ON MAXIMAL SUBGROUPS OF FINITE CLASSICAL GROUPS

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Abstract. In [1] M. Aschbacher described the structure of the possible maximal subgroups of a classical group G with natural module M. Each such subgroup is either a member of a canonical class of subgroups, or it is the normalizer of a quasi-simple, absolutely irreducible subgroup H of G. In the latter case suppose that H is a classical group whose defining characteristic is coprime to that of G. The aim of this article is to help classify those intermediate groups $H \leq K \leq G$, where K is again a classical group whose defining characteristic is equal to the one of G. If H is linear, then under mild hypotheses on the rank of H and the size of the field $\mathbb{F}_q$ of definition of H we show that there are no such proper K.

1. Introduction

In [2] M. Aschbacher and L. Scott showed that the maximal subgroup problem for finite groups can be reduced to the following two problems:

1. Determine the maximal subgroups of the almost simple groups.
2. Determine $H^1(G, V)$, where G is almost simple and V is an irreducible G-module.

This paper is a contribution to part (1) of this project. For most non-classical almost simple groups reasonably complete information on maximal subgroups exists. For classical groups M. Aschbacher proved in [1]:

Theorem 1.1. *If G is a classical group defined over a finite field F of characteristic l acting on its minimal dimensional FG-module M and H is a subgroup of G, then one of the following holds:*

1. *$F^*(H)$ stabilizes one of the "natural" structures of M.*
2. *$F^*(H)$ is quasi-simple acting absolutely irreducibly and tensor indecomposably on M.*

This structure theorem reduces the study of the maximal subgroups of the classical groups to the following two problems:

1. Determine the stabilizers of the *natural* structures and the maximality among such stabilizers.
2. Determine maximality among subgroups H appearing in part (2) of Theorem 1.1 above.

Under the assumption that $\dim M \geq 13$ the first problem was solved by P. Kleidman and M. Liebeck in [9]. In connection with the second, one would like to prove the following:

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Conjecture 1.2. *Let G be a finite classical group with natural module M. Let H be a subgroup of G as in part (2) of Theorem 1.1 above. Then generically H is maximal, i.e., there is a small list of specified cases when H fails to be maximal.*

Using Theorem 1.1 one can show that if a subgroup H as in (2) of Theorem 1.1 is not maximal, then either K is a member of $\mathcal{C}_2 \cup \mathcal{C}_4 \cup \mathcal{C}_6 \cup \mathcal{C}_7$ (Aschbacher classes, cf. [1]) or there exists a quasi-simple subgroup K of G with H in K and both H and K act absolutely irreducibly and tensor indecomposably on M. Thus, to establish Conjecture 1.2, one has to classify the list of such triples (H, K, G). We shall refer to this as the *general problem*.

From the classification of finite simple groups we know that H and K are either alternating, sporadic, classical or of exceptional Lie type. In case K is alternating there are partial results by J. Saxl [16] in characteristic zero, P. Kleidman and D. Wales [10], J.C. Jantzen and G. Seitz [8], and B. Ford [3]. If K is sporadic or of exceptional Lie type, enough is known about the maximal subgroups of K, i.e., the possible choices of H, to put a solution to the general problem within reach. For example, when K is of exceptional Lie type it is easy to show that there are very few possible configurations; in each case $F^*(H)$ is the centralizer in K of an outer automorphism of order at most 3. The remaining and most difficult cases of the general problem are those where H is alternating or of Lie type and K is a classical group.

W. Husen studied the case where H is of alternating type [7]. For example, he has shown that there are at most twenty possible triples (H, K, G) provided K is not alternating or of Lie type and the characteristic of the field of definition of K is equal to l. Moreover, he obtained some partial results in case these two characteristics coincide.

In our work, we will consider the case where both H and K are of Lie type. For a classical group X we denote its natural module by N_X. Suppose that H and K are defined over fields of order $q = p^a$ and $s = r^b$, respectively.

If the defining characteristic l of K is distinct from those of G and H, then using [12] twice, we get that roughly $s^{(q^{\dim N_H/2})/2} \leq \dim N_G$. On the other hand $\dim N_G \leq r^{\frac{1}{2}(\dim N_H)^2} \leq |H|^{\frac{1}{2}}$, where H is defined over $\mathbb{F}_q$, i.e.,

$$\frac{1}{2} q^{\dim N_H/2} \leq \log_s \dim N_G \leq \frac{(\dim N_H)^2}{4} \log_s q.$$

Both inequalities can certainly not be satisfied for infinitely many values of q and r. Thus this case seems easy to handle. Work of M. Liebeck, G. Seitz, and D. Testerman ([18], [12], [22]) establishes the conjecture when H, K, and G are all defined over fields of the same characteristic subject to mild restrictions on the characteristic. Work in progress by Liebeck and Seitz aims at removing these restrictions. In [19] G. Seitz verified Conjecture 1.2 for the case where H and K are defined over fields of characteristic l, K is classical, and G is defined over a field of characteristic $p \neq l$. The situation when all three groups are classical, H is defined over a field of characteristic p, while K and G are defined over fields of characteristic $l \neq p$, is a last major open case. This is the one under consideration here.

Our basic strategy is to obtain upper and lower bounds for $\dim N_G$ such that generically, there are no such proper triples of groups, as these bounds contradict each other; thus leading to a small list of possible configurations. When H is linear we actually show that there are no such triples provided $q \geq 17$ and $n \geq 16$, see Corollary 1.6. To obtain a lower bound we use the Landazuri-Seitz-Zalesskii bound of H and the representation theory of K. Whereas for the upper bound we use the subgroup structure and representation theory of H. Since the restriction of N_G to H is tensor indecomposable, we may assume that N_G is an l-restricted highest weight module for K. Thus we may obtain it as a section of an e-fold tensor product of of copies of the natural module N_K of K or its dual (or its spin module, in case K is orthogonal). Our first estimates show that e is small compared to r_K, the rank of K. This implies the existence of a certain subdominant weight which in turn yields the inequality

$$\binom{r_K}{e} \leq \dim N_G.$$

On the other hand we can show that there exists an H_e-invariant subspace Y of N_G, where H_e is a suitably chosen subgroup of H (depending on e) of relatively small index in H. This yields an upper bound

$$\dim N_G \leq [H : H_e] \dim Y.$$

Observe that $r_K = c \dim Y$, where c is at least the Landazuri-Seitz-Zalesskii bound from Table 1.

Following [9, §1.2] by $\mathcal{S}(G)$ we denote the collection of almost simple subgroups of G satisfying certain "irreducibility" conditions.

Our principal result is:

Theorem 1.3. *Let H be a quasi-simple finite classical group of Lie type of characteristic p. Let K and G be classical groups of Lie type of characteristic $l \neq p$. Suppose that $H \in \mathcal{S}(G)$ and $H \leq K \leq G$. Assume that and $q \geq 17$ and $n \geq 16$.*

(i) *If H is linear or orthogonal, then* $\dim N_G \leq (\dim N_K)^2$.
(ii) *If either H is unitary, or symplectic and $q \equiv 1 \mod 4$, then* $\dim N_G \leq (\dim N_K)^6$.
(iii) *If H is symplectic and $q \equiv 3 \mod 4$, then* $\dim N_G \leq (\dim N_K)^{10}$.

Remark 1.4. Using our methods the exponents in Theorem 1.3 are optimal. For all values of q and all choices of H our methods lead to an exponent of at most 24, see Proposition 1.7 below. In case H is symplectic or unitary and the natural module of K affords a linear nontrivial character for the unipotent radical of the parabolic subgroup P in H that stabilizes a singular one-dimensional subspace of the natural module of H we can improve the exponent in the statement of Theorem 1.3 to 2.

