(1-q^{-(b+2i)})$, and the second assertion holds.

In the unitary case, we have $\frac{\beta_{\mathbf{U}}(a)}{\beta_{\mathbf{U}}(b)} = \prod_{i=1}^{h}(1-(-q^{-1/2})^{(b+i)})$. Since $q^{-1/2} \le 1/2$, it follows from (3.2) that this product lies between $1-q^{-(b+1)/2}$ and $1+q^{-(b+1)/2}$. This establishes the assertion for the unitary ratios.

To prove the statement about the orthogonal ratios, it suffices to prove the following result.

5.3. *Define* $\xi(m) = \begin{cases} \beta_{\mathbf{S}}(m), & m \textit{ even};\\ \beta_{\mathbf{S}}(m-1)(1-q^{-(m+1)})^{1/2}, & m \textit{ odd}.\end{cases}$

Then

1. $\frac{1}{1+q^{-m/2}}\xi(m) \le \beta_{\mathbf{O}^{\epsilon}}(m) \le \frac{1}{1-q^{-m/2}}\xi(m)$ *for all* $m>0$.
2. $(1-q^{-(b+1)})^{(a-b)/2} \le \frac{\xi(a)}{\xi(b)} \le (1-q^{-(a+1)})^{(a-b)/2}$.

Proof of (5.3): The first statement follows from the definition of $\beta_{\mathbf{O}^{\epsilon}}(m)$ if m is even. If m is odd, then $\beta_{\mathbf{O}^{\epsilon}}(m) = \beta_{\mathbf{S}}(m-1) = \frac{\xi(m)}{(1-q^{-1}r^2)^{1/2}}$ where $r = q^{-m/2}$. This implies that $\xi(m) < \beta_{\mathbf{O}^{\epsilon}}(m)$. Since $(1-r)^2 < 1-r < 1-q^{-1}r < 1-q^{-1}r^2$, we have $\frac{\xi(m)}{(1-q^{-1}r^2)^{1/2}} < \frac{\xi(m)}{1-r}$, and (1) follows.

If m is even then $\frac{\xi(m+1)}{\xi(m)} = (1-q^{-(m+2)})^{1/2}$. If m is odd, then $\frac{\xi(m+1)}{\xi(m)} = \frac{\beta_{\mathbf{S}}(m+1)}{\beta_{\mathbf{S}}(m-1)(1-q^{-(m+1)})^{1/2}} = (1-q^{-(m+1)})^{1/2}$. This implies that (2) holds if $a-b=1$. The general case follows by induction on $a-b$. This completes the proof of (5.2) and (5.3).

5.4. *Assume that x and y are positive integers with $x+y \le n$*

1. $1 - xq^{-(n-x-y+1)} < \dfrac{\beta_{\mathbf{L}}(n-x)\beta_{\mathbf{L}}(n-y)}{\beta_{\mathbf{L}}(n-x-y)\beta_{\mathbf{L}}(n)} < 1.$
2. *If x, y, and n are even, then* $1 - \dfrac{x}{2}q^{-(n-x-y+2)} < \dfrac{\beta_{\mathbf{S}}(n-x)\beta_{\mathbf{S}}(n-y)}{\beta_{\mathbf{S}}(n-x-y)\beta_{\mathbf{S}}(n)} < 1.$
3. $1 - 2q^{-(n-x-y+1)/2} < \dfrac{\beta_{\mathbf{U}}(n-x)\beta_{\mathbf{U}}(n-y)}{\beta_{\mathbf{U}}(n-x-y)\beta_{\mathbf{U}}(n)} < 1 + 3q^{-(n-x-y+1)/2}.$
4. *If $x+y=n$, assume that $\theta = +$. Then*
$$1 - (\frac{9}{4} + \frac{x}{4})q^{-(n-x-y)/2} < \frac{\beta_{\mathbf{O}^\eta}(n-x)\beta_{\mathbf{O}^\xi}(n-y)}{\beta_{\mathbf{O}^\theta}(n-x-y)\beta_{\mathbf{O}^\epsilon}(n)} < 1 + 9q^{-(n-x-y)/2}.$$

Proof. To prove the first statement, it suffices to assume that $x \le y$. By (5.2) we have
$$\frac{\beta_{\mathbf{L}}(n-x)\beta_{\mathbf{L}}(n-y)}{\beta_{\mathbf{L}}(n-x-y)\beta_{\mathbf{L}}(n)} = \frac{\beta_{\mathbf{L}}(n-y)}{\beta_{\mathbf{L}}(n-y-x)}\frac{\beta_{\mathbf{L}}(n-x)}{\beta_{\mathbf{L}}(n)} \le \frac{(1-q^{-(n-y)})^x}{(1-q^{-(n-x+1)})^x} < 1$$
because $n - y < n - x + 1$.

Also,
$$\frac{\beta_{\mathbf{L}}(n-y)}{\beta_{\mathbf{L}}(n-y-x)}\frac{\beta_{\mathbf{L}}(n-x)}{\beta_{\mathbf{L}}(n)} \ge \left(\frac{1-q^{-(n-x-y+1)}}{1-q^{-n}}\right)^x$$
$$\ge (1-q^{-(n-x-y+1)})^x > 1 - xq^{-(n-x-y+1)}$$
by (3.3).(3).

The argument for the second statement is completely similar.

To prove the third statement, set $r = q^{-(n-x-y+1)/2}$ and $s = q^{-(n-y+1)/2}$. Statement (5.2) implies that
$$\frac{1-r}{1+s} \le \frac{\beta_{\mathbf{U}}(n-x)}{\beta_{\mathbf{U}}(n-x-y)}\frac{\beta_{\mathbf{U}}(n-y)}{\beta_{\mathbf{U}}(n)} \le \frac{1+r}{1-s}.$$

Noting that $q \ge 4$ in the unitary case, we have $s \le q^{-1/2}r \le r/2$, so $\dfrac{1+r}{1-s} \le (1+r)^2 < 1 + 3r$ by (3.4) and $\dfrac{1-r}{1+s} > (1-r)(1-s) > 1 - 2r$ by (3.3) and the third statement follows.

For the fourth statement, assume that $x \le y$. We first consider the case $x = 1$. By (5.5), proved below, we have
$$\frac{1-q^{-(n-1-y)/2}}{1+q^{-(n-1)/2}} \le \frac{\beta_{\mathbf{O}^\eta}(n-1)}{\beta_{\mathbf{O}^\epsilon}(n)}\frac{\beta_{\mathbf{O}^\xi}(n-y)}{\beta_{\mathbf{O}^\theta}(n-1-y)} \le \frac{1+q^{-(n-1-y)/2}}{1-q^{-(n-1)/2}}.$$
It follows from (3.3) that
$$\frac{1-q^{-(n-1-y)/2}}{1+q^{-(n-1)/2}} > (1-q^{-(n-1-y)/2})(1-q^{-(n-1)/2}) > 1 - 2q^{-(n-1-y)/2}.$$
It follows from (3.4) that
$$\frac{1+q^{-(n-1-y)/2}}{1-q^{-(n-1)/2}} < (1+q^{-(n-1-y)/2})(1+2q^{-(n-1)/2}) \le 1 + 5q^{-(n-1-y)/2}.$$

Assume now that $x \ge 2$ and set $r = q^{-(n-x-y)/2}$, $s = q^{-(n-x)/2}$, $t = q^{-(n-y)/2}$, and $u = q^{-n/2}$. Note that $s \le r/2$, $t \le r/2$, and $u \le r/4$ because $y \ge x$.

Set $\zeta(n,x,y) = \dfrac{\xi(n-x)\xi(n-y)}{\xi(n-x-y)\xi(n)}$. By 5.3.(1), we have

$$\frac{(1-r)(1-u)}{(1+s)(1+t)}\zeta(n,x,y) \le \frac{\beta_{\mathbf{O}^\eta}(n-x)\beta_{\mathbf{O}^\xi}(n-y)}{\beta_{\mathbf{O}^\theta}(n-x-y)\beta_{\mathbf{O}^\epsilon}(n)} \le \frac{(1+r)(1+u)}{(1-s)(1-t)}\zeta(n,x,y).$$

By 5.3.(2) with $a = n-y$ and $b = n-x-y$ we have

$$(1-q^{-(n-x-y+1)})^{x/2} \le \frac{\xi(n-y)}{\xi(n-x-y)} \le (1-q^{-(n-y+1)})^{x/2}.$$

By the same result with $a = n$ and $b = n-x$ we have

$$(1-q^{-(n-x+1)})^{x/2} \le \frac{\xi(n)}{\xi(n-x)} \le (1-q^{-(n+1)})^{x/2}.$$

This implies that

$$\begin{aligned}(1-q^{-(n-x-y+1)})^{x/2} &< \frac{(1-q^{-(n-x-y+1)})^{x/2}}{(1-q^{-(n+1)})^{x/2}}\\ &\le \zeta(n,x,y)\\ &\le \frac{(1-q^{n-y+1})^{x/2}}{(1-q^{-(n-x+1)})^{x/2}}\\ &\le 1.\end{aligned}$$

Since $q^{-(n-x-y+1)} = q^{-1}r^2$, we have

$$\begin{aligned}\zeta(n,x,y) &\ge (1-q^{-(n-x-y+1)})^{x/2}\\ &= (1-q^{-1}r^2)^{x/2}\\ &> 1-\frac{xr}{2q}r \ge 1-\frac{x}{4}r\end{aligned}$$

by (3.3).

We have

$$\begin{aligned}\frac{(1-r)(1-u)}{(1+s)(1+t)} &> (1-r)(1-u)(1-s)(1-t)\\ &\ge (1-r)(1-\frac{r}{4})(1-\frac{r}{2})^2\\ &\ge 1-\frac{9}{4}r\end{aligned}$$

by (3.3). Therefore

$$\frac{\beta_{\mathbf{O}^\eta}(n-x)\beta_{\mathbf{O}^\xi}(n-y)}{\beta_{\mathbf{O}^\theta}(n-x-y)\beta_{\mathbf{O}^\epsilon}(n)} > (1-\frac{9}{4}r)(1-\frac{x}{4}r) > (1-\frac{9+x}{4}r).$$

We also have, by (3.4), $\dfrac{1}{1-s} \le 1+r$ and $\dfrac{1}{1-t} \le 1+r$. Therefore

$$\frac{(1+r)(1+u)}{(1-s)(1-t)} \le (1+r)^3(1+r/4) \le (1+7r)(1+r/4) \le 1+9r$$

by (3.4).

This completes the proof of (5.4) subject to establishing (5.5). □

5.5. $1-q^{-m/2} \le \dfrac{\beta_{\mathbf{O}^\epsilon}(m+1)}{\beta_{\mathbf{O}^\delta}(m)} \le 1+q^{-m/2}$.

Proof. It follows directly from the definitions that if m is odd, then

$$\frac{\beta_{\mathbf{O}^\epsilon}(m+1)}{\beta_{\mathbf{O}^\delta}(m)} = \frac{\beta_{\mathbf{S}}(m+1)}{(1+\epsilon q^{-(m+1)/2})\beta_{\mathbf{S}}(m-1)} = 1-\epsilon q^{-(m+1)/2},$$

while if m is even, then

$$\frac{\beta_{\mathbf{O}^\epsilon}(m+1)}{\beta_{\mathbf{O}^\delta}(m)} = \frac{\beta_{\mathbf{S}}(m)(1+\delta q^{-m/2})}{\beta_{\mathbf{S}}(m)} = 1+\delta q^{-m/2}.$$

□

The following result is a consequence of (5.4) and the definitions of γ_i.

5.6. *Let d and k be positive integers such that $d \le n/2$ and $k \le n/2$.*

1. *If $h = \min(d,k)$, then*
$$1 - hq^{d+k}q^{-(n+1)} < \frac{\gamma_1(n-d,k)}{\gamma_1(n,k)} < 1.$$
2. *Suppose V is non-degenerate symplectic, $t \le d$, and $d+t+k \le n$.*
   (a) *If $h = \min(2d,k)$, and k is even, then*
$$1 - \frac{h}{2}q^{2d+k}q^{-(n+2)} < \frac{\gamma_2(n-d-t,k)}{\gamma_2(n,k)} < 1.$$
   (b) *If $h = \min(d,k)$, then*
$$1 - hq^{2d+2k}q^{-(n+2)} < \frac{\gamma_3(n-d-t,k)}{\gamma_3(n,k)} < 1.$$
3. *If V is non-degenerate unitary, $t \le d$, and $d+t+k \le n$, then*
   (a)
$$1 - 2q^{(2d+k)/2}q^{-(n+1)/2} < \frac{\gamma_2(n-d-t,k)}{\gamma_2(n,k)} < 1 + 3q^{(2d+k)/2}q^{-(n+1)/2}.$$
   (b)
$$1 - 2q^{d+k}q^{-(n+1)/2} < \frac{\gamma_3(n-d-t,k)}{\gamma_3(n,k)} < 1 + 3q^{d+k}q^{-(n+1)/2}.$$
4. *Suppose V is non-degenerate orthogonal, $t \le d$, and $d+t+k \le n$.*
   (a) *If $h = \min(2d,k)$, then*
$$1 - (\frac{9}{4}+\frac{h}{4})q^{(2d+k)/2}q^{-n/2} < \frac{\gamma_2(n-d-t,k,\eta,\delta)}{\gamma_2(n,k,\epsilon,\delta)} < 1 + 9q^{(2d+k)/2}q^{-n/2}.$$
   (b) *If $h = \min(d,k)$, then*
$$1 - (\frac{9}{4}+\frac{h}{4})q^{d+k}q^{-n/2} < \frac{\gamma_3(n-d-t,k)}{\gamma_3(n,k)} < 1 + 9q^{d+k}q^{-n/2}.$$

Proof. For the first statement, use $x = \min(d,k)$, $y = \max(d,k)$. For the first parts of the remaining statements, use $x = \min(d+t,k)$, $y = \max(d+t,k)$; for the second parts, use $x = \min(d+t,2k)$, $y = \max(d+t,2k)$. In all cases, $d+t \le 2d$ and $x \le h$. □

5.7. *Assume that $1 \le d \le n/2$ and $1 \le k \le n/2$.*

1. *If $h = \min(2d,k)$, then $hq^{2d+k-dk} \le q^{1+n/2}$.*
2. *If $h = \min(d,k)$, then $hq^{2d+2k-dk} \le q^{2+n/2}$.*
3. *If $h = \min(2d,k)$, then $hq^{d+k/2-dk} \le 2q^{1/2}$.*

4. $q^{d+k/2-dk} \le q^{1/2}$.
5. *If* $h = \min(d,k)$, *then* $hq^{d+k-dk} \le q$.
6. $q^{d+k-dk} \le q$.
7. *If* $h = \min(d,k)$, *then* $hq^{d+2k-dk} \le q^{1+n/2}$.

Proof. We first establish two claims concerning functions ξ and ζ that depend on parameters a, b, c that can be specialized to the cases of the statement.

Set $\xi(x,y) = \min(x,y)q^{ax+by-cxy}$, and $\zeta(z) = \xi(z,z)$.

Claim 1: If $cx \ge b$ and $cy \ge a$, then either $\xi(x,y) \le \zeta(x)$ or $\xi(x,y) \le \zeta(y)$. To establish the claim, we may assume by symmetry, that $x \ge y$. Then $\xi(x,y) = yq^{ax+by-cxy} = yq^{by+(a-cy)x}$, so $\partial\xi/\partial x \le 0$, and $\xi(x,y) \le \xi(y,y) = \zeta(y)$.

Claim 2: $\zeta(z)$ decreases for $z > (a+b+2)/2c$. This is so because if $z \ge 1$ then $z\log q \ge \log 2 > 1/2$, so that $\zeta'(z) = \zeta(z)(a+b-2cz+1/(z\log q))\log q < \zeta(z)(a+b-2cz+2)\log q$. This shows that $\zeta'(z) < 0$ for $z > (a+b+2)/2c$.

(1): If $k = 1$, then $hq^{2d+k-dk} = q^{d+1} \le q^{1+n/2}$. If $k \ge 2$, then $n \ge 4$. Setting $x = 2d$, $y = k$, $a = b = 1$, and $c = 1/2$, we have $\xi(x,y) = hq^{2d+k-dk}$. Claim 1 implies that $hq^{2d+k-dk} \le \zeta(z) = zq^{2z-z^2/2}$, for some z. Claim 2 implies that $\zeta(z)$ decreases for $z > 4$. We have $\zeta(q) = q^{3/2}$, $\zeta(2) = 2q^2$, $\zeta(3) = 3q^{3/2}$, and $\zeta(4) = 4$. When $n > 4$ each of these is less than or equal to $q^{n/2+1}$. If $n = 4$, the inequality is easily verified. This establishes (1).

(2): Assume by symmetry that $k \le d$. If $k = 1$, then $hq^{2d+2k-dk} = q^{d+2} \le q^{2+n/2}$. If $k \ge 2$, then $n \ge 4$. Setting $x = d$, $y = k$, $a = b = 2$, and $c = 1$, we have $\xi(x,y) = hq^{2d+2k-dk}$. Claim 1 implies that $hq^{2d+2k-dk} \le \zeta(z) = zq^{4z-z^2}$ for some $z \ge 2$. Claim 2 implies that $\zeta(z)$ decreases for $z > 3$. We have $\zeta(2) = 2q^4$ and $\zeta(3) = 3q^3$. Since $n \ge 4$, it follows that each of these expressions is bounded by $2q^{2+n/2}$. This establishes (2).

(3): If $k = 1$, then $hq^{d+k/2-dk} = q^{1/2}$, so assume that $k \ge 2$. Set $x = 2d$, $y = k$, $a = 1/2$, $b = 1/2$, and $c = 1$. Then $\xi(x,y) = hq^{d+k/2-dk}$ and $\zeta(z) = zq^{z-z^2/2}$. Claim 1 implies that $hq^{d+k/2-dk} \le \zeta(z)$ for some z. Claim 2 implies that $\zeta(z)$ decreases for $z > 3$. We have $\zeta(1) = q^{1/2}$, $\zeta(2) = 2$, and $\zeta(3) = 3q^{-3/2}$, so $hq^{d+k/2-dk} \le 2q^{1/2}$. The inequality (3) holds in all cases.

To prove (4) note that $d + k/2 - dk = 1/2 - (k-1)(d-1/2) \le 1/2$.

(5): Set $x = d$, $y = k$, and $a = b = c = 1$. We have $\xi(x,y) = hq^{d+k-dk}$, and $\zeta(z) = zq^{2z-z^2}$. Claim 1 implies that $hq^{d+k-dk} \le \zeta(z)$ for some z. Claim 2 implies that $\zeta(z)$ decreases for $z > 2$. We have $\zeta(1) = q$ and $\zeta(2) = 2$. In either case, $\zeta(z) \le q$. This establishes (5), and (6) follows immediately.

(7): If $d = 1$, then $hq^{d+2k-dk} = q^{k+1} \le q^{1+n/2}$, so assume that $d \ge 2$, whence $n \ge 4$. Setting $x = d$, $y = k$, $a = 1$, $b = 2$, and $c = 1$, we have $\xi(x,y) = hq^{d+2k-dk}$ and $\zeta(z) = zq^{3z-z^2}$. By Claim 1, $\xi(x,y) \le \zeta(z)$ for some $z \ge 2$. By Claim 2, $\zeta(z)$ decreases for $z > 2.5$. We have $\zeta(1) = q^2$, $\zeta(2) = 2q^2$, and $\zeta(3) = 3$. Since $n \ge 4$, all of these values are bounded by $q^{1+n/2}$, so the conclusion holds in all cases. □

We can now complete the proof of Proposition 5.1.

In the linear case, by (5.6.1) and (5.7.5), we have

$$q^{-dk} - q \cdot q^{-(n+1)} < \frac{\gamma_1(n-d,k)}{\gamma_1(n,k)} q^{-dk} < q^{-dk}.$$

In the symplectic case, by (5.6.2) and (5.7.1) we have

$$q^{-dk} - \frac{1}{2}q^{1+n/2}q^{-(n+2)} < \frac{\gamma_2(n-d-t,k)}{\gamma_2(n,k)}q^{-dk} < q^{-dk}$$

and, by (5.6.2) and (5.7.2),

$$q^{-dk} - q^{2+n/2}q^{-(n+2)} < \frac{\gamma_3(n-d-t,k)}{\gamma_3(n,k)}q^{-dk} < q^{-dk}.$$

In the unitary case, by (5.6.3) and (5.7.4) we have

$$q^{-dk} - 2q^{1/2}q^{-(n+1)/2} < \frac{\gamma_2(n-d-t,k)}{\gamma_2(n,k)}q^{-dk} < q^{-dk} + 3q^{1/2}q^{-(n+1)/2}$$

and, by (5.6.3) and (5.7.6),

$$q^{-dk} - 2q \cdot q^{-(n+1)/2} < \frac{\gamma_3(n-d-t,k)}{\gamma_3(n,k)}q^{-dk} < q^{-dk} + 3q \cdot q^{-(n+1)/2}.$$

In the orthogonal case, by (5.6.4), (5.7.3), and (5.7.4)

$$q^{-dk} - (\frac{9}{4} + \frac{2}{4})q^{1/2} \cdot q^{-n/2} < \frac{\gamma_2(n-d-t,k,\eta,\delta)}{\gamma_2(n,k,\epsilon,\delta)}q^{-dk} < q^{-dk} + 9q^{1/2} \cdot q^{-n/2}.$$

and, by (5.6.4), (5.7.5), and (5.7.6),

$$q^{-dk} - (\frac{9}{4} + \frac{1}{4})q \cdot q^{-n/2} < \frac{\gamma_3(n-d-t,k,\eta,\delta)}{\gamma_3(n,k,\epsilon,\delta)}q^{-dk} < q^{-dk} + 9q \cdot q^{-n/2}.$$

The assertions in Proposition 5.1 follow in all cases.

