SCHOTTKY UNIFORMIZATIONS OF BORDERED COMPACT RIEMANN SURFACES WITH A SPECIAL GROUP OF AUTOMORPHISMS

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ABSTRACT. Given a compact bordered Riemann surface of algebraic genus $p \geq 2$, it was shown by C.L. May that the order of its group of automorphisms is bounded by $12(p-1)$ and that on each genus $p \geq 2$ there are compact bordered Riemann surfaces of algebraic genus p admitting a group of automorphisms of order $4p$ and $4(p+1)$, respectively. In this note, we produce Schottky uniformizations of these type of compact bordered Riemann surfaces.

1. Introduction

Let us consider a compact bordered Riemann surface S of genus g with $k \geq 1$ bordered components. The number $p = 2g + k - 1$ is called the algebraic genus of S, and it denotes the genus of a compact Riemann surface S^d admitting an anticonformal involution $\tau : S^d \to S^d$, with k fixed point components, so that $S = S^d/\tau$.

Let H denotes a group of automorphisms of a compact bordered surface S of algebraic genus $p \geq 2$ (admitting reversing-orientation transformations). It was shown by C.L. May [11] that the order of H is bounded by $12(p-1)$. Moreover, for every $p \geq 2$ there are compact bordered Riemann surface of algebraic genus p with a group of automorphisms of order $4p$ and $4(p+1)$. In [2] E. Bujalance studied the compact bordered Klein surfaces of algebraic genus $p \geq 2$ with group of automorphisms of orders $4p$ and $4(p+1)$.

Theorem [2]**.** *Let X be a Klein surface with boundary of algebraic genus $p \geq 2$ and $p \neq 5, 11, 29$ admitting a group G of order $4(p+1)$. Then G is the full group of automorphisms of X and*

$$G = \langle x, y, z : x^2 = y^2 = z^2 = (zy)^2 = (xy)^{p+1} = 1, zxz = y(xy)^r \rangle,$$

for some r, $r^2 \equiv 1$ mod $(p+1)$ and $1 \leq r \leq p$. Moreover, X is orientable with $k = (p+1, r+1)$ boundary components, where $(p+1, r+1)$ denotes the greatest common divisor of $p+1$ and

1991 *Mathematics Subject Classification.* Primary 30F40.
Key words and phrases. Kleinian groups, Schottky groups, Riemann surfaces.
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$r+1$. *Conversely, for each group G as above and for $k=(p+1, r+1)$, there exists an orioentable Klein surface of algebraic genus p and k boundary components having G as group of automorphisms.*

It is also shown that the quotient S/G, in the case of the theorem, is the closed disc with four branch values in the boundary, one of order $(p+1)$ and the other three of order 2.

A simmilar result concerning the case that X has a group G of order $4p$ is also obtained in Bujalance's paper [2].

The aim of this note is to construct by means of Schottky uniformizations the compact bordered Riemann surfaces of every algebraic genus $p \geq 2$ admitting a group of automorphisms of both orders $4p$ and $4(p+1)$ so that this group is reflected by the Schottky uniformization.

2. Schottky Groups

Let p be some positive integer. Assume we have a collection of $2p$ pairwise disjoint simple loops, say α_1,..., α_p, β_1,..., β_p, bounding a common region $\mathcal{D}$ of connectivity $2p$. Let us also assume we have a collection of Moebius transformations A_1,..., A_g (necessarilly loxodromic ones) so that $A_j(\alpha_j)=\beta_j$ and $A_j(\mathcal{D}) \cap \mathcal{D} = \emptyset$, for all $j=1,...,p$. The group J generated by A_1,..., A_p turns out to be a Kleinian group, purely loxodromic, isomorphic to a free group of rank p and with connected region of discontinuity [9]. The group J is called a Schottky group of genus p.

For another equivalent definitions of Schottky groups one can check [10] (also see [3]).

Let J be a Schottky group of genus p with region of discontinuity Ω. Then Ω/J turns out to be a closed Riemann surface of genus p. Moreover, the classical retrosection theorem states that every closed Riemann surface of genus p can be obtained in this way. A simple proof of this fact was also given by L. Bers in [1] using quasiconformal mappings.

