$\mathcal{A}_4$, $\mathcal{A}_5$, $\mathcal{S}_4$ AND $\mathcal{S}_5$ OF SCHOTTKY TYPE

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ABSTRACT. Let H be a group of conformal automorphisms of a closed Riemann surface S, isomorphic to either of the alternating groups $\mathcal{A}_4$ or $\mathcal{A}_5$ or the symmetric groups $\mathcal{S}_4$ or $\mathcal{S}_5$. We provide necessary and sufficient conditions for the existence of a Schottky uniformization of S for which H lifts.

1. INTRODUCTION

Given a closed Riemann surface S, retrosection theorem asserts the existence of Schottky uniformizations $(\Omega, G, P : \Omega \to S)$ of S, that is, G is a Schottky group with region of discontinuity Ω and $P : \Omega \to S$ is a Galois covering with G as covering group.

Let H be a group of conformal automorphism of the closed Riemann surface S and $(\Omega, G, P : \Omega \to S)$ be a Schottky uniformization of S. We say that H lifts if for every $h \in H$ there exists some $\widehat{h} \in \mathrm{Aut}(\Omega)$ so that $P\widehat{h} = hP$. Since a conformal automorphisms of the region of discontinuity of any Schottky group is necessarilly the restriction of a Möbius transformation, we have that $\widehat{h}$ is a Möbius transformation.

A general problem is to determine when a group H of conformal automorphisms of a closed Riemann surface S lifts for some Schottky uniformization of S. If this is the case, then we say that H is of Schottky type.

In [2] we have given a necessary condition for H to be of Schottky type, called the condition (A) (see section 2). This condition turns out to be sufficient if H is an abelian group [2] [3]. If H is isomorphic to a dihedral group, then condition (A) holds trivially. In [4] we show that dihedral actions are also of Schottky type.

It is important to remark that condition (A) is not sufficient in general. A. Miller showed us an example of an $\mathcal{S}_5$ (the symmetric group on five letter) action on a closed Riemann surface of genus 56 which satisfies condition (A) but not of Schottky type. We recall this example in the last section and we use Schottky groups arguments to see why it is not of Schottky type.

A particular class of finite groups are those that appear as group of Möbius transformations: (i) cyclic groups, (ii) dihedral groups, (iii) the alternating groups $\mathcal{A}_4$, $\mathcal{A}_5$ and (iv) the symmetric group $\mathcal{S}_4$. As observed above, for cyclic and dihedral

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groups condition (A) turns out to be sufficient. The purpose of this note is to get a similar result for the other three groups.

Theorem 1. *Let H be a group of conformal automorphisms of a closed Riemann surface S, isomorphic to either $\mathcal{A}_4$, $\mathcal{A}_5$ or $\mathcal{S}_4$. Then condition (A) is necessary and sufficient for H to be of Schottky type.*

The same arguments of the proof of theorem 1 for the special case of a group isomorphic to the symmteric group $\mathcal{S}_4$ applies for $\mathcal{S}_5$ under some special assumptions. More precissely:

Theorem 2. *Let H be a group of conformal automorphisms of a closed Riemann surface S, isomorphic to the symmetric group in five letters $\mathcal{S}_5$. Let K be the index two subgroup, isomorphic to $\mathcal{A}_5$, $R = S/K$, $P : S \to R$ the natural holomorphic (branched) covering induced by K and $\widehat{\sigma} : R \to R$ the conformal involution induced by H on R. If either:*

(i) *$\widehat{\sigma}$ acts free fixed points; or*
(ii) *$\widehat{\sigma}$ has exactly two fixed points, each one not a branched value of P,*

then condition (A) is sufficient for H to be of Schottky type.

Theorem 1 and 2 give necessary and sufficient conditions for the action H to be realized as a finite normal extension of a Schottky group.

We may write theorem 1 in the context of handlebodies. Let M^3 be the connected sum of g copies of $D^2 \times S^1$, where D^2 is a closed 2-dimensional disc and S^1 is the unit circle. The 3-manifold M^3 is called a handlebody of genus g. Its boundary $S = \partial(M^3)$ is an orientable closed surface of genus g.

Theorem 3. *Let H be a group of orientation-preserving homeomorphisms of the boundary $S = \partial(M^3)$, where M^3 is a handlebody of genus g. If H is isomorphic to either $\mathcal{A}_4$, $\mathcal{A}_5$ or $\mathcal{S}_4$, then condition (A) is necessary and sufficient for the existence of an orientation-preserving homeomorphism $f : S \to S$ so that fHf^{-1} can be extended as a group of orientation-preserving homeomorphisms of M^3.*

The equivalence between theorems 1 and 3 is consequence of the Nielsen's realization theorem [5] [10]. This is also related to the works of B. Zimmermann [11] and A. McCullough, A. Miller and B. Zimmermann [9].

Related to closed Riemann surfaces are Riemann matrices, which are constructed as follows. Let S be a closed Riemann surface of genus g, and $\alpha_1,..., \alpha_g, \beta_1,..., \beta_g$ be a homology basis of it. We say that such a homology basis is symplectic if the intersection matrix is given by

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I denotes the identity $g \times g$ matrix.

Given a symplectic basis $\alpha_1,\ldots, \alpha_g, \beta_1,\ldots, \beta_g$, we may find a basis of holomorphic 1-forms on S, say $\omega_1,\ldots, \omega_g$, such that [1]

$$\int_{\alpha_i} \omega_j = \delta_{ij}.$$

The matrix $Z = (\int_{\beta_i} \omega_j)$, symmetric and with positive definite imaginary part (Riemann's conditions) [1], is called the Riemann matrix for S associated to the above symplectic basis.

The Siegel space of genus g, denoted by $\mathcal{H}_g$, is by definition the space of $g \times g$ symmetric matrices with complex coefficients and with positive definite imaginary part.

The symplectic group of genus g, denoted by $\mathrm{Sp}_{2g}(\mathbb{Z})$, consists of all $2g \times 2g$ matrices with integer coefficients satisfying the condition $AJA^t = J$.

The symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts on the Siegel space $\mathcal{H}_g$ by the rule

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}(Z) = (A + ZC)^{-1}(B + ZD).$$

Torelli's theorem asserts that the Riemann surface S is determined, up to conformal equivalence, by the Riemann matrix associated to a given symplectic basis.

