A Note on Existence and Path Properties of Space-Time Processes corresponding to Time Dependent Fleming-Viot Operators

-revised version-

Jörg Kienitz, University of Bielefeld, Germany

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Abstract

We study a class of time dependent Dirichlet forms $(\mathcal{E},\, D(\mathcal{E}))$ having a space of measures as state space E and derive some general results about them. Especially we prove that under certain conditions they "generate" space-time diffusion processes. All of this is applied to construct time-dependent Fleming-Viot processes and derive some path properties using parabolic capacitary methods. We establish a general method to compare the parabolic and the elliptic case.

1 Introduction

The main purpose of this paper is to construct time-dependent Fleming-Viot processes using the theory of time-dependent Dirichlet forms which is included in the so called "Theory of generalized Dirichlet Forms" and derive some path properties by proving a general result about capacities. Here we develope a general method for carrying over results from the elliptic to the parabolic case and vice versa.

The theory of generalized Dirichlet forms was developed in the work of Stannat, [11] and [12]. It extends the classical (symmetric) theory of Fukushima and the non-symmetric (coercive) theory of Ma and Röckner. A generalized Dirichlet form is a bilinear form defined on some L_2-space which is in general neither symmetric nor coercive but still posseses an associated C_0-resovent,

C_0-semigroup and a generator on L_2. It also covers the time-dependent case. In Section 2 we recall this case in detail, as well as some general results from [6] and [12]. The general results are used to construct space-time Fleming-Viot processes in Section 3. First these diffusions arose as high-density limits of certain Markov chain models, e.g. the Wright-Fisher or the Moran-model, see [1] or [4] for details. There, each individual is of some type. The number of occuring types is a natural number d. The underlying type space, the state space of the process, is nothing but the $d-1$-dimensional simplex $\overline{\Delta}_d$:

$$\overline{\Delta}_d := \left\{ x \in \mathbb{R}^d \mid \sum_{i=1}^{d} x_i = 1,\ x_i \geq 0 \right\}. \tag{1}$$

The component x_i of an element $x \in \overline{\Delta}_d$ can be considered as the fraction of type i in the population. The operator generating the diffusion is given by

$$Lf := \frac{1}{2} \sum_{i,j=1}^{d} x_i(\delta_{ij} - x_j) \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \left(\sum_{j=1}^{d} q_{ij} x_j \right) \frac{\partial f}{\partial x_i}, \tag{2}$$

which can be defined on the twice differentiable functions restricted to $\overline{\Delta}_d$. Here $Q = (q_{ij})_{1 \leq i,j \leq d}$ is a $d \times d$-matrix which is the generator of a Markov-chain. The entries can be viewed as the intensity of a mutation from type i to type j and the diffusion part describes the sampling rate. The main idea of Fleming and Viot, [5], for studying the limiting diffusion processes was to take as state space the space $\mathcal{M}_1(S)$ of all probability measures on a Polish space S equipped with the weak topology. Now the operator to be studied is as follows:

$$\begin{aligned}
(3) L^{FV} u(\mu) &= \frac{1}{2} \int_S \int_S \mu(dy) \left(\delta_y(dx) - \mu(dx)\right) \nabla_x (\nabla_y u(\mu)) \\
&+ \int_S \mu(dy)\, A \left(\nabla_x u(\mu)\right) \\
&+ \int_S \int_S \mu(dx)\mu(dy)\sigma(x,y) \nabla_x u(\mu) \\
&- \int_S \int_S \mu(dx)\mu(dy) \left(\int_{S \times S} \sigma(x,y)\, \mu(dx) \otimes \mu(dy) \right) \nabla_x u(\mu).
\end{aligned}$$

The function u is an element of an appropriate space of test functions. Here ∇ denotes the "gradient" defined below, A is the mutation operator and the last two summands represent the selection part. In Section 3 the geometry needed to define Dirichlet forms of gradient type introduced in [9],[10] is recalled. Furthermore, we introduce bilinear forms for which it is shown that

they are time-dependent Dirichlet forms. At this stage we use some facts about Dirichlet forms. For the crucial notions as sector condition, closability or associated processes we refer to the excellent book [8]. The results obtained are used to study the special case where the underlying measure m is the reversible measure m^{FV} of the Fleming-Viot process which exists for specific mutation operators A. The first approach via bilinear forms to this subject is due to Overbeck, Röckner and Schmuland in [10]. In this framework it is possible to construct Fleming-Viot processes with singular coefficients and interactive selection. The selection is modelled using a density function, which is only a positive maybe unbounded measurable function in the domain of the underlying Dirichlet form. These processes so far could be constructed by using Girsanov transform as done in [1]. However, the price to pay was the assumption of neutral mutation (see below) and that the process can only start at $\mathcal{E}$-q.e. point of the state space. In the more general case here being under consideration, the diffusion operators can be time-dependent, the mutation operator A can depend on time and on the space variable μ, and nevertheless changing the underlying geometry, i.e. the sampling rate, is allowed. Therefore some results of Dawson and March [2] are generalized. First of all we can model time-dependent interactive selection by considering time-dependent density function which may be unbounded. And as described above singular coefficients are allowed so that the Girsanov transform does not work. In the last Section 4, specific path properties of the constructed processes are proved. For example if one looks at the state space in the finite dimensional case it is proved that as in the symmetric case the process does not hit the edges and vertices of $\overline{\Delta}$ if the effect of the mutation is not too strong. We do not only derive path properties of the constructed processes, but prove a general method to employ parabolic capacities in Section 4. Hence it is possible to carry over all the results obtained by using exceptional sets from the elliptic case from [10].
As in [10] there are also some general results where the underlying measure is not necessarily the stationary reversible measure of the Fleming-Viot process. First of all if m-a.e. the measure μ is purely atomic then X_t is purely atomic for all t. That means the associated process only hits purely atomic measures. In the special case where the underlying measure is the stationary reversible measure of the Fleming-Viot process exceptional sets are determined, necessary and sufficient conditions in terms of the mutation measure ν_0 and the mutation intensity θ are given in [10] to describe certain sets which are not hit by the process. Because all these are results about exceptional sets or capacities they carry over to the more general situation being under consideration here by using the methods developed in Section 4.

