STABLE EQUIVALENCE AND GENERIC MODULES

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Abstract. Let Λ and Γ be finite dimensional algebras. It is shown that any stable equivalence $f\colon \overline{\mathrm{mod}}\,\Lambda \to \overline{\mathrm{mod}}\,\Gamma$ between the categories of finitely generated modules induces a bijection $M \mapsto M_f$ between the sets of isomorphism classes of generic modules over Λ and Γ such that the endolength of M_f is bounded by the endolength of M up to a scalar which only depends on f. Using Crawley-Boevey's characterization of tame representation type in terms of generic modules, one obtains as a consequence a new proof for the fact that a stable equivalence preserves tameness. This proof also shows that polynomial growth is preserved.

Generic modules were introduced by Crawley-Boevey in order to give a conceptual description of the representation type of a finite dimensional algebra [1]. In this note we study the relation between the generic modules over two finite dimensional algebras Λ and Γ under the assumption that there exists an equivalence $\overline{\mathrm{mod}}\,\Lambda \to \overline{\mathrm{mod}}\,\Gamma$ between the stable categories of finitely generated modules.

Recall that a Λ-module is *generic* if it is indecomposable and endofinite but not of finite length over Λ. A Λ-module M is *endofinite* if the *endolength* $\mathrm{endol}(M)$ which is the length of M over its endomorphism ring $\mathrm{End}_\Lambda(M)$ is finite.

The endolength of a module is *not* a categorical invariant. In other words, a Morita equivalence $F\colon \mathrm{Mod}\,\Lambda \to \mathrm{Mod}\,\Gamma$ between two module categories does not necessarily preserve the endolength. However, it is not hard to see that there exists a constant c_F such that $\mathrm{endol}(FM) \le c_F.\mathrm{endol}(M)$ for every Λ-module M. The purpose of this paper is to show that this fact remains true for a stable equivalence.

Using Crawley-Boevey's characterization of tame representation type in terms of generic modules, we obtain in this way a simple proof for the fact that a stable equivalence preserves tame representation type [2]. In fact, we improve the results obtained in [2] and show that a stable equivalence preserves polynomial growth.

In [2], it was shown that an equivalence $f\colon \overline{\mathrm{mod}}\,\Lambda \to \overline{\mathrm{mod}}\,\Gamma$ extends to an equivalence $F\colon \overline{\mathrm{Pinj}}\,\Lambda \to \overline{\mathrm{Pinj}}\,\Gamma$ between the full subcategories of (not necessarily finitely generated) pure-injective modules in $\overline{\mathrm{Mod}}\,\Lambda$ and $\overline{\mathrm{Mod}}\,\Gamma$. Any endofinite module is pure-injective, and our main result states that the functor F controls the endolength as follows.

Theorem. *Let Λ and Γ be artin algebras and suppose that $f\colon \overline{\mathrm{mod}}\,\Lambda \to \overline{\mathrm{mod}}\,\Gamma$ is an equivalence. Denote by $F\colon \overline{\mathrm{Pinj}}\,\Lambda \to \overline{\mathrm{Pinj}}\,\Gamma$ the equivalence between the full subcategories of pure-injective modules in $\overline{\mathrm{Mod}}\,\Lambda$ and $\overline{\mathrm{Mod}}\,\Gamma$ which extends f (see [2]). Then there exists a constant $c_f \in \mathbb{N}$ such that*

$$\mathrm{endol}(FM) \le c_f.\mathrm{endol}(M)$$

for every endofinite non-zero Λ-module M.

Corollary 1. *An equivalence $f\colon \overline{\mathrm{mod}}\,\Lambda \to \overline{\mathrm{mod}}\,\Gamma$ induces a bijection $M \mapsto M_f$ between the sets of isomorphism classes of indecomposable endofinite and non-injective modules over Λ and Γ such that $\mathrm{endol}(M_f) \le c_f.\mathrm{endol}(M)$ for all M.*

Before we can give the proof of the main theorem we need some preparations. Let us start with some notation. We fix an artin algebra Λ and denote by $\operatorname{Mod}\Lambda$ the category of all Λ-modules. The full subcategory of finitely generated modules is denoted by $\operatorname{mod}\Lambda$. The *stable category* $\overline{\operatorname{Mod}}\,\Lambda$ has the same objects as $\operatorname{Mod}\Lambda$ but $\overline{\operatorname{Hom}}_\Lambda(M,N)$ is the group $\operatorname{Hom}_\Lambda(M,N)$ of Λ-maps modulo the subgroup of maps which factor through some injective module. Analogously, the stable category $\overline{\operatorname{mod}}\,\Lambda$ of finitely generated modules, and the stable category $\overline{\operatorname{Pinj}}\,\Lambda$ of pure-injective modules are defined.