Let us fix a symplectic basis for the homology of S and let $Z \in \mathcal{H}_g$ be the Riemann matrix of S computed in such a basis. Assume now that we have a group H of conformal automorphisms of S. We have a faithful representation at the level of homology

$$\theta : H \to \theta(H) \subset \mathrm{Sp}_{2g}(\mathbb{Z})$$

and a faithful representation at the level of holomorphic 1-forms

$$\eta : H \to \eta(H) \subset \mathrm{GL}(g, \mathbb{C}).$$

The group $\theta(H)$ fixes the matrix Z, that is, $\theta(h)(Z) = Z$ for all $h \in H$. This makes more easy to compute Riemann matrices for Riemann surfaces with large group of automorphisms. Now, if the representation $\theta(H)$ is more or less easy, then we are able to compute explicitly the fixed points of it in the Siegel space.

Let us assume the group H to be of Schottky type. It follows the existence of a symplectic basis $\alpha_1,\ldots, \alpha_g, \beta_1,\ldots, \beta_g$, so that the action of H, at the level of homology, keeps invariant the submodule generated by the loops $\alpha_1,\ldots, \alpha_g$. In particular, there is a representation $\theta(H)$ in the symplectic group such that

$$\theta(h) = \begin{pmatrix} A_h & B_h \\ 0 & C_h \end{pmatrix}, \qquad \text{for each } h \in H.$$

The above matrices belong to to the symplectic group if and only if $B_h A_h^t$ is symmetric and $C_h = (A_h^{-1})^t$, for each $h \in H$.

If H is isomorphic to the alternating group A_n, where $n = 4, 5$, then there is a presentation $H = \langle \tau, \sigma : \tau^{2(n-4)+3} = \sigma^2 = (\tau\sigma)^3 = 1\rangle$. If H is isomorphic to the symmetric group $\mathcal{S}_4$, then H has a presentation $H = \langle \tau, \sigma : \tau^4 = \sigma^2 = (\tau\sigma)^3 = 1\rangle$. As a consequence of theorem 1 we have then the following:

Corollary. *Let $H = \langle \tau, \sigma : \tau^k = \sigma^2 = (\tau\sigma)^3 = 1\rangle$, where $k \in \{3,4,5\}$, be a group of conformal automorphisms of a closed Riemann surface S of genus g. If H satisfies condition (A), then there is a symplectic basis for the homology of S so that*

$$\theta(\tau) = \begin{pmatrix} x_1 & x_2 \\ 0 & (x_1^{-1})^t \end{pmatrix} \qquad \theta(\sigma) = \begin{pmatrix} y_1 & y_2 \\ 0 & (y_1^{-1})^t \end{pmatrix},$$

such that

(i) $x_1^k = y_1^2 = (x_1y_1)^3 = 1$, *that is, x_1 and y_1 gives a faithful representation of H in $SL(g,\mathbb{Z})$;*
(ii) $x_2x_1^t$ *and* $y_2y_1^t$ *are both symmetric;*
(iii) $x_1y_1(x_1y_2+x_2(y_1^{-1})^t)-(x_1y_2+x_2(y_1^{-1})^t)(x_1^{-1})^t(y_1^{-1})^t = -y_1^{-1}x_1^{-1}(x_1y_2+x_2(y_1^{-1})^t)y_1^tx_1^t.$

2. Schottky Uniformizations and Condition (A)

A group G of Möbius transformations is called a Schottky group of genus g if it is a purely loxodromic Kleinian group isomorphic to a free group of rank g [6]. Retrosection theorem asserts that for each closed Riemann surface S of genus g there is a Schottky group G, with region of discontinuity Ω, and a regular covering $P:\Omega \to S$ with G as covering group. We say that $(G, \Omega, P:\Omega\to S)$ is a Schottky uniformization of the surface S.

Assume now that H is a finite group (this is the case if the genus of S is greater or equal to two) of conformal automorphisms of the surface S.

We say that H is of Schottky type if there is a Schottky uniformization of S, say $(G,\Omega,P:\Omega\to S)$ so that H lifts, that is, for each $h\in H$ there is some $\widehat{h}$ in $\mathrm{PSL}(2,\mathbb{C})$ satisfying $P\widehat{h} = hP$.

A necessary condition, called condition (A), for H to be of Schottky type is given in [2]. To recall condition (A) we need some notation. Given a conformal automorphism $f:S\to S$ of finite order on a Riemann surface S and a fixed point p of it, then we have a well defined number $\alpha(h,p)\in[-\pi,\pi)$, called the rotation number of f at p.

Assume now that H is a finite group of conformal automorphisms of a closed Riemann surface S. Let us denote by $F(H)$ the set of fixed points of the non-trivial elements of H. For each point $p\in S$ we set $H(p)=\{h\in H : h(p)=p\}$. In particular, $F(H)=\{p\in S : H(p)\neq\{I\}\}$.

We say that H satisfies condition (A) if we are able to find a collection of pairs of points $\mathcal{C}=\{\{p_i,q_i\} : p_i,q_i\in F(H)\}$ satisfying the following properties:

(1) For every point $p\in F(H)$ there exists a unique $q\in F(H)$ such that $\{p,q\}$ belongs to the collection $\mathcal{C}$;
(2) $p\neq q$, for all $\{p,q\}\in\mathcal{C}$;
(3) For each $\{p,q\}\in\mathcal{C}$, we have:
  (3.1) $H(p)=H(q)$,
  (3.2) $\alpha(h,p)=-\alpha(h,q)$ for each $h\in H(p)=H(q)$ of order greater than two,
  (3.3) if there is $t\in H$ satisfying $t(p)=q$, then $t^2=1$.

The above conditions are obtained assuming the existence of a Schottky uniformization of S for which H lifts. The lifting of H will give a finite normal extension of a Schottky group. In particular, it will be a geometrically finite Kleinian group with connected region of discontinuity with no parabolic transformations. The details can be found in [2].

Remark. It is important to note that once we have a collection $\mathcal{C}$ as above, we may re-arrange it so that their pairs satisfy the following extra property:

(4) If $\{p, q\}$ belongs to $\mathcal{C}$ and $h \in H$, then $\{h(p), h(q)\}$ also belongs to $\mathcal{C}$.

Examples of groups that do not satisfy condition (A) are given by: (i) the cyclic group of order three acting on a genus one surface with fixed points; (ii) a cyclic group of order five acting on a surface of genus two; and (iii) a cyclic group of order seven acting on the Klein's curve (the genus three surface with total group of automorphisms of order 168).

Condition (A) turns out to be sufficient for abelian groups [3] and dihedral groups [4].