2 Time-Dependent Dirichlet Forms

In this section we recall parts of the framework from [11] and [12]. Let $I \subseteq \mathbb{R}$ be an interval and $(\mathcal{E}^{(t)}, V)_{t\in I}$ a family of Dirichlet forms with continuity constant K independent of t and common domain V which is a linear subspace of some L_2-space $H = L_2(X, m)$. V is a real Hilbert space with norm $\|\cdot\|_V$.
Assume
(a) $t \longmapsto \mathcal{E}^{(t)}(u, v)$ is $\mathcal{B}(I)$-measurable for all $u \in V, t \in I$.
(b) There exists a constant $c > 0$ such that

$$c\|u\|_V^2 \geq \mathcal{E}_1{}^{(t)}(u, u) \geq c^{-1}\|u\|_V^2 \quad \forall u \in V, t \in I.$$

For $\bar{u}, \bar{v} \in \mathcal{V} := L_2(I; V, dt)$ let $\mathcal{A}(\bar{u}, \bar{v}) := \int_I \mathcal{E}^{(t)}(\bar{u}(t), \bar{v}(t))\, dt$. Here $L_2(I; V, dt)$ denotes the Hilbert space consisting of (classes) of Lebesgue-measurable functions which take values in V and

$$\|\bar{u}\|_{L_2(I;V,dt)} := \left(\int_I \|\bar{u}(t,\cdot)\|_V^2\, dt\right)^{\frac{1}{2}} < +\infty.$$

Here $\|\cdot\|_V$ denotes the Hilbert space norm on V. By definition $(\mathcal{A}, \mathcal{V})$ is a coercive closed form on $\mathcal{H} := L_2(I; H, dt)$ with continuity constant K. Let (U_t) be the C_0-semigroup of contractions on $\mathcal{H}$ given by:

$$U_t h(s) := \begin{cases} h(s+t) & , \text{ if s, s+t } \in I \\ 0 & , \text{ else.} \end{cases} \tag{4}$$

and the corresponding adjoint semigroup:

$$\hat{U}_t h(s) := \begin{cases} h(s-t) & , \text{ if s-t, s } \in I \\ 0 & , \text{ else.} \end{cases}$$

$(U_t)_{t\geq 0}, (\hat{U}_t)_{t\geq 0}$ define C_0-semigroups of contractions on $\mathcal{H}$ and $\hat{\mathcal{H}}$ respectively such that $(U_t)_{t\geq 0}$ and $(\hat{U}_t)_{t\geq 0}$ can be extended to C_0-semigroups of contractions on $\hat{\mathcal{V}}' := L_2(I; V', dt)$. The corresponding generators are denoted by $(\frac{d}{dt}, D(\frac{d}{dt}, \mathcal{V}'))$ and $(\widehat{\frac{d}{dt}}, D(\widehat{\frac{d}{dt}}, \mathcal{V}'))$. If one only considers $I = \mathbb{R}$ then one only has one possibility for the extension. The domains of the generators are $D(\widehat{\frac{d}{dt}}, \mathcal{V}') = D(\frac{d}{dt}, \mathcal{V}') = H^{1,2}(R; V')$. Here $H^{1,2}(\mathbb{R}, V')$ as usual denotes the Sobolev space of square-integrable functions having square-integrable weak derivatives of first order given by $\{\bar{u}(t,\cdot) \in \mathcal{V}', \frac{d\bar{u}}{dt}(t,\cdot) \in \mathcal{V}'\}$. For the generators $\hat{\frac{d}{dt}} = -\frac{d}{dt}$ holds. For $I \neq \mathbb{R}$ boundary conditions are fixed by (4). For a detailed discussion we refer to [6]

Let $(\mathcal{G}_\alpha)_{\alpha\geq 0}, (\hat{\mathcal{G}}_\alpha)_{\alpha\geq 0}$ be the associated (co-)resolvent of $(U_t)_{t\geq 0}$, $(\hat{U}_t)_{t\geq 0}$ on $\mathcal{H}$ and $\mathcal{F} := D(\frac{d}{dt}, \mathcal{V}') \cap \mathcal{V}$, $\widehat{\mathcal{F}} := D(\widehat{\frac{d}{dt}}, \mathcal{V}') \cap \mathcal{V}$. The time-dependent Dirichlet form associated with the family $(\mathcal{E}^{(t)}, V)_{t\in I}$ is now given by:

$$\mathcal{E}(\bar{u}, \bar{v}) := \begin{cases} \mathcal{A}(\bar{u}, \bar{v}) - \langle \frac{d\bar{u}}{dt}, \bar{v}\rangle & , \quad \text{if } \bar{u} \in \mathcal{F}, \bar{v} \in \mathcal{V} \\ \mathcal{A}(\bar{u}, \bar{v}) - \langle \widehat{\frac{d\bar{v}}{dt}}, \bar{u}\rangle & , \quad \text{if } \bar{u} \in \mathcal{V}, \bar{v} \in \hat{\mathcal{F}}. \end{cases} \tag{5}$$

Some Potential Theory

Definition 2.1 [12,III 1.1] In the following $\mathcal{P}_1$ denotes the set of all 1-excessive functions in $\mathcal{H}$, i.e. $u \in \mathcal{P}_1$ is equivalent to $\beta\mathcal{G}_{\beta+1}u \leq u$ for all $\beta \geq 0$. Furthermore if $\mathcal{L}_h := \{v \mid v \in \mathcal{H},\ v \geq h\}$ and $\beta_h(v) := (v-h)^-$, then by [12,III Proposition 1.6] there are uniquely determined $h_\alpha \in \mathcal{F}$, $\hat{h}_\alpha \in \hat{\mathcal{F}}$ such that

$$\mathcal{E}_1(h_\alpha, v) = \alpha(\beta_h(h_\alpha), v)_{\mathcal{H}},\ \ \mathcal{E}_1(v, \hat{h}_\alpha) = \alpha(\beta_h(\hat{h}_\alpha), v)_{\mathcal{H}},$$

for all $v \in \mathcal{V}$. For $h \in \mathcal{H}$ such that $\mathcal{L}_h \cap \mathcal{F} \neq \emptyset$ there exists $e_h \in \mathcal{L}_h \cap \mathcal{P}_1$ and

(i) $\lim_{\alpha\to\infty} h_\alpha = e_h$ strongly in $\mathcal{H}$ and weakly in $\mathcal{V}$,

(ii) $e_h \leq u$ for all $u \in \mathcal{L}_h$, u 1-excessive,

(iii) $\mathcal{E}_1(e_h, v) \geq \mathcal{A}_1(e_h, e_h)$ for all $v \in \mathcal{L}_h \cap \hat{\mathcal{F}}$.

e_h is called the 1-reduced function of h. The following notation will be used in the sequel: If U is an open subset and $h \in \mathcal{H}$ then $h_U := e_{h\cdot 1_U}$ denotes the 1-reduced function of $h \cdot 1_U$.

Definition 2.2 [12,III 2.3](Nests and exceptional sets)

(i) An increasing sequence of closed subsets $(E_k)_{k\geq 1}$ is called $\mathcal{E}$-Nest if for all $u \in \mathcal{F} \cap \mathcal{P}_1$: $\lim_{k\to\infty} u_{E\setminus E_k} = 0$ in $\mathcal{H}$.