6. Linear subspaces

In this section, V is a linear space of dimension n, and g is a linear transformation on V.

Lemma 6.1. *One of the following holds.*

1. *There is a polynomial ϕ over $\mathbf{F}$ such that $\frac{n}{3} \leq \dim\ker\phi(g) \leq \frac{2n}{3}$.*
2. *g has an eigenspace W such that codim $W < \frac{n}{3}$.*
3. *V contains a subspace W of codimension less than $n/3$ such that g acts homogeneously on W with blocks of size at least 2, and $W = \ker\phi(g)$ for some polynomial ϕ.*

Proof. Let μ be the minimal polynomial for g on V. Suppose μ has an irreducible factor ϕ such that $\dim\ker\phi(g) \geq n/3$. If $\dim\ker\phi(g) \leq 2n/3$, then (1) holds. If $\dim\ker\phi(g) > 2n/3$, then either (2) or (3) holds for $W = \ker\phi(g)$, depending on whether ϕ is linear or of degree at least 2.

We may therefore assume that $\dim(\ker\tau(g)) < n/3$ for every irreducible factor τ of μ. Choose polynomials $\{\mu_i\}$ such that $\mu_0 = 1$, $\mu_m = \mu$, and $\tau_i = \mu_i/\mu_{i-1}$ is irreducible for $i = 1, \ldots, m$. Set $W_i = \ker\mu_i(g)$. Then $W_i\tau_i(g) \subseteq W_{i-1}$, whence $\dim W_i \leq \dim W_{i-1} + \dim\ker\tau_i(g) < \dim W_{i-1} + n/3$. As i goes from 0 to m the sequence $\langle \dim W_i \rangle$ increases from 0 to n with each step smaller than $n/3$. It follows that there is an i for which $n/3 \leq \dim W_i \leq 2n/3$. Therefore condition (1) holds.

Lemma 6.2. *Suppose ϕ is a polynomial over $\mathbf{F}$, $W = \ker\phi(g)$, and k is a positive integer.*

1. *For $i \le k$, let $\mathcal{L}_i = \{U \in \mathcal{L}(V,k)_g \,:\, \dim U\rho = i\}$.*
$$|\mathcal{L}_i| \le |\mathcal{L}(V/W,i)|\;|\mathcal{L}(W,k-i)|.$$
2. $|\mathcal{L}(V,k)_g| \le |\mathcal{L}(W,k)_g| + \sum_i |\mathcal{L}(V/W,i)|\;|\mathcal{L}(W,k-i)|$ *where every i in the sum satisfies $1 \le i \le k$ and $i \le$ codim W.*

Proof. Set $\rho = \phi(g)$ and $X = V\rho$. If U is a g-invariant k-space and $U \cap W$ is a $k-i$ subspace of W, then $U\rho$ is an i-subspace of X.

Given $Y \in \mathcal{L}(X,i)$, let Y^* be the complete inverse image of Y under ρ. If $U \in \mathcal{L}(V,k)_g$ and $U\rho = Y$, then $U/Y \in \mathcal{L}(Y^*/Y, k-i)$. Therefore
$$|\mathcal{L}_i| \le \sum_{Y \in \mathcal{L}(X,i)} |\mathcal{L}(Y^*/Y,k-i)|.$$
Because $\dim Y^* = \dim Y + \dim W$, we have $|\mathcal{L}(Y^*/Y,k-i)| = |\mathcal{L}(W,k-i)|$, and $|\mathcal{L}_i| \le |\mathcal{L}(X,i)|\;|\mathcal{L}(W,k-i)|$. The first statement holds because $\dim X = \dim V/W$. The second statement is an immediate consequence.

6.3. *Suppose W is a subspace of V and $2 \le k \le n/2$. If $\dim W \ge n/2$, then*
$$\sum_{0<i\le k,\ i\le codim\ W} |\mathcal{L}(W,k-i)|\;|\mathcal{L}(V/W,i)| < 3C_1 q^{-n/2}|\mathcal{L}(V,k)|.$$

Proof. Set $d =$ codim W and $x_i = \dfrac{|\mathcal{L}(W,k-i)|\;|\mathcal{L}(V/W,i)|}{|\mathcal{L}(V,k)|}$.

If $x_i \ne 0$ then $0 \le i \le d$ and $0 \le k-i \le n-d$. It follows from (2.5) that $x_i = y_i|\mathcal{L}(V/W,i)|q^{(n-d-k+i)(k-i)-(n-k)k}$ where
$$y_i = \frac{\beta_{\mathbf{L}}(n-d)\beta_{\mathbf{L}}(n-k)\beta_{\mathbf{L}}(k)}{\beta_{\mathbf{L}}(n)\beta_{\mathbf{L}}(n-d-k+i)\beta_{\mathbf{L}}(k-i)}.$$

Statement (5.4) shows that if $r+s = t+z$, $r \le t$, and $s \le t$, then $\dfrac{\beta_{\mathbf{L}}(r)\beta_{\mathbf{L}}(s)}{\beta_{\mathbf{L}}(t)\beta_{\mathbf{L}}(z)} < 1$. If $k \le n-d-k+i$, then $n-d-k \ge 0$, and we have
$$y_i = \frac{\beta_{\mathbf{L}}(n-d)\beta_{\mathbf{L}}(n-k)}{\beta_{\mathbf{L}}(n)\beta_{\mathbf{L}}(n-d-k)} \cdot \frac{\beta_{\mathbf{L}}(n-d-k)\beta_{\mathbf{L}}(k)}{\beta_{\mathbf{L}}(n-d-k+i)\beta_{\mathbf{L}}(k-i)} < 1.$$
If $k > n-d-k+i$, then $n-k < d+k-i$ and we have
$$y_i = \frac{\beta_{\mathbf{L}}(n-d)\beta_{\mathbf{L}}(d+k-i)}{\beta_{\mathbf{L}}(n)\beta_{\mathbf{L}}(k-i)} \cdot \frac{\beta_{\mathbf{L}}(n-k)\beta_{\mathbf{L}}(k)}{\beta_{\mathbf{L}}(d+k-i)\beta_{\mathbf{L}}(n-d-k+i)} < 1.$$

Because $|\mathcal{L}(V/W,i)| < C_1 q^{(d-i)i}$ it follows that $\dfrac{|\mathcal{L}(W,k-i)|\;|\mathcal{L}(V/W,i)|}{|\mathcal{L}(V,k)|} < C_1 q^{e_1(n,d,k,i)}$ where
$$e_1(n,d,k,i) = (n-d-k+i)(k-i) + (d-i)i - k(n-k).$$
Since $k \le n/2$, we must have $n \ge 4$. By Lemma 11.9,
$$\sum_{1\le i\le \min(d,k)} q^{e_1(n,d,k,i)} < 3q^{-n/2}.$$
Therefore $\displaystyle\sum_{1\le i\le k,\ i\le \text{codim } W} x_i < 3C_1 q^{-n/2}$, as required.

For future reference we record the following consequence of Lemma 6.2 and (6.3).

6.4. *If $W = \ker \phi(g)$ for some polynomial ϕ and $\dim W \geq n/2$ then*

$$|\mathcal{L}(V,k)_g| - |\mathcal{L}(W,k)_g| < 3C_1 q^{-n/2} |\mathcal{L}(V,k)|.$$

More specifically, we have.

6.5. *Suppose condition (1) of Lemma 6.1 holds and $k \geq 2$. Then*

$$|\mathcal{L}(V,k)_g| < (3C_1 + 1) q^{-n/2} |\mathcal{L}(V,k)|.$$

Proof. Set $W = \ker \phi$. By Lemma 6.2, we have

$$|\mathcal{L}(V,k)_g| \leq |\mathcal{L}(W,k)_g| + \sum_{i>0} |\mathcal{L}(V/W,i)|\ |\mathcal{L}(W,k-i)|.$$

Suppose $\dim W = n - d$ where $d \leq n/2$. Then $|\mathcal{L}(W,k)| \leq q^{-dk} |\mathcal{L}(V,k)|$ by Proposition 5.1. It follows from (6.4) that $|\mathcal{L}(V,k)_g| < (q^{-dk} + 3C_1 q^{-n/2}) |\mathcal{L}(V,k)|$. Since $n \geq n/3$ by condition (1) and $k \geq 2$, it follows that $|\mathcal{L}(V,k)_g| < (3C + 1) q^{-n/2} |\mathcal{L}(V,k)|$.

If $\dim W < n/2$, let W' be a subspace of V with $\dim W' = \operatorname{codim} W$. Then

$$\begin{aligned}\sum_{i\geq 0} |\mathcal{L}(V/W,i)|\ |\mathcal{L}(W,k-i)| &= \sum_{i \geq 0} |\mathcal{L}(W',i)|\ |\mathcal{L}(V/W',k-i)| \\ &< (3C_1 + 1) q^{-n/2} |\mathcal{L}(V,k)|\end{aligned}$$

by the previous argument.

6.6. *Suppose W is a subspace of V and g acts homogeneously on W with blocks of size at least 2. If $0 < k \leq n/2$ then*

$$|\mathcal{L}(W,k)_g| \leq 1.5 q^{2-n} |\mathcal{L}(V,k)| \leq 1.5 q^{-n/2} |\mathcal{L}(V,k)|.$$

Proof. Let b be the block size, and set $\mathbf{K} = \mathbf{F}[g]$, the subring of $\operatorname{End}(V)$ generated by g and the scalars in $\mathbf{F}$. Then $\mathbf{K} \cong \mathbf{F}_{q^b}$. Let $\tilde{W}$ denote W viewed as a $\mathbf{K}$-space. Then the g-invariant k-subspaces of W are precisely the k/b-subspaces of $\tilde{W}$. In particular, there are no g-invariant k-subspaces if b k.

Set $m = \dim W$, $m' = m/b$, and $k' = k/b$. By Lemma 2.1 we have $|\mathcal{L}(W,k)_g| = |\mathcal{L}(\tilde{W},k')| \leq \dfrac{q^{2b}}{q^{2b} - q^b - 1} q^{b \cdot k'(m'-k')} \leq \dfrac{16}{11} q^{k(m-k)/b}$ and $q^{k(n-k)} \leq |\mathcal{L}(V,k)|$. Therefore

$$\frac{|\mathcal{L}(W,k)_g|}{|\mathcal{L}(V,k)|} \leq \frac{16}{11} q^{k(m-k)/b - k(n-k)} \leq 1.5 q^{-k(n-k)(b-1)/b}.$$

Since $b \geq 2$, we have $-k(n-k)(b-1)/b \leq -k(n-k)/2$. Because $b \mid k$, we have $2 \leq k$. By assumption, $k \leq n/2$. The function $x \mapsto -x(n-x)$ decreases on the interval $[2, n/2]$. Therefore $-k(n-k)(b-1)/b \leq -2(n-2)/2 = 2-n$. It follows that $|\mathcal{L}(W,k)_g| \leq 1.5 q^{2-n} |\mathcal{L}(V,k)|$. Since $n \geq 4$, we have $2 - n \leq -n/2$, and the last inequality follows.

Proposition 6.7. *If g is a linear transformation on V and $k \leq n/2$ then one of the following holds.*

1. $$\frac{|\mathcal{L}(V,k)_g|}{|\mathcal{L}(V,k)|} \leq (1.5 + 3C_1) q^{-n/2} < 12 q^{-n/2}.$$

2. *g has an eigenspace of codimension d where $dk < n/2$, and*

$$q^{-dk} - q^{-n} \leq \frac{|\mathcal{L}(V,k)_g|}{|\mathcal{L}(V,k)|} < q^{-dk} + 3C_1 q^{-n/2} < q^{-dk} + 11q^{-n/2}.$$

Proof. The second statement holds for avery linear transformation if $k = 0$. If $k = 1$, then the assertion follows from Proposition 4.5. So assume that $k \geq 2$.

If condition (1) of Lemma 6.1 holds, then statement (1) holds by (6.5).

If condition (2) of Lemma 6.1 holds, then $\mathcal{L}(W,k) = \mathcal{L}(W,k)_g$, and statement (2) follows from Proposition 5.1 and (6.4).

If neither condition (1) nor condition (2) of Lemma 6.1 holds, then there is a subspace W with $\dim W > 2n/3$ such that g acts homogeneously on W with blocks of size at least 2 and $W = \ker \phi(g)$ for some polynomial ϕ. By (6.6), $|\mathcal{L}(W,k)_g| < 1.5q^{-n/2}\mathcal{L}(V,k)$. Conclusion (1) now follows from (6.4).

7. Non-degenerate subspaces

In this section, V admits a non-degenerate form, g is a linear transformation acting on V, and U is a non-degenerate k-subspace of V.

7.1. *If g does not have an eigenspace of codimension less than $n/2k$, then*

$$\frac{|\mathcal{N}(V,U)_g|}{|\mathcal{N}(V,U)|} < \frac{C_1}{c_2}\,(1.5 + 3C_1)\, q^{-n/2} < 340q^{-n/2}.$$

Proof. We have $|\mathcal{N}(V,U)| \geq c_2 q^{k(n-k)}$ by Proposition 2.2 and $|\mathcal{L}(V,k)| < C_1 q^{k(n-k)}$ by Lemma 2.1. Therefore $|\mathcal{N}(V,U)| > \dfrac{c_2}{C_1}|\mathcal{L}(V,k)|$. Since $|\mathcal{N}(V,U)_g| \leq |\mathcal{L}(V,k)_g|$ the first inequality now follows from Lemma 6.7. The second inequality follows from the fact that $C_1 \leq 3.5$ and $c_2 \geq 1/8$.

7.2. *If g has an eigenspace W of codimension d, $d < n/2$, then*

$$\begin{aligned} q^{-dk} - 3q^{-(n-1)/2} < \frac{|\mathcal{N}(V,U)_g|}{|\mathcal{N}(V,U)|} &< q^{-dk} + (9 + \frac{3C_1^2}{c_2\sqrt{q}})q^{-(n-1)/2} \\ &\leq q^{-dk} + 220q^{-(n-1)/2}. \end{aligned}$$

Proof. The lower bound for $\dfrac{|\mathcal{N}(V,U)_g|}{|\mathcal{N}(V,U)|}$ follows from Proposition 5.1.

If $\mathcal{N}'$ is the collection of g-invariant subspaces U' isometric to U with $U' \not\subseteq W$, then $|\mathcal{N}'| \leq |\mathcal{L}(V,k)_g \setminus \mathcal{L}(W,k)_g| < 3C_1 q^{-n/2}|\mathcal{L}(V,k)|$ by (6.4).

Because $|\mathcal{L}(V,k)| < (C_1/c_2)|\mathcal{N}(V,U)|$, this shows that

$$\begin{aligned} |\mathcal{N}'| &< \frac{3C_1^2}{c_2} q^{-n/2}|\mathcal{N}(V,U)| \\ &= \frac{3C_1^2}{c_2\sqrt{q}} q^{-(n-1)/2}|\mathcal{N}(V,U)|. \end{aligned}$$

By Proposition 5.1.(2), we have $|\mathcal{N}(W,U)_g| < q^{-dk} + 9q^{-(n-1)/2}$. Since $|\mathcal{N}(V,U)_g| = |\mathcal{N}(W,U)_g| + |\mathcal{N}'|$, the upper bounds follow.

By (6.4), there are at most $3C_1 q^{-n/2}|\mathcal{L}(V,k)| \leq \dfrac{3C_1^2}{c_2} q^{-n/2}|\mathcal{N}(V,U)|$ g-invariant k-subspaces U of V such that U is not contained in W. From Proposition 5.1 we have that

$$(q^{-dk} - 3q^{-(n-1)/2})|\mathcal{N}(V,U)| \le |\mathcal{N}(W,U)| \le (q^{-dk} + 9q^{-(n-1)/2})|\mathcal{N}(V,U)|.$$

The assertion follows.

Combining the results above, we have.

Proposition 7.3. *If g is a linear transformation on V, U is a non-degenerate k-subspace of V, and $k \le n/2$ then one of the following holds.*

1. $$\frac{|\mathcal{N}(V,U)_g|}{|\mathcal{N}(V,U)|} < 340q^{-n/2}.$$
2. *g has an eigenspace of codimension d where $d \le n/2k$, and*
$$q^{-dk} - 3q^{-(n-1)/2} \le \frac{|\mathcal{N}(V,U)_g|}{|\mathcal{N}(V,U)|} \le q^{-dk} + 220q^{-(n-1)/2}.$$

8. The subspace W

We introduce here the subspace W that will play a crucial role in computing the fixed point ratios of linear transformations on totally singular subspaces. Recall that a subspace X of V is totally isotropic with respect to the bilinear form $\mathbf{f}$ if $\mathbf{f}(x,y) = 0$ for all $x, y \in X$.

Lemma 8.1. *Assume that V is non-degenerate with respect to the bilinear form $\mathbf{f}$ and that g is a linear transformation on V. Then there is a subspace W of V such that $\dim W \ge n/2$, $W = \ker \phi(g)$ where ϕ is a polynomial over $\mathbf{F}$, and one of following further conditions holds.*

1. *W is non-degenerate and $\dim W \le 2n/3$.*
2. *W is non-degenerate and $W = W_1 + W_2$, where $W_i = \ker \phi_i(g)$, ϕ_i a polynomial over $\mathbf{F}$, and W_i is totally isotropic of dimension $\frac{1}{2}\dim W$, $i = 1, 2$.*
3. *W is non-degenerate and $\phi = \mu^c$ for some irreducible polynomial μ of degree at least 2 and some positive integer c.*
4. *ϕ is a linear polynomial.*
5. *$\phi = \mu^2$ where μ is linear and $\dim \ker \mu(g) < n/2$.*
6. *$\phi = \mu^c$ where μ is linear, c is a positive integer, and $\dim W < 3n/4$.*

Furthermore, in cases (4), (5), and (6) the space $\ker \phi(g)^s$ is non-degenerate for some positive integer s.

Proof. Let $\prod \mu_i^{c_i}$ be the minimal polynomial for g, where μ_i is irreducible for each i and μ_i and μ_j are relatively prime when $i \ne j$. Then $V = \sum W_i$ where $W_i = \ker \mu_i^{c_i}(g)$. For each i, there is a unique i' such that $W_{i'}$ is not in the orthogonal complement of W_i. Set $X_i = \langle W_i, W_{i'} \rangle$. If $i = i'$, then $X_i = W_i$ is non-degenerate. If $i \ne i'$, then W_i and $W_{i'}$ are totally isotropic and $X_i = W_i + W_{i'}$ is non-degenerate. We can number so that $V = \sum_{i=1}^r X_i$, where $\{X_i\}$ are distinct, $i = 1, \ldots, r$. Set $m_i = \dim X_i$. Then $\sum m_i = n$.

Suppose $m_i \le 2n/3$ for all i. For each $J \subseteq \{1, \ldots, r\}$ set $X_J = \sum_{j \in J} X_j$, and set $s(J) = \dim X_J$. Choose J so that $s(J)$ is as large as possible subject to $s(J) \le 2n/3$, and set $W = X_J = \sum_{j \in J} m_j$. We claim that W satisfies condition (1).

We have $\dim W = s(J)$. If $s(J) < n/3$ and $i \notin J$ then there exist at least two indices i, l that lie in $\{1, \ldots, r\} \setminus J$. Without loss $m_i \le m_l$, so that $s(J) + 2m_i \le$

$s(J)+m_i+m_l \le n$. Therefore $m_i \le (n-s(J))/2$ and $s(J) < s(J\cup\{i\}) = m_i+s(J) \le (n+s(J))/2 < 2n/3$, contradicting the choice of J. Therefore $s(J) \ge n/3$. Let $J' = \{1,\dots,r\} \setminus J$. Since $s(J') = n - s(J)$ it follows that $s(J') \le 2n/3$, whence $s(J') \le s(J)$ by choice of J. This implies that $s(J) \ge n/2$. This shows that $n/2 \le \dim W \le 2n/3$. Since $W = \ker \phi(g)$ where $\phi = \prod_{i \text{ or } i' \in J} \mu_i^{c_i}$, this establishes the claim that W satisfies condition (1).

Now assume that $m_1 > 2n/3$. If W_1 is totally isotropic, then so is $W_{1'}$ and $W = W_1 + W_{1'}$ satisfies condition (2) with $\phi = \phi_1\phi_2$ where $\phi_1 = \mu_1^{c_1}$ and $\phi_2 = \mu_{1'}^{c_1}$. Assume therefore that W_1 is not totally isotropic. Then $W_1 = X_1$, and W_1 is non-degenerate.

Set $\mu = \mu_1$. If μ has degree at least 2, then $\phi = \mu^{c_1}$ satisfies (3), so assume that μ is linear.

Let c be the smallest positive integer such that $\dim\ker \mu(g)^c \ge n/2$. If $c = 1$, then (4) holds with $\phi = \mu$. If $c = 2$, then (5) holds. Assume that $c \ge 3$. Set $\sigma = \mu(g)$. For $0 \le i \le c$, set $K_i = \ker \sigma^i$. Then $\langle \dim K_i - \dim K_{i-1}\rangle$ is a decreasing sequence because σ induces an injection from K_{i+1}/K_i to K_i/K_{i-1}. Set $k_i = \dim K_i$, $i \ge 0$, and $d_i = k_i - k_{i-1}$, $i \ge 1$. We claim that $k_c < 3n/4$. By choice of c, $k_{c-1} < n/2 \le k_c$. Since $\langle d_i\rangle$ is decreasing we have that $cd_c \le \sum_{i=1}^c d_i = k_c$, so $d_c \le k_c/c \le k_c/3$. Therefore $k_c = k_{c-1} + d_c < n/2 + k_c/3$. It follows that $2k_c/3 < n/2$, whence $k_c < 3n/4$. This establishes the claim. Statement (6) must therefore hold when $\phi = \mu^c$ and $W = \ker\phi(g)$.

The final assertion holds because, in each case, $\phi = \mu^c$ and $W_1 = \ker \mu(g)^{c_1}$ is non-degenerate.