For the Weil modules our exponents are optimal.

Corollary 1.5. *Let H, K, and G be as in Theorem 1.3 and assume that H is linear. Then $\dim N_G \leq (\dim N_K)^2$ and N_G regarded as a K-module is the heart of one of the following: N_K, $S^2(N_K)$, $\Lambda^2(N_K)$, $N_K \otimes N_K^*$, their duals or Galois conjugates.*

Proof. This follows directly from the theorem, Lemma 3.5, and the fact that $e = 2$. ☐

Since the modules listed in the conclusion of Corollary 1.5 are dealt with in [13] and [14], we have

Corollary 1.6. *Let H, K, and G be as in Theorem 1.3 and assume that H is linear. Then $K = G$, i.e., there are no such proper triples in this instance.*

The major step towards proving Theorem 1.3 is the following

Proposition 1.7. *Let H, K, and G be as in Theorem 1.3.*

(i) *If H is linear or orthogonal, then* $\dim N_G \leq (\dim N_K)^4$ *with the following exceptions $H = \mathrm{L}_n(2)$ and K is orthogonal and* $\dim N_G \leq (\dim N_K)^6$, *or $H = \mathrm{O}_n(2)$ and K is orthogonal and* $\dim N_G \leq (\dim N_K)^6$.

(ii) *If H is unitary, then* $\dim N_G \leq (\dim N_K)^{12}$ *with the following exceptions $H = \mathrm{U}_n(q)$ for $q = 2$ or $q = 4$ and $l = 5$.*

(iii) *If H is symplectic, then* $\dim N_G \leq (\dim N_K)^{14}$ *with the following exceptions $H = \mathrm{Sp}_n(q)$ and $q \equiv 3 \mod 4$, $q \leq 19$ and K is linear, N_K is Harish Chandra induced from a Weil module of the $\mathrm{Sp}_{n-2}(q)$ Levi factor of H. Then* $\dim N_G \leq (\dim N_K)^{24}$.

Once more information about small dimensional cross characteristic modules for classical groups of non-linear type becomes available, further cases should be within reach of settlement.

2. Notation and Preliminaries

Definition 2.1. Let X be a classical group with natural module N_X of dimension $n_X = \dim N_X$. We denote a maximal torus of X by T_X, its Weyl group by W_X, and its rank by r_X. A highest weight module of X is denoted by $M_X(\lambda_X)$.

Fix a vector space N of dimension n over a finite field of characteristic p. Endow N either with the trivial bilinear form, a nondegenerate alternating bilinear form, a nondegenerate quadratic form of Witt defect at most 1, or a nondegenerate unitary form. If the form on N is not unitary, we assume that the finite field has order q, else q^2. Let $I(N)$ denote the full group of isometries of N. We assume that $F^*(I(N))$ is a quasi-simple group.

Definition 2.2. A group X is said to be a *quasi-simple finite classical group of type* **L**, **S**, **O**, respectively **U**, provided X is a central extension of the factor $F^*(I(N_X))/Z(F^*(I(N_X)))$, where N_X is endowed as above with a trivial, alternating, quadratic, respectively unitary form. Since $\mathrm{Sp}(n,q) \simeq \Omega(n+1,q)$ when $p = 2$, we regard symplectic groups over fields of characteristic 2 as orthogonal groups. Thus if X is of type **S**, we always assume that p is odd.

Lemma 2.3. *Upper and lower bounds for H-representations in non-defining characteristic are given in Table 1 below.*

In the following table we collect upper and lower bounds for the H-representations in non-defining characteristic. Our lower bounds b are rough estimates of the Landazuri-Seitz-Zalesskii bounds from [11] and [21], which are convenient for our purposes. The upper bounds for $\dim N_G$ are upper bounds for the degrees of the Deligne-Lusztig characters $\deg R_{T,\Theta}$. They are taken from [19, Theorem 2.2].

H	$n_K \geq b$	$n_G \leq \deg R_{T,\Theta}$	$s := \log_q(\deg R_{T,\Theta})/\log_q(b)$
$A_r(q)$	q^r	$q^{r(r+1)/2+1}$	$(r+2)/2$
$B_r(q)$	q^{2r-3}	q^{r^2+1}	$(r^2+1)/(2r-3)$
$C_r(q)$	q^{r-2}	q^{r^2+1}	$(r^2+1)/(r-2)$
$D_r(q)$	q^{2r-3}	$q^{r(r-1)+1}$	$(r^2-r+1)/(2r-3)$
${}^2A_r(q)$	q^{r-1}	$q^{r(r+1)/2+1}$	$(r^2+r+1)/2(r-1)$
${}^2D_r(q)$	q^{2r-3}	$q^{r(r-1)+1}$	$(r^2-r+1)/(2r-3)$

Table 1: Upper and lower bounds for the dimensions of N_K and N_G.

Lemma 2.4. *In the following cases Proposition 1.7 is valid:*

H	$n \leq m$
$\mathrm{SL}_n(q)$	7
$\mathrm{O}_n(q)$	11
$\mathrm{U}_n(q)$	15
$\mathrm{Sp}_n(q)$	14

Proof. Comparing the dimensions of the minimal and maximal degree of irreducible H-representations yields the desired conclusion. □

The following definition is motivated by Proposition 3.2.

Definition 2.5. We call a triple $H \leq K \leq G$ *generic* provided K is linear or unitary and $n_K \geq 21$, or $n_K \geq 128$ and K is orthogonal or symplectic. Else, we call the triple *non-generic.*

Because of Lemma 2.4 we only need to consider generic triples in our investigation.

Before discussing our bounds in the next sections, we require an elementary observation used in these estimates.

Lemma 2.6. *For integers $m \geq 2^6$ and $j \leq 5\log_2 m$ we have $\binom{m}{j} \geq m^{j/4}$. If $30 \leq m \leq 64$ and $10 \leq j \leq m/2$, then $\binom{m}{j} \geq m^{j/4}$. If $j \leq 10$, then $\binom{m}{j} \geq m^{j/2}$.*

Proof. Note that $\binom{m}{j} \geq (\frac{m}{j})^j$. Now $\frac{m}{j}$ is minimal for the maximal possible value of j, i.e., when $j = \lfloor 5\log_2 m \rfloor$. One checks that $\frac{m}{5\log_2 m} \geq m^{1/4}$ provided $m \geq 2^6$. □

By $\overline{F}$ we dentoe the algebraic closure of the field of definition F of G.

3. Initial bounds for $\dim N_K$ and $\dim N_G$

In view of Lemma 2.4 we may assume that throughout this section the triple $H \leq K \leq G$ is generic. Let $k := r_K = \operatorname{rank} K$ be the rank of K, $d := n_K = \dim N_K$, and $W := W_K$ is the Weyl group of K. Moreover, $\lambda = \sum_{i=1}^{k} a_i\lambda_i$ is an l-restricted dominant weight for T_K. Define

$$\begin{aligned} l_\lambda &:= \max\{c \mid c \leq k/2, a_c \neq 0\}; \\ r_\lambda &:= \max\{c \mid c < k/2, a_{k-c} \neq 0\}. \end{aligned}$$

Also, we define

$$e(l_\lambda) := \sum_{i=1}^{l_\lambda} i a_i; \qquad e(r_\lambda) := \sum_{i=1}^{r_\lambda} i a_{k+1-i}.$$

Remark 3.1. Proposition 3.2 and our assumption that the triple $H \leq K \leq G$ is generic guarantee that $e(r_\lambda) = 0$ in case K is symplectic or orthogonal.