The main idea in the proof of theorem 1 is to find on S a collection of pairwise disjoint simple loops that satisfies the following two properties:

(1) they dissect S into genus zero surfaces; and
(2) the collection is invariant under the action of H.

If we have such a collection of loops, then we can find a subcollection consisting on g (= genus of S) simple loops that dissect S into one genus zero surface of connectivity $2g$. Such a set defines a Schottky uniformization of S. Since the normalizer of the subcollection is the same as for the total collection, we have that it is invariant under the action of H, that is, the group H can be lifted to such a Schottky covering.

Remark.

(1) The case $H = \mathcal{S}_4$ is essentially consequence of the following two facts:
  (1.1) If H is a group of conformal automorphisms of a closed Riemann surface, isomorphic to the symmetric group $\mathcal{S}_4$, then H satisfies condition (A) if and only if its alternating subgroup $\mathcal{A}_4$ does it. In fact, it is clear that every subgroup K of a group H that satisfies condition (A) must also satisfy it. To see the reciprocal situation, in our particular case, we observe that a presentation of H is given by generators x, y and z, restricted to the relations $x^3 = y^2 = (xy)^3 = z^4 = (yz)^2 = 1$. We have that the relation $yzy = z^{-1}$ permits us to obtain a pairing of the fixed points of z as desired. We also need to observe that an equation of the form $tzt = z^{\pm 1}$, for the unknown t, has as only solutions in H the elements $t = z^a$, and elements of order two (this takes care of property (3.3) in the definition of condition (A)).
  (1.2) There is no genus one Riemann surface admitting $\mathcal{A}_4$ as group of conformal automorphisms.

(2) For the case of H isomorphic to $\mathcal{S}_5$, we have that the equivalent to (1.1) also holds. But the existence of Riemann surfaces of genus 13 admitting $\mathcal{A}_5$ as group of conformal automorphisms makes an obstruction for condition (A) to be sufficient in this case. The construction of surfaces of genus 13 as above can be done as follows. Let F be the kernel of the homomorphism

$$\phi : \langle u, v : u^5 = v^5 = (uv)^5 = 1\rangle \to \langle x, y : x^5 = y^2 = (xy)^3 = 1\rangle,$$

defined by $\phi(u) = x$ and $\phi(v) = yx^{-1}y$. The group F defines a torsion free fuchsian group uniformizing a closed Riemann surface of genus 13 admitting a group $H = \mathcal{A}_5$ of conformal automorphisms so that S/H is of genus zero with exactly three branch values of order 5.

3. Torsion Free Actions of $\mathcal{A}_4$ and $\mathcal{A}_5$

In this section, we proceed to prove theorem 1 for free fixed point actions of $\mathcal{A}_n$, for $n = 4, 5$. As said before, we only need to find a collection of pairwise disjoint simple loops with the above desired properties.

Let us consider a group H, isomorphic to either $\mathcal{A}_4$ or $\mathcal{A}_5$ of conformal automorphisms of a closed Riemann surface S acting free fixed points.

The Riemann-Hurwitz's formula [1] asserts that the genus of the quotient surface $R = S/H$ satisfies

(i) $\gamma = \frac{g+11}{12}$, for $\mathcal{A}_4$;
(ii) $\gamma = \frac{g+59}{60}$, for $\mathcal{A}_5$.

Since $g \geq 2$, we must have $\gamma \geq 2$. Let us denote by $P : S \to R$ the holomorphic regular covering induced by the action of H on S.

Let us first consider a collection $\mathcal{F} = \{\alpha_1, ..., \alpha_{3\gamma-3}\}$ consisting of $3\gamma - 3$ pairwise disjoint simple loops on R cutting off R into $2\gamma - 2$ three-holed spheres Σ_1,..., $\Sigma_{2\gamma-2}$ as it is shown in figure 1.

Let Σ be one of these three holed spheres and $\widehat{\Sigma}$ be one of the connected components of $P^{-1}(\Sigma)$. Let us denote by β_1, β_2 and β_3 the three boundaries of Σ (it may happen that two of these boundaries are produced by the same loop in $\mathcal{F}$).

We have that $P : \widehat{\Sigma} \to \Sigma$ is a regular unbranched covering, with covering group K a subgroup of H. It follows that either K is trivial, a cyclic group of order 2, 3, 5 or $K = H$.

Case A. If K is trivial, then $P : \widehat{\Sigma} \to \Sigma$ is an homeomorphism.

Case B. If K is a cyclic group of order two, then we have that β_i either lifts to one loop or two loops. By gluing discs to $\widehat{\Sigma}$ and Σ we get closed orientable surfaces of genus g_1 and zero, respectively. The regular covering P extends to a possible branched covering with at most 3 branch values of order 2. Applying Riemann-Hurwitz's formula, we have that the only possibility is $g_1 = 0$ and exactly two branch values of order 2. In this case, $\widehat{\Sigma}$ is a four-holed sphere.

Case C. If K is a cyclic group of order $k \in \{3,5\}$. We have that each β_i either lifts to exactly k loops or one loop. By gluing discs to $\widehat{\Sigma}$ and Σ we get closed orientable surfaces of genus g_1 and zero, respectively. The regular covering P extends to a possible branched covering with at most 3 branch values of order k. Applying Riemann-Hurwitz's formula, we have that the only possibilities are (i) $g_1 = 0$ and exactly two branched values of order k (in which case $\widehat{\Sigma}$ is a $(k+2)$ holed sphere), (ii) $k = 3$, $g_1 = 1$ and exactly three branched values of order 3, or (iii) $k = 5$, $g_1 = 2$ and exactly three branched values of order 5.

Case D. If $K = H$. By gluing discs to $\widehat{\Sigma}$ and Σ we get closed orientable surfaces of genus g_1 and zero, respectively. The regular covering P extends to a possible branched covering with at most 3 branch values of possible orders 2 and k, where $k \in \{3,5\}$. Riemann-Hurwitz's formula asserts in this case

$$2(g_1 - 1) = |H|(-2 + \frac{r}{2} + (1 - \frac{1}{k})s),$$

where $r, s \in \{0,1,2,3\}$ and $r + s \leq 3$.

If we have that $g_1 \geq 1$, then $(-2 + \frac{r}{2} + (1 - \frac{1}{k})s) \geq 0$ and, inparticular, we have that the only possibilities (in this case) are given by:

(i) $r = 1$, $s = 2$, $k = 5$, in which case $g_1 = 4$ and $H = \mathcal{A}_5$; and
(ii) $r = 0$, $s = 3$. For $k = 3$ we obtain $g_1 = 1$, which is impossible since there is no $\mathcal{A}_4$ or $\mathcal{A}_5$ actions on genus one Riemann surfaces. For $k = 5$ (necessarily, $H = \mathcal{A}_5$), we obtain $g_1 = 13$. As already remarked (see end remark in section 2) there are Riemann surfaces of genus 13 admitting $\mathcal{A}_5$ as group of conformal automorphism with quotient a genus zero surface with exactly three branched values of order 5.