9. Totally singular subspaces, Semisimple action

We assume in this section that V admits a non-degenerate form, g is a linear transformation on V, and W satisfies one of the first three conclusions of Lemma 8.1. Set $W' = W^\perp$. Then W' is non-degenerate and $V = W \perp W'$. There is a polynomial ϕ' such that $W' = \ker\phi'$ and $\phi(g) + \phi'(g) = 1_V$.

9.1. *Suppose* $\dim W' = d \le n/2$. *Then*

$$|\mathcal{S}(V,k)_g| \le |\mathcal{S}(W,k)_g| + \sum_{j>0} |\mathcal{S}(W,k-j)|\ |\mathcal{S}(W',j)|$$

where every j in the sum satisfies $2j \le d$ and $j \le k$.

Proof. The hypotheses for this section imply that $W = V\phi'(g)$ and $W' = V\phi(g)$. If $U \in \mathcal{S}(V,k)_g$ then $U = U\phi'(g) + U\phi(g)$, so the map $\alpha : U \to (U\phi'(g), U\phi(g))$ is an injection from $\mathcal{S}(V,k)_g$ to $\mathcal{S}(W) \times \mathcal{S}(W')$. The image of this map is contained in $\mathcal{S}(W,k-j) \times \mathcal{S}(W',j)$ for some $j > 0$. If $U \in \mathcal{S}(V,k)_g \setminus \mathcal{S}(W,k)_g$, then $\alpha(U) \in \mathcal{S}(W,k-j) \times \mathcal{S}(W',j)$ for some $j > 0$. Therefore $|\mathcal{S}(V,k)_g| \le |\mathcal{S}(W,k)_g| + \sum_{j>0} |\mathcal{S}(W,k-j)|\ |\mathcal{S}(W',j)|$. For every non-zero term we have $j \le k$ and $2j \le \dim W' = d$. The assertion now follows.

9.2. *If* $\dim W' = d \le n/2$, *then*

$$|\mathcal{S}(V,k)_g| - |\mathcal{S}(W,k)_g| \le \sum_{j=1}^{h} |\mathcal{S}(W,k-j)|\ |\mathcal{S}(W',j)| \le \frac{3C_3^2}{c_3} q^{-(n-2)/2}\ |\mathcal{S}(V,k)|$$

where $h = \min(k, d/2)$.

Proof. It follows from Lemma 2.3 that $|\mathcal{S}(W,k-j)| \le C_3 q^{(n-d-k+j)(k-j)-\psi(k-j)}$ and $|\mathcal{S}(W',j)| \le C_3 q^{(d-j)j-\psi(j)}$. Also, $|\mathcal{S}(V,k)| \ge c_3 q^{(n-k)k-\psi(k)}$. Therefore

$$\sum_{0<j\le k; 2j\le d} |\mathcal{S}(W,k-j)|\ |\mathcal{S}(W',j)| \le \frac{C_3^2}{c_3}|\mathcal{S}(V,k)| \sum_{0<j\le k; 2j\le d} q^{e_2(n,d,k,j)}$$

where $e_2(n,d,k,j) = (n-d-k+j)(k-j)-\psi(k-j)+(d-j)j-\psi(j)-(n-k)k+\psi(k)$. It follows from Lemma 11.10 that $\sum_{0<j\le k; 2j\le d} q^{e_2(n,d,k,j)} < 3q^{-(n-2)/2}$. This completes the proof.

9.3. *If W has codimension d in V, $d \le n/2$, and W is non-degenerate, then*

$$|\mathcal{S}(W,k)| \le \frac{C_3}{c_3} q^{-dk} |\mathcal{S}(V,k)|.$$

In particular, if $d \ge n/4$ and $k \ge 2$, then

$$|\mathcal{S}(W,k)| \le \frac{C_3}{c_3} q^{-n/2} |\mathcal{S}(V,k)|.$$

Proof. We have $|\mathcal{S}(W,k)| \le C_3 q^{(n-d-k)k-\psi(k)}$ and $|\mathcal{S}(V,k)| \ge c_3 q^{(n-k)k-\psi(k)}$ by Lemma 2.3.

9.4. *If $W = W_1 + W_2$ where W_1 and W_2 are totally isotropic and $W_i = \ker \phi_i(g)$ for polynomials ϕ_i, and $k \ge 2$, then*

$$|\mathcal{S}(W,k)_g| < \frac{4C_1^2}{c_3} q^{-(n-4)/2} |\mathcal{S}(V,k)|.$$

Proof. Let $m = \dim W$. We may assume that $\mathcal{S}(W,k) \ne \emptyset$, so $k \le m/2$. We have $W_1 = W\phi_2(g)$ and $W_2 = W\phi_1(g)$.

Consider the function $\alpha : \mathcal{S}(W,k)_g \to \mathcal{L}(W_1) \times \mathcal{L}(W_2)$. defined by $\alpha(U) = (U\phi_2(g), U\phi_1(g))$. This function is injective because $U = U\phi_2 + U\phi_1$. If $X = U\phi_2(g)$, then $U\phi_1(g) \in W_2 \cap X^\perp$ because $U\phi_1 \subseteq U \subseteq X^\perp$ and $\dim U\phi_2(g) = k - \dim X$.

It follows that $|\mathcal{S}(W,k)_g| \le \sum_{j=0}^k \sum_{X \in \mathcal{L}(W_1,j)} |\mathcal{L}(W_2 \cap X^\perp, k-j)|$. Since $W_2 \cap X^\perp$ has dimension $m/2-j$ whenever X is a j-subspace of W_1, we have $|\mathcal{S}(W,k)_g| \le \sum_{j=0}^k |\mathcal{L}(W_1,j)|\ |\mathcal{L}(Y_j,k-j)|$ where Y_j has dimension $m/2-j$.

By Lemma 2.1, $|\mathcal{L}(W_1,j)| < C_1 q^{j(m/2-j)}$ and $|\mathcal{L}(W_2,k-j)| < C_1 q^{(k-j)(m/2-k)}$. By Lemma 2.3, $|\mathcal{S}(V,k)| \ge c_3 q^{k(n-k)-\psi(k)}$.

Therefore

$$\frac{|\mathcal{S}(W,k)_g|}{|\mathcal{S}(V,k)|} \le \frac{C_1^2}{c_3} \sum_{j=0}^k q^{e_4(n,m,k,j)}$$

where

$$e_4(n,m,k,j) = j(\frac{m}{2}-j) + (k-j)(\frac{m}{2}-k) - k(n-k) + \psi(k).$$

By Lemma 11.11 we have $\sum_j q^{e_4(n,m,k,j)} < 4q^{-(n-4)/2}$, and the statement follows.

9.5. *If g acts on W with minimal polynomial μ^c where μ is irreducible of degree at least 2 and c is a positive integer, then*

$$|\mathcal{S}(W,k)_g| < \frac{C_3}{c_3} q^{-(n-2)/2} |\mathcal{S}(V,k)|.$$

Proof. Choose an integer l so that $q^l > c$, and set $r = q^l$ and $h = g^r$. In $\mathrm{End}(V)$ we have $\mu(h) = \mu(g^r) = (\mu(g))^r = 0$, so h acts homogeneously on V with blocks of size larger than 1. Since every g-invariant subspace of W is evidently also h-invariant we have $|\mathcal{S}(W,k)_g| \le |\mathcal{S}(W,k)_h|$, and it suffices to bound $|\mathcal{S}(W,k)_h|$. We may therefore assume that $\mu(g) = 0$, so that g acts homogeneously on W with blocks of size b, $b > 1$.

W is a non-degenerate space of dimension $m' = m/b$ over $\mathbf{K} = \mathbf{F}[g] \cong \mathbf{F}_{q^b}$. Choose a $\mathbf{K}$-basis $\langle w_i \rangle$ for W, and define the form $\tilde{\mathbf{f}}$ on W so that $\tilde{\mathbf{f}}(\sum_i w_i p_i(g), w) = \sum_i p_i(g)\mathbf{f}(w_i, w)$ and $\tilde{\mathbf{f}}$ is of the same type as $\mathbf{f}$.

If $k' = k/b$, and U is a g-invariant totally singular subspace of W, then U is a k'-subspace of the m'-space $\tilde{W}$ over $\mathbf{K}$. This shows that $|\mathcal{S}(W,k)_g| = |\mathcal{S}(\tilde{W},k')|$.

By Lemma 2.3, $|\mathcal{S}(\tilde{W},k')| < C_3 q^{b\cdot(k'(m'-k')-\psi(k'))}$ and $|\mathcal{S}(V,k)| > c_3 q^{k(n-k)-\psi(k)}$. Therefore

$$\frac{|\mathcal{S}(W,k)_g|}{|\mathcal{S}(V,k)|} < \frac{C_3}{c_3} q^{e_3(n,m,k,b)}$$

where $e_3(n,m,k,b) = bk'(m'-k') - b\psi(k') - k(n-k) + \psi(k)$. It follows from the definition of ψ that $\psi(k) - b\psi(k/b) = (b-1)k^2/(2b)$. Since $m \le n$ we have

$$\begin{aligned} e_3(n,m,k,b) &= \frac{k(m-k)}{b} - k(n-k) + \frac{k^2}{2} - \frac{k^2}{2b} \\ &= -kn + \frac{km}{b} + \frac{3(b-1)}{2b}k^2 \\ &= -kn + \frac{kn}{b} + \frac{3(b-1)}{2b}k^2 \\ &\le -\frac{(b-1)k}{2b}(2n-3k) \le -\frac{k}{4}(2n-3k). \end{aligned}$$

Set $\alpha(k) = -k(2n-3k)/4$. Because $\alpha''(k) > 0$, the maximum value of α on the interval $[2, n/2]$ occurs at one of the endpoints. We have $\alpha(2) = 3 - n$ and $\alpha(n/2) = -n^2/16$. We have $b \le k \le n/2$, so $n \ge 4$. Since $3 - n \le -(n-2)/2$ and $-n^2/16 \le -(n-2)/2$ for $n \ge 4$, it follows that $e_3(n,m,k,b) \le -(n-2)/2$. The assertion follows.

Proposition 9.6. *Assume that V is non-degenerate and that g is a linear transformation on V that preserves κ up to scalar multiplication and that $k \ge 2$. Assume further that one of conditions (1), (2), or (3) of Lemma 8.1 holds. Then* $\dfrac{|\mathcal{S}(V,k)_g|}{|\mathcal{S}(V,k)|} \le \dfrac{4C_1^2}{c_3} q^{-(n-4)/2} + \dfrac{3C_3^2}{c_3} q^{-(n-2)/2} < 200 q^{-(n-4)/2}$.

Proof. Let W be the subspace of Lemma 8.1. By (9.1) and (9.2) we have $|\mathcal{S}(V,k)_g| \le |\mathcal{S}(W,k)_g| + \dfrac{3C_3^2}{c_3} q^{-(n-2)/2} |\mathcal{S}(V,k)|$.

From (9.3), (9.4), and (9.5), we have

$$\frac{|\mathcal{S}(W,k)_g|}{|\mathcal{S}(V,k)|} \leq \begin{cases} \frac{C_3}{c_3} q^{-n/2}, & \text{if (1) holds;} \\ \frac{4C_1^2}{c_3} q^{-(n-4)/2}, & \text{if (2) holds;} \\ \frac{C_3}{c_3} q^{-(n-2)/2}, & \text{if (3) holds.} \end{cases}$$

Since $C_3 < 4C_1^2 q$, the inequality follows.

10. Totally singular subspaces, Unipotent action

We consider here the remaining cases [(4), (5), and (6)] of Lemma 8.1. In this section, we assume that $W = \ker \phi(g)$ where $\phi = \mu^c$ for some linear polynomial μ, and that $\ker \mu(g)^s$ is non-degenerate for some positive integer s. Set $m = \dim W$ and $d = \text{codim } W$.

Suppose $\mu(x) = x - \omega$, where $\omega \in \mathbf{F}$. Let z be scalar multiplication of V by ω, and set $h = gz^{-1}$. Then the g-invariant subspaces of V coincide with the h-invariant subspaces of V, and $\ker \mu(g)^i = \ker(h - I)^i$ for all i. Replacing g by h, we may therefore assume that $\mu(x) = x - 1$, so that $\ker \mu(g) = C_V(g)$ and $V\mu(g) = [V, g]$.

10.1. *$V\mu(g)^i \perp \ker \mu(g)^i$ for all positive integers i. In particular, $C_V(g) \perp [V, g]$.*

Proof. Set $W_i = \ker \mu(g)^i$.

By assumption W_s is non-degenerate, so there exist $v, w \in W_s$ with $\mathbf{f}(v, w) \neq 0$. Choose positive integers i, j and vectors $v \in W_i$, and $w \in W_j$ so that $\mathbf{f}(v, w) \neq 0$ and $i + j$ is as small as possible. Then $vg = v + v'$ and $wg = w + w'$ where $v' \in W_{i-1} \subseteq W_j^{\perp}$ and $w' \in W_{j-1} \subseteq W_i^{\perp}$. We have $\mathbf{f}(vg, wg) = \mathbf{f}(v, w) + \mathbf{f}(v, w') + \mathbf{f}(v', w) + \mathbf{f}(v', w') = \mathbf{f}(v, w)$. Therefore $\mathbf{f}(xg, yg) = \mathbf{f}(x, y)$, for all $x, y \in V$.

We claim that $\mathbf{f}(v\mu(g)^i, w) = (-1)^i \mathbf{f}(v, w\mu(g)^i g^{-i})$ for all $v, w \in V$ and every positive integer i. We have $\mathbf{f}(v\mu(g), w) = \mathbf{f}(vg, w) - \mathbf{f}(v, w) = \mathbf{f}(vg, w) - \mathbf{f}(vg, wg) = \mathbf{f}(vg, w - wg) = \mathbf{f}(v, wg^{-1} - w) = -\mathbf{f}(v, w\mu(g)g^{-1})$. The claim now follows by induction because $\mu(g)$ commutes with g^{-1}.

Let $x \in V\mu(g)^i$ and $y \in \ker \mu(g)^i$. Then $x = v\mu(g)^i$ for some v, and $\mathbf{f}(x, y) = (-1)^i \mathbf{f}(v, y\mu(g)^i g^{-i})$ by the claim. Since $W_i = \ker \mu(g)^i$ is g-invariant we have $y\mu(g)^i = 0$, whence $\mathbf{f}(x, y) = 0$, and the statement holds.

10.2. *Suppose $\rho = \mu(g)^i$ for some integer i. For each g-invariant subspace X of V, set*

$$\mathcal{Y}(X) = \{Y \subseteq X\rho : Y \textit{ is totally singular and } Y \subseteq \textit{rad } Y\rho^{-1}\}$$
$$\mathcal{Y}_j = \mathcal{Y}_j(X) = \mathcal{Y}(X) \cap \mathcal{L}(X\rho, j)$$
$$\mathcal{S}_j = \mathcal{S}_j(X, k) = \{S \in \mathcal{S}(X, k)_g \,:\, \dim S\rho = j\}$$

Then

1. $|\mathcal{S}_j(X, k)| \leq \sum_{Y \in \mathcal{Y}_j(X)} |\mathcal{S}(Y\rho^{-1}/Y, k - j)|$.
2. $|\mathcal{S}(X, k)_g| \leq \sum_j |\mathcal{S}_j(X, k)| \leq \sum_j \sum_{Y \in \mathcal{Y}_j} |\mathcal{S}(Y\rho^{-1}/Y, k - j)|$.

Proof. Let $U \in \mathcal{S}(X, k)_g$ and let $Y = U\rho$. Then $Y \perp \ker \rho$ by (10.1). Since U is totally singular and $Y \subseteq U$, we have $Y \perp U$. Therefore $Y \perp U + \ker \rho$. Since $Y\rho^{-1} = U + \ker \rho$ we have $Y \subseteq \text{rad } Y\rho^{-1}$. Thus, $Y \in \mathcal{Y}(X)$.

We have $\rho : \mathcal{S}(X,k)_g \to \mathcal{Y}(X) = \cup_j \mathcal{Y}_j$ where the union is over all j with $0 \le j \le k$ and $j \le \dim X\rho$. If $Y \in \mathcal{Y}_j$ and $U\rho = Y$, then $U/Y \in \mathcal{S}(Y\rho^{-1}/Y, k-j)$. The assertions follow.

Lemma 10.3. $|\mathcal{S}(V,k)_g| - |\mathcal{S}_0(W,k)_g| \le \dfrac{14C_1^2C_3}{c_3} q^{-(n-6)/2}$. *If either* $n \ge 16$ *or* V *is not orthogonal, then* $|\mathcal{S}(V,k)_g| - |\mathcal{S}_0(W,k)_g| \le \dfrac{15C_1^2C_3}{c_3} q^{-(n-4)/2}$.

Proof. We apply (10.2) with $X = V$ and $\rho = \phi(g)$. Since $\mathcal{Y}_0 = \{0\}$ and $0\rho^{-1} = W$ we have $|\mathcal{S}(V,k)_g| \le |\mathcal{S}(W,k)_g| + \sum_{j=1}^d \sum_{Y\in\mathcal{Y}_j} |\mathcal{S}(Y\rho^{-1}/Y, k-j)|$. For $Y \in \mathcal{Y}_j$, we have $\dim Y\rho^{-1} = \dim Y + \dim\ker\rho = n-d+j$ and $\dim T(Y\rho^{-1}) \le \dim Y\rho^{-1} = d-j$. Therefore $\dim T(Y\rho^{-1}/Y) \le d-2j$. By (2.15), we have $\mathcal{S}(Y\rho^{-1}/Y, k-j) \le \sigma^*(Z_j, k-j)$ where Z_j has dimension $n-d$ and $\dim T(Z_j) = d-2j$. This implies that

$$\begin{aligned}\sum_{j=1}^{d}\sum_{Y\in\mathcal{Y}_j} |\mathcal{S}(Y\rho^{-1}/Y, k-j)| &\le \sum_{j=1}^{d/2} |\mathcal{Y}_j|\sigma^*(Z_j, k-j)\\ &\le \sum_{j=1}^{d/2} |\mathcal{L}(V\rho, j)|\sigma^*(Z_j, k-j)\\ &\le \sum_{j=1}^{d} C_1 q^{j(d-j)} \sum_{i\in\mathcal{I}_j} C_1C_3 q^{e(n-d,d-2j,k-j,i)}\end{aligned}$$

where $\mathcal{I}_j = \{i : 0 \le i \le k-j, i \le d-2j\}$.

Evidently,

$$\begin{aligned}|\mathcal{S}(W,k)_g| &\le |\mathcal{S}_0(W,k)_g| + \sum_{i=1}^{k} |\mathcal{S}_i(W,k)|\\ &\le |\mathcal{S}_0(W,k)| + C_1C_3 q^{e(n-d,d,k,i)}.\end{aligned}$$

This implies that

$$\begin{aligned}|\mathcal{S}(V,k)_g| - |\mathcal{S}_0(W,k)_g| \le C_1C_3 &\sum_{i\in\mathcal{I}_0,\ i>0} q^{e(m,d,k,i)}\\ &+ C_1^2C_3 \sum_{j=1}^{k}\sum_{i\in\mathcal{I}_j} q^{j(d-j)+e(m,d-2j,k-j,i)}.\end{aligned}$$

Because $1 < C_1$ and $|\mathcal{S}(V,k)| \ge c_3 q^{k(n-k)-\psi(k)}$ we have

$$|\mathcal{S}(V,k)_g| - |\mathcal{S}_0(W,k)_g| < \frac{C_1^2C_3}{c_3} \sum_{i+j>0,\ i\in\mathcal{I}_j} q^{e_5(n,d,k,i,j)} |\mathcal{S}(V,k)|$$

where

$$e_5(n,d,k,i,j) = j(d-j) + e(n-d, d-2j, k-j, i) - k(n-k) + \psi(k).$$

By Lemma 11.12, $\sum_{i+j>0, i\in\mathcal{I}_j} q^{e_5(n,d,k,i,j)} \le 14q^{-(n-6)/2}$. If either $n \ge 16$ or V is not orthogonal, then $\sum_{i+j>0, i\in\mathcal{I}_j} q^{e_5(n,d,k,i,j)} \le 15q^{-(n-4)/2}$. The result follows.

10.4. 1. *If* $d < n/2$ *then* $q^{-dk} - 3q^{-(n-2)/2} < \dfrac{|\mathcal{S}_0(W,k)|}{|\mathcal{S}(V,k)|} < q^{-dk} + 9q^{-(n-2)/2}$.

2. *If* $n/4 \leq d < n/2$ *and* $k \geq 2$, *then* $|\mathcal{S}_0(W,k)| < 10q^{-(n-2)/2}|\mathcal{S}(V,k)|$.

3. *If* $d = n/2$, $n \geq 4$, *and* $k \geq 1$, *then* $\dfrac{|\mathcal{L}(W,k)|}{|\mathcal{S}(V,k)|} < \dfrac{C_1}{c_3}q^{-(n-2)/2} < 8q^{-(n-2)/2}$.

Proof. The first two statements follow immediately from Proposition 5.1.(3).

Under the assumption of the third statement, we have $|\mathcal{L}(W,k)| < C_1 q^{k(n/2-k)}$ and $|\mathcal{S}(V,k)| > c_3 q^{k(n-k)-\psi(k)}$ by Lemmas 2.1 and 2.3. This implies that $|\mathcal{L}(W,k)| \leq (C_1/c_3)q^{-kn/2+\psi(k)}|\mathcal{S}(V,k)|$. Because $\psi(k) = k(k+a)/2$ where $|a| \leq 1$, we have $-kn/2 + \psi(k) \leq -n/2 + \psi(1) \leq -(n-2)/2$, and the third statement must hold.