If J is a Schottky group with region of discontinuity Ω and $P : \Omega \to S$ is a normal covering map onto a Riemann surface of genus p, with J as covering group, then we say that $(J, \Omega, P : \Omega \to S)$ is a Schottky uniformization of S.

A lifting problem concerning Schottky uniformizations of closed Riemann surfaces is the following. Assume we have a closed Riemann surface S and a group H of conformal automorphisms of it. In general, there is no Schottky uniformization of S for which the group H

lifts. It is then natural to ask when we may find a Schottky uniformization of S for which this lifting is possible. When this is the case, we say that H is of Schottky type.

In [5] we have found easy necessary conditions to be satisfied by H in order to be of Schottky type. These conditions are trivilally satisfied by groups isomorphic to a dihedral group and for groups acting free fixed points. In [5], [6], [7] and [8] we have shown that these necessary conditions are in fact sufficient for abelian groups, dihedral groups, the alternating groups A_4 and A_5 and for the symmetric group S_4.

In the case of groups admitting orientation-reversing transformations, we have some results for the cyclic case in [4]. From now on, Δ will be the unit disc and S^1 its boundary circle. In this note then we get a collection of closed real Riemann surfaces with a group of order $4P$ and $4(p+1)$, commuting the real structure of the surface, and of Schottky type.

3. Groups of order $4p$

Let Δ be the unit disc and S^1 its boundary circle. Fix an integer $p \geq 2$ and denote by L_1 the real line, L_2 the line through zero and $e^{\frac{i\pi}{2p}}$ and L_3 a circle orthogonal to both S^1 and L_2 and disjoint from L_1. Let θ, σ, τ and η be the reflections on S^1, L_1, L_2 and L_3, respectively.

If we denote by $\widehat{K}$ the group generated by the above reflections, then we have that

$$\widehat{K} = \langle \theta, \sigma, \tau, \eta : \theta^2 = \sigma^2 = \tau^2 = \eta^2 = (\tau\sigma)^{2p} = (\tau\eta)^2 = (\theta\sigma)^2 = (\theta\tau)^2 = (\theta\eta)^2 = 1 \rangle.$$

Denoting by K the index two subgroup generated by the reflections σ, τ and η, we have

$$K = \langle \sigma, \tau, \eta : \sigma^2 = \tau^2 = \eta^2 = (\tau\sigma)^{2p} = (\tau\eta)^2 = 1 \rangle.$$

Set $W = \tau\sigma$ the rotation of order $2p$, $Q = W^p$ the antipodal map and $E = \eta\tau$ (a conformal involution).

Let J be the smallest normal subgroup of K containing QE. It turns out that J is a hyperelliptic Schottky group of genus p with free generators QE, $WQEW^{-1}$,..., $W^{p-1}QEW^{-(p-1)}$ (see figure 1 for the case $p = 2$). Since θ commutes with all elements in K, we have that J is also normal subgroup of $\widehat{K}$.

The quotient $G = K/J$ has presentation

$$G = \langle x, y, z : x^2 = y^2 = z^2 = (yx)^{2p} = (yz)^2 = (yx)^p zy = 1 \rangle = \langle x, y : x^2 = y^2 = (yx)^{2p} \rangle,$$

and, in particular, G is isomorphic to the dihedral group of order $4p$.

If we denote by Ω the region of discontinuity of $\widehat{K}$, then it is also the region of discontinuity of each of the above normal subgroups. Denoting by H the group generated by θ (a normal subgroup of $\widehat{K}$), we have that Ω/H is the closed disc $\overline{\Delta}$. We have that $S = \overline{\Delta}/J$ is a compact bordered Riemann surface of algebraic genus p, its double is the closed Riemann surface of genus p uniformized by the Schottky group J, that is, Ω/J.

The surface S admits the group $G = K/J$ as group of automorphisms. Moreover, S/G is uniformized by $\Omega/\widehat{K} = \overline{\Delta}/K$ which is a closed disc with four branche values in the border, one of order $2p$ and three of order 2.

4. Groups of order $4(p+1)$

Fix an integer $p \geq 2$ and denote by L_1 the real line, L_2 the line through zero and $e^{\frac{i\pi}{(p+1)}}$ and L_3 a circle orthogonal to both S^1 and L_1 and disjoint from L_2. Let θ, σ, τ and η be the reflections on S^1, L_1, L_2 and L_3, respectively. Let us also fix some integer $k \in \{1, ..., p\}$.