3.1. **The alternating group $\mathcal{A}_4$.** Let us assume H isomorphic to $\mathcal{A}_4$. Under this restriction, we have that the lifting of each Σ_i, for $i = 1, ..., 3\gamma - 3$ consists of exactly:

(1) 12 three-holed spheres, if we are in case A;
(2) 6 four-holed spheres, if we are in case B;
(3) 4 five-holed spheres or 4 three-holed tori, if we are in case C.

If none of the surfaces Σ_i has as lifting three-holed tori, then we have that the lifting loops under P of the collection α_1,..., $\alpha_{3\gamma-3}$, defines on S a collection of pairwise disjoint simple loops that cut off S into genus zero surfaces. In this way, we have the existence of a collection of simple loops on the surface S that is invariant under the action of H and cut off S into genus zero surfaces. In particular, we have shown theorem 1 for the case $H = \mathcal{A}_4$ acting free fixed points. The following lemma finish the proof in this case.

Lemma 1. *There is a collection of loops α_j as before so that neither of the surfaces Σ_i lifts to a tori.*

Proof. We first consider a collection of simple loops β_1,..., β_γ as shown in figure 2.

Let us take a surface Σ_i for some $i \in \{1,...,\gamma\}$. If the loop α_i does not lift to exactly 4 loops, then (by case A.(ii) above) the liftings of Σ_i cannot be given by tori. If α_i lifts to exactly 4 loops and the loop β_i does not lift to exactly 4 loops, then we replace α_i by β_i.

Assume that both α_i and β_i lift to exactly 4 loops. Consider the intersection point $z_i \in \Sigma_i$ of the loops α_i and β_i. Let w_i be a point in one of the components of $P^{-1}(\Sigma_i)$ such that $P(w_i) = z_i$. We have in this way a natural homomorphism $\phi : \langle \alpha_i, \beta_i \rangle \to \mathcal{A}_4$, given by lifting loops based at z_i. Set $x = \phi(\alpha_i)$. Then x is an element of order 3 in $\mathcal{A}_4$. There is an element $y \in \mathcal{A}_4$ of order 2 such that $\mathcal{A}_4 = \langle x, y : x^3 = y^2 = (xy)^3 = 1 \rangle$. Since $\phi(\beta_i)$ is also an element of order 3, we have that $\phi(\beta_i) \in \{x^a, yx^a y, xyx^a yx^{-1}, x^{-1}yx^a yx\}$, for $a = \pm 1$. In either case, we replace the loop α_i by a simple loop free homotopic to $\alpha_i^{-a}\beta_i$.

We have that this new loop α_i does not lift to exactly 4 different loops. Moreover, since the new loop α_i is interior to Σ_i, each of the connected components in $P^{-1}(\Sigma_i)$ has two boundaries determined by the same new loop α_i. In particular, this implies that the liftings of the loop $\alpha_{\gamma+i}$ cannot be given by exactly 4 loops (see case 1 and case 2 above).

The above also asserts that the other surfaces $\Sigma_{\gamma+1}$,..., $\Sigma_{2\gamma-2}$ cannot have as lifting surfaces of genus one. □

3.2. **The alternating group $\mathcal{A}_5$.** Let us assume H isomorphic to $\mathcal{A}_5$. Under this restriction, we have that the lifting of each Σ_i, for $i = 1, ..., 3\gamma - 3$ consists of exactly:

(1) 60 three-holed spheres, if we are in case A;
(2) 30 four-holed spheres, if we are in case B;
(3) 12 seven-holed spheres or 12 three-holed genus two surfaces, if we are in case C;
(4) one surface of genus 13 with 36 holes, if we are in case D with $r = 0$, $s = 3$ and $k = 5$;
(5) one surface of genus 4 with 54 holes, if we are in case D with $r = 1$, $s = 2$ and $k = 5$.

If may choose the loops α_i so that none of the surfaces Σ_i has as lifting either a three-holed genus two surface or a genus 13 surface with 36 holes or a genus four surface with 54 holes, then we have that the lifting loops under P of such a collection α_1,..., $\alpha_{3\gamma-3}$, defines on S a collection of pairwise disjoint simple loops that cut off S into genus zero surfaces. We have then constructed on S a collection of pairwise disjoint simple loops satisfying that they are invariant under the group H and cut off S into genus zero surfaces. In particular, we have shown theorem 1 for the case $H = \mathcal{A}_5$ acting free fixed points. The following lemma finish the proof in this case.

Lemma 2. *We may choose the loops α_j with the property that the liftings of each surface Σ_j is neither of genera 2, 4 or 13 .*

Proof. Let $i \in \{1, ..., \gamma\}$ be fixed and assume that α_i lifts to exactly 12 loops. Draw a simple loop β_i as in figure 2.

If the loop β_i has the property that does not lift to exactly 12 loops, then we replace α_i by β_i.

Assume now that the loop β_i also lifts to 12 loops. In this case, we consider the intersection point $z_i \in \Sigma_i$ of the loops α_i and β_i. Let w_i be a point in one of the components of $P^{-1}(\Sigma_i)$ such that $P(w_i) = z_i$. We have in this way a natural homomorphism $\phi : \langle \alpha_i, \beta_i \rangle \to \mathcal{A}_5$, given by lifting loops based at z_i. Set $x = \phi(\alpha_i)$. Then x is an element of order 5 in $\mathcal{A}_5$. There is an element $y \in \mathcal{A}_5$ of order 2 such that $\mathcal{A}_5 = \langle x, y : x^5 = y^2 = (xy)^3 = 1 \rangle$. The elements of order 5 in $\mathcal{A}_5$ are of the form x^l or $x^m y x^l y x^{-m}$, for some $l \in \{1, 2, 3, 4\}$ and some $m \in \{0, 1, 2, 3, 4\}$. In particular, $\phi(\beta_i)$ has one of the above forms.

Let us consider the simple loops $\theta_b = \alpha_i^b \beta_i$, for $b \in \{1, 2, 3, 4\}$. If for some b we have $\phi(\theta_b)$ is not of order 5, then we replace α by this new loop and we are done. To proceed to see how to find such a value of b.