Let $s := \log_q(\deg R_{T,\Theta})/\log_q(b)$ from Table 1 above.

Proposition 3.2. *If K is linear or unitary, then $r_\lambda + l_\lambda \leq 4s$ provided $d \geq 21$. If K is symplectic or orthogonal, then $r_\lambda = 0$ and $l_\lambda \leq 4s$ provided $d \geq 128$.*

Proof. Set $t := r_\lambda + l_\lambda$. First suppose that K is linear or unitary. Observe that the stabilizer W_λ is of the form $\prod \mathrm{Sym}_{d_i} \times \mathrm{Sym}_{d-t}$ for some positive integers d_i such that $\sum d_i \leq t \leq d$. Consequently, we have $[W : W_\lambda] \geq \binom{d}{t}$. Observe that, since K is linear or unitary, Lemma 2.6 is satisfied for $m = d$ and $j = t$. Therefore, using Lemmas 2.3 and 2.6, we see that

$$d^s \geq \dim N_G \geq |W \cdot \lambda| \geq \binom{d}{t} \geq d^{t/4}.$$

Consequently, we must have $t = r_\lambda + l_\lambda \leq 4s$, as claimed.

Next suppose that K is symplectic or orthogonal. Then we claim that $r_\lambda = 0$. For, if $r_\lambda \neq 0$, then

$$d^s \geq \dim N_G \geq |W \cdot \lambda| \geq 2^{k-1},$$

since then the index $[W : W_\lambda] \geq 2^{k-1}$. One checks that this inequality is satisfied only if $d \leq 128$. Thus $r_\lambda = 0$. Then the stabilizer W_λ has order at most $2^{k-l_\lambda-\epsilon}(k-l_\lambda)!l_\lambda!$, and since $|W| = 2^{k-\epsilon}k!$, we obtain

$$(2k+1)^s \geq d^s \geq \dim N_G \geq |W \cdot \lambda| \geq \binom{k}{t} 2^t \geq k^{t/4}2^t,$$

where ϵ is either 0 or 1 depending on the type of K, and where we apply Lemma 2.6 for the last inequality again. Now observe that this inequality fails if $t > 4s$. Consequently, since $r_\lambda = 0$, we must have $l_\lambda \leq 4s$, as claimed. ☐

Lemma 3.3. *Fix an l-restricted dominant weight λ of K. Set $c_l := e(l_\lambda)$ and $c_r := e(r_\lambda)$. If $r_\lambda + l_\lambda \leq c_l + c_r \leq k$, then the sum of the fundamental dominant weights $\lambda_{c_l} + \lambda_{c_r}$ is subdominant to λ.*

Proof. Note that in case K is symplectic or orthogonal our assumption on $\dim N_K$ ensures that $r_\lambda = 0$, by Proposition 3.2.

It suffices to observe that $\lambda - (\lambda_{c_l} + \lambda_{c_r})$ is a positive linear combination of positive roots. We indicate the idea of the proof. Let $\{x_1, \ldots, x_d\}$ be a basis of N_K of T_K-weight vectors, i.e., every element from T_K acts via a diagonal matrix, say $\mathrm{diag}(t_1, \ldots, t_d)$. If $\lambda = \sum_{i=1}^k a_i\lambda_i$ and m_λ is a highest weight vector of $M_K(\lambda)$, then T_K acts via the character

$$t_1^{b_1}t_2^{b_2}\cdots t_{l_\lambda}^{b_{l_\lambda}} \cdot t_{k-r_\lambda}^{-b_{k-r_\lambda}} \cdots t_{k+1}^{-b_{k+1}},$$

where $b_j = \sum_{i=j}^k a_i$ for each $1 \leq j \leq k$. If $\alpha_{i,j}$ is a root of K with respect to T_K, then $\alpha_{i,j}(\mathrm{diag}(t_1, \ldots, t_d)) = t_i t_j^{-1}$ for every $i < j \leq k$.

Next we observe that for example $\lambda - \alpha_{1,l_\lambda+1}$ is the weight which is obtained from λ by reducing the exponent of t_1 by 1 and increasing the exponent of $t_{\lambda+1}$ from 0 to 1. If $\lambda - \alpha_{1,l_\lambda+1}$ has the property that the exponent of t_1 is strictly larger than 1, we may continue subtracting the root $\alpha_{1,l_\lambda+2}$, etc. Continuing in this fashion yields the claim. ☐

Corollary 3.4. *If $r_\lambda + l_\lambda \leq 4s \leq k$, then $e(l_\lambda) + e(r_\lambda) \leq 4s$.*

Proof. By Lemma 3.3 we know that the weight $\lambda_{c_l} + \lambda_{c_r}$ is subdominant to λ. By [15] the multiplicity of every subdominant weight is nonzero. Using the estimates from the proof of Proposition 3.2 we get the desired conclusion. ☐

Lemma 3.5. *Let λ be an l-restricted dominant weight such that $e := e(l_\lambda)+e(r_\lambda) \leq k$. Then*

$$\binom{k}{e} \leq \dim M_K(\lambda) \leq (\dim N_K)^e.$$

Proof. Observe that $\lambda_{c_l} + \lambda_{c_r}$ is a subdominant weight of λ by Lemma 3.3 and by [15] the dimension of its weight space is not zero. The size of the orbit of $\lambda_{c_l} + \lambda_{c_r}$ under the Weyl group W_K is bounded below by $\binom{k}{e}$; whence the lower bound.

By general theory we know that $M_K(\lambda)$ is a section of a tensor product module

$$T(\lambda) := V^{\otimes e(l_\lambda)} \otimes (V^*)^{\otimes e(r_\lambda)}.$$

Hence the upper bound follows as well. □

Suppose that we have a decomposition

$$N_K = V = V_0 \oplus \bigoplus_{i=1}^{m} V_i,$$

where each V_i is a sum of T_K-weight spaces and the various V_i's for $i \geq 1$ are permuted by the action of W_K. Set $a := [m/2]$. Assume furthermore that the indexing of the V_i's for $i \geq 1$ is chosen in such a way that $\sum_{i=1}^{a} V_i$ is a totally singular subspace of V with respect to the form defining K. Let R be the stabilizer in K of this direct sum decomposition. In this setup we have

Lemma 3.6. *Let λ be an l-restricted dominant weight such that $e := e(l_\lambda)+e(r_\lambda) \leq k$. Suppose that $a \geq e$. Then there is an R-submodule X of $M_K(\lambda)$ which is isomorphic to the R-submodule $Y := V_1 \otimes V_2 \otimes \cdots \otimes V_e$ of $T(\lambda)$.*

Proof. Once again observe that $\nu = \lambda_{c_l} + \lambda_{c_r}$ is a subdominant weight of λ by Lemma 3.3 and by [15] the dimension of its weight space is not zero. We choose a (non-zero) weight vector v_i in one of the T_K-weight spaces of V_i. Let $\mathbf{t} := v_1 \otimes v_2 \otimes \cdots \otimes v_e$ be in Y and let $S := \mathrm{Sym}_{e_l} \times \mathrm{Sym}_{e_r}$ be the group permuting the factors of $T(\lambda)$, where Sym_{e_l} permutes the first e_l factors, etc. Note that S preserves the T_K-weight spaces of $T(\lambda)$. Observe that, by construction, the Weyl module of highest weight λ maps onto a submodule U of $T(\lambda)$; and U maps onto $M_K(\lambda)$. Dualizing yields a K-module homomorphism from $T(\lambda)^*$ onto U^*. Then the image of $\mathbf{t}^*$ in U^*, denoted by $\bar{\mathbf{t}}$, generates the ν-weight space of U^* as an S-module. Taking duals once again, we obtain a map of S-modules from U_μ onto $M_K(\lambda)_\mu$. Hence the image of $(\bar{\mathbf{t}})^*$ is the S-generator of $M_K(\lambda)_\mu$.