The previous result is useful if either $\dim W \leq 3n/4$ or $W = C_V(g)$. We now consider the situation where g has class 2 action on W, and $\dim W \geq 3n/4$. In this case, we bound $|\mathcal{S}_0(W,k)_g|$ by bounding $|\mathcal{S}(W,k)_g|$.

Lemma 10.5. *Assume that* $2 \leq k \leq n/2$, $W = \ker(g-I)^2$, $\dim W \geq 3n/4$, *and* $\dim C_V(g) < n/2$. *Then*

$$|\mathcal{S}(W,k)_g| \leq \left(\frac{3C_1^2}{c_3} + \frac{3C_1^2C_3}{2c_3}n\right) q^{-(n-3)/2}|\mathcal{S}(V,k)|.$$

Proof. Set $X = C_V(g)$. Then $X = C_W(g)$ and $[W,g] \subseteq X$. Set $h = \dim[W,g]$. Then $h = \dim[W,g] = \dim W - \dim X = n - d - \dim X$ so that $\dim X = n - d - h$. Because $[W,g] \subseteq X$, we have $h \leq n - d - h$; therefore, $h \leq n/2 - d/2$. Furthermore $d + h > n/2$ because $\dim X < n/2$. Since $h \leq n/2 - d/2 \leq n/2 < h + d$, we must have $d \geq 1$. By assumption, $d \leq n/4$.

10.6. *Set*

$$\mathcal{J} = \{j : 0 \leq j \leq h, 0 \leq k - j \leq n - d - h\},$$
$$\mathcal{J}_1 = \{j \in \mathcal{J} : j \leq d + h - n/2\}, and$$
$$\mathcal{J}_2 = \{j \in \mathcal{J} : j > d + h - n/2\}.$$

Then

$$|\mathcal{S}(W,k)_g| \leq \sum_{j\in\mathcal{J}_1} |\mathcal{L}([W,g],j)|\ |\mathcal{L}(Z,k-j)| + \sum_{j\in\mathcal{J}_2} |\mathcal{L}([W,g],j)|\ \sigma^*(Z_j,k-j)$$

where Z *and* Z_j *have dimension* $n-d-h$ *and* $T(Z_j)$ *has dimension* $d+h-2j$.

Proof. Applying (10.2) with $S = W$ and $\rho = g - I$, we have

$$|\mathcal{S}(W,k)_g| \leq \sum_{j\in\mathcal{J}_1}\sum_{Y\in\mathcal{Y}_j(W)} |\mathcal{S}(Y\rho^{-1}/Y,k-j)| + \sum_{j\in\mathcal{J}_2}\sum_{Y\in\mathcal{Y}_j(W)} |\mathcal{S}(Y\rho^{-1}/Y,k-j)|.$$

For all j, we have $|\mathcal{Y}_j| \leq |\mathcal{L}(W\rho,j)|$.

If $j \in \mathcal{J}_1$, and $Y \in \mathcal{Y}_j$, then $|\mathcal{S}(Y\rho^{-1}/Y,k-j)| \leq |\mathcal{L}(Z,k-j)|$. where $\dim Z = \dim Y\rho^{-1}/Y = n-d-h$.

For $j \in \mathcal{J}_2$, and $Y \in \mathcal{Y}_j$ we have $\dim(Y\rho^{-1})^\perp \leq n - \dim Y\rho^{-1} \leq d+h-j$. Because $Y \subseteq Y\rho^{-1}$ we also have $\dim T(Y\rho^{-1}/Y) \leq d+h-2j$. Therefore

$|\mathcal{S}(Y\rho^{-1}/Y, k-j)| \le \sigma^*(Z_j, k-j)$ where $\dim Z_j = n-d-h$ and $\dim T(Z) = d+h-2j$. The statement follows. □

We now bound the two sums.

10.7.

$$\sum_{j\in\mathcal{J}_1} |\mathcal{L}([W,g],j)|\ |\mathcal{L}(Z,k-j)| < \frac{3C_1^2}{c_3} q^{-(n-3)/2} |\mathcal{S}(V,k)|.$$

Proof. We have $|\mathcal{L}([W,g],j)| \le C_1 q^{j(h-j)}$ and $|\mathcal{L}(Z,k-j)| \le C_1 q^{(k-j)(n-d-h-k+j)}$ by Lemma 2.1, and $|\mathcal{S}(V,k)| \ge c_3 q^{k(n-k)-\psi(k)}$ by Lemma 2.3. Therefore

$$|\mathcal{L}([W,g],j)|\ |\mathcal{L}(Z,k-j)| \le \frac{C_1^2}{c_3} q^{e_6(n,d,h,k,j)} |\mathcal{S}(V,k)|$$

where $e_6(n,d,h,k,j) = j(h-j) + (k-j)(n-d-h-k+j) - k(n-k) + \psi(k)$. By Lemma 11.14, $\sum_{j\in\mathcal{J}_1} q^{e_6(n,d,h,k,j)} \le 3q^{3/2-n/2}$. The result follows. □

10.8. $\displaystyle\sum_{j\in\mathcal{J}_2} |\mathcal{L}([W,g],j)|\ \sigma^*(Z_j,k-j) < \frac{3C_1^2 C_3}{2c_3} n q^{-(n-3)/2} |\mathcal{S}(V,k)|.$

Proof. We have $|\mathcal{L}([W,g],j)| \le C_1 q^{j(h-j)}$ by Lemma 2.1. By definition, $\sigma^*(Z_j, k-j) = C_1 C_3 \sum_{i\in\mathcal{I}_j} q^{e(n-d-h,d+h-2j,k-j,i)}$ where $e(m,t,k,i) = (m-k)(k-i) + i(t-i) - \psi(k-i)$ and $\mathcal{I}_j = \{i : 0 \le i \le k-j, i \le d+h-2j\}$.

Since $|\mathcal{S}(V,k)| \ge c_3 q^{k(n-k)-\psi(k)}$ it follows that

$$\sum_{j\in\mathcal{J}_2} |\mathcal{L}(Y,j)|\ \sigma^*(Z_j,k-j) \le \frac{C_1^2 C_3}{c_3} \sum_{j\in\mathcal{J}_2, i\in\mathcal{I}_j} q^{e_7(n,d,h,k,i,j)} |\mathcal{S}(V,k)|$$

where

$$e_7(n,d,h,k,i,j) = j(h-j) + e(n-d-h, d+h-2j, k-j, i) - k(n-k) + \psi(k)$$

and $\mathcal{I}_j = \{i : 0 \le i \le k-j, i \le d+h-2j\}$.

By Lemma 11.15, we have

$$\sum_{j\in\mathcal{J}_2,\ i\in\mathcal{I}_j} q^{e_7(n,d,h,k,i,j)} \le \frac{3}{2} n q^{(n-3)/2}.$$

Statement (10.8) follows. □

The assertion in Lemma 10.5 now follows from (10.6), (10.7), and (10.8).

Proposition 10.9. *Assume that V is non-degenerate and that g is a linear transformation on V that preserves κ up to scalar multiplication. Then one of the following holds*

1.
$$\begin{aligned}\frac{|\mathcal{S}(V,k)_g|}{|\mathcal{S}(V,k)|} &\le \left(\frac{3C_1^2}{c_3} + \frac{3C_1^2 C_3}{2c_3} n\right) q^{-(n-3)/2} + \frac{14 C_1^2 C_3}{c_3} q^{-(n-6)/2} \\ &< (2000 + 75n) q^{-(n-6)/2}.\end{aligned}$$

2. *g has an eigenspace of codimension $d < n/2k$ and*

$$q^{-dk} - 3q^{-(n-2)/2} \leq \frac{|\mathcal{S}(V,k)_g|}{|\mathcal{S}(V,k)|} \leq q^{-dk} + \left(\frac{9}{q^2} + \frac{14C_1^2C_3}{c_3}\right) q^{-(n-6)/2}$$
$$< q^{-dk} + 2000q^{-(n-6)/2}.$$

Proof. Statement (2) holds whenever $k = 0$. If $k = 1$, then the conclusion follows from Proposition 4.6, so assume that $k \geq 2$. If one of conditions (1), (2), (3) of Lemma 8.1 holds then the statement follows from Proposition 9.6, so we may assume that (4), (5), or (6) of Lemma 8.1 holds. Let $d = \operatorname{codim} W$ where W is the space satisfying one of those conditions. Then $d \leq n/2$ and $|\mathcal{S}(V,k)_g| \leq |\mathcal{S}_0(W,k)_g| + \frac{14C_1^2C_3}{c_3}q^{-(n-6)/2}$ by Lemma 10.3.

If $dk \geq n/2$, in particular, if $d \geq n/4$, then $|\mathcal{S}_0(W,k)_g| \leq |\mathcal{S}_0(W,k)| < 14q^{-(n-2)/2}$ by (10.4).(2).

If (5) of Lemma 8.1 holds, then

$$|\mathcal{S}_0(W,k)_g| \leq \left(\frac{3C_1^2}{c_3} + \frac{3C_1^2C_3}{2c_3}n\right) q^{-(n-3)/2}$$

by Lemma 10.5 so (1) holds. We may therefore assume that $d < n/2k \leq n/4$ and that (5) of Lemma 8.1 does not hold.

Then (4) of Lemma 8.1 must hold, so W is an eigenspace for g of codimension less than $n/2k$. Since

$$|\mathcal{S}_0(W,k)| \leq |\mathcal{S}(V,k)_g| \leq |\mathcal{S}_0(W,k)| + \frac{14C_1^2C_3}{c_3}q^{-(n-6)/2}$$

it follows from Proposition 5.1.(3) that (2) holds.

11. Calculations

This section contains the technical calculations that produce the bounds on the exponents used in various sums.

11.1. *Suppose $\xi(x,y)$ satisfies $\frac{\partial \xi}{\partial x} = A - y$ and $\frac{\partial \xi}{\partial y} = B - x$. Suppose further that*

$$x_* \leq x \leq x^* \quad \text{and} \quad y_* \leq y \leq y^*.$$

1. *If $y \leq A$ and $x^* \leq B$ then $\xi(x,y) \leq \xi(x^*,y^*)$.*
2. *If $y \leq A$ and $B \leq x^*$ then $\xi(x,y) \leq \xi(x^*,y_*)$.*
3. *If $A \leq y$ and $x_* \leq B$ then $\xi(x,y) \leq \xi(x_*,y^*)$.*
4. *If $A \leq y$ and $B \leq x_*$ then $\xi(x,y) \leq \xi(x_*,y_*)$.*

Proof. The conditions on partial derivatives imply that

$$\xi(x,y) \leq \begin{cases} \xi(x^*,y), & y \leq A \\ \xi(x_*,y), & A \leq y \end{cases}$$

and

$$\xi(x,y) \leq \begin{cases} \xi(x,y^*), & x \leq B \\ \xi(x,y_*), & B \leq x. \end{cases}$$

Applying the first criterion, followed by the second with $x = x^*$ or x_*, as appropriate, yields the result.

11.2. *Let $a, b, c,$ and r be non-negative numbers such that $rb \leq c$. Then $ab - bc \leq a^2/4r$.*

Proof. The assumptions imply that $-bc \leq -rb^2$, so $ab - bc \leq ab - rb^2$. The maximum value of the quadratic function $ax - rx^2$ is $a^2/4r$.

11.3. *Suppose a is an integer and $aj + b \leq K$ for all $j \in [c, d]$. Then*

$$\sum_{j=c}^{d} q^{aj+b} \leq \begin{cases} \frac{q}{q-1} q^K, & a \neq 0 \\ (d - c + 1) q^K, & a = 0. \end{cases}$$

Proof. This is an easy consequence of the formula for the sum of a geometric series.

11.4. *Let ϕ be a quadratic function such that $\phi''(x) = 2a$ where a is a negative constant. If $\phi(j) \leq K$ for all integers j in the interval $[c, d]$ then $\sum_{j \in [c,d]} q^{\phi(j)} < \frac{2}{1 - q^a} q^K$. If the critical value of ϕ does not lie in the open interval (c, d), then $\sum_{j \in [c,d]} q^{\phi(j)} < \frac{1}{1 - q^a} q^K$.*

Proof. By symmetry of the graph of ϕ about its axis it suffices to prove the second statement. Without loss, assume that ϕ is decreasing on $[c, d]$. By the Taylor expansion, we have $\phi(c + x) = \phi(c) + \phi'(c)x + \phi''(c)x^2/2$ for all values of x. Since $\phi(c) \leq K$, $\phi'(c) \leq 0$, and $\phi''(c) = 2a < 0$, it follows that $\phi(c + x) \leq K + ax$ for all integers $c + x$ in $[c, d]$. Therefore

$$\sum_{j \in [c,d]} q^{\phi(j)} < \sum_{m=0}^{\infty} q^{K+am} = \frac{q^K}{1 - q^a}.$$

11.5. *Let ϕ be a quadratic function such that $\phi''(x) = 2a$ where a is a positive constant. Then $\sum_{j \in [c,d]} q^{\phi(j)} < \frac{1}{1 - q^{-a}} (q^{\phi(c)} + q^{\phi(d)})$. If $\phi'(x) \geq 0$ for all x in $[c, d]$ then $\sum_{j \in [c,d]} q^{\phi(j)} < \frac{1}{1 - q^{-a})} q^{\phi(d)}$. If $\phi'(x) \leq 0$ for all x in $[c, d]$ then $\sum_{j \in [c,d]} q^{\phi(j)} < \frac{1}{1 - q^{-a}} q^{\phi(c)}$.*

Proof. By symmetry, it suffices to prove the second statement. For each $x \in [c, d]$ we have $\phi(x + 1) = \phi(x) + \phi'(x) + \phi''(x)/2 \geq \phi(x) + a$. It follows that $\phi(y + m) \geq \phi(y) + ma$ whenever m is a positive integer such that y and $y + m$ are both in $[c, d]$. Applying this to $y = d - m$, we have $\phi(d - m) \leq \phi(d) - ma$ whenever $d - m \in [c, d]$. Therefore $\sum_{j \in [c,d]} q^{\phi(j)} < q^{\phi(d)} \sum_{m > 0} q^{-ma} = \frac{1}{1 - q^{-a}} q^{\phi(d)}$.

11.6. *Suppose $\phi(x) = ax^2 + bx + c$ where $a > 0$. If $c + d \leq -b/a$ then $\phi(x) \leq \phi(c)$ for all $x \in [c, d]$. If $c + d \geq -b/a$ then $\phi(x) \leq \phi(d)$ for all $x \in [c, d]$.*

Proof. The graph of ϕ is a parabola which is concave up. On a closed interval it attains its maximum value at the endpoint further from the axis of symmetry, $x = -b/2a$.

11.7. *If* $a < 0$ *then* $ax^2 + bx + c \le c - \dfrac{(2b+1)^2}{16a} - \dfrac{x}{2}$.

Proof. This follows from the identity $ax^2+bx+c = a(x+\dfrac{2b+1}{4a})^2+c-\dfrac{(2b+1)^2}{16a}-\dfrac{x}{2}$.

11.8. *If* m *is a real number, then* $\sum_{i=0}^{\lfloor m \rfloor} q^{-mi} \le (q+1)/q \le 3/2$.

Proof. If $m < 1$, then the sum is 0 or 1. If $1 \le m < 2$, then the sum is $1 + q^{-m} \le 1 + q^{-1}$. If $m \ge 2$, then the sum is bounded by $\sum_{i=0}^{\infty} q^{-mi} = 1/(1-q^{-m}) < (1-q^{-2})^{-1} \le 1 + q^{-1}$.

Lemma 11.9. *Suppose* $e_1(n,d,k,i) = (n-d-k+i)(k-i) + (d-i)i - k(n-k)$. *If* $n \ge 4$, $d \le n/2$, $k \le n/2$, *and* $h = \min(d,k)$, *then* $\sum_{1 \le i \le h} q^{e_1(n,d,k,i)} < 3q^{-n/2}$.

Proof. We claim that $e_1(n,d,k,i) \le -n/2$ for every i in $[1,h]$. Set $\xi(x,y) = e_1(n,x,y,i)$. To establish the claim, it suffices to show that $\xi(x,y) \le -n/2$ for all integers x and y with $i \le x \le n/2$ and $i \le y \le n/2$. Pick x and y in that range. Since $\partial\xi/\partial x = 2i - y$ and $\partial\xi/\partial y = 2i - x$, the hypotheses of (11.1) hold with $A = B = 2i$, $x_* = y_* = i$, and $x^* = y^* = n/2$.

If $2i \le y$, then $\xi(x,y) \le \xi(i, n/2) = -in/2 \le -n/2$ because $x_* \le 2i = B$.

If $y \le 2i = A$ and $2i = B \le n/2 = x^*$, then $\xi(x,y) \le \xi(n/2, i) = -in/2 \le -n/2$.

To establish the claim, we may therefore assume that $y < 2i$ and $n/2 < 2i$. By (11.1) we have $\xi(x,y) \le \xi(n/2,n/2) = -n^2/4 + 2i(n/2-i)$. Since $i > n/4$, we have $2i(n/2-i) < n^2/8$, so that $\xi(x,y) < -n^2/8$. Since $n \ge 4$, we have $\xi(x,y) < -n/2$, and the claim is established.

Because $\partial^2 e_1(n,d,k,i)/\partial i^2 = -4$, (11.4) now implies that

$$\sum_i q^{e_1(n,d,k,i)} \le \frac{2q^{-n/2}}{1-q^{-2}} < 3q^{-n/2}.$$

Remark . The argument shows that $\sum_i q^{e_1(n,d,k,i)} < 2D_2 q^{-n/2}$.

Lemma 11.10. *Suppose*

$e_2(n,d,k,j) = (n-d-k+j)(k-j) - \psi(k-j) + (d-j)j - \psi(j) - (n-k)k + \psi(k)$.

If $d \le n/2$, $k \le n/2$, $h \le k$, *and* $h \le d/2$. *then*

$$\sum_{j=1}^{h} q^{e_2(n,d,k,j)} < 3q^{-(n-2)/2}.$$

Proof. Since $\partial^2 e_2/\partial j^2 = -4$, it follows from (11.4) that if $e_2(n,d,k,j) \le K$ for all j with $1 \le j \le h$, then $\sum_{j=1}^{h} q^{e_2(n,d,k,j)} \le \dfrac{2}{1-q^{-2}} q^K < 3q^K$. Set $\xi(x,y) = e_2(n,x,y,j)$. It suffices to show that $\xi(x,y) \le -(n-2)/2$ whenever $2j \le x \le n/2$ and $j \le y \le n/2$. Since $\partial\xi/\partial x = 2j - y$ and $\partial\xi/\partial y = 3j - x$, the hypotheses of (11.1) hold with $A = 2j$, with $B = 3j$, $x_* = 2j$, $y_* = j$, and $x^* = y^* = n/2$.

If $y \le 2j$ and $3j \le n/2$, then $\xi(x,y) \le \xi(n/2, j) = -jn/2 \le -n/2$.

If $2j \le y$ then $\xi(x,y) \le \xi(j,n/2) = -jn/2+j^2$ because $y_* \le A$. Since $h \le d/2 \le n/4$, the function $\alpha(x) = -xn/2+x^2$ is decreasing on $[1,h]$ so $-jn/2+j^2 \le -n/2+1$. Therefore $\xi(x,y) \le -n/2+1$ in this case.

Finally, if $y \le 2j$ and $n/2 \le 3j$, then $\xi(x,y) \le \xi(n/2,n/2) = -n^2/4 + 3j(n/2-j) \le -n^2/4 + 3(n/4)^2 = -n^2/16$. By (11.7) we have $-n^2/16 \le 1 - n/2$.

This completes the proof.

Remark . The argument shows that $\sum_i q^{e_2(n,d,k,i)} \le 2D_2 q^{-n/2}$, where $D_j = \dfrac{1}{1-q^{-j}}$.

Lemma 11.11. *Set* $e_4(n,m,k,j) = j(\dfrac{m}{2} - j) + (k-j)(\dfrac{m}{2} - k) - k(n-k) + \psi(k)$. *Assume that* $m \le n$ *and* $2 \le k \le m/2$. *Then* $\sum_{j=0}^{k} q^{e_4(n,m,k,j)} < 4q^{-(n-4)/2}$.

Proof. Because $\partial^2 e_4/\partial j^2 = -2$ and $2/(1-q^{-1}) \le 4$ it suffices, by (11.4), to show that $e_4(n,m,k,j) \le -(n-4)/2$ whenever $0 \le j \le k$. Since $\partial e_4(n,m,k,j)/\partial m > 0$ we have $e_4(n,m,k,j) \le e_4(n,n,k,j)$, so we may assume that $m = n$. Set $\alpha(k,j) = e_4(n,n,k,j)$.

Since $\partial^2\alpha/\partial k^2 = \psi''(k) = 1$, it suffices to show that $\alpha(n/2,j) \le -(n-4)/2$ and $\alpha(b,j) \le -(n-4)/2$ where $b = \max(2,j)$.

By direct evaluation, $\alpha(n/2,j) = j(n/2-j) - n^2/4 + \psi(n/2)$. By (11.2), $j(n/2-j) \le n^2/16$. Therefore $\alpha(n/2,j) \le -3n^2/16 + n^2/8 + n/4 = -n^2/16 + n/4$. By (11.7), we have $-n^2/16 + n/4 \le 9/4 - n/2$. Since $\alpha(k,j)$ is a half-integer it follows that $\alpha(n/2,j) \le -(n-4)/2$.

If $j \ge 2$, then $b = j$ and $\alpha(b,j) = \alpha(j,j) = -nj/2 + \psi(2) \le -n + \psi(2)$. We have $\psi(2) = 2 + a \le 3$, so $\alpha(b,j) \le -n+4$ when $j \ge 2$. If $j < 2$, then $b = 2$ and $\alpha(b,j) = \alpha(2,j) = -n - j^2 + 2j + \psi(2) \le -n+1+\psi(2) \le -n+4$. Since $n \ge 2k \ge 4$, it follows that $\alpha(b,j) \le -n/2+2$ in either case, so $e_4(n,m,k,j) \le -(n-4)/2$, as required. We have $-n+1+\psi(2) \le -n+4 \le 2-n/2$ in either case. Since $q/(q-1) \le 2$, the assertion follows.