(1) If $\phi(\beta_i) = x^l$, then for $b = 5 - l$ we have that $\phi(\theta_b)$ is trivial.
(2) If $\phi(\beta_i) = x^m y x^l y x^{-m}$, then $\phi(\theta_b) = x^m x^b y x^l y x^{-m}$. The order of this element is the same as for $x^b y x^l y$.
  (2.1) if $l = 1$, then for $b = 1$ we see that $x^b y x^l y$ must have order 3;
  (2.2) if $l = 2$, then for $b = 1$ we see that $x^b y x^l y = (yx^{-1}) y (yx^{-1})^{-1}$ has order 2;
  (2.3) if $l = 3$, then for $b = 1$ we see that $x^b y x^l y = (yx^{-1})(yx)(yx^{-1})^{-1}$ has order 3; and
  (2.4) if $l = 4$, then for $b = 4$ we see that $x^b y x^l y = yx$ has order 3.

As a consequence of the above, we may assume now that each of the loops α_1,..., α_γ lifts to either 30 or 20 different loops. In particular, the lifting components of each Σ_i, for $i = 1, ..., \gamma$ are genus zero surfaces.

It follows from the possibilities given in cases (A)-(D) above that each loop α_γ,..., $\alpha_{2\gamma}$ lifts to exactly 60 loops. Now it follows that the other loops also lift to exactly 60 loops each one. In particular, each surface Σ_j, where $j = \gamma + 1, ..., 2\gamma - 2$, lifts to genus zero surfaces. □

4. Proof of Theorem 1 in the general situation for $\mathcal{A}_4$ and $\mathcal{A}_5$

Let us assume H is a group of conformal automorphisms of some closed Riemann surface S. Set $R = S/H$ and let us denote by $P : S \to R$ the natural holomorphic branched covering induced by the action of H on S. As before, we denote by $F(H)$ the set of fixed points of the non-trivial elements of H.

If H is isomorphic to the alternating group $\mathcal{A}_k$, for $k = 4, 5$, then we have a presentation

$$H = \langle x, y : x^{2(k-4)+3} = y^2 = (xy)^3 = 1 \rangle$$

If we assume that H satisfies condition (A), then we get a collection $\mathcal{C} = \{\{p_i, q_i\} : p_i, q_i \in F(H)\}$ satisfying the conditions described in section 2. Moreover, it is not hard to assume that this collection has the extra property that if

$\{p,q\} \in \mathcal{C}$ and $h \in H$, then $\{h(p), h(q)\} \in \mathcal{C}$, that is, the collection is H invariant (see [2]).

Since there is no dihedral subgroup in either $\mathcal{A}_4$ and $\mathcal{A}_5$, condition (A) part (3.3) asserts that for each pair $\{p,q\}$ in $\mathcal{C}$ we have that $P(p) \neq P(q)$.

It follows that we can draw pairwise disjoint simple loops, say η_1,..., η_m, each one bounding a topological disc containing exactly two branched values which are the projections of paired fixed points as above.

The condition imposed on the rotation number, by condition (A), of each pair asserts that each of these loops η_i must lift to exactly $\frac{k!}{2}$ simple loops. The topological subsurfaces bounded by such loops lift to genus zero surfaces, each one invariant under a cyclic subgroup of H.

Choose a simple loop η, disjoint from the all the above ones such that η dissects R into two subsurfaces. One of this surfaces is of genus zero and contains all the branching of the (branched) cover $P : S \to R$. The other subsurface, say R_1, is a closed orientable surface of genus γ, with a boundary given by η.

Clearly, such a loop η lifts to exactly $\frac{k!}{2}$ simple loops. This is consequence that η is free homotopic to the product $\eta_1\eta_2\cdots\eta_m$.

If $\gamma = 0$, then we are done. If $\gamma > 0$, then the previous section (free fixed point actions) ensures the existence of a collection of pairwise disjoint simple loops on R_1 so that their liftings, together the liftings of all the above simple loops, dissect S into genus zero surfaces. Such a collection of loops on S will be invariant under the action of H. Since they cut off S into genus zero surfaces, they define a Schottky uniformization for which H lifts and, the proof of theorem 1, for the alternating situation, is done.

5. The proof of theorem 1 for $\mathcal{S}_4$ and theorem 2

5.1. **Proof of Theorem 1 for $\mathcal{S}_4$.** We need to find a collection of pairwise disjoint simple loops on the surface S such that (i) cut off S into planar surfaces and (ii) it is invariant under the action of H.

Set K the index two subgroup of H, which is isomorphic to $\mathcal{A}_4$.

We denote by $P : S \to R = S/K$ the canonical (branched) Galois covering induced by the alternating group K.

The automorphisms of order 4 in H induce the same automorphism $\widehat{\sigma} : R \to R$ of order two. The topological action of the involution $\widehat{\sigma} : R \to R$ is well understand and it is shown in figure 3. We have two possibilities:

(1) Fix($\widehat{\sigma}$) is non-empty; or

(2) Fix($\widehat{\sigma}$) is empty.

The main fact is that no branched value of $P : S \to R$ can be fixed point of $\widehat{\sigma}$. In fact, assume there is a pint $t \in S$ fixed by some non-trivial element in K so that $P(t)$ is fixed by $\widehat{\sigma}$. The group H is generated by K and an element $z \in H - K$ of order four. The above asserts that there is some element $x \in K$, different from the identity, such that $x(t) = t$, and there is some $y \in K$ such that $yz(t) = t$. Since $yz \in H - K$, we will have that the stabilizer of t in H is not cyclic, a contradiction.

We proceed to show the existence of a collection of loops as desired in each situation.

Case 1: Let us assume $\mathrm{Fix}(\widehat{\sigma}) \neq \emptyset$. In this case, we proceed to draw a collection of simple loops as shown in figure 4. Observe that if K acts free fixed points, then the loops η_j and the regions A_j are not there.

Each of the loops η_j lifts to exactly 12 simple loops, permuted by K. In fact, the loop η_1 bounds a disc containing only the branch values p_1 and $\widehat{\sigma}(p_1)$. The rotation numbers at these points are the same but with opposite sign (condition (A)). In a similar fashion we see that the other loops η_2,..., η_m, each one lifts to exactly 12 loops.

On the subsurface Σ_1 we may construct a collection of loops that cut off it into three-holed spheres. By the arguments in the above sections, we may assume that each of these three-holed sphere lifts to genus zero surfaces. We translates these loops to the subsurface $\widehat{\sigma}(\Sigma_1)$.