(ii) A subset $N \subset E$ is called $\mathcal{E}$-exceptional if there exists an $\mathcal{E}$-nest $(E_k)_{k\geq 1}$ such that $N \subset \bigcap_{k\geq 1} E_k^c$.

(iii) A property holds $\mathcal{E}$-quasi-everywhere ($\mathcal{E}$-q.e.) if the property holds outside some $\mathcal{E}$-exceptional set.

Definition 2.3 Let $U \subset E$, U open, and $0 < \varphi \in L_2(E, m)$. The capacity is given by

$$Cap_\varphi(U) := (h_U, \varphi)_{\mathcal{H}} \tag{6}$$

and for arbitray $A \subset E$,

$$Cap_\varphi(A) := \inf_{\substack{A \subset U \\ U open}} Cap_\varphi(U). \tag{7}$$

By the general theory one has the following important result about Capacities and exceptional sets:

Proposition 2.4 [12, III Prop. 2.10] Let $(E_k)_{k\geq 0}$ be an increasing sequence of closed subsets of E. $(E_k)_{k\geq 1}$ is an $\mathcal{E}$-Nest iff $\lim_{k\to\infty} Cap_\varphi(E_k^c) = 0$.

Remark 2.5 [12,III 2.6] Assume $(E_k)_{k\geq 1}$ to be an $\mathcal{E}$-nest. Then $\bigcup_{k\geq 1} \mathcal{V}_{E_k}$ is a dense set in $\mathcal{V}$. Here we set $\mathcal{V}_{E_k} = \{u | u \in \mathcal{V}, u = 0 \ m\text{-a.e on } E\backslash E_k\}$ Therefore, if $(\mathcal{A}, \mathcal{V})$ is a Dirichlet form $(E_k)_{k\geq 1}$ is an $\mathcal{A}$-nest in the sense of [7, III.2.1].

Definition 2.6 [12,III 3.1](Quasi continuity)
An $\mathcal{E}$-q.e. defined function $f : E \to \mathbb{R}$ is called $\mathcal{E}$-quasi-continuous if $f \in C(\{E_k\})$ for some $\mathcal{E}$-nest $(E_k)_{k\geq 1}$ where

$$C(\{E_k\}) = \left\{ f : A \longrightarrow \mathbb{R} \mid \bigcup_{k\geq 1} E_k \subset A \subset E, f|_{E_k} \text{ is continuous for all } k \right\}.$$

Definition 2.7 A right process $\mathbb{M} = (\Omega, \mathcal{M}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ with state space E, filtration $(\mathcal{M}_t)_{t\geq 0}$ and associated resolvent $(R_\alpha)_{\alpha>0}$ is called $\mathcal{E}$-associated (in the resolvent sense, or r.s. for short) if $R_\alpha f$ is an m-version of $\mathcal{G}_\alpha f$ for all $\alpha > 0$ and $f \in \mathcal{B}_b \cap \mathcal{H}$. $\mathbb{M}$ is called properly associated to $\mathcal{E}$ (r.s.) if additionally $R_\alpha f$ is $\mathcal{E}$-q.c. for all $\alpha > 0$, $f \in \mathcal{B}_b \cap \mathcal{H}$.

Definition 2.8 A time-dependent Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ associated with $(\mathcal{A}, \mathcal{V})$ is called quasi-regular if:

(i) There exists an $\mathcal{E}$-nest $(E_k)_{k\geq 1}$ consisting of compact sets.

(ii) There exists a dense subset of $\mathcal{F}$ whose elements have $\mathcal{E}$-q.c. m-versions.

(iii) There are u_n, $n \in \mathbb{N}$, having $\mathcal{E}$-q.c. m-versions $\tilde{u}_n$, $n \in \mathbb{N}$, and there exists an $\mathcal{E}$-exceptional set $N \subset E$ such that $\{\tilde{u}_n \mid n \in \mathbb{N}\}$ separates the points of $E\backslash N$.

Remark 2.9 If $(\mathcal{E}^{(t)}, V)$ is quasi-regular for all t, the time-dependent Dirichlet form $\mathcal{E}$ is quasi-regular. (cf [12])

The main theorem now is a special case of the main theorem of the theory of generalized Dirichlet forms connecting the analytic and the probabilistic part of the theory.

Theorem 2.10 [12, IV Corollary 3.2]Let $\mathcal{E}$ be a time-dependent Dirichlet form. $\mathcal{E}$ is quasi-regular iff there exists an m-tight special-standard process $(X_t)_t$ which is properly associated to $\mathcal{E}$ (r.s.).

We need the following version of [8, Prop. 5.30] about cadlag sample paths.

Proposition 2.11 Let $\mathbb{M} = (\Omega, \mathcal{M}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ be a right process with state space E, filtration $(\mathcal{M}_t)_{t\geq 0}$ and lifetime ζ. Furthermore, let $\mathbb{M}$ be properly associated to the time-dependent Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L_2(E, m)$.

(i) If $(E_k)_{k\geq 1}$ is an $\mathcal{E}$-nest, then

$$P_x[\lim_{k\to\infty} \sigma_{E\setminus E_k} < \zeta] = 0 \quad \text{for } \mathcal{E} - q.e.\ z \in E. \tag{8}$$

Let $N \in \mathcal{B}(E)$ be $\mathcal{E}$-exceptional. Then

$$P_x[\sigma_N < \infty] = P_x[\sigma_N < \zeta] = 0 \quad \text{for } \mathcal{E} - q.e.\ z \in E. \tag{9}$$

Here $\sigma_A := \inf\{t > 0 \mid X_t \in A\}$.

(ii) Assume $\mathbb{M}$ is a special standard process and $u : E \longrightarrow \mathbb{R}$ is $\mathcal{E}$-q.c. then for $\mathcal{E} - q.e.\ x \in E$ we have that $u(X_{t-}) = u(X_t)_-$ for all $t \in (0, \zeta)$ P_x-a.e. and

$$\begin{aligned} P_x[t \mapsto u(X_t) &\text{ is right-continuous on } [0,\infty) \text{ and} \\ t \mapsto u(X_{t-}) &\text{ is left-continuous on } (0,\zeta)] = 1. \end{aligned} \tag{10}$$

Proof:(cf. [12, Lemma IV.3.10] and [8, Proof of Prop. 5.30])
If (i) holds than (ii) follows immediately. Therefore we show (i).
Let $\varphi > 0, h := R_1\varphi$ and $U_k := E\setminus E_k$. Now

$$\begin{aligned} Cap_\varphi(U_k) &= \int_E \mathbb{E}_x\left[e^{-\sigma_{U_k}} R_1\varphi(X_{\sigma_{U_k}})\right] \varphi(x)\, m(dx) \\ &= \int_E \mathbb{E}_x\left[\int_{\sigma_{U_k}}^{\zeta} e^{-s}\varphi(X_s)\, ds\right] \varphi(x)\, m(dx) \end{aligned}$$

holds. Define

$$h_k(z) := \mathbb{E}_x \left[\int_{\sigma_{U_k}}^{\zeta} e^{-s} \varphi(X_s)\, ds \right], \; k \in \mathbb{N}.$$