Remark . The argument shows that $\sum_i q^{e_4(n,d,k,i)} < 2D_1 q^{-(n-4)/2}$.

Lemma 11.12. *Suppose* $e_5(n,d,k,i,j) = j(d-j) + e(n-d, d-2j, k-j, i) - k(n-k) + \psi(k)$ *where* $e(m,t,h,i) = (m-h)(h-i) + i(t-i) - \psi(h-i)$. *Suppose also that* $2d \le n$, $2k \le n$, *and* $\mathcal{I}_j = \{i : 0 \le i \le k-j, i \le d-2j\}$. *If* $\mathcal{I} = \{(i,j) : 0 \le j, 0 < i+j,$ *and* $i \in \mathcal{I}_j\}$, *then*

$$\sum_{(i,j)\in\mathcal{I}} q^{e_5(n,d,k,i,j)} \le 14q^{-(n-6)/2}.$$

If V *is symplectic or unitary, or if* $n \ge 16$, *then*

$$\sum_{(i,j)\in\mathcal{I}} q^{e_5(n,d,k,i,j)} \le 15q^{-(n-4)/2}.$$

Proof. Set $l = i+j$, $\alpha_1(l) = -ln/2 + \psi(l) + l^2$, and $\alpha_2(l) = -l^2 + \psi(l)$. We claim that

$$e_5(n,d,k,i,j) \le \begin{cases} \alpha_1(l) - li, & l \le n/4 \\ \alpha_2(l) - (n/2 - l)t, & n/4 < l \end{cases}$$

where $t = \lfloor n/2 \rfloor - l - j$. Consider the function $\xi(x,y) = e_5(n,y,x,i,j)$ defined when $l \le x \le n/2$ and $l + j \le y \le n/2$. We have $\partial\xi/\partial x = 2l + j - y$ and $\partial\xi/\partial y = 2l - x$, and ξ satisfies the hypotheses of (11.1) with $A = 2l + j$, with $B = 2i$, $x_* = l$, $y_* = l + j$, and $x^* = y^* = n/2$.

Suppose $l \leq n/4$. If $y \leq 2l + j$, then (11.1) implies that $\xi(x, y) \leq \xi(n/2, l + j) = -l\frac{n}{2} + \psi(l) + lj = \alpha_1(l) - li$. If $2l + j \leq y$, then (11.1) implies that $\xi(x, y) \leq \xi(l, n/2) = -l\frac{n}{2} + \psi(l) \leq \alpha_1(l) - li$. This establishes the claim for $l \leq n/4$.

Suppose $l > n/4$. Then $d < 2l < 2l + j$. We conclude from (11.1) that

$$\begin{aligned}\xi(x, y) \leq \xi(n/2, n/2) &= -\frac{n^2}{4} - 2l^2 + \psi(l) + ln + \frac{1}{2}nj - lj \\ &= -l^2 + \psi(l) - (\frac{n}{2} - l)(\frac{n}{2} - l - j) \\ &\leq -l^2 + \psi(l) - (\frac{n}{2} - l)t\end{aligned}$$

So the claim holds for $l > n/4$. Note that because $l + j \leq d \leq n/2$ and $j \geq 0$, we have $0 \leq n/2 - l - j \leq n/2 - l$. This implies that $0 \leq t \leq \lfloor n/2 - l \rfloor$.

Set $s_l = \sum_{i+j=l} q^{e_5(n,d,k,i,j)}$. The claim and the observation about t imply that

$$s_l \leq \begin{cases} q^{\alpha_1(l)} \sum_{i=0}^{l} q^{-li}, & l \leq n/4 \\ q^{\alpha_2(l)} \sum_{t=0}^{\lfloor n/2-l \rfloor} q^{-(n/2-l)t}, & l > n/4. \end{cases}$$

By (11.8), we have $s_l \leq 3q^{\alpha_1(l)}/2$, $l \leq n/4$, and $s_l \leq 3q^{\alpha_2(l)}/2, l > n/4$.

Since $\alpha_1''(l) = 3$, and $(q + 1)/q \leq 3/2$ we have

$$\sum_{l=1}^{n/4} s_l \leq \frac{3}{2} \sum_{l=1}^{n/4} q^{\alpha_1(l)} \leq \frac{3}{2} \left(\frac{1}{1 - q^{-3/2}} \right) (q^{\alpha_1(1)} + q^{\alpha_1(\lfloor n/4 \rfloor)})$$

by (11.5).

We have $\alpha_1(l) \leq -(n-4)/2$. By (11.13), proved below, $\alpha_1(\lfloor n/4 \rfloor) \leq -(n-6)/2$ so that

$$\begin{aligned}\sum_{l \in [1,n/4]} s_l &\leq \frac{3}{2(1 - q^{-3/2})}(q^{-(n-4)/2} + q^{-(n-6)/2}) \\ &\leq \frac{3}{2(1 - q^{-3/2})}(\frac{q + 1}{q})q^{-(n-6)/2} \\ &< 4q^{-(n-6)/2}.\end{aligned}$$

We have $\alpha_2'(l) = -l + \mathrm{a}/2 \leq -1/2$ and $\alpha''(l) = -1$. By (11.13), $\alpha_2(x) \leq -(n - 6)/2$ for all x with $n/4 < x \leq n/2$. Therefore

$$\sum_{l \in (n/4,n/2]} s_l \leq \frac{3}{2} \sum_{l \in (n/4,n/2]} q^{\alpha_2(l)} < \frac{3}{(1 - q^{-1/2})} q^{-(n-6)/2} < 10q^{-(n-6)/2}.$$

Thus

$$\sum_{l \in [1,n/2]} s_l \leq 10q^{-(n-6)/2} + 4q^{-(n-6)/2} = 14q^{-(n-6)/2}.$$

If $n \geq 16$ or if V is not orthogonal, then $\alpha_1(\lfloor n/4 \rfloor) \leq -(n-4)/2$ by (11.13), and

$$\sum_{l \in [1,n/4]} s_l \leq \frac{3}{1 - q^{-3/2}} q^{-(n-4)/2} < 5q^{-(n-4)/2}.$$

Also, $\displaystyle\sum_{l\in(n/4,n/2]} s_l \leq \frac{3}{(1-q^{-1/2})}q^{-(n-4)/2} < 10q^{-(n-4)/2}$.

Therefore $\sum_{l\in[1,n/2]} s_l < 5q^{-(n-4)/2} + 10q^{-(n-4)/2} = 15q^{-(n-4)/2}$.

Remark . The argument shows that

$$\sum_{(i,j)\in\mathcal{I}} q^{e_5(n,d,k,i,j)} < (E_1^2 D_{3/2} + 2E_1 D_{1/2})q^{-(n-6)/2},$$

and if either $n \geq 16$ or V is not orthogonal then

$$\sum_{(i,j)\in\mathcal{I}} q^{e_5(n,d,k,i,j)} < (2E_1 D_{3/2} + 2E_1 D_{1/2})q^{-(n-4)/2},$$

where $E_1 = \dfrac{q+1}{q}$.

11.13. *Assume that n and x are positive integers, $x \leq n/2$, and*

$$\alpha(x) = \begin{cases} -nx/2 + x^2 + \psi(x), & x \leq n/4 \\ -x^2 + \psi(x), & n/4 \leq x. \end{cases}$$

Then either

1. $\alpha(x) \leq -(n-4)/2$ *or*
2. V *is orthogonal,* $n < 16$*, and* $\alpha(x) \leq -(n-6)/2$.

Proof. Suppose first that V is not orthogonal. Then $\psi(x) \leq x^2/2$. If $x \geq n/4$, then $\alpha(x) \leq -x^2/2$, which decreases. So $\alpha(x) \leq \alpha(n/4) = -n^2/32$. By (11.7), we have $-n^2/32 \leq 1\cdot 2 - n/2$, so $\psi(x) \leq -(n-4)/2$ for $x \geq n/4$. If $x \leq n/4$, then $\alpha(x) \leq -nx/2 + 3x^2/2$ and $\alpha(x) \leq \max(\alpha(1), \alpha(n/4)) \leq \max(3/2 - n/2, -n^2/32) \leq -(n-4)/2$.

If V is orthogonal, then $\psi(x) = x(x+1)/2$. If $x \geq n/4$, then $\alpha(x) = -x^2/2 + x/2$. We have $\alpha'(x) = 1/2 - x < 0$, so $\alpha(x) \leq \alpha(\lceil n/4\rceil) \leq -n^2/32 + n/8$. By (11.7), we have $-n^2/32 + n/8 \leq (5/4)^2 \cdot 2 - n/2$. Since $\alpha(x)$ is an integer, this implies that $\alpha(x) \leq -(n-6)/2$ If $x \leq n/4$, then $\alpha(x) = -nx/2 + 3x^2/2 + x/2$. We have $\alpha(1) = -(n-4)/2$ and $\alpha(n/4) = -n^2/32 + n/8$. Since the graph of $\alpha(x)$ is concave up this implies that $\alpha(x) \leq -(n-6)/2$ in all cases.

The function $y \mapsto -y^2/32 + 5y/8$ decreases for $y \geq 10$, and takes on the value 2 at $y = 16$. Therefore $-y^2/32 + 5y/8 \leq 2$ whenever $y \geq 16$. This implies that $-y^2/32 + y/8 \leq -y/2 + 2$ for all $y \geq 16$, so $\alpha(x) \leq -(n-4)/2$ for $n \geq 16$. □

Lemma 11.14. *Suppose* $e_6(n,d,h,k,j) = j(h-j) + (k-j)(n-d-h-k+j) - k(n-k) + \psi(k)$, $\mathcal{J}_1 = \{j : j \leq d + h - n/2,\ 0 \leq j \leq h,\ 0 \leq k - j \leq n - d - h\}$, $2 \leq k \leq n/2$*, and* $h \leq n/2 - d/2$*. Then* $\sum_{j\in\mathcal{J}_1} q^{e_6(n,d,h,k,j)} \leq 3q^{-(n-3)/2}$.

Proof. We first argue that $e_6(n,d,h,k,j) \leq -kn/2 + \psi(k) + k^2/8$ for all $j \in \mathcal{J}_1$.

Since $\dfrac{\partial e_6}{\partial d} = j - k \leq 0$ and $d \geq n/2 + j - h$ it follows that $e_6(n,d,h,k,j) \leq e_6(n, n/2+j-h, h, k, j) = -kn/2 + \psi(k) - j(n/2 - h - k + j)$. We have $n/2 + j - h \leq$

$d \le n-2h$. Therefore $h \le n/2 - j$. This implies that

$$\begin{aligned} e_6(n, n/2+j-h, h, k, j) &\le -kn/2 + \psi(k) - j(n/2 - (n/2-j) - k + j) \\ &= -kn/2 + \psi(k) + j(k-2j) \\ &\le -\frac{kn}{2} + \psi(k) + \frac{k^2}{8} \end{aligned}$$

by (11.2).

Set $\alpha(x) = -xn/2 + \psi(x) + x^2/8$. Since $\alpha''(x) > 0$, it follows that if $2 \le x \le n/2$ then either $\alpha(x) \le \alpha(2) = -n + \psi(2) + 1/2$ or $\alpha(x) \le \alpha(n/2) = -7n^2/32 + \psi(n/2)$. We have $-n/2 \le -2$ and $\psi(2) \le 3$. Therefore $\alpha(2) \le -n/2 + 3/2$.

Since $\psi(n/2) \le n^2/8 + n/4$, we have $\alpha(n/2) \le -3n^2/32 + n/4 \le (\frac{3}{2})^2 \frac{2}{3} - \frac{n}{2}$ by (11.7). It follows that $\alpha(k) \le 3/2 - n/2$ for all k in $[2, n/2]$.

Therefore $e_6(n,d,h,k,j) \le -(n-3)/2$ for all $j \in J$. Since $\partial^2 e_6(n,d,h,k,j)/\partial j^2 = -4$, it follows from (11.4) that $\sum_{j \in J} q^{e_6(n,d,h,k,j)} \le \frac{2}{1-q^{-2}} q^{-(n-3)/2} < 3q^{-(n-3)/2}$.

Remark . The argument shows that $\sum_{j \in \mathcal{J}_1} q^{e_6(n,d,h,k,j)} \le 2D_2 q^{-(n-3)/2}$.

Lemma 11.15. *Set*

$$e_7(n,d,h,k,i,j) = j(h-j) + e(n-d-h, d+h-2j, k-j, i) - k(n-k) + \psi(k)$$

where $e(m,t,k,i) = (m-k)(k-i) + i(t-i) - \psi(k-i)$. *Also, set*

$$\mathcal{J}_2 = \{j : j > d+h-n/2,\ 0 \le j \le h,\ 0 \le k-j \le n-d-h\},$$
$$\mathcal{I}_j = \{i : 0 \le i \le k-j, i \le d+h-2j\},$$
$$\mathcal{I} = \{(i,j) : j \in \mathcal{J}_2, i \in \mathcal{I}_j\}.$$

If $d \le n/4$, $k \le n/2$, $d+h > n/2$, $d+2h \le n$, *and* $2 \le k$ *then* $\sum_{(i,j)\in\mathcal{I}} q^{e_7(n,d,h,k,i,j)} < \frac{3}{2} n q^{-(n-3)/2}$.

Proof. Set $l = i+j$ and $t_l = \sum_{i+j=l} q^{e_7(n,d,h,k,i,j)}$.

Assume that $e_7(n,d,h,k,i,j) \le K$ for all $i,j \in \mathcal{I}$. Then

$$t_l \le \begin{cases} \frac{q}{q-1} q^K, & l \ne k-d \\ (k-d+1)q^K, & l = k-d. \end{cases}$$

by (11.3). Since $d + 2h \le n < 2(d+h)$ we have $d > 0$, whence $d \ge 1$. Therefore

$$\begin{aligned} \sum_{(i,j)\in\mathcal{I}} q^{e_7(n,d,h,k,i,j)} &= \sum_{l=0}^{k} t_l \le \left((\frac{q}{q-1})k + (k-d+1) \right) q^K \\ &< \frac{2q-1}{q-1} k q^K \\ &\le 3\frac{n}{2} q^K \end{aligned}$$

because $q \ge 2$ and $k \le n/2$. To complete the proof of the lemma, it suffices to show that $e_7(n,d,h,k,i,j) \le -(n-3)/2$ for all $(i,j) \in \mathcal{I}$. Set $\xi(x,y) = e_7(n,d,y,x,i,j)$. Since $e_7(n,d,h,k,i,j)$ is a half-integer, this will follow from the following.

11.16. *$\xi(x,y) < -(n-4)/2$ for all integers x, y with $l \le x \le n/2$ and $n/2 - d \le y \le n/2 - d + s$, $s = \min(j, d/2)$.*

Proof of (11.16): We have $\dfrac{\partial \xi}{\partial x} = 2l + j - d - y$ and $\dfrac{\partial \xi}{\partial y} = 2l - x$.

If $2l + j \le d + y$. Then $\xi(x,y) \le \xi(l,y) \le \xi(l, n/2 - d + s)$.

If $d + y < 2l + j$ and $4l \le n$, then $\xi(x,y) \le \xi(n/2, y) \le \xi(n/2, n/2 - d)$.

If $d + y < 2l + j$ and $n < 4l$, then $\xi(x,y) \le \xi(n/2, y) \le \xi(n/2, n/2 - d + s)$.

Letting $\phi(x,t) = \xi(x, n/2 - d + t)$. It therefore suffices to bound $\phi(n/2, 0)$ when $4l \le n$, $\phi(n/2, s)$ when $n < 4l$, and $\phi(l, s)$ in general.

We compute that

$$\phi(x,t) = -\frac{n}{2}x + 2xl - 2l^2 + \psi(l) - t(x - 2l) - j(d - x + l).$$

Therefore $\phi(l,s) = -nl/2 + \psi(l) + ls - jd$. If $s = j$, then $j \le d/2$ and $ls - jd = lj - dj \le l^2/8$ by (11.2). If $s = d/2$, then $d/2 \le j$ and $ls - jd = ld/2 - dj \le l^2/8$ by (11.2). This implies that $\phi(l,s) \le -nl/2 + \psi(l) + l^2/8$.

Set $\alpha(l) = -nl/2 + \psi(l) + l^2/8$. Then $\alpha(l) = 5l^2/8 - (n - \mathrm{a})l/2$. By assumption, $n \ge 2k \ge 4$. If V is orthogonal, then $n \ge 6$. Therefore $n \ge (10 + 8\mathrm{a})/3$ where a is the constant in the formula $\psi(x) = x(x + \mathrm{a})/2$. This implies that $1 + n/2 \le 4(n - \mathrm{a})/5$. Applying (11.6), we have $\alpha(l) \le \alpha(1) = -n/2 + 1/8 + \psi(1)$ for all l in $[1/n/2]$. Since $\phi(l,s)$ is a half-integer, we have $\phi(l,s) \le 1 - n/2$.

Suppose $4l \le n$, and set $z = \phi(n/2, 0)$. Then $z = -n^2/4 + ln - 2l^2 + \psi(l) - j(d - n/2 + l)$, and $z \le -n^2/4 + ln - 2l^2 + \psi(l) - j(1 - n/2 + l)$ because $d \ge 1$. Since $l + 1 \le 2l \le n/2$, we have $\partial z/\partial j = n/2 - 1 - l > 0$. Since $j \le l$, this implies that $z \le -n^2/4 + ln - 2l^2 + \psi(l) - l(1 - n/2 + l) = -n^2/4 + 3ln/2 - 3l^2 + \psi(l) - l$.

Since $\psi(l) \le l^2/2 + l/2$, we have $\dfrac{3}{2}ln - 3l^2 + \psi(l) - l \le \dfrac{3n-1}{2}l - \dfrac{5}{2}l^2$. By (11.2), $\dfrac{3n-1}{2}l - \dfrac{5}{2}l^2 \le (\dfrac{3n-1}{2})^2 \cdot (\dfrac{1}{4}) \cdot \dfrac{2}{5} = \dfrac{9n^2 - 6n + 1}{40}$. We therefore have

$$\begin{aligned} -\frac{n^2}{4} + \frac{3}{2}ln - 3l^2 + \psi(l) - l &\le -\frac{n^2}{40} - \frac{3}{20}n + \frac{1}{40} \\ &\le \frac{1}{40} + (\frac{7}{10})^2\frac{5}{2} - \frac{n}{2} = \frac{5}{4} - \frac{n}{2} < 2 - \frac{n}{2} \end{aligned}$$

by (11.7). This shows that $\phi(n/2, 0) < 2 - \dfrac{n}{2}$ when $4l \le n$.

Now assume that $n < 4l$, and set $z = \phi(n/2, s)$. Then

$$z = -\frac{n^2}{4} + ln - 2l^2 + \psi(l) - s(n/2 - 2l) - j(d - n/2 + l).$$

If $s = j$, then $j \le d/2$ and

$$\begin{aligned} z &= -n^2/4 + ln - 2l^2 + \psi(l) + lj - dj \\ &\le -\frac{n^2}{4} + ln - 2l^2 + \psi(l) + \frac{l^2}{8} \end{aligned}$$

by (11.2).

Statement (11.2) and the definition of ψ imply that $ln - 2l^2 + \psi(l) + l^2/8 \le ln - \dfrac{11}{8}l^2 + \dfrac{1}{2}l = \dfrac{1}{2}(2n+1)l - \dfrac{11}{8}l^2 \le \dfrac{(2n+1)^2}{4} \cdot \dfrac{1}{4} \cdot \dfrac{8}{11} = \dfrac{(2n+1)^2}{22}$. Therefore, by (11.7),

$$\begin{aligned}
-\frac{n^2}{4} + ln - 2l^2 + \psi(l) + \frac{l^2}{8} &\leq -\frac{3n^2}{44} + \frac{2}{11}n + \frac{1}{22} \\
&\leq \frac{1}{22} + (\frac{15}{11})^2 \frac{11}{12} - \frac{n}{2} \\
&= \frac{7}{4} - \frac{n}{2} < 2 - \frac{n}{2}.
\end{aligned}$$

If $s = d/2$, then $d \leq 2j$ and

$$z = -n^2/4 + ln - 2l^2 + \psi(l) - jd - (\frac{n}{4} - l)d + (\frac{n}{2} - l)j.$$

Set $\phi(j) = -jd - (\frac{n}{4} - l)d + (\frac{n}{2} - l)j$. Then ϕ is linear in j, so $\phi(j)$ is bounded on the interval $[d/2, l]$ by the $\max(\phi(d/2), \phi(l))$. Since $\phi(d/2) = ld/2 - d^2/2 \leq l^2/8$ and $\phi(l) = -nd/4 + ln/2 - l^2 \leq -n/4 + ln/2 - l^2$, it follows that either $z \leq \frac{-n^2}{4} + ln - 2l^2 + \psi(l) + \frac{l^2}{8}$ or $z \leq -\frac{n^2}{4} + \frac{3}{2}ln - 3l^2 - \frac{n}{4} + \psi(l) = -\frac{n^2}{4} - \frac{n}{4} + \frac{3}{2}ln - 3l^2 + \psi(l)$. As shown above, the former is less than $-(n-4)/2$. It remains only to show that $-\frac{n^2}{4} + \frac{3}{2}ln - 3l^2 - \frac{n}{4} + \psi(l) < -(n-4)/2$.