Next observe that every weight of Y occurs with multiplicity 1 and is W_K-conjugate to ν. Consider the linear $\overline{F}$-span X of the members in the $(W_K \cap R)$-orbit of the image of $(\bar{\mathbf{t}})^*$ in $M_K(\lambda)$.

Note that our composition of taking duals and projections is a K-homomorphism. Thus, restricting to Y, yields a nontrivial R-homomorphism from Y to X.

By construction the image of $\mathbf{t}$ is not zero and hence every weight space of Y has a nontrivial image in X. Consequently, as the weight multiplicities in Y are all equal to 1, we see that $\dim Y = \dim X$; so our map is an isomorphism. □

4. Proof of Proposition 1.7

Throughout this section we continue with the same assumptions as in Section 3, in particular, the triple $H \leq K \leq G$ is generic and λ is an l-restricted dominant weight for T_K. Moreover, we abbreviate $V = N_K$, $M = M_K(\lambda) = N_G$, $n := n_H =$

$\dim N_H$, and $k = r_K = \operatorname{rank} K$. Recall from Section 3 that for our given λ we define $e := e(r_\lambda) + e(l_\lambda)$.

Recall that a p-group Q is special if $Q' = Z(Q) = F(Q)$; sometimes also called a Heisenberg group.

Case 1: First we suppose that H is of type **L** or **O**. Let $P = LQ$ be the stabilizer in H of a singular point in N_H. Observe that there exists a nontrivial Q-linear character χ such that $V_\chi = \{v \in V \mid z \cdot v = \chi(z)v \text{ for } z \in Z(Q)\}$ is nontrivial. We consider the action of of H_χ, the stabilizer of χ in H, on V_χ.

Case 1.1: We assume that for every Q-linear character ψ the group H_ψ does not have a linear constituent in the socle of V_ψ. Let χ be a nontrivial linear character of Q. We observe that V_χ is singular with respect to the form defining K if q if odd. If q is even, V_χ may be non-degenerate. Let $P_\chi(K)$ be the parabolic subgroup of K stabilizing V_χ. We observe that the Levi factor of $P_\chi(K)$ is isomorphic to a quotient of $\mathrm{GL}(V_\chi) \circ I(V/(V_\chi \oplus V_{-\chi}))$ by some central subgroup. We have an injection $H_\chi \hookrightarrow P_\chi(K)$ and the projection of the image of H_χ in $P_\chi(K)$ onto the first factor $\mathrm{GL}(V_\chi)$ of the Levi factor of $P_\chi(K)$ is nontrivial; in fact the kernel of this map contains no more than $O_p(H_\chi)$.

Let V_{lin} be the Q-submodule of $V|_Q$ consisting of the nontrivial linear constituents V_ψ and set $d_{lin} := \dim V_{lin}$. Note, if Q is abelian, then V_{lin} equals $[V, Q]$, the commutator space of V with respect to Q. This is the case when H is linear or orthogonal. Let $V = V_{lin} \perp \overline{V}$. The stabilizer of this decomposition in K is $I(V_{lin}) \times I(\overline{V})$.

Lemma 4.1. *We have $I(\overline{V}) \subseteq C_K(M_\lambda)$.*

Proof. Observe that $I(V_{lin}) \times I(\overline{V})$ is a reductive ***subsystem subgroup***, i.e., its root system is obtained by removing some nodes from the extended Dynkin diagram of K. In view of Proposition 3.2 we see that the weight λ is supported on at most n nodes. Moreover, the support of λ is located at the end of the diagram. Without loss of generality we may assume that λ is supported only on the nodes corresponding to the component associated to $I(V_{lin})$. Since $r_\lambda = 0$ by Proposition 3.2 and by the construction of $I(V_{lin}) \times I(\overline{V})$, the simple roots of $I(\overline{V})$ are ***strongly orthogonal*** to the simple roots in $I(V_{lin})$, i.e., $(\alpha, \beta) = 0$ and $\alpha \pm \beta$ is not a root of K, where α and β are simple roots of $I(\overline{V})$ and $I(V_{lin})$, respectively. Thus the semisimple subgroup generated by $\{s_\alpha, X_\alpha \mid \alpha \text{ a simple root of } I(\overline{V})\}$ centralizes M_λ. But this subgroup is just $I(\overline{V})$. Thus the claim follows. $\square$

Lemma 4.2. *An element of H_χ centralizes an element of M_λ if and only if its projection into $I(V_\chi)$ does.*

Proof. If there exists a P-invariant subspace V^c complementing V_χ, then we argue as in Lemma 4.1. Otherwise, let $R = DR_u$ be the parabolic subgroup of K stabilizing V_χ in V. After possibly replacing R by a K-conjugate, we may suppose that $R_u \subseteq C_K(M_\lambda)$. Observe that D is of the form $\mathrm{GL}(V_\chi) \circ I(V_\chi^\perp / V_\chi)$. We may choose D in such a way that the second factor lies in $C_K(M_\lambda)$. Let $x = y_1 y_2 r$ be in $P \subseteq R$ such that $r \in R_u$ and y_i is in the i-th factor of D. Since we have already observed that $y_2 r \in C_K(M_\lambda)$, the claim follows. $\square$

Let M_0 be the smallest $I(V_\chi)$-invariant submodule of M which contains M_λ. Observe that M_0 is a highest weight module for $I(V_\chi)$, where the highest weight

λ_χ is the restriction of λ to a suitably chosen maximal torus of $I(V_\chi)$. Let b be the Landazuri-Seitz-Zalesskii bound from the second column from Table 1 above.

Lemma 4.3. *If* $e \geq 3$, *then* $\dim M \geq b^{2.5} \dim M_0$.

Proof. Note that in our case the hypotheses of Lemma 3.5 are satisfied. By this lemma we have $\dim M_0 \leq (\dim V_\chi)^e$ as well as $\dim M \geq \binom{k}{e}$. Now we observe that $k \geq \frac{b}{2} \dim V_\chi$. Thus we compute that

$$\frac{\dim M}{\dim M_0} \geq \frac{1}{e!} \prod_{i=0}^{e-1} \frac{\frac{b}{2}\dim V_\chi - i}{\dim V_\chi} \geq \frac{1}{2^e}\binom{b}{e} \leq b^{2.5}.$$

□

To complete the argument we observe that by Lemma 4.3 and Frobenius reciprocity

$$b^{2.5} \dim M_0 \leq \dim M \leq \dim[(M_0)^H_{H_\chi}] = \dim M_0[H : H_\chi] \leq b^{2.4} \dim M_0.$$

This is a contradiction to our assumption that $e \geq 3$, thus completing the analysis of case 1.1.

Case 1.2: Next we suppose that V_χ has a linear H_χ-character in its socle.