Then $h_k \downarrow 0$ m-a.e. for $k \to \infty$. Because we have that $\mathbb{M}$ is properly associated with $\mathcal{E}$ in the resolvent sense, $R_\alpha f$ is $\mathcal{E}$-q.c. for all $\alpha > 0$, $f \in \mathcal{B}_b \cap \mathcal{H}$. Now one obtains that $h_k \overset{k\to\infty}{\longrightarrow} 0$ $\mathcal{E}$-quasi uniformly which means (i) holds. □

3 Time-Dependent Dirichlet Forms and Fleming-Viot Processes

Framework

In this subsection we recall the definition of an appropriate tangent bundle to define Dirichlet forms. The interested reader is refered to [6], [9] and [10] for more details on the geometry of the Fleming-Viot process. We consider the diffusion operator L^{FV} mentioned in the introduction acting on a nice space of test functions. Let us recall some notations and definitions from [9] and [10]. At first set $\mu(f) := \int_S f\, d\mu$ for $f \in \mathcal{B}_b(S)$, where $\mathcal{B}_b$ denotes bounded measurable functions. Furthermore we take as a space of test functions the finitely based smooth cylinder functions $\mathcal{FC}^\infty$ given by

$$\mathcal{FC}^\infty = \{\mu \mapsto \varphi(\mu(f_1), ..., \mu(f_n)) \mid f_1, ..., f_n \in C_b(S), \varphi \in C^\infty(\mathbb{R}^n), n \in \mathbb{N}\}.$$

Finally let m be a finite positive measure with $supp[m] = \mathcal{M}_1(S)$.

The Tangent Bundle

First let us recall the definition of a measurable field of Hilbert spaces (see e.g. [3]):
Let $(A, \mathcal{A}, \nu)$ be a measure space. A collection $H = (H_\alpha)_{\alpha\in A}$ of Hilbert spaces together with a linear space $S(A \to H)$ consisting of maps ω (called measurable sections) from A to the disjoint union $\bigcup_{\alpha\in A} H_\alpha$ such that $\omega(\alpha) \in H_\alpha$ is called measurable field of Hilbert spaces iff

(a) $\omega \in S(A \to H)$ iff the map $\alpha \mapsto (\omega(\alpha), \sigma(\alpha))_{H_\alpha}$ is measurable for each $\sigma \in S(A \to H)$.

(b) There exists a countable subset $\{l_i \mid i \in \mathbb{N}\}$ of $S(A \to H)$ such that the linear span of $\{l_i(\alpha) \mid i \in \mathbb{N}\}$ is dense in H_α for each $\alpha \in A$.

Let $\tilde{H}$ be the set of all square-integrable measurable sections. $\tilde{H}$ is a real vector space with inner product $(\omega_1, \omega_2)_{\tilde{H}} = \int_A (\omega_1(\alpha), \omega_2(\alpha))_{H_\alpha}\, \nu(d\alpha)$. If one identifies two sections ω_1, ω_2 with $\|\omega_1 - \omega_2\| = 0$, $\tilde{H}$ is a Hilbert space, which is separable if H_α is for all α. $\tilde{H}$ is called the direct integral of $(H_\alpha)_\alpha$ and is denoted by $\int_A^\oplus H_\alpha\, \nu(d\alpha)$. For further details see [3].

Definition 3.1 (Tangent Bundle)
Let $T\mathcal{M}_1(S) := (T_\mu \mathcal{M}_1(S))_{\mu\in\mathcal{M}_1(S)} := (L_2(S,\mu))_{\mu\in\mathcal{M}_1(S)}$. Then $T\mathcal{M}_1(S)$ is a measurable field of Hilbert spaces over $\mathcal{M}_1(S)$. $\mathcal{FC}^\infty$ is an algebra of functions on $\mathcal{M}_1(S)$ that generate the Borel-σ-algebra w.r.t. the weak topology (by definition!). As in [10] we consider the following map:

$$\nabla : \mathcal{FC}^\infty \longrightarrow S\left(\mathcal{M}_1(S) \to T\mathcal{M}_1(S)\right) \tag{11}$$

given by

$$\nabla u(\mu) := \left(\nabla_x u(\mu)\right)_{x\in S} \tag{12}$$

where

$$\nabla_x u(\mu) := \frac{d}{dt} u((1-t)\mu + t\delta_x) \mid_{t=0} \tag{13}$$

∇ is called the (intrinsic) gradient.

Remark 3.2 1)We have an explicit representation of ∇u for test functions $u \in \mathcal{FC}^\infty$, $u(\mu) = u(\mu(f_1), ..., \mu(f_n))$:

$$\nabla_x u(\mu) = \sum_{l=1}^{n} \frac{\partial\varphi}{\partial y_l}(\mu(f_1), ..., \mu(f_n))(f_l(x) - \mu(f_l)), \quad x \in S. \tag{14}$$

2) We can think of $\mathcal{M}_1(S)$ with inner product $\langle\cdot,\cdot\rangle_\mu$ as being a non-flat manifold with the corresponding tangent bundle $T\mathcal{M}_1(S)$.

In this paper we restrict ourselves to Fleming-Viot processes with neutral mutation and without selection until it is otherwise stated. The mutation operator is defined as

$$Af(x) := \frac{\theta}{2} \int_S (f(\xi) - f(x))\, \nu_0(d\xi), \tag{15}$$

where $f \in \mathcal{B}_b(S)$, $\theta > 0$ and ν_0 is a probability measure on S.

Bilinear forms

Let $E := \mathcal{M}_1(S)$ and let

$$b(\cdot,\cdot,\cdot),\ d(\cdot,\cdot,\cdot) : I \times S \times E \longrightarrow \mathbb{R}, \tag{16}$$

be measurable functions, such that

$$\sup_{t\in I}\sup_{\mu\in E} \|b_t(\mu)\|_\mu\ ,\ \sup_{t\in I}\sup_{\mu\in E} \|d_t(\mu)\|_\mu < +\infty, \tag{17}$$

where we set $b_t(\mu)(\cdot) := b(t,\cdot,\mu)$ and $d_t(\mu)(\cdot) := d(t,\cdot,\mu)$.
In the following let

$$N : I \times E \longrightarrow \bigcup_{\mu\in E} L(L_2(S,d\mu)) = \bigcup_{\mu\in E} L(T_\mu), \tag{18}$$

$$N(t,\mu) \in L(L_2(S,\mu)),$$

be a measurable mapping in the space of all linear operators on the underlying tangent bundle, denoted by $\bigcup_{\mu\in E} L(L_2(S,\mu))$ or as $T\mathcal{M}_1(S)$.