Because $\psi(l) \leq l^2/2 + l/2$, we have $\frac{3}{2}ln - 3l^2 + \psi(l) \leq \frac{3}{2}ln - \frac{5}{2}l^2 + \frac{l}{2} = \frac{3n+1}{2}l - \frac{5}{2}l^2 \leq \frac{(3n+1)^2}{16} \cdot \frac{2}{5} = \frac{(3n+1)^2}{40}$ by (11.2). (11.7) now implies that

$$\begin{aligned}
\frac{-n^2}{4} - \frac{n}{4} + \frac{(3n+1)^2}{40} &= -\frac{n^2}{40} - \frac{n}{10} + \frac{1}{40} \\
&\leq \frac{1}{40} + (\frac{4}{5})^2 \cdot \frac{1}{16} \cdot 40 - \frac{n}{2} < 2 - \frac{n}{2}.
\end{aligned}$$

This completes the proof of (11.16), and therefore the proof of Lemma 11.15.

Remark . The argument shows that

$$\sum_{j \in \mathcal{J}_2} \sum_{i \in \mathcal{I}_j} q^{e_7(n,d,h,k,i,j)} \leq (1 + \frac{D_1}{2q}) n q^{-(n-3)/2}.$$

12. Orthogonal subgroups of symplectic groups in even characteristic

We prove the following in this section.

Proposition 12.1. *Let V be an odd-dimensional orthogonal space in even characteristic and assume that rad V is one-dimensional and nonsingular. Let U be a complement to rad V, and let $\mathcal{H} = \mathcal{H}(V, U)$ be the collection of non-degenerate hyperplanes of V isometric to U. Let $\mathcal{H}_g = \mathcal{H}(V, U)_g$ be the collection of g-invariant members of $\mathcal{H}$. Then one of the following holds.*

1.
$$\frac{|\mathcal{H}_g|}{|\mathcal{H}|} < 4q^{-(n+1)/2}.$$

2. *The eigenspace for g containing rad V has codimension $d < n/2$ and*

$$q^{-d} - 2q^{-(n-1)/2} < \frac{|\mathcal{H}_g|}{|\mathcal{H}|} < q^{-d} + 5q^{-(n-1)/2}$$

Proof. If λ is the scalar such that $\mathbf{q}(v^g) = \lambda\mathbf{q}(v)$ for all $v \in V$, then there is a $\mu \in \mathbf{F}_q$ such that $\mu^2 = \lambda$ because q is even. If $h = \mu^{-1}g \in I(V)$, then $h \in I(V)$ and h and g have the same eigenspaces and the same invariant subspaces. Replacing g by h, it suffices to assume that $g \in I(V)$. In particular, g acts trivially on rad V. It follows that $\mathcal{H}_g = \{Z \in \mathcal{H} : [V, g] \subseteq Z\}$.

Set $W = [V, g]$, and let $\mathcal{W}$ be the set of all subspaces W' of V such that W' is isometric to W and rad $V \not\subseteq W'$. Let $\mathcal{W}_0$ be the set of all subspaces of U that are isometric to W. Assume that U has type δ.

Let $\mathcal{X}$ be the set of all ordered pairs (U', W') such that $U' \in \mathcal{U}$, $W' \in \mathcal{W}$, and $W' \subseteq U'$. By Witt's theorem, $I(V)$ is transitive on $\mathcal{H}$, so we have $|\mathcal{X}| = |\mathcal{H}||\mathcal{W}_0|$. On the other hand, $|\mathcal{X}| = |\mathcal{W}||\mathcal{H}_g|$ because $I(V)$ is also transitive on $\mathcal{W}$. Therefore $|\mathcal{H}_g|/|\mathcal{H}| = |\mathcal{W}_0|/|\mathcal{W}|$.

Assume that W has dimension d, and that the totally singular radical T of W has dimension t.

Suppose rad W is totally singular. Consider the injective function $\phi : X \mapsto (T(X), X/T(X))$ defined on $\mathcal{W}$. If $X \in \mathcal{W}$ then $T(X) \in \mathcal{S}(V, t)$ and $X/T(X) \in \mathcal{N}(T(X)^\perp/T(X), W/T)$. If $X \in \mathcal{W}_0$ then $T(X) \in \mathcal{S}(U, t)$ and $X/T(X) \in \mathcal{N}((U \cap T(X)^\perp)/T(X), W/T)$.

For each $Y \in \mathcal{S}(V, t)$ and $Z/Y \in \mathcal{N}(Y^\perp/Y, W/T)$ we have $T(Z) = Y$ and $Z \in \mathcal{W}$. If $Z/Y \in \mathcal{N}((Y^\perp \cap U)/T, W/T)$, then $Z \in \mathcal{W}_0$. This implies that

$$|\mathcal{W}| = |\mathcal{S}(V,t)||\mathcal{N}(T^\perp/T, W/T)| \text{ and}$$

$$|\mathcal{W}_0| = |\mathcal{S}(U,t)|\ |\mathcal{N}((U \cap T^\perp)/T, W/T)|.$$

Therefore

$$\frac{|\mathcal{H}_g|}{|\mathcal{H}|} = \frac{|\mathcal{S}(U,t)|}{|\mathcal{S}(V,t)|}\frac{|\mathcal{N}((U \cap T^\perp)/T, W/T)|}{|\mathcal{N}(T^\perp/T, W/T)|}.$$

Assuming as we may, that $\mathcal{W}_0 \neq \emptyset$, we must have $d - t = \dim W/T \leq \dim(U \cap T^\perp)/T = n - 1 - 2t$, whence $n - d - t - 1 \geq 0$. From (2.9) and Lemma 2.12.(1) we have

$$\frac{|\mathcal{S}(U,t)|}{|\mathcal{S}(V,t)|} = \frac{\beta_{\mathbf{O}^\delta}(n-1)}{\beta_{\mathbf{O}^\circ}(n)}\frac{\beta_{\mathbf{O}^\circ}(n-2t)}{\beta_{\mathbf{O}^\delta}(n-2t-1)}q^{-t}.$$

Since $\dim T^\perp/T = n - 2t$ and $\dim(U \cap T^\perp)/T = n - 2t - 1$, it follows from (2.7) and Lemma 2.12.(2) that

$$\frac{|\mathcal{N}((U \cap T^\perp)/T, W/T)|}{|\mathcal{N}(T^\perp/T, W/T)|} = \frac{\beta_{\mathbf{O}^\delta}(n-2t-1)}{\beta_{\mathbf{O}^\circ}(n-2t)}\frac{\beta_{\mathbf{O}^\circ}(n-d-t)}{\beta_{\mathbf{O}^\eta}(n-d-t-1)}q^{-(d-t)}$$

where η is the type of the orthogonal complement to W/T in $(U \cap T^\perp)/T$. Note that $\eta = +$ if $n - d - t - 1 = 0$.

It follows that

$$\frac{|\mathcal{H}_g|}{|\mathcal{H}|} = \frac{\beta_{\mathbf{O}^\delta}(n-1)}{\beta_{\mathbf{O}^\circ}(n)}\frac{\beta_{\mathbf{O}^\circ}(n-d-t)}{\beta_{\mathbf{O}^\eta}(n-d-t-1)}q^{-d} = \frac{1 + \eta q^{-(n-d-t-1)/2}}{1 + \delta q^{-(n-1)/2}}q^{-d}.$$

If $d \geq n/2$, then $d \geq (n+1)/2$ because n is odd. Since $n-d-t-1 \geq 0$ we have $\frac{|\mathcal{H}_g|}{|\mathcal{H}|} < \frac{2}{1-q^{-(n-1)/2}} q^{-(n+1)/2} \leq 4q^{-(n+1)/2}$ because $1-q^{-(n-1)/2} \geq 1-q^{-1} \geq 1/2$.

Since $t \leq d$ we have $(d+t)/2 - d \leq 0$. If $d < n/2$, it follows that

$$\frac{q^{-d} - q^{-(n-1)/2}}{1+q^{-(n-1)/2}} < \frac{|\mathcal{H}_g|}{|\mathcal{H}|} < \frac{q^{-d} + q^{-(n-1)/2}}{1-q^{-(n-1)/2}}.$$

By 3.3.(1) and (2), we have

$$\frac{q^{-d} - q^{-(n-1)/2}}{1+q^{-(n-1)/2}} > (q^{-d} - q^{-(n-1)/2})(1-q^{-(n-1)/2}) > q^{-d} - 2q^{-(n-1)/2}.$$

By (3.4),

$$\frac{q^{-d} + q^{-(n-1)/2}}{1-q^{-(n-1)/2}} q^{-d} < (q^{-d} + q^{-(n-1)/2})(1+2q^{-(n-1)/2}) < 1 + 5q^{-(n-1)/2}.$$

This establishes the proposition where rad W is totally singular.

We may therefore assume that rad W contains a nonsingular 1-space. Set $R =$ rad W. For $A \subseteq V$ set $A^+ = A + \mathrm{rad}\ V$. Then R/T is a nonsingular 1-subspace of $T^\perp/T$ and $R/T \neq T^+/T$. Furthermore W is a complement to R^+/R in $R^\perp/R$. This implies that the map $X \mapsto (T(X), \mathrm{rad}\ X/T(X), X^+/(\mathrm{rad}\ X)^+, X/\mathrm{rad}\ X)$ defines a bijection between $\mathcal{W}$ and $\{(S, Y/S, Z/Y^+, H/Y)\}$ where S is a totally singular t-subspace of U, Y/S is a nonsingular 1-space of $S^\perp/S$ different from S^+/S, Z/Y^+ is a non-degenerate $d-t-1$-subspace of the symplectic space $Y^\perp/Y^+$, and H/Y is a complement to Y^+/Y in Z/Y.

Since there are q^{d-t-1} complements to Y^+/Y in Z/Y this implies that

$$|\mathcal{W}| = |\mathcal{S}(V,t)|\ (|\mathcal{P}(T^\perp/T)| - 1)\ |\mathcal{N}(R^\perp/R^+, W^+/R^+)|\ q^{d-t-1}.$$

To compute $|\mathcal{W}_0|$, consider the function $X \mapsto (T(X), \mathrm{rad}\ X/T(X), X/\mathrm{rad}\ X)$. This defines a bijection between $\mathcal{W}_0$ and the collection of triples $(S, Y/S, Z/Y)$ where S is a totally singular t-subspace of U, Y/S is a non-singular 1-subspace of $(U \cap S^\perp)/S$, and Z/Y is a non-degenerate (symplectic) $d-t-1$-subspace of $(U \cap Y^\perp)/Y$. Letting $\mathcal{P}(H)$ denote the set of non-singular 1-subspaces in H, we have

$$|\mathcal{W}_0| = |\mathcal{S}(U,t)|\ |\mathcal{P}((U \cap T^\perp)/T)|\ |\mathcal{N}((U \cap R^\perp)/R, W/R)|.$$

Since $(U \cap R^\perp)/R$ and $R^\perp/(R + \mathrm{rad}\ V)$ are isometric, it follows that

$$\frac{|\mathcal{W}_0|}{|\mathcal{W}|} = \frac{|\mathcal{S}(U,t)|}{|\mathcal{S}(V,t)|} \frac{|\mathcal{P}(U \cap T^\perp)/T)|}{(|\mathcal{P}(T^\perp/T)| - 1)} q^{-(d-t-1)}.$$

Every coset of $(\mathrm{rad}\ V + T)/T$ in the nonsingular (but not non-degenerate) $n-2t$-space $T^\perp/T$ contains a unique vector $v+T$ with $\tilde{\mathbf{q}}(v+T) = 1$. Therefore $|\mathcal{P}(T^\perp/T)| = q^{n-2t-1}$. By (4.8), we have

$$|\mathcal{P}((U \cap T^\perp)/T)| = (1 - \delta q^{-(n-2t-1)/2}) q^{n-2t-2}.$$

In view of Lemmas 2.3 and 2.12 this implies that

$$\begin{aligned}\frac{|\mathcal{H}_g|}{|\mathcal{H}|} &= \frac{|\mathcal{S}(U,t)|}{|\mathcal{S}(V,t)|}\frac{(1-\delta q^{-(n-2t-1)/2})q^{n-2t-2}}{q^{n-2t-1}-1}q^{-(d-t-1)} \\ &= \frac{\beta_{\mathbf{O}^\delta}(n-1)}{\beta_{\mathbf{O}^\circ}(n)}\frac{\beta_{\mathbf{O}^\circ}(n-2t)}{\beta_{\mathbf{O}^\delta}(n-2t-1)}q^{-t}\frac{1}{1+\delta q^{-(n-2t-1)/2}}q^{-1}q^{-(d-t-1)} \\ &= \frac{1}{1+\delta q^{-(n-1)/2}}q^{-d}.\end{aligned}$$

Therefore

$$1-q^{-(n-1)/2} < \frac{|\mathcal{H}_g|}{|\mathcal{H}|} < 1+2q^{-(n-1)/2}$$

by (3.3) and (3.4).

13. Field automorphisms

The purpose of this section is to prove the following.

Proposition 13.1. *Assume that $g \in \Gamma(V)\setminus\Delta(V)$. Then*
1. $|\mathcal{L}(V,k)_g| < 6q^{-(n-1)/2}|\mathcal{L}(V,k)|$.
2. *If U is a non-degenerate k-space then* $|\mathcal{N}(V,U)_g| < 20q^{-(n-1)/2}|\mathcal{N}(V,U)|$.
3. $|\mathcal{S}(V,k)_g| < (83+1.4n)q^{-(n-6)/2}|\mathcal{S}(V,k)|$.
4. $|\mathcal{P}(V)_g| < 7q^{-(n-1)/2}|\mathcal{P}(V)|$.
5. $|\mathcal{H}(V,U)_g| < 13q^{-(n-1)/2}|\mathcal{H}(V,U)|$.

Proof. Since $\mathcal{L}(V,k)_g \subseteq \mathcal{L}(V,k)_{g^r}$ for all positive integers k and r, it suffices to assume that $g^r \in \Delta(V)$ for some prime r. Set $h = g^r$.

Since $q \geq 4$ we have $C_1 \leq 16/11$, $C_2 < 13/12$, $c_2 \geq 9/32$, $C_3 \leq 11/4$, and $c_3 \geq 5/7$.

Because $\mathcal{L}(V,k)_g \subseteq \mathcal{L}(V,k)_h$, we have, as a consequence of Propositions 4.6, 6.7, 7.3, 10.9, and 12.1:

13.2. *If h has no eigenspace of dimension at least $n/2$, then*
1. $|\mathcal{L}(V,k)_g| < (1.5+3C_1)q^{-n/2}|\mathcal{L}(V,k)| \leq 6q^{-n/2}|\mathcal{L}(V,k)|$.
2. *If V is non-degenerate and U is a non-degenerate k-space then*
$$|\mathcal{N}(V,U)_g| < \frac{C_1}{c_2}(1.5+3C_1)q^{-n/2}|\mathcal{N}(V,U)| \leq 32q^{-n/2}|\mathcal{N}(V,U)|.$$
3. *If V is non-degenerate then*
$$\begin{aligned}\frac{|\mathcal{S}(V,k)_g|}{|\mathcal{S}(V,k)|} &\leq (\frac{3C_1^2}{c_3}+\frac{3C_1^2C_3}{2c_3}n)q^{-(n-3)/2}+\frac{10C_1^2C_3}{c_3}q^{-(n-6)/2} \\ &< (72+1.4n)q^{-(n-6)/2}.\end{aligned}$$
4. *If V is a non-degenerate orthogonal spaces and q is even then* $|\mathcal{P}(V)_g| < 12q^{-n/2}|\mathcal{P}(V)|$.
5. *If V is an odd-dimensional orthogonal spaces and q is even then* $|\mathcal{H}(V,U)_g| < 4q^{-(n+1)/2}|\mathcal{H}(V,U)|$.

Because $q^{-1/2} \leq 1/2$, the inequalities of (13.2) are stronger than those of Proposition 13.1. Assume therefore that h has an eigenspace W of codimension d where $d \leq n/2$.

When computing $|\mathcal{P}(V)_g|$, assume further that W contains nonsingular vectors.

13.3. 1.

$$|\mathcal{L}(V,k)_g| < |\mathcal{L}(W,k)_g| + 5q^{-n/2}|\mathcal{L}(V,k)|.$$

2. *If V is non-degenerate and U is a non-degenerate k-space, then*

$$|\mathcal{N}(V,U)_g| < |\mathcal{N}(W,U)_g| + 23q^{-n/2}|\mathcal{N}(V,U)|.$$

3. *If V is non-degenerate, then*

$$|\mathcal{S}(V,k)_g| \leq |\mathcal{S}(W,k)_g| + 82q^{-(n-6)/2}|\mathcal{S}(V,k)|.$$

4. *If V is an even-dimensional orthogonal space in characteristic 2, then*

$$|\mathcal{P}(V)_g| \leq |\mathcal{L}(W,1)_g|.$$

5. *If V is an odd-dimensional orthogonal space in characteristic 2, then*

$$|\mathcal{H}(V,U)_g| \leq |\mathcal{L}(W',1)_{g'}|$$

where g' is a semilinear transformation on V but not a linear transformation, $(g')^r$ is a linear transformation, and W' is an eigenspace for ${g'}^r$ with $\dim W' = \dim W$.

Proof. (1) If U is a g-invariant k-space then either $U \subseteq W$, or U is an h-invariant k-space that is not contained in W. Therefore $|\mathcal{L}(V,k)_g| \leq |\mathcal{L}(W,k)_g| + |\mathcal{L}(V,k)_h| - |\mathcal{L}(W,k)_h|$. By (6.4) we have $|\mathcal{L}(V,k)_h| - |\mathcal{L}(W,k)_h| < 3C_1q^{-n/2}|\mathcal{L}(V,k)|$.

(2) Similarly, $|\mathcal{N}(V,U)_g| \leq |\mathcal{N}(W,U)_g| + |\mathcal{L}(V,k)_h| - |\mathcal{L}(W,k)_h| \leq |\mathcal{N}(W,k)_g| + 3C_1q^{-n/2}|\mathcal{L}(V,k)|$. By Lemmas 2.1 and 2.2, we have $|\mathcal{L}(V,k)| \leq \dfrac{C_1}{c_2}|\mathcal{N}(V,U)|$. Therefore $|\mathcal{N}(V,U)_g| \leq |\mathcal{N}(W,U)_g| + \dfrac{3C_1^2}{c_2}q^{-n/2}|\mathcal{N}(V,U)|$. By numerical computation, $3C_1^2/c_2 < 23$.

(3) We have $|\mathcal{S}(V,k)_g| \leq |\mathcal{S}_0(W,k)_g| + |\mathcal{S}(V,k)_h| - |\mathcal{S}_0(W,k)_h|$ where W is the space corresponding to h described in Lemma 8.1. If W is non-degenerate then $|\mathcal{S}(V,k)_h| - |\mathcal{S}_0(W,k)_h| \leq \dfrac{3C_3^2}{c_3}q^{-(n-2)/2}|\mathcal{S}(V,k)|$ by (9.2). Otherwise, $|\mathcal{S}(V,k)_h| - |\mathcal{S}_0(W,k)_h| \leq \dfrac{10C_1^2C_3}{c_3}q^{-(n-6)/2}|\mathcal{S}(V,k)|$ by Lemma 10.3. By numerical computation $10C_1^2C_3/c_3 < 82$.

(4) In the case of orthogonal spaces in characteristic 2, at most one eigenspace can contain nonsingular eigenvectors. This implies that every g-invariant nonsingular point is contained in W. Therefore $|\mathcal{P}(V)_g| \leq |\mathcal{L}(W,1)_g|$.

(5) Since rad V is h-invariant, we have $[\text{rad } V, \lambda h] = 1$ for some scalar λ. Set $X = [V, \lambda h]$. Then $\mathcal{H}(V,U)_h$ consists of the elements of $\mathcal{H}(V,U)$ that contain X. For every $Z \in \mathcal{H}(V,U)_g$ we have $X \subseteq Z$ and $Z^g = Z$. Regarding g as an element of $\Gamma L(V)$ and applying the transpose-inverse automorphism ι, we have $Z^\iota \subseteq C_V(\lambda^{-1}h^\iota)$ and $(Z^\iota)^{g^\iota} = Z^\iota$. Setting $W' = C_V(\lambda^{-1}h)$ and $g' = g^\iota$, we have $Z^\iota \in \mathcal{L}(W',1)_{g'}$ and $\dim W' = \dim W$. Furthermore g' is semilinear and $(g')^r = h^\iota$ is linear. $\square$

13.4. 1. $|\mathcal{L}(W,k)_g| < 3.5q^{-(n-1)/2}|\mathcal{L}(V,k)|$.

2. $|\mathcal{N}(W,U)_g| < 7q^{-(n-1)/2}|\mathcal{N}(V,U)|$ *whenever U is a non-degenerate k-space.*
3. $|\mathcal{S}(W,k)_g| < 4q^{-(n-2)/2}|\mathcal{S}(V,k)|$.
4. $|\mathcal{L}(W,1)_g| < 7q^{-(n-1)/2}|\mathcal{P}(V)|$.

5. $|\mathcal{L}(W,1)_g| < 13q^{-(n-1)/2}|\mathcal{H}(V,U)|$ *whenever* U *is a non-degenerate hyperplane of* V.

Proof. By assumption, g is an additive homomorphism of V and there is a nontrivial field automorphism τ of $\mathbf{F}_q$ of prime order r such that $(\alpha v)g = \alpha^\tau(v)g$ for every $v \in V$ and $\alpha \in \mathbf{F}_q$.

We claim that there is a scalar ω and an $\mathbf{F}$-basis $\{w_i\}$ for W such that $(w_i)g = \omega w_i$ for all i. To prove this, let $H = GL(W)$, and let $\bar{g}$ be the image of g in $\mathrm{Aut}H$ under the map $G_W \to \mathrm{Aut}H$. Then $\bar{g}$ has prime order, so by (7-2) of [9], it follows that $\bar{g}$ is conjugate to $\bar{\phi}$ for some standard field automorphism ϕ of V. This implies that there is a scalar ω and there is a basis $\{w_i\}$ of W such that $(w_i)g = \omega w_i$ for all i, so the claim holds.