Now, the loops θ_1, θ_2, $\widehat{\sigma}(\theta_2)$,..., θ_{2k}, determine three-holed spheres in the surface R. By section 3, we have that each component of the lifting of each of these three-holed sphere is either of genus zero or a three-holed genus one surface. If we are in the genus one situation, the invariance of the subsurface in R by $\widehat{\sigma}$ will asserts that we have an action of $\mathcal{A}_4$ as group of conformal automorphisms in a genus one surface. This is known to be impossible.

Since the loop η_m lifts to a exactly 12 loops, we have that we may glue a disc to the boundary of X corresponding to η_m and discs to the corresponding boundaries of the liftings $P^{-1}(X)$. In this way, we may use same arguments as above.

As a consequence, the collection of loops constructed in R has the property that (i) it is invariant under $\widehat{\sigma}$ and (ii) their liftings cut off S into genus zero surfaces. From this, we obtain the desired collection of loops.

Case 2: Let us assume $\mathrm{Fix}(\widehat{\sigma}) = \emptyset$. In this case, we proceed to draw a simple loop γ that is invariant under $\widehat{\sigma}$, disjoint from the branched values of the (branched) covering $P : S \to R$ and homologically non-trivial as shown in figure 5. If we show that the loop γ lifts to exactly 12 loops, then we can proceed as in the above case (think of such a loop as the fixed point of $\widehat{\sigma}$ enclosed by the loops η_1 in that situation).

To show the lifting property of γ, we consider a connected component η of $P^{-1}(\gamma)$. If our claim is not true, then the stabilizer in K of η is either (i) a cyclic group of order 2 or 3, or (ii) K, in all situations acting free fixed points. This and invariance of γ under $\widehat{\sigma}$ asserts that there is some element in $H - K$ that keeps invariant η. It follows that its stabilizer in H contains the alternating group K acting free fixed points. This will assert that $\mathcal{A}_4$ is a surjective homomorphism image of $\mathbb{Z}$, a contradiction.

Let us consider the collection of loops as shown in figure 6. Now we can proceed as in case 1 to construct the desired collection of loops. The proof of theorem 1, for $\mathcal{S}_4$, is now finished.

5.2. **Proof of Theorem 2.** If we consider a group H of conformal automorphisms, isomorphic to $\mathcal{S}_5$, of a closed Riemann surface S, and let us denote by K the index two subgroup, isomorphic to $\mathcal{A}_5$. Set $P : S \to R = S/K$ the branched covering induced by the action of K on S and $\widehat{\sigma} : R \to R$ the conformal involution induced by H on R. If we have that no branched value of P is fixed point of $\widehat{\sigma}$ and either (i) $\widehat{\sigma}$ acts free fixed points or (ii) $\widehat{\sigma}$ has exactly two fixed points, then we may follow the same arguments as in the above section to obtain that H is of Schottky type if K satisfies condition (A).

6. An example of an $\mathcal{S}_5$ not of Schottky type

We describe the explicit example, due to A. Miller, of a group H of conformal automorphisms, isomorphic to $\mathcal{S}_5$, satisfying condition (A) but not of Schottky type. The reason of why the arguments done for $\mathcal{S}_4$ fail in this situation is that we may have branched values in $R = S/\mathcal{A}_5$ that are fixed points of the involution induced on R by the elements in $\mathcal{S}_5 - \mathcal{A}_5$.

The symmetric group in five letters $\mathcal{S}_5$ is generated by the permutations $a = (123)$, $b = (154)$, $c = (1234)$ and $d = (123)(45)$, satisfying the relation $abcd = 1$.

Consider a Fuchsian group F, acting on the upper-half plane $\mathbb{H}$, of signature $(0, 4; 3, 3, 4, 6)$. Such a group has a presentation:

$$F =< S, T, U, V : S^3 = T^3 = U^4 = V^6 = STUV = 1 >.$$

We have a surjective homomorphism $\Phi : F \to \mathcal{S}_5$ defined by $\Phi(S) = a$, $\Phi(T) = b$, $\Phi(U) = c$ and $\Phi(V) = d$.

If N denotes the kernel of the homomorphism $\Phi : F \to \mathcal{S}_5$, then N is easily seen to be torsion–free.

The Riemann–Hurwitz formula [1] asserts that the surface $S = \mathbb{H}/N$ is a closed Riemann surface of genus 56. The surjective homomorphism $\Phi : F \to \mathcal{S}_5$ produces a group $H = F/N$, isomorphic to $\mathcal{S}_5$, as group of conformal automorphisms of S. This group acts on S with exactly 130 fixed points.

The set of fixed points fall out into 65 pairs, say (p_i, q_i), $i = 1, ..., 65$, such that the pairing satisfies the properties of Condition (A). Moreover, for each pair (p_i, q_i) there is an involution j_i in H such that, $j_i(p_i) = q_i$.

The quotient Riemann surface $S/H = \mathbb{H}/F$ is an sphere with four branch values of order 3, 3, 4 and 6, respectively.

Next, we proceed to show that H is not of Schottky type. For it, we assume it is of Schottky type and let us get a contradiction. Our assumption asserts that there is a Schottky uniformization of S $(G, \Omega, \pi : \Omega \to S)$ such that, for every h in H there exists an automorphism $\tilde{h}$ of Ω with $\pi \circ \tilde{h} = h \circ \pi$. Let us denote by $\widehat{G}$ the group of automorphisms of Ω generated by the liftings of all the elements of H. We obtain an uniformization of S/H given by $(\widehat{G}, \Omega, p : \Omega \to S/H)$.

Since the $\widehat{G}$ has the Schottky group G as a subgroup of index two, it satisfies the following properties.

(1) $\widehat{G}$ is finitely generated.

(2) $\widehat{G}$ is a function group ([6]).

(3) $\widehat{G}$ has no parabolic elements.

(4) $\widehat{G}$ is geometrically finite ([6]).

Maskit's classification of finitely generated function groups [8] asserts that $\widehat{G}$ is constructed, by use of the Maskit–Klein Combination Theorems [6], from the following basics function groups:

(i) Finite groups.

(ii) Euclidean groups.

(iii) Finite extensions of cyclic loxodromic groups.

(iv) Quasi-Fuchsian groups.

(v) Degenerated groups.

See [6] for the definition of the above groups. The above properties of $\widehat{G}$ imply that we cannot use the groups of type (ii), (iv) and (v) in the construction of such a group. Thus, $\widehat{G}$ is constructed from groups of type (i) and (iii).