We denote by $S_1,\dots,S_p$ a representative set of simple Λ-modules, and $T_1,\dots,T_q$ denotes a representative set of simple Γ-modules. Keeping the equivalence $f\colon \overline{\operatorname{mod}}\,\Lambda\to\overline{\operatorname{mod}}\,\Gamma$ fixed, we choose Λ-modules $U_1,\dots,U_q$ sucht that $f(U_j)\cong T_j$ in $\overline{\operatorname{mod}}\,\Gamma$ for all j. Now let M and X be Λ-modules and suppose that X has finite length. We denote by $[X,M]$ the length of $\operatorname{Hom}_\Lambda(X,M)$ as $\operatorname{End}_\Lambda(M)$-module, and $[\![X,M]\!]$ denotes the length of $\overline{\operatorname{Hom}}_\Lambda(X,M)$ as $\overline{\operatorname{End}}_\Lambda(M)$-module. The multiplicity of S_i in a composition series of X is denoted by $m_i(X)$. We note the following elementary fact.

Lemma 1. *Let M and X be Λ-modules and suppose that X has finite length. Then*

$$[\![X,M]\!]\le [X,M]\le \sum_{i=1}^{p} m_i(X)[S_i,M].$$

Proof. The first inequality follows from the fact that the length of $\overline{\operatorname{Hom}}_\Lambda(X,M)$ as an $\overline{\operatorname{End}}_\Lambda(M)$-module equals the one as an $\operatorname{End}_\Lambda(M)$-module. In order to prove the second inequality, observe that any exact sequence $0\to X'\to X\to X''\to 0$ of Λ-modules gives an exact sequence

$$0\longrightarrow \operatorname{Hom}_\Lambda(X'',M)\longrightarrow \operatorname{Hom}_\Lambda(X,M)\longrightarrow \operatorname{Hom}_\Lambda(X',M)$$

and so $[X,M]\le[X',M]+[X'',M]$. By induction on the length of X we get the second inequality. $\square$

The next lemma is simple but extremely useful.

Lemma 2. *Let M be a Λ-module without any non-zero injective submodule, and let S be a simple Λ-module. Then $\operatorname{Hom}_\Lambda(S,M)=\overline{\operatorname{Hom}}_\Lambda(S,M)$ and therefore $[S,M]=[\![S,M]\!]$.*

Proof. Any non-zero map $S\to M$ which factors through some injective module, also factors through the injective envelope $E(S)$ of S, and induces therefore a monomorphism $E(S)\to M$. This is impossible by our assumption on M. Thus $\operatorname{Hom}_\Lambda(S,M)=\overline{\operatorname{Hom}}_\Lambda(S,M)$. $\square$

Proof of the Theorem. Let M be an endofinite non-zero Λ-module and let $N=FM$. We denote by I_N a maximal injective submodule of N which exists by Zorn's lemma since direct limits of injective modules are injective. This gives a direct decomposition $N=N_I\coprod I_N$. Using the preceding lemmas we obtain the following bound for the endolengh of N_I:

$$\begin{aligned}
\operatorname{endol}(N_I) = [\Gamma, N_I] \leq \sum_{j=1}^{q} m_j(\Gamma)[T_j, N_I] &= \sum_{j=1}^{q} m_j(\Gamma)[\![T_j, N_I]\!] \\
&= \sum_{j=1}^{q} m_j(\Gamma)[\![U_j, M]\!] \\
&\leq \sum_{j=1}^{q} m_j(\Gamma)[U_j, M] \\
&\leq \sum_{j=1}^{q} m_j(\Gamma)\Big(\sum_{i=1}^{p} m_i(U_j)[S_i, M]\Big) \\
&= \sum_{i=1}^{p}\Big(\sum_{j=1}^{q} m_j(\Gamma)m_i(U_j)\Big)[S_i, M] \\
&\leq n_f \sum_{i=1}^{p}[S_i, M] \\
&\leq n_f[\Lambda/\operatorname{rad}\Lambda, M] \\
&\leq n_f[\Lambda, M] = n_f.\operatorname{endol}(M)
\end{aligned}$$

where $n_f = \max\{\sum_{j=1}^{q} m_j(\Gamma)m_i(U_j) \mid 1 \leq i \leq p\}$. The endolength of any injective Γ-module I is bounded by the composition length $\ell(\Gamma)$ of Γ since I is isomorphic to a direct factor of a product of copies of the injective cogenerator $D\Gamma$ (D denotes the usual duality between $\operatorname{mod}\Gamma$ and $\operatorname{mod}\Gamma^{\mathrm{op}}$), and

$$\operatorname{endol}(\prod_{\kappa} D\Gamma) = \operatorname{endol}(D\Gamma) = \operatorname{endol}(\Gamma) = \ell(\Gamma)$$

for any cardinal $\kappa > 0$. Defining $c_f = n_f + \ell(\Gamma)$, we obtain

$$\operatorname{endol}(FM) = \operatorname{endol}(N_I) + \operatorname{endol}(I_N) \leq n_f.\operatorname{endol}(M) + \ell(\Gamma) \leq c_f.\operatorname{endol}(M).$$