Replacing g by $\omega^{-1}g$ we may assume that $w_i g = w_i$ for all i. If U is a g-invariant k-subspace of W, then applying the claim to U implies that there is a scalar μ and there is a basis $\{u_j\}$ of U with $(u_j)g = \mu u_j$ for all j. Without loss, we may assume that $u_1 = w_{i_0} + \sum_{i \neq i_0} \beta_i w_i$ for scalars β_i. Therefore $(u_1)g = w_{i_0} + \sum_{i \neq i_0} \beta_i^\tau w_i$. It follows that $\mu = 1$. Letting $\mathbf{k}$ be the fixed subfield of τ, every g-invariant k-subspace of V must have a basis consisting of $\mathbf{k}$-linear combinations of $\{w_i\}$.

By Lemma 2.1 we have $|\mathcal{L}(V,k)| \geq q^{k(n-k)}$ and $|\mathcal{L}(W,k)_g| < 3.5q_0^{k(n-k)} \leq 3.5q^{k(n-k)/2}$. Therefore

$$\frac{|\mathcal{L}(W,k)_g|}{|\mathcal{L}(V,k)|} < 3.5q^{-k(n-k)/2} \leq 3.5q^{-(n-1)/2}.$$

By Lemma 2.2 we have $|\mathcal{N}(V,U)| \geq c_2 q^{k(n-k)} \geq \dfrac{9}{32}q^{k(n-k)/2}$. Also, $|\mathcal{N}(W,U)_g| < \dfrac{15}{8}q_0^{k(n-k)} \leq \dfrac{15}{8}q^{k(n-k)/2}$. Therefore

$$\frac{|\mathcal{N}(W,U)_g|}{|\mathcal{N}(V,U)|} < \frac{15}{8}\cdot\frac{32}{9}q^{-k(n-k)/2} < 7q^{-k(n-k)/2} \leq 7q^{-(n-1)/2}.$$

By Lemma 2.3, $|\mathcal{S}(V,k)| \geq \dfrac{5}{7}q^{k(n-k)-\psi(k)}$, and $|\mathcal{S}(W,k)_g| \leq \dfrac{11}{4}q_0^{k(n-k)-\psi(k)}$. Therefore

$$\frac{|\mathcal{S}(W,k)_g|}{|\mathcal{S}(V,k)|} \leq \frac{11}{4}\cdot\frac{7}{5}q^{(\phi(k)-k(n-k))/2} < 4q^{(\phi(k)-k(n-k))/2} \leq 4q^{-(n-2)/2}.$$

This proves the first three statements.

By (1), $|\mathcal{L}(W,1)_g| \leq 3.5q^{-(n-1)/2}|\mathcal{L}(V,1)|$. By (4.1), $|\mathcal{L}(V,1)| < (1-q^{-1})^{-1}q^{n-1} \leq 4q^{n-1}/3$. We have $|\mathcal{P}(V)| \geq q^{n-1}(1-q^{-n/2}) \geq 3q^{n-1}/4$ by (4.8) because $q \geq 4$. Therefore $|\mathcal{L}(V,1)| \leq (4/3)^2|\mathcal{P}(V)|$ and $|\mathcal{L}(W,1)_g| \leq (4/3)^2(3.5)q^{-(n-1)/2}|\mathcal{P}(V)| < 7q^{-(n-1)/2}|\mathcal{P}(V)|$.

To prove (5), assume that U has type ϵ. Then $|\mathcal{H}(V,U)| = [I(V):I(U)] = \frac{1}{2}q^{n-1}(1+\epsilon q^{-(n-1)/2})$. Therefore $|\mathcal{H}(V,U)| \geq \frac{1}{2}q^{n-1}(1-q^{-(n-1)/2})$. Since $|\mathcal{L}(V,1)| \leq 4q^{n-1}/3$, it follows that

$$\frac{|\mathcal{L}(V,1)|}{|\mathcal{H}(V,U)|} \leq \frac{8}{3(1-q^{-(n-1)/2})} < \frac{32}{9}$$

because $q \geq 4$. By (1), we have $|\mathcal{L}(W,1)_g| \leq 3.5q^{-(n-1)/2}|\mathcal{L}(V,1)|$. Therefore $|\mathcal{L}(W,1)_g| < 13q^{-(n-1)/2}|\mathcal{H}(V,U)|$. $\square$

Proposition 13.1 is a straightforward consequence of the three previous results because $q^{-n/2} \leq q^{-(n-1)/2}/2$.

14. Graph automorphisms

Assume in this section that V is a linear space and that G contains 'graph' automorphisms lying outside $\Gamma(V)$. that is, G contains elements that are not semilinear or linear transformations of V.

Set $W = V + V^*$ where $V^* = \mathrm{Hom}_{\mathbf{F}}(V, \mathbf{F})$ is the dual of V. For $h \in \Gamma(V)$ and $\phi \in V^*$ define $\phi^h \in V^*$ by $\phi^h(v) = \phi(vh^{-1})$ for $v \in V$. There is an embedding $\pi : \Gamma(V) \to GL(W)$ which extends to a permutation action on $\mathcal{L}(V) \cup \mathcal{L}(V^*)$. Using π, elements of $\Gamma(V)$ act on $\mathcal{L}(V)$ and on $\mathcal{L}(V^*)$ while elements of $A(V) \setminus \Gamma(V)$ interchange members of $\mathcal{L}(V)$ and $\mathcal{L}(V^*)$. In particular, if $U \in \mathcal{L}(V, k)$ and $g \in A(V) \setminus \Gamma(V)$, then $U^{\pi(g)} \in \mathcal{L}(V^*, k)$.

For $X \in \mathcal{L}(V^*)$ let $X^\vee = \cap_{x \in X} \ker x$. Then $\dim X^\vee = n - \dim X$. For $g \in A(V) \setminus \Gamma(V)$ define $U^g = (U^{\pi(g)})^\vee$. Then $U^g \in \mathcal{L}(V, n-k)$ whenever $U \in \mathcal{L}(V, k)$. Therefore $A(V)$ acts on $\mathcal{L}(V, n/2)$ when n is even. If $k \leq n/2$ and $g \in \Gamma(V)$, then g acts component wise on $\mathcal{L}(V, k) \times \mathcal{L}(V, n-k)$. If $g \in A(V) \setminus \Gamma(V)$ then g acts on $\mathcal{L}(V, k) \times \mathcal{L}(V, n-k)$ via $(U, U') \mapsto (U'^g, U^g)$.

For $k \leq n/2$ set

$$\mathcal{K}(V, k) = \{(U, U') \in \mathcal{L}(V, k) \times \mathcal{L}(V, n-k) : U \cap U' = 0\}$$

and

$$\mathcal{Q}(V, k) = \{(U, U') \in \mathcal{L}(V, k) \times \mathcal{L}(V, n-k) : U \subseteq U'\}.$$

Lemma 14.1. 1. *If U is a minimal nontrivial $M \cap \Gamma(V)$-invariant subspace of V, $\dim U = k$, and $h \in M \setminus \Gamma(V) \cap M$ then either $(U, U^h) \in \mathcal{K}(V, k)$ or $(U, U^h) \in \mathcal{Q}(V, k)$.*

2. *For every integer k with $0 \leq k \leq n/2$, G acts transitively on $\mathcal{K}(V, k)$ and on $\mathcal{Q}(V, k)$.*

Proof. (1) We have $U^h \in \mathcal{L}(V, n-k)$ and $U \cap U^h$ is $M \cap \Gamma(V)$-invariant. Minimality of U implies that either $U \cap U^h = 0$ or $U \cap U^h = U$. In the former case, $(U, U^h) \in \mathcal{K}(V, k)$; in the latter, $(U, U^h) \in \mathcal{Q}(V, k)$.

(2) The stablizer of U in $SL(V)$ is transitive on complements to U as well as on m-spaces containing U for all m. Therefore G is transitive because $SL(V) \subseteq G$.

When n is even the action of G on $\mathcal{Q}(V, n/2)$ is equivalent to an action of G on $\mathcal{L}(V, n/2)$.

Proposition 14.2. *Suppose n is even and $g \in G \setminus G \cap \Gamma(V)$, then*

1. *If $n = 4$ then $|\mathcal{L}(V, n/2)_g| \leq q^{-1}|\mathcal{L}(V, n/2)|$.*
2. *If $n \geq 6$, then $|\mathcal{L}(V, n/2)_g| \leq C_1 q^{-n/2}|\mathcal{L}(V, n/2)|$.*

Proof. V admits the bilinear form ϕ where $\phi(v, w) = (v)w^g$. We have $U = U^\perp$ with respect to this form. Since $\dim U^\perp = n - \dim U + \dim \mathrm{rad}\, \phi$ this implies that $\mathrm{rad}\, \phi = 0$. By counting ordered bases, the number of $n/2$-spaces U with $U = U^\perp$ is at most

$$\frac{(q^n - 1)(q^{n-1} - q)(q^{n-2} - q^2) \dots (q^{n/2+1} - q^{n/2-1})}{(q^{n/2} - 1)(q^{n/2} - q)(q^{n/2} - q^2) \dots (q^{n/2} - q^{n/2-1})} = \frac{\beta_{\mathbf{S}}(n)}{\beta_{\mathbf{L}}(n/2)} q^{n(n+2)/8}.$$

If $n = 4$, then $|\mathcal{L}(V,2)_g| \leq \dfrac{(q^4-1)(q^3-q)}{(q^2-1)(q^2-q)}$ while $|\mathcal{L}(V,2)| \leq \dfrac{(q^4-1)(q^4-q)}{(q^2-1)(q^2-q)}$ so

$$\frac{|\mathcal{L}(V,2)_g|}{|\mathcal{L}(V,2)|} \leq \frac{q^3-q}{q^4-q} = \frac{q+1}{q^2+q+1} < q^{-1}.$$

If $n \geq 6$, then $\dfrac{\beta_{\mathbf{S}}(n)}{\beta_{\mathbf{L}}(n/2)} < \dfrac{1}{\beta_{\mathbf{L}}(n/2)} < C_1$ by (2.4), so $|\mathcal{L}(V,n/2)_g| < C_3 q^{n^2/8+n/4}$. On the other hand, $|\mathcal{L}(V,n/2)| \geq q^{n^2/4}$ by Lemma 2.1. Therefore $\dfrac{|\mathcal{L}(V,n/2)_g|}{|\mathcal{L}(V,n/2)|} < C_1 q^{-n^2/8+n/4}$. Since $n \geq 6$, it follows that $-n^2/8+n/4 \leq -n/2$, and the statement follows.

For $g \in A(V)$, let $\mathcal{K}(V,k)_g$ be the set of g-invariant pairs in $\mathcal{K}(V,k)$ and let $\mathcal{Q}(V,k)_g$ be the set of g-invariant pairs in $\mathcal{Q}(V,k)$. The balance of this section is devoted to proving the following proposition.

Proposition 14.3. *Assume that $g \in A(V)$ and $1 \leq k \leq n/2$. In the case of $\mathcal{Q}$, assume that $k < n/2$. Then one of the following holds.*

1. $$|\mathcal{Q}(V,k)_g| \leq 12q^{-(n-1)/2}|\mathcal{Q}(V,k)| \text{ and}$$
   $$|\mathcal{K}(V,k)_g| \leq 12q^{-n/2}|\mathcal{K}(V,k)|; \text{ or}$$
2. *g is a linear transformation on M, g has an eigenspace of codimension d, $dk < n/2$,*

$$(q^{-2dk} - 2q^{-n/2})|\mathcal{Q}(V,k)| \leq |\mathcal{Q}(V,k)_g| \leq (q^{-2dk} + 35q^{-(n-1)/2})|\mathcal{Q}(V,k)| \text{ and}$$
$$(q^{-2dk} - q^{-n})|\mathcal{K}(V,k)| \leq |\mathcal{K}(V,k)_g| \leq (q^{-2dk} + 21q^{-n/2})|\mathcal{K}(V,k)|.$$

Proof. We first compute the orders of $\mathcal{Q}(V,k)$ and $\mathcal{K}(V,k)$.

14.4. 1. *Let U be a k-subspace of V. Then*

$$|\mathcal{Q}(V,k)| = |\mathcal{L}(V,k)|\ |\mathcal{L}(V/U, n-2k)|.$$

2. $$|\mathcal{K}(V,k)| = |\mathcal{L}(V,k)|\ q^{k(n-k)}.$$

Proof. For each k-subspace U there are $|\mathcal{L}(V/U, n-2k)|$ subspaces of V of dimension $n-k$ that contain U and $q^{k(n-k)}$ complements to U in V. □

14.5. *Assume that $g \notin \Gamma(V)$. Then*

1. $|\mathcal{Q}(V,k)_g| \leq q^{-(n-1)/2}|\mathcal{Q}(V,k)|$.
2. $|\mathcal{K}(V,k)_g| \leq q^{-(n-1)}|\mathcal{K}(V,k)|$.

Proof. If (U,U') is g-invariant, then $U' = (U^{\pi(g)})^{\vee}$. This implies that $|\mathcal{Q}(V,k)_g| + |\mathcal{K}(V,k)_g| \leq |\mathcal{L}(V,k)|$, so $|\mathcal{Q}(V,k)_g| \leq |\mathcal{L}(V,k)|$ and $|\mathcal{K}(V,k)_g| \leq |\mathcal{L}(V,k)|$.

By Lemma 2.1 and (14.4) we have

$$|\mathcal{Q}(V,k)| = |\mathcal{L}(V,k)|\ |\mathcal{L}(V/U, n-k)| > |\mathcal{L}(V,k)| q^{k(n-2k)}.$$

Thus $|\mathcal{Q}(V,k)_g|/|\mathcal{Q}(V,k)| \leq q^{-k(n-2k)}$. By assumption, $k \in [1,(n-1)/2]$. The maximum value of the quadratic function $x \mapsto -(n-2x)x$ on that interval is

$-(n-1)/2$, when $x = (n-1)/2$. Therefore $-(n-2k)k \le -(n-1)/2$. The first assertion now follows.

Again by (14.4) we have $|\mathcal{K}(V,k)_g| \le |\mathcal{L}(V,k)| = q^{-k(n-k)}|\mathcal{K}(V,k)|$. Because $1 \le k \le n/2$ we have $-k(n-k) \le -(n-1)$, and the second assertion also follows. □

Assume henceforth that $g \in \Gamma(V)$.

14.6. 1. *If $g \in \Gamma(V) \setminus \Delta(V)$ then*

$$\frac{\mathcal{Q}(V,k)_g}{\mathcal{Q}(V,k)} < 6q^{-(n-1)/2} \quad \textit{and} \quad \frac{\mathcal{K}(V,k)_g}{\mathcal{K}(V,k)} < 6q^{-(n-1)/2}.$$

2. *If $g \in \Delta(V)$ and g does not have an eigenspace of codimension d with $dk < n/2$ then*

$$\frac{\mathcal{Q}(V,k)_g}{\mathcal{Q}(V,k)} < 12q^{-n/2} \quad \textit{and} \quad \frac{\mathcal{K}(V,k)_g}{\mathcal{K}(V,k)} < 12q^{-n/2}.$$

Proof. Evidently,

$$|\mathcal{Q}(V,k)_g| \le \sum_{U \in \mathcal{L}(V,k)_g} |\mathcal{L}(V/U, n-2k)| \le |\mathcal{L}(V,k)_g|\,|\mathcal{L}(V/U,n-2k)|.$$

Therefore $\dfrac{|\mathcal{Q}(V,k)_g|}{|\mathcal{Q}(V,k)|} \le \dfrac{|\mathcal{L}(V,k)_g|}{|\mathcal{L}(V,k)|}$ by (14.4). The assertions about $\mathcal{Q}$ now follow from Proposition 13.1 and 6.7

Since $|\mathcal{K}(V,k)_g| \le \sum_{U \in \mathcal{L}(V,k)_g} |\mathcal{C}(V,U)| \le |\mathcal{L}(V,k)_g||\mathcal{C}(V,U)|$ where $\mathcal{C}(V,U)$ is the set of complements to U in V and $|\mathcal{C}(V,U)| = q^{k(n-k)}$, we have $\dfrac{|\mathcal{K}(V,k)_g|}{|\mathcal{K}(V,k)|} \le \dfrac{|\mathcal{L}(V,k)_g|}{|\mathcal{L}(V,k)|}$ by (14.4), and the $\mathcal{K}$ statements follow also follow from the same results. □

Assume now that $k \ge 1$ and that $C_V(g)$ has codimension d in V and $dk < n/2$. Set

$$\mathcal{Q}_i = \{(U,U') \in \mathcal{Q}(V,k)_g : U \subseteq C_V(g), [V,g] \subseteq U', \text{ and } \dim U \cap [V,g] = i\}.$$

$$\mathcal{Q}_A = \{(U,U') \in \mathcal{Q}(V,k)_g : U \not\subseteq C_V(g)\}.$$

$$\mathcal{Q}_B = \{(U,U') \in \mathcal{Q}(V,k)_g : [V,g] \not\subseteq U'\}.$$

$$\mathcal{K}_0 = \{(U,U') \in \mathcal{K}(V,k)_g : U \subseteq C_V(g) \text{ and } [V,g] \subseteq U'\}.$$

$$\mathcal{K}_A = \{(U,U') \in \mathcal{K}(V,k)_g : U \not\subseteq C_V(g)\}.$$

$$\mathcal{K}_B = \{(U,U') \in \mathcal{K}(V,k)_g : [V,g] \not\subseteq U'\}.$$

It is evident that $\mathcal{Q}_0 \subseteq \mathcal{Q}(V,k)_g \subseteq \cup_i \mathcal{Q}_i \cup \mathcal{Q}_A \cup \mathcal{Q}_B$ and $\mathcal{K}_0 \subseteq \mathcal{K}(V,k)_g \subseteq \mathcal{K}_0 \cup \mathcal{K}_A \cup \mathcal{Q}_B$. To complete the proof of Proposition 14.3 we shall establish upper bounds for $|\mathcal{Q}_i|$, $|\mathcal{Q}_A|$, $|\mathcal{Q}_B|$, $|\mathcal{K}_0|$, $|\mathcal{K}_A|$, and $|\mathcal{K}_B|$, and lower bounds for $|\mathcal{Q}_0|$ and $|\mathcal{K}_0|$.

14.7. 1. $|\mathcal{K}_A| < 3C_1 q^{-n/2}|\mathcal{K}(V,k)|$ *and* $|\mathcal{K}_B| < 3C_1 q^{-n/2}|\mathcal{K}(V,k)|$.

2. $|\mathcal{Q}_A| < 3C_1 q^{-n/2}|\mathcal{Q}(V,k)|$ *and* $|\mathcal{Q}_B| < 3C_1 q^{-n/2}|\mathcal{Q}(V,k)|$.

Proof. If $(U, U') \in \mathcal{K}_A$, then $U \in \mathcal{L}(V,k)_g \setminus \mathcal{L}(C_V(g),k)_g$ and $U' \in \mathcal{C}(V,U)$. Since $|\mathcal{C}(V,U)| = q^{k(n-k)}$ we have

$$|\mathcal{K}_A| \le (\ |\mathcal{L}(V,k)_g| - |\mathcal{L}(C_V(g),k)_g| \ ) \ q^{k(n-k)} < 3C_1 q^{-n/2} \ |\mathcal{L}(V,k)| \ q^{k(n-k)}$$

by (6.4). It follows from (14.4) that $|\mathcal{K}_A| < 3C_1 q^{-n/2}|\mathcal{K}(V,k)|$.

If $(U,U') \in \mathcal{K}_B$, then $(U'^{\iota}, U^{\iota}) \in \mathcal{K}_A^{\iota}$ where $\mathcal{K}_A^{\iota}$ is the set of all g^{ι}-invariant pairs (X, X') with X' a complement to X, $\dim X = k$, and $X \not\subseteq C_V(g^{\iota})$. Therefore $|\mathcal{K}_B| < 3C_1 q^{-n/2}|\mathcal{K}(V,k)|$ by the previous paragraph.

If $(U,U') \in \mathcal{Q}_A$, then $U \in \mathcal{L}(V,k)_g \setminus \mathcal{L}(C_V(g),k)_g$ and $U'/U \in \mathcal{L}(V/U, n-2k)$. Therefore

$$\begin{aligned} |\mathcal{Q}_A| &\le (|\mathcal{L}(V,k)_g| - |\mathcal{L}(C_V(g),k)_g|) \ |\mathcal{L}(V/U, n-2k)| \\ &< 3C_1 q^{-n/2}|\mathcal{L}(V,k)| \ |\mathcal{L}(V/U,n-2k)| \\ &= 3C_1 q^{-n/2}|\mathcal{Q}(V,k)| \end{aligned}$$

by (14.4).

Dually,

$$|\mathcal{Q}_B| < 3C_1 q^{-n/2}|\mathcal{L}(V,n-k)| \ |\mathcal{L}(U',k)| = 3C_1 q^{-n/2}|\mathcal{Q}(V,k)|$$

by 14.4.(1). □

14.8. *If $g \in \Delta(V)$ and g has an eigenspace of codimension d such that $dk < n/2$, then*

$$(q^{-2dk} - q^{-n}) < \frac{|\mathcal{K}(V,k)_g|}{|\mathcal{K}(V,k)|} < q^{-2dk} + 6C_1 q^{-n/2}.$$

Proof. Let $U \in \mathcal{L}(C_V(g),k)$. Then the map $X \mapsto X/[V,g]$ defines a bijection between $\{U' : (U,U') \in \mathcal{K}_0\}$ and the set $\mathcal{C}$ of complements to $(U + [V,g])/[V,g]$ in $V/[V,g]$. Because $|\mathcal{C}| = q^{k(n-d-k)}$ it follows that $|\mathcal{K}_0| = |\mathcal{L}(C_V(g),k)| \cdot q^{k(n-d-k)}$. Since $q^{-dk} - q^{-n} < \dfrac{|\mathcal{L}(C_V(g),k)|}{|\mathcal{L}(V,k)|} \le q^{-dk}$ by Proposition 5.1, we have

$$(q^{-dk} - q^{-n})q^{(n-k-d)k}|\mathcal{L}(V,k)| \le |\mathcal{K}_0| \le q^{-dk}q^{(n-k-d)k}|\mathcal{L}(V,k)|,$$

so

$$(q^{-2dk} - q^{-n-dk})q^{(n-k)k}|\mathcal{L}(V,k)| \le |\mathcal{K}_0| \le q^{-2dk}q^{(n-k)k}|\mathcal{L}(V,k)|.$$

It follows from 14.4.(1) that

$$(q^{-2dk} - q^{-n-dk}) \le \frac{|\mathcal{K}_0|}{|\mathcal{K}(V,k)|} \le q^{-2dk}.$$

Because $\mathcal{K}(V,k)_g = \mathcal{K}_0 \cup \mathcal{K}_A \cup \mathcal{K}_B$ the statement now follows from (14.7). □

Set $S = C_V(g) \cap [V,g]$, and let $s = \dim S$.