We observe that $\dim V \geq b$, where the bound b is taken from Table 1 above. Thus by Lemma 3.5 and Lemma 2.6 we see that $\dim N_G \geq b^{e/4}$. In order to get an upper bound we want to apply Lemma 3.6. Here we need two separate arguments depending on whether a weight space V_χ is singular or not. If the weight spaces are singular, then we can apply Lemma 3.6 directly, provided we ensure that we pick e characters that are pairwise not inverse to one another. If the weight spaces are not singular, then we construct a subgroup L_S of L which centralizes a nondegenerate subspace S of N_K of dimension at least $2e+1$. We take a basis of a totally isotropic e-dimensional subspace S_e of S and apply Lemma 3.6 to it. As S_e is L_S-invariant, this yields a 1-dimensional L_S-invariant subspace of N_G.

First we consider the instance when there is a nonzero singular, H_χ-invariant vector v_χ in the socle of V_χ, where "singular" refers to the form defining K. Consider the subspace spanned by the members of the P-orbit of v_χ in V. We pick e characters χ of Q in Q^* so that the direct sum of the associated spaces $\overline{F}v_\chi$ is totally singular. Moreover, we pick the χ's in a nondegenerate Q-subspace of smallest possible dimension in Q^*. If H is linear, then we require that this subspace of Q^* contains at least $2e$ nonzero elements if q is odd, else we require that it contains at least e nonzero elements. If H is of type **O**, the character χ has one of three possible types, to which we refer to as "singular", "plus", or "minus" type characters. The smallest nontrivial orbit of characters in any nondegenerate orthogonal space is the one of the "singular" type characters. In order to handle all these cases simultaneously, we pick a nondegenerate subspace of Q^* that contains at least $2e$ nonzero singular vectors.

Using this approach we get an upper bound for $\dim N_G$ from Lemma 3.6 of $\dim N_G \leq [H : R]$, where R is the stabilizer in P of the chosen $Q/[Q,Q]$ space containing the chosen e characters.

We present the arguments in the cases when H is of type **L** and **O** separately.

1.2.1. First suppose that H is linear. Here we may assume that $n \geq 8$ by Lemma 2.4. In order to invoke Lemma 3.6, we need e pairwise orthogonal subspaces V_χ of

V; this can be accomplished by choosing e characters so that no two of which are inverse to each other. Thus, if q is odd, then we pick j minimal so that the number of nonzero vectors in a j-subspace of Q is at least $2e$, whereas if q is even, we pick j minimal so that the number of nonzero vectors in a j-subspace of Q is at least e; i.e., $q^j - 1 \geq 2e \geq q^{j-1} - 1$ in the first instance, and $q^j - 1 \geq e \geq q^{j-1} - 1$ in the second. Note that we have

$$\dim N_G \leq [H : R] \leq 4q^{\dim H - \dim R}.$$

Thus, as H is linear, we obtain

$$\dim N_G \leq [H : R] \leq 4q^{n(j+1)-j}.$$

From [5] we see that $r \geq \frac{1}{2}(\frac{q^n-1}{q-1} - 3) \geq \frac{1}{2}q^{n-1} \geq q^{n-2}$. For the lower bound we employ the bound of Lemma 3.5. By Lemma 2.6 we know that

$$\binom{q^{n-2}}{e} \geq q^{e(n-2)/4}.$$

Observe that for $j = 1$ we always have a contradiction, as we assume that $e \geq 5$, and thus the lower bound is at least $q^{3(n-1)}$. One checks that for $j = 2, 3, 4, 5$ and $q \geq 32, 7, 4, 3$, respectively, as well as for $j \geq 7$ and $q \geq 2$ we already get a contradiction.

Thus the only cases that remain are $q = 2$ and $2 \leq j \leq 6$, $q = 3$ and $2 \leq j \leq 4$, $q = 4$ or 5 and $j = 2$ or 3, or $7 \leq q \leq 31$ and $j = 2$. For these cases we evaluate the binomial coefficient from Lemma 3.5 directly. First we consider the case when $3 \leq q \leq 31$ and $j = 2$. Since $e \geq 5$, evaluating the first 5 terms of $\binom{\frac{1}{2}q^{n-1}}{e}$ shows that in this instance this binomial coefficient is larger than $4(q^{3n-2})$. If $e = 3, 4$ and $q \geq 11$, then $j = 1$ and thus we are done. The values $q = 2$ and $j = 2$ lead to $e \leq 3$ contrary to our hypothesis on e. The remaining cases for $j = 3$ and $3 \leq q \leq 5$ readily lead to a contradiction, since $n \geq 8$ and $e \geq 5$; similarly, the case $q = 3$ and $j = 4$. Finally, we consider the cases where $q = 2$ and $3 \leq j \leq 6$. Here we get a contradiction provided $e \geq 6$. Now consider the case $e = 5$ and thus $j = 3$. Our lower bound is $\binom{2^{n-1}-2}{5}$. Computing the index $[H : R]$ directly in this instance yields $8 \cdot (2^n - 1)(2^{n-1} - 1)(2^{n-2} - 1)(2^{n-3} - 1)$. The normalizer of R in $\mathrm{L}_{n-1}(2)$ contains an $\mathrm{L}_3(2)$ that centralizes an $\mathrm{L}_{n-4}(2)$. Let B be a Borel subgroup of this $\mathrm{L}_3(2)$. Now we observe that B has 3 orbits on the $\mathrm{L}_{n-4}(2)$-invariant characters of the unipotent radical of P of lengths 1, 2, and 4. We pick the 5 characters in Q^* in the union of the two orbits of lengths 1 and 4 respectively. Consequently, this set of vectors is B-invariant; and thus the stabilizer of these vectors is the order of B times larger than in our initial setup. Comparing the bounds for $n = 8$ already yields the desired contradiction.

1.2.2. Now suppose that H is of type $\mathbf{O}$. Here the subgroup R of H is of the form $Q : \mathrm{O}_{n-2-j}(q)$. In case q is odd and j is even, we have $q^j - 1 \geq 2e \geq q^{j-1} + q^{j/2-1} - 1$ and in case j is odd we have $q^j + q^{(j-1)/2} - 1 \geq 2e \geq q^{j-1} - 1$. Now if q is even then we may drop the coefficient 2 of e in these inequalities.

For our choice of the subgroup R of P we choose a j-dimensional nondegenerate subspace whose stabilizer in P is R. Moreover, we pick j minimal such that this subspace contains at least $2e$ singular vectors. We obtain an upper bound for $\dim N_G$ of

$$\dim N_G \leq [H : R] \leq [H : P][P : R] \leq q^{n-1}q^{n-2} \cdots q^{n-1-j} \leq q^{jn-j(j+1)/2}.$$