Definition 3.3 For $u, v \in \mathcal{FC}^\infty$ let

$$\mathcal{E}^0(u,v) \ := \ \int_E \langle \nabla u(\mu), \nabla v(\mu)\rangle_\mu \, m(d\mu), \tag{19}$$

$$\begin{aligned} \mathcal{E}^{(t)}_{N,b,d}(u,v) \ &:= \ \int\limits_E \langle N_t(\mu)\nabla u(\mu), \nabla v(\mu)\rangle_\mu \, m(d\mu) \\ &+ \ \int\limits_E \langle b_t(\mu), \nabla u(\mu)\rangle_\mu v(\mu)\, m(d\mu) \\ &+ \ \int\limits_E \langle d_t(\mu), \nabla v(\mu)\rangle_\mu u(\mu)\, m(d\mu), \end{aligned} \tag{20}$$

and with the notations of Section 2

$$\mathcal{E}(\bar u, \bar v) = \int_I \mathcal{E}^{(t)}_{N,b,d}(\bar u, \bar v)\, dt \pm \int_I\int_E \frac{\partial \bar u}{\partial t}\bar v \, dm dt, \tag{21}$$

where $\bar u \in \mathcal{F}$, $\bar v \in \mathcal{V}$ or $\bar v \in \hat{\mathcal{F}}$, $\bar u \in \mathcal{V}$.

Theorem 3.4 Let $\mathcal{E}^0$ be closable in $L_2(E,m)$ with closure $(\mathcal{E}, D(\mathcal{E}))$. Assume the conditions in (17) for b and d and let

$$c \in \mathcal{B}_b(I \times E). \tag{22}$$

Assume the following condition:

$$(23)\quad \int\limits_E u^2(\mu)c_t(\mu)\,m(d\mu) + \int\limits_E \langle b_t(\mu)+d_t(\mu), \nabla u(\mu)\rangle_\mu u(\mu)\,m(d\mu) \geq 0,$$

for all $u \in \mathcal{FC}^\infty$, $t \in I$. For N there exists $\nu, M > 0$ such that for all $(t,\mu) \in I \times E$:

$$(24)\qquad \nu^{-1}\langle h,h\rangle_\mu \leq \langle N_t(\mu)h,h\rangle_\mu \leq \nu\langle h,h\rangle_\mu$$

and

$$(25)\qquad \|\check{N}_t(\mu)\|_{L(T_\mu)} \leq M,$$

where $\check{N} := \frac{1}{2}(N-\hat{N})$ and $\hat{N}$ is the adjoint of N. Then $(\mathcal{E}^{(t)}_{N,b,d,c}, \mathcal{FC}^\infty)$ satisfies a weak sector condition with sector constant K independent of t. In particular it can be extended to a coercive closed form $(\mathcal{E}^{(t)}, D(\mathcal{E}^{(t)}))$ on $L_2(E,m)$ which is even a Dirichlet form.

Proof:
In the following let me write $\mathcal{E}^{(t)}$ instead of $\mathcal{E}^{(t)}_{N,b,d,c}$ and $\mathcal{E}^{(t)}_N$ instead of $\mathcal{E}^{(t)}_{N,0,0,0}$. The theorem is proved if we can show that the weak sector condition and closability hold. Also the existence of a constant as in (Section 2 (b)) must be shown.

1. Weak sector condition:
Let $\check{\mathcal{E}}^{(t)}(u,v) := \frac{1}{2}(\mathcal{E}^{(t)}(u,v) - \mathcal{E}^{(t)}(v,u))$, then one obtains for the nonsymmetric part of the form $\mathcal{E}^{(t)}$

$$(26)\qquad \left|\check{\mathcal{E}}^{(t)}(u,v)\right| \leq 3K\nu^{-1}\mathcal{E}^{(t)}_{N,1}(u,u)^{\frac{1}{2}}\mathcal{E}^{(t)}_{N,1}(v,v)^{\frac{1}{2}},$$

where $K := \sup_\mu\{\|\check{N}_t(\mu)\|_{L(T_\mu)} \vee \|b_t - d_t\|_\mu\}$. Because of [12, I.2.1 (iv)] this means the form $(\mathcal{E}^{(t)}, \mathcal{FC}^\infty)$ satisfies the weak sector condition with continuity constant independent of t.

2. Closability:
Let $u_n \in \mathcal{FC}^\infty_b, n \in \mathbb{N}$, such that $u_n \overset{n\to\infty}{\longrightarrow} 0$ in $L_2(E,m)$ and $(u_n)_{n\in\mathbb{N}}$ is a $\mathcal{E}^{(t)}_N$-Cauchy sequence. Henceforth by (23), (24) and (25) $(u_n)_{n\in\mathbb{N}}$ is also $\mathcal{E}^0$-Cauchy. According to the assumption $\mathcal{E}^0$ is closable,therefore

$$\lim_{n\to\infty} \mathcal{E}^0(u_n,u_n) = 0$$

and thus by (24)

$$\lim_{n\to\infty}\int_E \langle N_t(\mu)\nabla u_n(\mu), \nabla u_n(\mu)\rangle_\mu\, m(d\mu) \leq \lim_{n\to\infty} \nu\mathcal{E}^0(u_n,u_n) = 0$$

Hence

$$\int_E u_n(\mu)(\langle d_t(\mu) + b_t(\mu), \nabla u_n(\mu)\rangle_\mu + cu_n(\mu))\, m(d\mu) \overset{n\to\infty}{\longrightarrow} 0.$$

Therefore $\mathcal{E}_N^{(t)}(u_n, u_n) \overset{n\to\infty}{\longrightarrow} 0$, which proves the closability of $\mathcal{E}^{(t)}$.