In the use of the first Klein–Maskit Combination Theorem, we cannot use groups of type (iii), except by the one which is obtained at the free product of two elements of order two (this follows from the fact that $\widehat{G}$ must uniformize a surface of genus zero). The exceptional group of type (iii) is obtained from two cyclic groups of order two by the first Klein–Maskit Combination Theorem. As a consequence, we only need the finite groups in the use of the above Combination Theorem. Observe that this operation is not enough to get a group uniformizing an sphere with signature $(0,4;3,3,4,6)$. Thus we also need the second Klein–Maskit Combination Theorem. Now, with the second Klein–Maskit Combination Theorem we either produces groups which uniformize either surfaces of positive genus or surfaces with signatures $(0,4;2,2,2,n)$ $(n \geq 2)$. As a consequence, we cannot produce a group $\widehat{G}$ as above uniformizing a surface with signature $(0,3;3,3,4,6)$. This proves the non–existence of a Schottky uniformization as desired.

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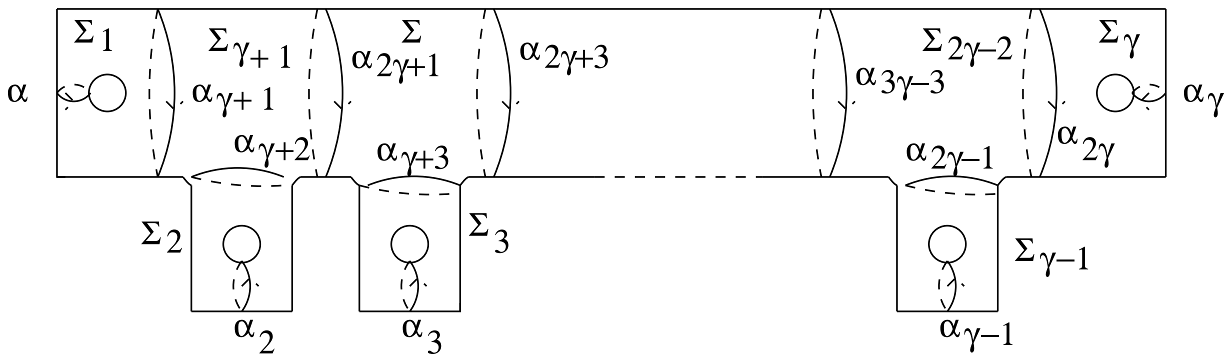