As above, using the information from Table 1, we have $k \geq q^{n-4}$. By Lemma 2.4 we may assume that $n \geq 12$. Once again, for the lower bound we employ the bound of Lemma 3.5. By Lemma 2.6 we know that

$$\binom{q^{n-4}}{e} \geq q^{e(n-4)/4}.$$

Observe that we always achieve our desired contradiction provided $j \leq 2$, since $e \geq 5$. So, now suppose that $j \geq 3$. Then $e \geq (q-1)/2$. Since $n \geq 12$ we derive a contradiction for $j = 3, 4, 5, 6$ and $q \geq 17, 7, 3, 2$, respectively. This leaves the following open cases: $2 \leq q \leq 16$ and $j = 3$, or $2 \leq q \leq 5$ and $j = 4$, or $q = 2$ and $j = 5$. For these cases we again evaluate the binomial coefficient from Lemma 3.5 directly. First we consider the case when $2 \leq q \leq 16$ and $j = 3$. Since $e \geq 5$, evaluating the first 5 terms of $\binom{q^{n-4}}{e}$ shows that in this instance this binomial coefficient is larger than q^{3n-6} and we get a contradiction. Next we study the instances when $2 \leq q \leq 5$ and $j = 4$. If $q = 4$ or 5 and $j = 4$ then $e \geq 12$ and this readily leads to a contradiction. For $q = 2$ and $j = 5$ we have $e \geq 9$. This again leads to a contradiction. The only cases that remain are the ones for $q = 2$, $j = 4$, and $5 \leq e \leq 8$, or $q = 3$, $j = 4$, and $5 \leq e \leq 16$. For both cases $q = 2, 3$, we derive a contradiction for $e \geq 6$, since $n \geq 12$. This only leaves the case $q = 2, 3$, $j = 4$, and $e = 5$ to be considered. If $q = 3$, $j = 4$, and $e = 5$, we can improve the lower bound for k to $\frac{1}{2}(3^9 - 1)$ and the upper bound for $\dim N_G$ by factor of $1/5$. For, $\mathrm{O}_4^-(3)$ contains an element of order 5 which has no fixed points on any 4-dimensional module over $\mathbb{F}_3$. Using the improved bounds leads to a contradiction in this instance as well. Finally, when $q = 2$, $j = 4$, and $e = 5$ we use the actual Landazuri-Seitz-Zalesskii bound from [21] for H, and thus we get the largest possible estimate for the rank of K. As in the case $q = 3$, $j = 4$ we improve the upper bound of $\dim N_G$ by a factor of $1/5$, leading once again to a contradiction.

We now turn to the cases where H is still of type **L** or **O** and for each χ of Q every nonzero H_χ-invariant vector in the socle of V_χ is nondegenerate. Note that this implies that q is even in these instances. Otherwise, V_χ is totally singular with respect to any nondegenerate bilinear form. Moreover, we observe that K must be of type **O**, for otherwise, Q embeds into a maximal split torus of K. As before we pick a nonzero vector v_χ in the socle of V_χ and consider the subspace of V spanned by the P-orbit of v_χ. We choose $2e+1$ (or possibly only $2e$) vectors in this subspace and consider their span S such that the form of K restricted to S has Witt index e. Now choose a basis s_1 up to s_e of a maximal singular subspace of S and apply Lemma 3.6. In order to get the lowest possible upper bound for $\dim N_G$, we pick $2e + 1$ characters in such a fashion that they belong to a nondegenerate subspace of Q^* of smallest possible dimension. We argue as above and again we separate the cases **L** and **O**.

1.2.3. Firstly, we study the case when H is linear. We refer the reader to the setup of (**1.2.1**). In this case e satisfies $q^j - 1 \geq 2e + 1 \geq q^{j-1}$. As opposed to (**1.2.1**), here R no longer contains the all of Q but rather the $(n-i-j)$-dimensional subspace centralizing S. Thus for our upper bound we obtain

$$\dim N_G \leq [H : R] \leq 4q^{n(j+1)}.$$

Our lower bound is as in (**1.2.1**), namely

$$q^{(n-2)e/4} \leq \dim N_G.$$

As $e \geq 5$, we see that $j \geq 2$. If $j = 2$, then $q \geq 8$, in which case we get a contradiction, since $\binom{\frac{1}{2}q^{n-1}}{5} \geq 4q^{3n}$. For $j = 3$ we see that $q \geq 4$ and $e \geq 7$. Once again we get our desired contradiction, since $(q^{n-2})^6 \geq 4q^{4n}$. Next consider $j = 4$; we get that $q = 2$ and $5 \leq e \leq 7$. First we consider the cases when $e = 6$ or 7. We sharpen our upper bound for $\dim N_G$ as follows. Observe that the centralizer in N_G of R is nontrivial. The group $2^4 : \mathrm{L}_4(2)$ normalizes R and together they generate a second maximal parabolic subgroup P_1 of H contained in P. The index of P_1 in H is

$$[H : P_1] = \frac{(2^n-1)(2^{n-1}-1)\cdots(2^{n-4}-1)}{(2^4-1)(2^3-1)(2^2-1)}.$$

The maximal degree of an irreducible representation of $2^4 : \mathrm{SL}_4(2)$ in characteristic not equal to 2 is at most the denominator of $[H : P_1]$. Thus, by Frobenius reciprocity, we obtain an upper bound for $\dim N_G$ equal to the numerator of $[H : P_1]$. Comparing this sharper upper bound with our lower bound yields a contradiction when $j = 4$ and $e \neq 5$. A similar argument eliminates the case when $j = 5$, as then $e \geq 8$. Finally, the case $q = 2$ and $e = 5$ is the one on our list.

1.2.4. Now suppose that H is of type **O**. Again by Lemma 2.4 we know that $n \geq 12$. Here we may take the same upper bound for $\dim N_G$ as in the (**1.2.2**). However, we do have to argue as in case (**1.2.3**). We decompose the unipotent radical Q of $P = LQ$ as an R-module $Q = [Q, R] \oplus C_Q(R)$. We define the subgroup R_1 of R as $R_1 := (L \cap R)[Q, R]$. By Lemma 3.6 and Frobenius reciprocity we obtain that $\dim N_G \leq [H : R_1]$. As in (**1.2.3**) we observe that $C_Q(R)$ normalizes R_1 and thus also $C_{N_G}(R_1)$. As every irreducible representation of $C_Q(R)$ is one-dimensional, we obtain that in fact $\dim N_G \leq [H : R]$.

In the present situation the ranges for e are as follows: if j is even, we have $q^j - 1 \geq 2e + 1 \geq q^{j-1} + q^{j/2-1} - 1$ and in case j is odd we have $q^j + q^{(j-1)/2} - 1 \geq 2e + 1 \geq q^{j-1} - 1$. Our initial bounds for $\dim N_G$ are identical to the ones from (**1.2.2**). Thus we get a contradiction when $j \leq 2$ or $j = 3, 4, 5, 6$ and $q \geq 32, 8, 4, 4$, respectively. First we consider the case when $j = 6$ and $q = 2$; then $e \geq 17$. Using Lemma 3.5 as a lower bound and the fact that $n \geq 12$ yields a contradiction. For $j = 5$ and $q = 2$ we get $e \geq 7$; then using the actual lower bound from [12] leads to a contradiction. Now we study the case $j = 4$ and $q = 2$, or 4, and $e \geq 5$ in the first case and $e \geq 33$ in the latter. For $q = 4$ this readily leads to a contradiction, as desired. For $q = 2$ and $j = 4$ we get also a contradiction once $e \geq 6$. But the case $q = 2$, $j = 4$ and $e = 5$ where H and K are orthogonal is one on our list of exceptions. For $j = 3$ we see that $q \geq 4$ and $e \geq 7$. Again, this is a contradiction using our bounds.

We have now completed the discussion in the case when H is orthogonal.

Case 2: Now we suppose that H is of type **U** or **S**. Let H_1 be the derived subgroup of the Levi subgroup of the stabilizer of a singular one-dimensional subspace of N_H. Note that H_1 is quasi-simple, again of type **U** or **S** and of corank 1 in H. We choose a parabolic subgroup $P = LQ$ of H_1 satisfying the following properties

(i) Q has nilpotency class 2;
(ii) every nontrivial orbit of L on $Z(Q)$ contains at least e elements;

(iii) $Z(Q)$ is minimal subject to condition (ii).

Observe that we can always find such a parabolic subgroup. Specifically, we may assume that P is the j-th maximal standard parabolic subgroup P_j of H_1, i.e., the stabilizer of a singular j-dimensional subspace of the natural module N_{H_1} of H_1.