This finishes the proof of the Theorem. $\square$

We are now in a position to prove that a stable equivalence preserves the representation type. To this end denote for an algebra Λ by $g_\Lambda(n)$ the number of isomorphism classes of generic Λ-modules of endolength n, and put $\gamma_\Lambda(n) = \sum_{i=1}^{n} g_\Lambda(i)$.

Corollary 2. *Let $\underline{\operatorname{mod}}\,\Lambda \to \underline{\operatorname{mod}}\,\Gamma$ be an equivalence. Then there exists a constant $c \in \mathbb{N}$ such that $\gamma_\Lambda(n) \leq \gamma_\Gamma(cn)$ for all $n \geq 1$. In particular, the following holds:*

1. *If Γ is generically tame, i.e., $g_\Gamma(n) < \infty$ for all $n \geq 1$, then Λ is generically tame.*
2. *If Γ is generically of polynomial growth, i.e., there exist $k \geq 1$ and $d \geq 1$ such that $g_\Gamma(n) \leq dn^k$ for all $n \geq 1$, then Λ is generically of polynomial growth.*
3. *If Γ is generically domestic, i.e., there exist only finitely many isomorphism classes of generic modules, then Λ is generically domestic.*

Proof. The formula $\gamma_\Lambda(n) \leq \gamma_\Gamma(cn)$ is an immediate consequence of Corollary 1. Observe that in the definition of generically tame algebras one can replace $g_\Lambda(n)$ by $\gamma_\Lambda(n)$. Hence,

(1) follows from the formula $\gamma_\Lambda(n) \leq \gamma_\Gamma(cn)$. Now assume that $g_\Gamma(n) \leq dn^k$ for some $k \geq 1$, $d \geq 1$, and all $n \geq 1$. Then

$$g_\Lambda(n) \leq \gamma_\Lambda(n) \leq \gamma_\Gamma(cn) = \sum_{i=1}^{cn} g_\Gamma(i) \leq \sum_{i=1}^{cn} di^k \leq (cn)d(cn)^k = (c^{k+1}d)n^{k+1},$$

for all $n \geq 1$. This implies (2). An algebra Γ is generically domestic if and only if there exists $N \in \mathbb{N}$ such that $\gamma_\Gamma(n) \leq N$ for all $n \geq 1$. Hence, also (3) is a consequence of the formula $\gamma_\Lambda(n) \leq \gamma_\Gamma(cn)$. □

Suppose that Λ is a finite dimensional algebra over an algebraically closed field and denote by $\mu_\Lambda(n)$ the minimal number of continuous one-parameter families which is needed to parametrize all but finitely many indecomposable Λ-modules of dimension n. In [1, Theorem 4.4 and 5.6], Crawley-Boevey has shown that $\mu_\Lambda(n)$ is finite for all $n \geq 1$ if and only if Λ is generically tame; moreover in this case $\mu_\Lambda(n) = \sum_{i|n} g_\Lambda(i)$. Therefore the 'generical' definition of the representation type coincides with the 'classical' one. More precisely:

- Λ is of tame representation type (i.e., $\mu_\Lambda(n) < \infty$ for all $n \geq 1$) if and only if Λ is generically tame.
- Λ is of polynomial growth (i.e., there exist $k \geq 1$ and $d \geq 1$ such that $\mu_\Lambda(n) \leq dn^k$ for all $n \geq 1$) if and only if Λ is generically of polynomial growth. Indeed, the inequality $g_\Lambda(n) \leq \mu_\Lambda(n)$ shows that polynomial growth implies 'generical' polynomial growth. On the other hand, if $g_\Lambda(n) \leq dn^k$ for some $k \geq 1$, $d \geq 1$, and all $n \geq 1$, then

$$\mu_\Lambda(n) \leq \sum_{i=1}^{n} g_\Lambda(i) \leq \sum_{i=1}^{n} di^k \leq ndn^k = dn^{k+1}.$$

- Λ is of domestic representation type (i.e., there exists $N \in \mathbb{N}$ such that $\mu_\Lambda(n) \leq N$ for all $n \geq 1$) if and only if Λ is generically domestic.

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References

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