FIGURE 1. A collection of simple loops on R

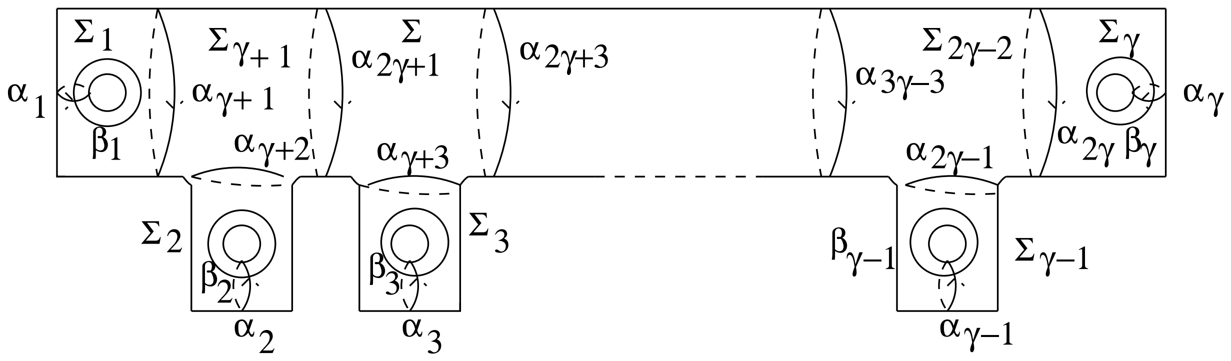

FIGURE 2. The loops β_i

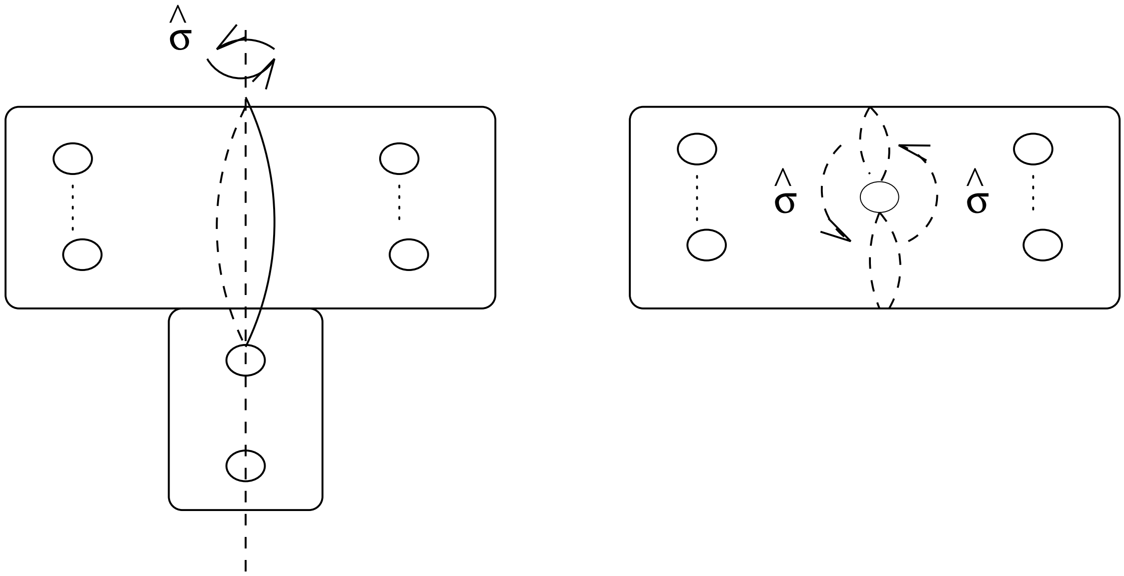

FIGURE 3. The topological action of $\widehat{\sigma}$

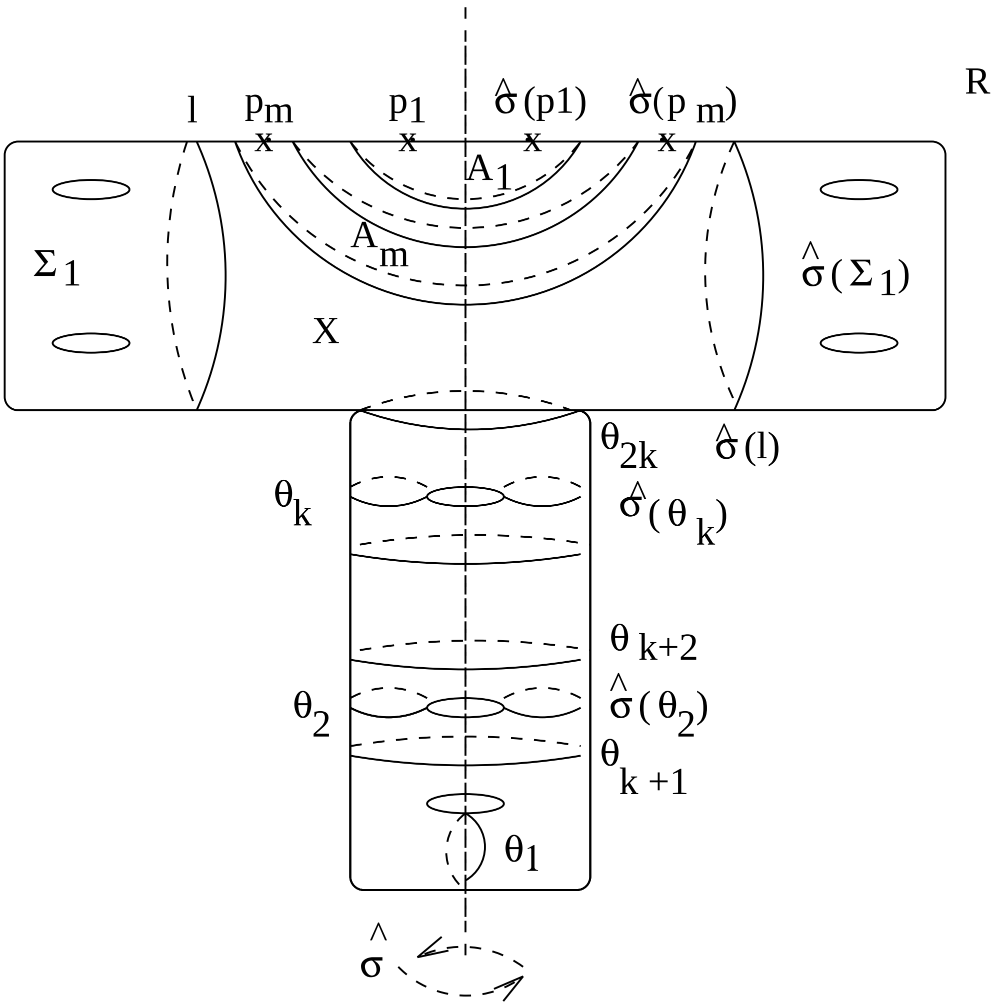

Figure 4. A set of loops on R

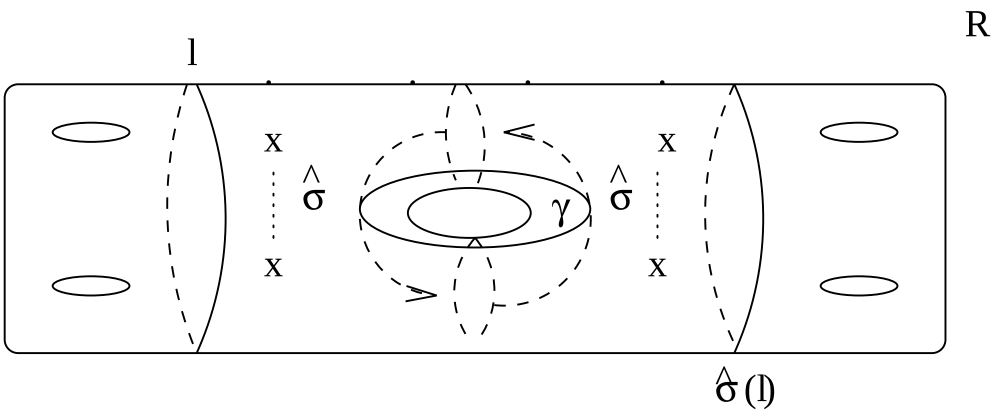

Figure 5. A set of loops on R

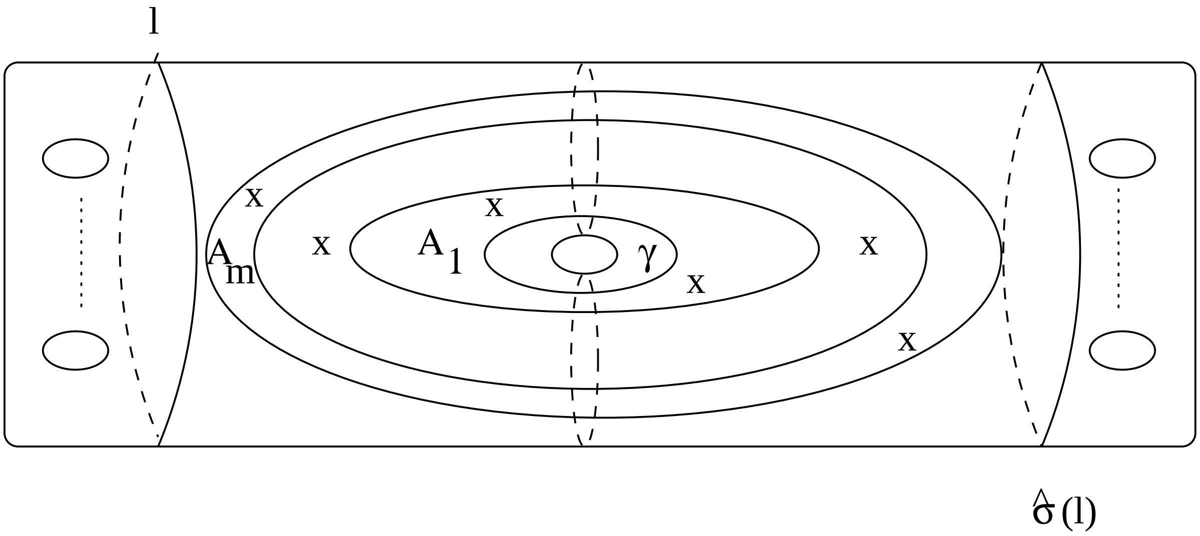

FIGURE 6. A set of loops on R