14.9. $|\mathcal{Q}_i| = |\mathcal{L}(S,i)| \ |\mathcal{L}(C_V(g)/S, k-i)| \ q^{(k-i)(s-i)} \ |\mathcal{L}(V/Z_i, n-2k-d+i)|$ *where* $\dim Z_i = k + d - i$.

Proof. Consider the injective map

$$(U,U') \mapsto (U \cap S, \ (U+S)/S, \ U/(U \cap S), \ U'/(U + [V,g]))$$

defined on $\mathcal{Q}_i$.

If $(U,U') \in \mathcal{Q}_i$ then

1. $U \cap S \in \mathcal{L}(S,i)$,

2. $(U+S)/S \in \mathcal{L}(C_V(g)/S, k-i)$,
3. $U/(U\cap S) \in \mathcal{C}((U+S)/(U\cap S), S/(U\cap S))$, and
4. $U'/(U+[V,g]) \in \mathcal{L}(V/(U+[V,g]), n-2k-d+i)$.

Suppose (Y, Z, X, X') satisfy the conditions

1. $Y \in \mathcal{L}(S,i)$,
2. $Z/S \in \mathcal{L}(C_V(g)/S, k-i)$,
3. $X/Y \in \mathcal{C}(Z/Y, S/Y)$, and
4. $X'/(Z+[V,g]) \in \mathcal{L}(V/(Z+[V,g]), n-2k-d+i)$.

Then $\dim Z = k-i+s$, $\dim X = \dim Z - \dim S + \dim Y = k$, and $X \subseteq Z \subseteq C_V(g)$. Furthermore, $S \subseteq Z \cap [V,g] \subseteq C_V(g) \cap [V,g] = S$, so $S = Z\cap [V,g]$, and $\dim(Z+[V,g]) = \dim Z + \dim[V,g] - \dim S = k-i+s+d-s = k-i+d$. Also, $\dim X' = \dim(Z+[V,g]) + n-2k-d+i = n-k$, $[V,g] \subseteq X'$, and $X \subseteq Z \subseteq X'$. Since $X \cap [V,g] = X \cap C_V(g) \cap [V,g] = X \cap S = Y$ is an i-space, it follows that $(X, X') \in \mathcal{Q}_i$.

The preceding argument shows that

$$|\mathcal{Q}_i| = |\mathcal{L}(S,i)|\ |\mathcal{L}(C_V(g)/S, k-i)|\ |\mathcal{C}(Z/Y, S/Y)|\ |\mathcal{L}(V/Z_i, n-2d-k+i)|$$

where $Y \in \mathcal{L}(S,i)$, $\dim Z/Y = k+s-2i$, and $\dim Z_i = k+d-i$. Since $\dim Z/Y = k+s-2i$ and $\dim S/Y = s-i$ we have $|\mathcal{C}(Z/Y, S/Y)| = q^{(s-i)(k-i)}$, and the statement follows. □

14.10. $(q^{-2dk} - 2q^{-n/2})|\mathcal{Q}(V,k)| < |\mathcal{Q}_0| < q^{-2dk}|\mathcal{Q}(V,k)|$.

Proof. By the previous statement, $|\mathcal{Q}_0| = |\mathcal{L}(C_V(g)/S, k)|\ |\mathcal{L}(V/Z, n-d-2k)|\ q^{sk}$ where Z has dimension $k+d$.

By (2.5), we have

$$\begin{aligned} |\mathcal{Q}_0| &= \gamma_1(n-d-s,k)q^{k(n-d-s-k)}\gamma_1(n-k-d,k)q^{k(n-d-2k)} \cdot q^{sk} \\ &= \gamma_1(n-d-s,k)\gamma_1(n-k-d,k)q^{k(2n-3k-2d)}. \end{aligned}$$

By (14.4) and (2.5), we have

$$\begin{aligned} |\mathcal{Q}(V,k)| &= \gamma_1(n,k)q^{k(n-k)}\gamma_1(n-k,n-2k)q^{k(n-2k)} \\ &= \gamma_1(n,k)\gamma_1(n-k,n-2k)q^{k(2n-3k)}. \end{aligned}$$

Set $\tau(n,d,k,s) = \dfrac{\gamma_1(n-d-s,k)\gamma_1(n-k-d,k)}{\gamma_1(n,k)\gamma_1(n-k,n-2k)}$. Then

$$\frac{|\mathcal{Q}_0|}{|\mathcal{Q}(V,k)|} = \tau(n,d,k,s)q^{-2dk}.$$

By (2.5), $\tau(n,d,k,s) = \dfrac{\beta_{\mathbf{L}}(n-d-s)\beta_{\mathbf{L}}(n-k)}{\beta_{\mathbf{L}}(n-d-s-k)\beta_{\mathbf{L}}(n)} \cdot \dfrac{\beta_{\mathbf{L}}(n-k-d)\beta_{\mathbf{L}}(n-2k)}{\beta_{\mathbf{L}}(n-d-2k)\beta_{\mathbf{L}}(n-k)}$.

It follows from (5.4) that $\tau(n,d,k,s) < 1$. Therefore $\dfrac{|\mathcal{Q}_0|}{|\mathcal{Q}(V,k)|} < q^{-2dk}$.

Also by (5.4),

$$\frac{\beta_{\mathbf{L}}(n-k-d)\beta_{\mathbf{L}}(n-2k)}{\beta_{\mathbf{L}}(n-d-2k)\beta_{\mathbf{L}}(n-k)} > 1 - \min(d,k)q^{-(n-d-2k+1)}.$$

Since $s \le d$, we have $\dfrac{\beta_{\mathbf{L}}(n-d-s)}{\beta_{\mathbf{L}}(n-d-s-k)} \ge \dfrac{\beta_{\mathbf{L}}(n-2d)}{\beta_{\mathbf{L}}(n-2d-k)}$. Therefore

$$\frac{\beta_{\mathbf{L}}(n-d-s)\beta_{\mathbf{L}}(n-k)}{\beta_{\mathbf{L}}(n-d-s-k)\beta_{\mathbf{L}}(n)} \ge \frac{\beta_{\mathbf{L}}(n-2d)\beta_{\mathbf{L}}(n-k)}{\beta_{\mathbf{L}}(n-2d-k)\beta_{\mathbf{L}}(n)} > 1-\min(2d,k)q^{-(n-2d-k+1)}.$$

This shows that

$$\begin{aligned}\frac{|\mathcal{Q}_0|}{|\mathcal{Q}(V,k)|} &> (1-\min(d,k)q^{-(n-d-2k+1)})(1-\min(2d,k)q^{-(n-2d-k+1)})q^{-2dk}\\ &= (q^{-dk}-q^{-(n+1)}q_1)(q^{-dk}-q^{-(n+1)}q_2)\end{aligned}$$

where $q_1 = \min(d,k)q^{2k+d-dk}$ and $q_2 = \min(2d,k)q^{2d+k-dk}$.

By 5.7.(7) and 5.7.(1) we have $q_1 \le q^{1+n/2}$ and $q_2 \le q^{1+n/2}$. Therefore

$$\frac{|\mathcal{Q}_0|}{|\mathcal{Q}(V,k)|} > (q^{-dk}-q^{-n/2})^2 > q^{-2dk}-2q^{-n/2}.$$

□

14.11. $\displaystyle\sum_{i>0} |\mathcal{Q}_i| \le 4C_1 q^{-(n-1)}|\mathcal{Q}(V,k)|$.

Proof. If $i > s$ or if $i > k$ then $\mathcal{Q}_i = \emptyset$. If $i \le s$ and $i \le k$, then, by (14.9)

$$|\mathcal{Q}_i| = |\mathcal{L}(S,i)|\ |\mathcal{L}(C_V(g)/S,k-i)|\ |\mathcal{L}(V/Z_i,n-d-2k+i)|\ q^{(s-i)(k-i)}$$

where $\dim Z_i = k+d-i$. By (14.4),

$$|\mathcal{Q}(V,k)| = |\mathcal{L}(V,k)|\ |\mathcal{L}(V/U,n-2k)|.$$

It follows from (2.5).(2) that

$$\frac{|\mathcal{Q}_i|}{|\mathcal{Q}(V,k)|} = \gamma q^{e_8(n,d,k,s,i)}$$

where

$$\begin{aligned}\gamma = &\frac{\beta_{\mathbf{L}}(s)}{\beta_{\mathbf{L}}(i)\beta_{\mathbf{L}}(s-i)}\frac{\beta_{\mathbf{L}}(n-d-s)}{\beta_{\mathbf{L}}(k-i)\beta_{\mathbf{L}}(n-d-s-k+i)}\frac{\beta_{\mathbf{L}}(n-d-k+i)}{\beta_{\mathbf{L}}(n-d-2k+i)\beta_{\mathbf{L}}(k)}\\ &\frac{\beta_{\mathbf{L}}(k)\beta_{\mathbf{L}}(n-k)}{\beta_{\mathbf{L}}(n)}\frac{\beta_{\mathbf{L}}(k)\beta_{\mathbf{L}}(n-2k)}{\beta_{\mathbf{L}}(n-k)}\end{aligned}$$

and $e_8(n,d,k,s,i) = i(s-i)+(k-i)(n-d-s-k+i)+k(n-d-2k+i)+(s-i)(k-i)-k(2n-3k)$.

Since

$$\begin{aligned}\gamma =&\frac{\beta_{\mathbf{L}}(n-d-s)\beta_{\mathbf{L}}(n-k+i)}{\beta_{\mathbf{L}}(n)\beta_{\mathbf{L}}(n-d-s-k+i)}\cdot\frac{\beta_{\mathbf{L}}(n-k)\beta_{\mathbf{L}}(k)}{\beta_{\mathbf{L}}(n-k+i)\beta_{\mathbf{L}}(k-i)}\\ &\cdot\frac{\beta_{\mathbf{L}}(n-d-k+i)\beta_{\mathbf{L}}(n-2k)}{\beta_{\mathbf{L}}(n-k)\beta_{\mathbf{L}}(n-d-2k+i)}\cdot\frac{\beta_{\mathbf{L}}(s)}{\beta_{\mathbf{L}}(s-i)}\cdot\frac{1}{\beta_{\mathbf{L}}(i)},\end{aligned}$$

it follows from (5.4) that $\gamma < C_1$.

Since $s \le d$, and $\partial e_8/\partial s = i \ge 0$, we have $e_8(n,d,k,s,i) \le e_8(n,d,k,d,i) \le -2dk - i(n-2d-2k+i)$.

We claim that $-2dk-i(n-2d-2k+i) \le -(n-1)$ for all i with $1 \le i \le d$ and $i \le k$. Setting $\xi(x,y) = -2xy - i(n-2x-2y+i)$, it suffices to show that $\xi(x,y) \le -(n-1)$ whenever $1 \le i \le x \le n/2$, and $i \le y \le n/2$. Since $\partial\xi/\partial x = 2i-2y \le 0$ and $\partial\xi/\partial y = 2i-2x \le 0$ in for all x and y under consideration, we have $\xi(x,y) \le \xi(i,i) = -in + i^2$. The function $i \mapsto -in + i^2$ is decreasing on the interval $[1, n/2]$. Therefore $\xi(i,i) \le \xi(1,1) = -(n-1)$, and the claim follows.

This implies that $e_8(n,d,k,s,i) \le -(n-1)$ for all i. Because $\partial^2 e_8/\partial i^2 = -2$, it follows from (11.4) that

$$\sum_{i>0} \frac{|\mathcal{Q}_i|}{|\mathcal{Q}(V,k)|} \le C_1 \sum_{i>0} q^{e_8(n,d,k,s,i)} \le \frac{2C_1}{1-q^{-1}} q^{-(n-1)} \le 4C_1 q^{-(n-1)}.$$

□

14.12. *If $g \in \Delta(V)$ and g has an eigenspace of codimension d such that $dk < n/2$ then*

$$q^{-2dk} - 2q^{-n/2} < \frac{|\mathcal{Q}(V,k)_g|}{|\mathcal{Q}(V,k)|} < q^{-2dk} + 6C_1 q^{-n/2} + 4C_1 q^{-(n-1)} < q^{-2dk} + 35q^{-n/2}.$$

Proof. Since $\mathcal{Q}(V,k)_g = \cup_i \mathcal{Q}_i \cup \mathcal{Q}_A \cup \mathcal{Q}_B$. this follows from (14.7), (14.10), and and (14.11). □

Proposition 14.3 now follows from (14.5), (14.6), (14.8), and (14.12).

15. Proofs of Theorems 1-6

To complete the proofs of the Theorems, we need some transitivity results.

Lemma 15.1.
1. *If G is linear and $0 \le k \le n$, then G acts transitively on $\mathcal{L}(V,k)$.*
2. *If V is non-degenerate symplectic, orthogonal, or unitary, and U is a non-degenerate subspace of V, then G acts transitively on $\mathcal{N}(V,U)$.*
3. *If either (a) V is non-degenerate symplectic, orthogonal, or unitary, and $0 \le k < n/2$ or (b) V is non-degenerate symplectic or unitary and $k = n/2$, then G acts transitively on $\mathcal{S}(V,k)$.*
4. *If V is orthogonal of type $+$ and $k = n/2$, then either G acts transitively on $\mathcal{S}(V,k)$ or G has two orbits on $\mathcal{S}(V,k)$, which are fused by an outer automorphism of G.*
5. *If V is non-degenerate orthogonal in even characteristic, then G acts transitively on $\mathcal{P}(V)$.*
6. *If V is orthogonal in even characteristic, rad V is nonsingular, and U is a complement to rad V in V, then G acts transitively on $\mathcal{H}(V,U)$.*

Since G acts transitively on $\mathcal{L}(V,k)$, $\mathcal{N}(V,U)$, $\mathcal{P}(V)$, $\mathcal{H}(V,U)$, $\mathcal{K}(V,k)$, and $\mathcal{Q}(V,k)$, when appropriate, most of the Theorems stated in Chapter 1 follow immediately from the Propositions proved in subsequent sections.

Specifically, Theorem 1 follows from Propositions 6.7, 13.1, and 14.2, together with the observation that the fixed point ratio of g on k-spaces is the same as the fixed point ratio of g^ι on $n-k$-spaces. Theorem 2 follows from Propositions 7.3 and 13.1, because the stabilizer of U is the same as the stabilizer of $U^\perp$ when U is non-degenerate. Theorem 4 follows from Propositions 4.6 and 13.1. Theorem 5 follows from Propositions 12.1 and 13.1. Theorem 6 follows from Proposition 14.3.

Because G has at most two orbits on $\mathcal{S}(V,k)$, Theorem 3 follows from Propositions 10.9 and 13.1.

16. Table of notation

$\mathbf{F}$	a field of order q
V	a classical vector space of dimension n over $\mathbf{F}$.
$I(V)$	isometries of V
$\Delta(V)$	linear transformations preserving κ up to scalar mulitplication
$\Gamma(V)$	linear and semilinear transformations preserving κ up to scalar mulitplication
ι	transpose-inverse automorphism of $GL(V)$.
$A(V)$	$\Gamma(V)$ when V has a non-trivial form; $\Gamma(V)\langle\iota\rangle$ when V is a linear space
G	a subgroup of $A(V)$ containing $F^*(I(V))$
V^*	$\hom_{\mathbf{F}}(V,\mathbf{F})$, the dual of V
$X^\vee$	$\cap_{x\in X}\ker x$, $X\subseteq V^*$
$T(X)$	the largest totally singular subspace contained in rad X

Collections of subspaces

$\mathcal{L}(X,k)$	k-subspaces of the vector space X
$\mathcal{L}(X,k)_g$	g-invariant k-subspaces of X
$\mathcal{N}(X,U)$	non-degenerate subspaces of X isometric to U, U a non-degenerate space
$\mathcal{N}(X,U)_g$	g-invariant members of $\mathcal{N}(X,U)$
$\mathcal{S}(X,k)$	totally singular k-subspaces of X
$\mathcal{S}(X,k)_g$	g-invariant members of $\mathcal{S}(X,k)$
$\mathcal{S}_i(X,k)_g$	the elements U of $\mathcal{S}(X,k)$ such that $U\cap\operatorname{rad}X$ has dimension i
$\sigma^*(X,k)$	an upper bound for $\lvert\mathcal{S}(X,k)_g\rvert$
$\mathcal{C}(X,U)$	the set of complements to U in X
$\mathcal{P}(X)$	nonsingular 1-subspaces of X, X an orthogonal space in characteristic 2
$\mathcal{H}(V,X)$	subspaces of the odd-dimensional orthogonal space V in even characteristic that are isometric to X, X a complement to rad V
$\mathcal{K}(X,k)$	the set of pairs (U,U') with $U\in\mathcal{L}(V,k)$, $U'\in\mathcal{L}(V,n-k)$, and $U\cap U'=0$, $k\leq n/2$.
$\mathcal{K}(X,k)_g$	g-invariant members of $\mathcal{K}(X,k)$
$\mathcal{K}_0$	pairs (U,U') in $\mathcal{K}(V,k)_g$ with $U\subseteq C_V(g)$ and $[V,\omega g]\subseteq U'$
$\mathcal{K}_A$	pairs (U,U') in $\mathcal{K}(V,k)_g$ with $U\not\subseteq C_V(g)$
$\mathcal{K}_B$	pairs (U,U') in $\mathcal{K}(V,k)_g$ with $[V,\omega g]\not\subseteq U'$
$\mathcal{Q}(V,k)$	the set of pairs (U,U') with $U\in\mathcal{L}(V,k)$, $U'\in\mathcal{L}(V,n-k)$, and $U\subset U'$, $k\leq n/2$.
$\mathcal{Q}(X,k)_g$	g-invariant members of $\mathcal{Q}(X,k)$
$\mathcal{Q}_i$	g-invariant pairs (U,U') in $\mathcal{Q}(V,k)$ with $U\subseteq C_V(g)$, $[V,g]\subseteq U'$, and $\dim U\cap[V,g]=i$
$\mathcal{Q}_A$	pairs (U,U') in $\mathcal{Q}(V,k)_g$ with $U\not\subseteq C_V(g)$
$\mathcal{Q}_B$	pairs (U,U') in $\mathcal{Q}(V,k)_g$ with $[V,\omega g]\not\subseteq U'$

Functions

$\psi(k)$	$\dfrac{1}{2}k(k+\mathrm{a})$, where $\mathrm{a}=1,0,-1$ in the orthogonal, unitary, symplectic cases respectively
$\varphi(k)$	$k^2-\psi(k)=\frac{1}{2}k(k-\mathrm{a})$, a as above
$\beta_{\mathbf{L}}(n)$	$\displaystyle\prod_{i=1}^{n}(1-q^{-i})$

$\beta_{\mathbf{S}}(n)$	$\prod_{i=1}^{n/2}(1-q^{-2i})$, n even
$\beta_{\mathbf{U}}(n)$	$\prod_{i=1}^{n}(1-(-q^{-1/2})^i)$
$\beta_{\mathbf{O}^\circ}(n)$	$\beta_{\mathbf{S}}(n-1)$, n odd
$\beta_{\mathbf{O}^+}(n)$	$\frac{\beta_{\mathbf{S}}(n)}{1+q^{-n/2}}$, n even
$\beta_{\mathbf{O}^-}(n)$	$\frac{\beta_{\mathbf{S}}(n)}{1-q^{-n/2}}$, n even, $n>0$
$\gamma_1(n,k)$	$\frac{\beta_{\mathbf{L}}(n)}{\beta_{\mathbf{L}}(k)\beta_{\mathbf{L}}(n-k)}$
$\gamma_2(n,k)$	$\frac{\beta_{\mathbf{X}}(n)}{\beta_{\mathbf{X}}(k)\beta_{\mathbf{X}}(n-k)}$, $\mathbf{X}=\mathbf{S}$ or $\mathbf{U}$
$\gamma_2(n,k,\epsilon,\delta)$	$\frac{\beta_{\mathbf{O}^\epsilon}(n)}{\beta_{\mathbf{O}^\delta}(k)\beta_{\mathbf{O}^\eta}(n-k)}$, V orthogonal, η the type of the orthogonal complement
$\gamma_3(n,k)$	$\frac{\beta_{\mathbf{X}}(n)}{\beta_{\mathbf{L}}(k)\beta_{\mathbf{X}}(n-2k)}$

Constants

C_1	$7/2$, if $q=2$; $q^2/(q^2-q-1)$, if $q>2$.
c_2	$1/8$, $q=2$; $\frac{(q-1)^2}{2q^2}$, $q>2$.
C_2	$15/8$, $q=2$; $\frac{q^4}{q^4-q^2-1}$, $q>2$.
c_3	$11/24$, $q=2$; $\frac{q^4-q^2-1}{q^4+q^3}$, $q>2$
C_3	$45/8$, $q=2$; $2\frac{q^6-q^4-q^2+1}{q^6-q^5-q^4}$, $q>2$
D_i	$\frac{1}{1-q^{-i}}$
E_1	$\frac{q+1}{q}$

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