3. Existence of the claimed constant

Using (23) one has

$$\mathcal{E}^{(t)}(u,u) \geq \int_E \langle N_t(\mu)\nabla u(\mu), \nabla u(\mu)\rangle_\mu\, m(d\mu) \geq \nu^{-1}\mathcal{E}^0(u,u)$$

and taking into account that

$$\begin{aligned} |\ \langle \Lambda_t(\mu), \nabla u(\mu)\rangle_\mu v(\mu)\ | \ &\leq\ (1/2\langle \nabla u(\mu), \nabla u(\mu)\rangle_\mu + 2c_0^2 u^2(\mu))^{\frac{1}{2}} \\ &\quad \cdot(1/2\langle \nabla v(\mu), \nabla v(\mu)\rangle_\mu + 2c_0^2 v^2(\mu))^{\frac{1}{2}}. \end{aligned}$$

by Cauchy Schwarz inequality, where $\Lambda_t := b_t$ or d_t are uniformly bounded by $c_0 := max\{\sup_{t,\mu}\|b_t\|_\mu, \sup_{t,\mu}\|d_t\|_\mu\}$. Clearly

$$\left|\int_E u(\mu)v(\mu)c_t(\mu)\, m(d\mu)\right| \leq \|c_t\|_\infty \mathcal{E}_1^0(u,u)^{\frac{1}{2}}\mathcal{E}_1^0(v,v)^{\frac{1}{2}}.$$

because c is bounded. Finally this leads to

$$\mathcal{E}^{(t)}(u,u) \leq C\mathcal{E}_1^0(u,u), \quad \text{with } C := \nu + const + \|c_t(\mu)\|_\infty.$$

This implies the existence of a constant K as stated in the theorem. □

By the results of the last Theorem $(\mathcal{E}^{(t)}, D(\mathcal{E}^0))_{t\in I}$ is a family of Dirichlet forms satisfying (a) and (b) in Section 2. In particular, the associated time-dependent Dirichlet form $\mathcal{E}$ (cf. (5), Section 2) is defined.

Remark 3.5 Time-dependent density functions:
Let φ be a measurable function, $\varphi(t,\cdot) \in D(\mathcal{E}^{(t)})$ such that $\varphi(t,\cdot) > 0$ for all $t > 0$ and assume there is a positive constant c > 0 and t_0 such that

(27) $$c^{-1}\varphi(t_0,\cdot) \leq \varphi(t,\cdot) \leq c\,\varphi(t_0,\cdot), \text{ for all } t > 0$$

then

(28) $$\mathcal{E}_{\varphi_t}(u,v) := \int_E \langle \nabla u(\mu), \nabla v(\mu)\rangle_\mu\, \varphi_t^2\, m(d\mu)$$

where m is as in Theorem 3.4 and $\varphi_t(\cdot) := \varphi(t,\cdot)$, is closable on $L_2(E, \varphi_t^2\, m)$, again by [10]. Therefore we can use this bilinear form as the starting point in Theorem 3.4.

Since each form $(\mathcal{E}^{(t)}, \mathcal{FC}^\infty)$ is quasi-regular the associated time-dependent Dirichlet form $\mathcal{E}$ is quasi-regular too ([12, IV 4(a)]). Therefore, there exists an associated special standard process $(Y_t)_t$. To see that this process is in fact a Fleming-Viot process one has

Proposition 3.6 The process $(Y_t)_{t>0}$ solves the martingale problem for the operator $(\bar{L}^{FV}, D(\bar{L}^{FV}))$, where $\bar{L}^{FV}\bar{u}(t,\mu)$ $(= L^{FV}\bar{u}(t,\mu) + \partial_t\bar{u}(t,\mu)$ on appropriate testfunctions) in the sense that for every $\bar{u} \in D(\bar{L}^{FV})$

$$\tilde{\bar{u}}(Y_t) - \tilde{\bar{u}}(Y_0) - \int_0^t \bar{L}^{FV}\bar{u}(Y_s)\,ds, \quad t \geq 0 \tag{29}$$

is a martingale corresponding to the given filtration under $P_{(t,\mu)}$ for $\mathcal{E}$-q.e. $(t,\mu) \in \mathbb{R} \times E$. Here $\tilde{\bar{u}}$ denotes an $\mathcal{E}$-q.c. m-version of $\bar{u}$.

Proof: Let $\bar{u} \in D(\bar{L}^{FV})$. First note that $\tilde{\bar{u}}(Y_t) - \tilde{\bar{u}}(Y_0)$ does not depend on the $\mathcal{E}$-q.c. m-version of $\bar{u}$. Second, also $\int_0^t \bar{L}^{FV}\bar{u}(Y_s)\,ds$ exists and does not depend on the m-version of $\bar{L}^{FV}\bar{u}$. The rest follows using the Markov-property and Fubini's theorem. □

3.1 Applications

(i)In [2] Dawson and March considered sampling rates which depend on the point $\mu \in \mathcal{M}_1(S)$. Their operators are of the form

$$\begin{aligned}\mathcal{L}_1 u(\mu) \;&:=\; \frac{1}{2}\int_S\int_S \gamma(\mu,x,y)(\nabla_x - \nabla_y)^2 u(\mu)\,\mu(dx)\,\mu(dy) \\ &\quad + \int_S \tilde{A}(\mu)\nabla_x u(\mu)\,\mu(dx)\end{aligned}$$

where $(\tilde{A}(\mu), \tilde{\mathcal{A}})$ is the generator of a conservative Feller process and $\gamma(\mu,x,y) \geq 0$ is symmetric and continuous. The diffusion part of $\mathcal{L}_1$ is

$$\begin{aligned}&\frac{1}{2}\int_S\int_S \gamma(\mu,x,y)(\nabla_x - \nabla_y)^2 u(\mu)\,\mu(dx)\,\mu(dy) \\ &\overset{\gamma\ sym.}{=} \int_S\int_S \mu(dx)(\delta_y(dx) - \mu(dy))\,\gamma(\mu,x,y)\nabla_x\nabla_y u(\mu).\end{aligned}$$

Therefore, one sees that the classical case is given by $\gamma(\mu,x,y) \equiv \frac{1}{2}$. In the above framework we are able to modify the underlying geometry using the operators $(N_{t,\mu})_{t,\mu}$. This is nothing but changing the diffusion part of the Fleming-Viot operators, i. e. the sampling rate. In this model the sampling rate can also be time-dependent and need not be symmetric.

(ii) Other Perturbations
Consider a drift of the form $B(\mu, x)$, where B has to fulfill the conditions in Theorem 3.4.
For example one obtains for the special mutation operator B for a function $f \in \mathcal{B}_b(S)$, b measurable and bounded

$$B(\mu)f(x) := \frac{\theta}{2}\int_S (f(\xi) - f(x))b(\mu, \xi)\,\mu(d\xi),$$

that

$$\int_S B(\mu)\nabla_x u(\mu)\,\mu(dx) = \frac{\theta}{2}\langle b, \nabla u(\mu)\rangle_\mu. \tag{30}$$

In view of

$$\begin{aligned} |B(\mu)f(x)| &= \frac{\theta}{2}\int_S (f(\xi) - f(x))b(\mu, \xi)\,\mu(d\xi) \\ &\leq \theta\left(\|f\|_\infty \sup_{\mu\in\mathcal{M}_1(S)} \|b\|_\mu\right) < +\infty, \end{aligned}$$

This means the mutation operator can be time and space dependent. This is a perturbation in the sense of Theorem 3.5.