Lemma 4.4. *There exists a non-trivial L-invariant submodule V' of V that is singular with respect to the form defining K.*

Proof. If q is odd, then we may pick V' to be the eigenspace of a transvection centralizing L. Else, q is even and H is unitary. Here we may pick V' to be the eigenspace of a non-central element of order coprime to 2 and l centralizing L. Note that the centralizer of L in H contains a fundamental $\mathrm{SL}_2(q)$. So, unless $q = 2$ and $l \neq 3$ this supplies the desired element. If $q = 2$ and $l = 3$, then an element of order 3 is unipotent and we choose V' to be the radical of the centralizer of this 3-element in V. □

Corollary 4.5. *Suppose as in Lemma 4.4 and assume that q is even, and either $q \geq 8$, or $q = 4$ and $l \neq 5$. Then there exist non-trivial L-invariant isomorphic submodules V' and V'' of V such that their direct sum is singular with respect to the form defining K.*

Proof. From the proof of Lemma 4.4 H is unitary. Let U be the unipotent radical of a parabolic subgroup of H containing H_1 as a Levi subgroup. Here we may pick V' and V'' to be eigenspaces in the commutator $[V, U]$ of a non-central element of order coprime to 2 and l centralizing L. We pick the corresponding eigenvalues in such a way as to be not inverse to each other and nonidentity. One checks that V' and V'' are isomorphic. □

Corollary 4.6. *Suppose as in Lemma 4.4 and assume that H is of type* **S** *and $q \geq 5$ is odd. Then there exist non-trivial L-invariant submodules V' and V'' of V such that their direct sum is singular with respect to the form defining K. Moreover, if $q \geq 7$ and z is a transvection in H_1 such that every primitive p-th root of unity ζ is an eigenvalue of z on V, then we may pick V' and V'' in such a way that if ζ is an eigenvalue of z on V', then ζ^{-1} is an eigenvalue of z on V''.*

Proof. Here we choose V' and V'' as eigenspaces of a long root element centralizing H_1. If the eigenvalue on V' is ν, then V'' is from the orbit of ν^{-1} under the automizer of the long root group containing z. Note that when $q \geq 7$ each automizer orbit has at least 3 elements guaranteeing the existence of an element ν' not equal to ν or its inverse. □

Remark 4.7. Observe that we can apply Lemma 3.6 to any direct sum decomposition of any subspace of V' into e summands.

We continue to study the submodule V' from Lemma 4.4 and consider the decomposition of the restriction of V' to $Z(Q)$:

$$V'|_{Z(Q)} = \bigoplus_{i=1}^{s} V'_{\kappa_i},$$

where each κ_i is a character of $Z(Q)$ and $V'_\kappa = \{v \in V' \mid z \cdot v = \kappa(z)v \text{ for } z \in Z(Q)\}$. Observe that the derived subgroup $L' = L_1 L_2$ of L is a product of a linear group L_1 and one which is of the same type but smaller rank than H_1, denoted L_2. Observe that L_2 centralizes $Z(Q)$. Consider the subgroup $R := L_2 Q$ of P. Observe that Q

modulo the kernel of the action of Q on V'_κ is special for each V'_κ. By [19, (1.4)] we see that each V'_κ can be written as a tensor product of R-modules

$$V'_\kappa = E_\kappa \otimes Y,$$

where E_κ is an R-module with nontrivial non-linear Q-action of dimension $q^{(n-2j)/2}$ in case **S** and $q^{(n-2j)}$ in case **U** and Y is a projective L_2-module.

2.1. First we consider the case when H and so H_1 are of type **U**. In this instance E_κ is a selfdual L_2-module. For, in characteristic zero E_κ is the direct sum of all the Weil modules for L_2, and thus is selfdual [17]. This property is preserved when reducing modulo l because the dimensions of the Weil modules of the unitary group are at most one more than the Landazuri-Seitz-Zalesskii bounds. Here we distinguish the cases when q is even respectively odd.

In case q is odd the spaces V'_κ are totally singular with respect to the form defining K. By Remark 4.7 any direct sum of e pairwise distinct V'_κ's satisfy the hypotheses of Lemma 3.6. Thus we pick e characters that are pairwise inverse to each other, with one possible exception, namely when e is odd, all but one of the summands are paired. As the minimal L_1-orbit on $Z(Q)^*$ has size q^{j-1}, we set up so that $q^j - 1 \geq e \geq q^{j-1} - 1$. Now we observe that the module $\bigotimes_{i=1}^e E_{\kappa_i}$ contains a unique one-dimensional R-submodule in its socle provided e is even. If e is odd, then $\bigotimes_{i=1}^e E_{\kappa_i}$ contains an R-submodule of dimension $\dim E_\kappa$. Thus we obtain the following estimate

$$\dim N_G \leq (\dim Y)^e [H : R] (\dim E_{\kappa_i})^\delta,$$

where δ is the parity of e modulo 2.

If q is even, we consider the direct sum $V' \oplus V''$ from Corollary 4.5. We consider the restriction of $V' \oplus V''$ to $Z(Q)$. Then we have

$$V'|_{Z(Q)} \oplus V''|_{Z(Q)} = \bigoplus_{i \geq 1} E_{\kappa_i} \otimes Y' \oplus \bigoplus_{j \geq 1} E_{\kappa_j} \otimes Y''.$$

We apply Lemma 3.6 to the following collection of e subspaces of $(V' \oplus V'')|_{Z(Q)}$, by choosing $[e/2]$ summands from V' and the identically indexed set of summands from V''. If e is odd, we pick the last summand from V'. The R-module

$$E_{\kappa_i} \otimes Y' \otimes E_{\kappa_i} \otimes Y''$$

contains an R-submodule of dimension $(\dim Y)^2$ in its socle, where Y' and Y'' are isomorphic to Y as R-modules. Thus we get an upper bound for $\dim N_G$ of

$$\dim N_G \leq [H : R] (\dim Y)^e (\dim E_\kappa)^\delta,$$

where δ is the parity of e modulo 2. In particular, we see that $q^j - 1 \geq (e+\delta)/2 \geq q^{j-1} - 1$.

Observe that our upper bounds for $\dim N_G$ are identical independent of whether q is even and odd.

Now we compute the lower bound for $\dim N_G$ using Lemma 3.5, i.e., $\binom{k}{e}$. Note that for the rank of K we have $k \geq \frac{1}{2} q^{(n-2j-2)}(q^j - 1) \dim Y$. Therefore, for $k = c \dim Y$ we do get a contradiction once

$$\binom{c}{e} \geq [H : R] (\dim E_\kappa)^\delta,$$

where δ is the parity of e. Observe, that we may use the lower bound

$$\binom{c}{e} \geq (\dim E_\kappa)^e,$$

i.e., it suffices to show that

$$(\dim E_{\kappa_i})^{e-\delta} \geq [H:R].$$

We conclude this case by observing that this inequality holds in our situation: we estimate that $[H:R] \leq q^a$, where $a := 2n(j+2)-(j+1)(2j+1)-2$, and $\dim E_\kappa = q^{n-2j-2}$.