(iii) Time-Dependent Densities - Interactive Time-Dependent Selection
In the above model one can consider fitness functions as in [4]. The fitness function can be choosen time-dependent as it is shown in the following:
For test functions $u, v \in \mathcal{FC}^\infty$, $\varphi \in \mathcal{V}$, $\varphi > 0$ $m^{FV} \otimes dt$-a.e., satisfying (27) and $\varphi_t^n := \varphi_t \wedge n$ $\left(\in \mathrm{D}(\mathcal{E}^{(t)})\right)$ one has, in view of $(\varphi_t^n)^2 \in \mathrm{D}(\mathcal{E}^{(t)})$:

$$\int_{\mathbb{R}} \mathcal{E}^{\varphi_t^n}(u, v)\,dt \overset{Prodr.}{=} -\int_{\mathbb{R}}\int_E \bar{L}^{FV} uv(\varphi_t^n)^2 - \langle \nabla u, \nabla(\varphi_t^n)\rangle_\mu v\varphi_t^n\,dm^{FV}dt.$$

and with Lebesgue's convergence theorem:

$$\int_{\mathbb{R}} \mathcal{E}^{\varphi_t}(u, v)\,dt = -\int_{\mathbb{R}}\int_E \bar{L}^{FV} uv\varphi_t^2 - \varphi_t^{-1}\langle \nabla u, \nabla\varphi_t\rangle_\mu v\varphi_t^2\,dm^{FV}dt.$$

For the corresponding generator $(L^\varphi, D(L^\varphi))$ of $\mathcal{E}^\varphi$ one obtains formally

$$L^\varphi u(\mu) = \bar{L}^{FV} u(\mu) + \varphi_t^{-1}\langle \nabla u, \nabla\varphi_t\rangle_\mu.$$

Having the time-independent case in mind one can consider the operator

$$L^t u(\mu) = \bar{L}^{FV} u(\mu) + \langle \nabla u(\mu), V_t(\mu)\rangle_\mu.$$

Where we set $V_t(\mu,x) = \int_S v_t(x,y)\,\mu(dy)$ for all $t \in \mathbb{R}$. The function v_t is nothing but the fitness-function considered at any time $t \in \mathbb{R}$. It is usually assumed that the function v_t is bounded and symmetric, but in the above framework it may be unbounded.
The biological meaning is that v gives the relative fitness of an allele x to another allele y at any time t. The function V gives the fitness of an allele x to all alleles in the gene pool at any time t. In this model one can therefore describe the fitness-function as $V_t(\mu,x) = \frac{\nabla_x \varphi_t(\mu)}{\varphi_t(\mu)} + b_t(\mu,x)$ where the first summand may be unbounded and therefore is $V_t(\mu,x)$. In contrast to the classical case we only need $\sup_\mu \sup_t \sup_x |b_t(\mu,x)| < +\infty$.

4 Path Properties via Parabolic Capacities

In the following we describe some path properties for the time-dependent processes corresponding to time-dependent Fleming-Viot operators. Everything can be shown using the path properties for the processes associated to the closure of $(\mathcal{E}^{(t)}, \mathcal{FC}^\infty)_{t\in I}$ which can be found in [10, Chap. 4, 6 and 7]. Throughout this section let $\mathcal{E}^{(t)} := \mathcal{E}^{(t)}_{N,b,d,c}$, $t \in \mathbb{R}$, be a family of Dirichlet forms satisfying the assumptions of Theorem 3.4 of section 3. All these results are derived using exceptional sets or capacitary methods. In the following we establish a general method for carrying over results for capacities and exceptional sets from the elliptic situation to the parabolic situation and vice versa.
We start with a general result, where the underlying measure need not to be the time-reversible stationary measure of a Fleming-Viot process.

Theorem 4.1 Let μ be an m-a.e. purely atomic measure on S, $(Y_t)_t$ be the m-tight special standard process associated to the time-dependent Dirichlet form $\mathcal{E}$ as in (21) of Chapter 3 and X_t be the projection of Y_t onto the E-coordinate. Then we have the following:
(i) If $\mathbb{R} \times \{X_t$ is not purely atomic$\}$ is $\mathcal{E}$-exceptional
then $\{X_t$ is not purely atomic$\}$ is $\mathcal{E}^{(t)}$-exceptional for all t
and
(ii) If $\{X_t$ is not purely atomic$\}$ is $\mathcal{E}^{(t)}$-exceptional for all t
then $\mathbb{R} \times \{X_t$ is not purely atomic$\}$ is $\mathcal{E}$-exceptional.

To prove the theorem we need two lemmas to be proved first.

Lemma 4.2 Let $(\mathcal{E}^{(t)}, V)_{t\in\mathbb{R}}$ be a family of Dirichlet forms satisfying (a) and (b) of Section 2 and $\mathcal{E}$ the associated time-dependent Dirichlet form.

If $N \subset E$ is $\mathcal{E}^{(t_0)}$-exceptional for $t_0 \in \mathbb{R}$ (and therefore for all t $\in \mathbb{R}$), $(s, s') \times N$(even $\mathbb{R} \times N$) is an $\mathcal{E}$-exceptional set for all $s, s' \in \mathbb{R},\ s < s'$.

Proof:
Define the following coercive bilinear form:

$$\tilde{\mathcal{E}}(u, v) := \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}\right)_{\mathcal{H}} + \mathcal{A}(u, v) \quad , u, v \in D(\tilde{\mathcal{E}}) := D(\frac{\partial}{\partial t}, \mathcal{H}) \cap \mathcal{V}.$$

$\tilde{\mathcal{E}}$ is a Dirichlet form.

To show that $(s, s') \times N$ is $\mathcal{E}$-exceptional it is enough to show that $\bigcup_k \mathcal{F}_{K_k} \subset \mathcal{F}$ is dense. (cf. Remark III.2.11 in [12])
Let $(E_k)_{k \geq 1}$ be an $\mathcal{E}^{(t)}$-nest for a fixed $t \in \mathbb{R}$ (and therefore for all $t \in \mathbb{R}$), let $K_k := [-k, k] \times E_k$. To this end let $\tilde{D} := \operatorname{span}\{fu \mid f \in C_0^1(\mathbb{R}), u \in \bigcup_{k \geq 1} V_{E_k}\}$. $\tilde{D} \subset \mathcal{V}$ is dense because $\bigcup_{k \geq 1} V_{E_k}$ is a dense subset of V, $(E_k)_{k \geq 1}$ is an $\mathcal{E}^{(t)}$-nest, $U_t(\tilde{D}) \subset \tilde{D}$ for all $t \geq 0$ and $\tilde{D} \subset D(\frac{\partial}{\partial t}, \mathcal{V})$. One has that $\tilde{D}$ is a dense subset of $D(\frac{\partial}{\partial t}, \mathcal{V})$. Because of $\tilde{D} \subset \bigcup_{k \geq 1} D(\tilde{\mathcal{E}})_{K_k}$ it holds that $(K_k)_{k \geq 1}$ is an $\tilde{\mathcal{E}}$-nest. Furthermore,

$$\begin{aligned} \bigcap_{k \geq 1} K_k^c &= \bigcap_{k \geq 1} ([-k, k] \times E_k)^c \\ &= \bigcap_{k \geq 1} (([-k, k]^c \times E_k) \cup ([-k, k]^c \times E_k^c) \cup ([-k, k] \times E_k^c)) \\ &\supseteq \mathbb{R} \times \bigcap_{k \geq 1} E_k^c \supseteq (s, s') \times N. \end{aligned}$$

Therefore, $(s, s') \times N \subset \bigcup_{k \geq 1} K_k^c$ for an $\tilde{\mathcal{E}}$-nest. It is therefore $\tilde{\mathcal{E}}$-exceptional.$\square$

Lemma 4.3 Let $(\mathcal{E}^{(t)}, V)_{t \in \mathbb{R}}$ be a familiy of Dirichlet forms satisfying (a) and (b) of Sction 2 and $\mathcal{E}$ the associated time-dependent Dirichlet form. If the set $(s, s') \times N$ is $\mathcal{E}$-exceptional then N is $\mathcal{E}^{(t)}$-exceptional for all $t \in \mathbb{R}$.

Proof:
Let us fix $t_0 \in \mathbb{R}$ and let $0 < \varphi \in L_2(E, m)$. We define $w := \mathcal{G}_1^{(t_0)} \varphi$ and $h := 1_{(s,s')} \otimes w$.
Assumption: N is not $\mathcal{E}^{(t_0)}$-exceptional.
Then one has

$$\inf_{\substack{N \subset U \\ U\ open}} \mathcal{E}_1^{(t_0)}(w_U, w_U) \geq \epsilon > 0.$$

Let $W \subseteq \mathbb{R} \times E$ be open such that $(s, s') \times N \subset W$. Then

$$\mathcal{A}_1(h_W, h_W) \geq (s' - s)\epsilon K^{-2}$$

holds where h_W denotes the 1-reduced function with respect to $\mathcal{A}$, because for all $v \in D(\mathcal{E}^{(t_0)})$ such that $v \geq w$ on U, U open, we have that

$$\mathcal{E}_1^{(t_0)}(w_U, w_U) \leq \mathcal{E}_1^{(t_0)}(w_U, v) \leq K\mathcal{E}_1^{(t_0)}(w_U, w_U)^{\frac{1}{2}}\mathcal{E}_1^{(t_0)}(v, v)^{\frac{1}{2}}.$$

Therefore, $\mathcal{E}_1^{(t_0)}(w_U, w_U) \leq K^2\mathcal{E}_1^{(t_0)}(v, v)$. Now consider the set:$W_t = \{x \mid (t, x) \in W\}$. W_t is open for all t and $h_W(t, \cdot) \geq w$ on $W_t \quad dt-$a.e. $(t \in (s, s'))$. This means

$$\mathcal{E}_1^{(t_0)}(h_W(t, \cdot), h_W(t, \cdot) \geq K^{-2}\mathcal{E}_1^{(t_0)}(w_{W_t}, w_{W_t}) \geq K^{-2}\epsilon.$$

In this case w_{W_t} is of course the 1-reduced function with respect to $\mathcal{E}^{(t_0)}$. Finally we obtain :

$$\int_{\mathbb{R}} \mathcal{E}_1^{(t_0)}(h_W(t, \cdot), h_W(t, \cdot))\, dt \geq \int_{(s,s')} \mathcal{E}_1^{(t_0)}(h_W(t, \cdot), h_W(t, \cdot))\, dt \geq (s' - s)K^{-2}\epsilon.$$

We have shown that the set $(s, s') \times N$ is not $\mathcal{A}$-exceptional.
Regarding 2.5 one knows that each $\mathcal{E}$-exceptional set is an $\mathcal{A}$-exceptional set which is a contradiction to our assumption and therefore the set N must be $\mathcal{E}^{(t_0)}$-exceptional. $\square$

Proof of Theorem 4.2:
By Lemma 4.3 and Lemma 4.4 the claims of Theorem 4.2 follow from the fact that if μ is m-a.e. purely atomic $\{X_t$ is purely atomic$\}$ is $\mathcal{E}^{(t)}$-exceptional as shown in [10]. $\square$

The Classical Fleming-Viot Case

We now specialize to the case where the underlying measure $m = \varphi^2 m^{FV}$ (i.e., the time-dependent Fleming-Viot process with time-dependent interactive selection). However, we make the additional assumption that $c_1(t) \leq \varphi(\cdot, t)^2 \leq c_2(t)$, where $c_1, c_2 : I \to \mathbb{R}$ (cf. [10, Chapter 7]). This means that a set is $\mathcal{E}_{N,b,d,c}^{(t)}$-exceptional iff it is $\mathcal{E}^{(0)}$-exceptional. When we want to emphasize the dependence on θ and ν_0 we will use the notation $\Pi_{\theta,\nu_0} =: m$. We will now transfer the results from [10, Chapter 7] to the time-dependent situation.

Proposition 4.4 *For fixed $x_1, ..., x_k \in S$, if $\nu_0(\{x_1, ..., x_k\}) > 0$ then*

$$\left\{\mu | \mu = \sum_{i=1}^k p_i\epsilon_i, p_i \geq 0, \sum_{i=1}^k p_i = 1\right\} \times (s, s')$$

is $\mathcal{E}-$exceptional iff

$$\theta\nu_0(\{x_1,...,x_k\}) \geq 1.$$

Proof:
Use the two lemmas of this section and [10, Proposition 7.6, Corollary 7.7].
□

Theorem 4.5 *For every $k \geq 1$, the set*

$$\left\{\mu|\mu = \sum_{i=1}^{k} p_i\epsilon_i, p_i \geq 0, \sum_{i=1}^{k} p_i = 1\right\} \times (s, s')$$

is $\mathcal{E}-$exceptional iff

$$\theta \inf_{x_1,...x_k} \nu_0(\{x_1,...,x_k\}) \geq 1.$$

Proof:
Use the two lemmas of this section and [10, Theorem 7.9]. □

Corollary 4.6 *If ν_0 is atomless, then $\mathcal{E}-$q.e. $(\mu, t) \in E \times I$ has infinitely many atoms iff $\theta \geq 1$.*

Proof:
Use the the two lemmas of this section and [10, Corollary 7.10]. □

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