Note that in case q is even we suppose that $q \geq 8$, or $q = 4$ and $l \neq 5$ by Corollary 4.5. Then for $j \leq 4$ we always get a contradiction, since $n \geq 8$ in type **U**, cf. Lemma 2.4. Moreover, one checks that this is also the case for $j = 3$ and $q \geq 4$. In case $e \geq 8$, we get the desired contradiction already for $j = 1$ and also if $j = 2$ and $e \geq 10$. In particular, this is the case whenever $q \geq 11$. In summary, if $e \geq 12$ we always have a contradiction for the values of q that we are considering. Thus we may suppose that $e \leq 11$. Now we consider the cases $q = 2$ or $q = 4$ and $l = 5$. We find subspaces V' and V'' as in Corollary 4.5 by replacing the group H_1 with $\mathrm{U}_{n-4}(2)$ in case $q = 2$ and by $\mathrm{U}_{n-3}(4)$ otherwise.

2.2. Next we consider the case when H is of type **S**. Here q is odd and by Lemma 2.4 we have $n \geq 8$. Thus the spaces V_κ are totally singular with respect to the form defining K and as before we aim to choose e summands V_κ of $V|_{Z(Q)}$ such that the corresponding characters satisfy the hypotheses of Lemma 3.6.

We consider the restriction of the direct sum $V' \oplus V''$ from Corollary 4.6 to $Z(Q)$. Then we have

$$V'|_{Z(Q)} \oplus V''|_{Z(Q)} = \bigoplus_{i \geq 1} E_{\kappa_i} \otimes Y' \oplus \bigoplus_{j \geq 1} E_{\kappa_j} \otimes Y''.$$

We apply Lemma 3.6 to the following collection of e subspaces of $(V' \oplus V'')|_{Z(Q)}$, by choosing $[e/2]$ summands from V' and the inversely indexed set of summands from V''. With out loss of generality, we may suppose that $\dim Y' \leq \dim Y''$. If e is odd, we pick the last summand from V'. The R-module

$$E_{\kappa_i} \otimes Y' \otimes E_{-\kappa_i} \otimes Y''$$

contains an R-submodule of dimension $\dim Y' \dim Y''$ in its socle. Thus we get an upper bound for $\dim N_G$ of

$$\dim N_G \leq [H:R](\dim Y')^{(e+\delta)/2}(\dim Y'')^{(e-\delta)/2}(\dim E_\kappa)^\delta,$$

where δ is the parity of e. In particular, we see that $q^j - 1 \geq (e+\delta)/2 \geq q^{j-1} - 1$, because of our assumption on the existence of eigenvalues of z.

Next we compute the lower bound for $\dim N_G$ using Lemma 3.6. As above, in order to get a contradiction, it suffices to show that

$$(\dim E_{\kappa_i})^{e-\delta} \geq [H:R].$$

Note that here we have $\dim E_\kappa = q^{(n-2j-2)/2}$ and that $[H:R] \leq q^a$, where $a := (n-j-1)(j+2)-1$.

Note that $q \geq 7$ by Corollary 4.6 and by Lemma 2.4 we have $n \geq 8$. If $j \geq 2$, then we always derive a contradiction, likewise if $j = 1$ and $e \geq 8$.

Now for the cases $q = 3, 5$ not covered by Corollary 4.6 we argue as follows. Here we apply Lemma 4.4 directly and use the fact that each nonidentity eigenvalue of z occurs on V. We pick j so that $q^j - 1 \geq e + \delta \geq q^{j-1} - 1$. As before, we decompose $V'|_{Z(Q)}$ and pick e summands with the property that the indexing set of the summands contains as many pairs of the form (ν, ν^{-1}) as possible. In this fashion we derive the identical upper and lower bounds for $\dim N_G$ as before, the only difference being a weaker rate of increase of e as a function of j. In any event, we derive a contradiction whenever $e \geq 10$ and $q = 5$, or $e \geq 12$ and $q = 3$.

The argument here also applies for the cases $q \geq 7$ and $q \equiv 1 \mod 4$ where we drop the assumption on the existence of all eigenvalues of z on V. The condition $q \equiv 1 \mod 4$ ensures that with each occuring eigenvalue ν also its inverse appears. Here we get $q^j - 1 \geq 2e \geq q^{j-1} - 1$. This results in the same bounds for $\dim N_G$ as above. In any event, here we derive a contradiction whenever $e \geq 14$, or $e = 12, 13$ and $q \geq 5$ and $n \geq 28$.

Now we study the case $q \equiv 3 \mod 4$. By [6] we have that $\dim V \leq 3(q^{n/2} - 1)/2$. In particular, this implies that the R-composition factors of the spaces Y which occur in our decompositions above must be linear.

By Landazuri-Seitz-Zaleskii [12] we have

$$(q^{n/2} - 1)^e \leq \binom{(q^{n/2} - 1)/2}{e} \leq \dim V.$$

Let P_j be the j-th maximal standard parabolic subgroup of H with Levi subgroup $L_1 L_2$, where $L_2 = \mathrm{Sp}_{n-2j}(q)$ and unipotent radical Q_j. We pick j so that $q^j - 1 \geq 2e \geq q^{j-1} - 1$. Further, let R_1 be the first standard parabolic subgroup of L_2. Because of the congruence condition, any choice of a set of e summands of the decomposition of $V|_{Z(Q_j)}$ satisfies the hypotheses of Lemma 3.6, because here K must be of type $\mathbf{L}$. Each of these is of the form $E_\kappa \otimes Y$, where Y has a trivial L_2-module in its socle. So, without loss of generality, we may assume that each summand is of the form E_κ. Notice that the tensor product of two E_κ's satisfies the hypotheses on the existence of eigenvalues of transvections from above. Thus the tensor product of any four E_κ's contains a trivial R_1-submodule in its socle, and the tensor product of any three of them contains an R_1-submodule of dimension $\dim E_\kappa$. For our upper bound for $\dim N_G$ we get

$$\dim N_G \leq [H : R_1](\dim E_\kappa)^2.$$

The lower bound is as above and thus we have our desired contradiction once

$$(q^{n/2} - 1)^e \geq (\dim E_\kappa)^2 [H : R_1].$$

We compute that $(\dim E_\kappa)^2 [H : R_1] \leq q^a$, where $a = 2n(j+2) - j(j+3) - 4$. We always have a contradiction whenever $e \geq 24$, or if $q \geq 7$ and $e \geq 17$, or $q \geq 29$ and $e \geq 14$.

This completes the proof of Proposition 1.7.

Proof of Theorem 1.3. Here we use the notation and setup of the proof of Proposition 1.7. In case H is linear or orthogonal we observe that the desired statement is already proved whenever we are in the situation of (**1.1**) above. Moreover, in the situation of (**1.2**) we see that for every choice of q we may pick $j = 1$ whenever $2e + 1 \leq q - 1$. If $e = 3$ or 4, then $\binom{b}{e} \geq \binom{q^{n-4}}{3}$, whereas $\dim N_G \leq 4q^{2n}$ proving the first case of the theorem. In the second case of Theorem 1.3 we let R be the Levi factor of type $\mathrm{U}_{n-2}(q)$ respectively $\mathrm{Sp}_{n-2}(q)$. We decompose V with respect to a root subgroup centralizing R. Using the same arguments as in (**2.1**) and (**2.2**) we obtain a contradiction once

$$(\dim E_{\kappa_i})^{e-\delta} \geq [H : R].$$

In case H is of type $\mathbf{S}$ we have $\dim E_{\kappa_i} = q^{(n-2)/2}$ and $[H : R] \leq q^{2n}$, while in the unitary case we have $\dim E_{\kappa_i} = q^{n-2}$ and $[H : R] \leq q^{4n}$. If $e \geq 6$ we obtain the desired contradiction. Finally, in the last case of Theorem 1.3 we argue as above with the only difference being that here R is the largest parabolic subgroup of $\mathrm{Sp}_{n-2}(q)$. $\square$

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