If we denote by $\widehat{K}$ the group generated by the above reflections, then we have that

$$\widehat{K} = \langle \theta, \sigma, \tau, \eta : \theta^2 = \sigma^2 = \tau^2 = \eta^2 = (\tau\sigma)^{p+1} = (\sigma\eta)^2 = (\theta\sigma)^2 = (\theta\tau)^2 = (\theta\eta)^2 = 1 \rangle.$$

Denote also by K the index two subgroup generated by the reflections σ, τ and η. We have

$$K = \langle \sigma, \tau, \eta : \sigma^2 = \tau^2 = \eta^2 = (\tau\sigma)^{p+1} = (\sigma\eta)^2 = 1 \rangle.$$

Set $W = \tau\sigma$ the rotation of order $(p+1)$ and $\eta_j = W^j \eta W^{-j}$ the reflections on the translates of the circle L_3 by the rotation W. Let J_k be the smallest normal subgroup of K containing $T_k = \eta W^k \eta W^{-1} = \eta(\tau\sigma)^{k-1}\tau\eta\tau$.

The quotient $G_k = K/J_k$ has presentation

$$G = \langle x, y, z : x^2 = y^2 = z^2 = (xy)^{p+1} = (xz)^2 = 1, zxz = y(xy)^r \rangle,$$

where x, y and z are the classes representing τ, σ and η, respectively and $r = p + 1 - k$ (see Bujalance's theorem). This is a group of order $4(p+1)$ if and only if $r^2 \equiv 1 \mod (p+1)$.

If we denote by Ω the region of discontinuity of $\widehat{K}$, then it is also the region of discontinuity of each of the above normal subgroups. Denoting by H the group generated by θ (a normal subgroup of $\widehat{K}$), we have that Ω/H is the closed disc $\overline{\Delta}$.

In general, the subgroup J_k will have torsion and, in particular, will not be a Schottky group. For instance, if we take $p=3$ and $k=2$, then J_2 contains the transformations T_2 and $\sigma T_2^{-1}\sigma$. But $\sigma T_2^{-1}\sigma T_2 = W^2$ and the group J_2 has torsion. More precisely, we have the following:

Proposition. *J_k is a Schottky group of genus p if and only if $k^2 \equiv 1 \mod (p+1)$.*

Proof. Let us write down first some easy relations satisfied by the transformations τ, η_j, η, σ and W.

(1) $\tau\eta\tau = \eta_1$;
(2) $\tau\eta_j\tau = \eta_{p+2-j}$, for $j = 2, ..., p$;
(3) $\sigma\eta\sigma = \eta$;
(4) $\sigma\eta_j\sigma = \eta_{p+1-j}$, for $j = 2, ..., p$;
(5) $W^n\eta_j W^{-n} = \eta_{j+n}$, for all $j = 1, ..., p$.

For a fixed $k \in \{1, ..., p\}$, we consider the hyperbolic transformations $A_j = \eta W^{j(k-1)}\eta_j$, for $j = 1, ..., p$. For instance, if $k=1$, then $A_j = \eta\eta_j$ and if $k=p$, then $A_j = \eta W^{-2j}\eta_j$.

Let us denote by $\widehat{J}_k$ the group generated by the transformations A_1,..., A_p.

We have the following relations as consequence of (1)–(5):

(a) $\eta A_j \eta = \eta\eta W^{j(k-1)}\eta_j\eta = W^{j(k-1)}\eta_j\eta = \eta_{jk}W^{j(k-1)}\eta = \eta_{jk}W^{-jk(k-1)}\eta\eta W^{(k^2-1)j}\eta = A_{jk}^{-1}\eta W^{j(k^2-1)}\eta$, for $j = 1, ..., p$;
(b) $\tau A_1 \tau = \tau\eta W^{k-1}\eta_1\tau = \eta_1\tau(\tau\sigma)^{k-1}\tau\eta = \eta_1(\sigma\tau)^{k-1}\eta = \eta_1 W^{-(k-1)}\eta = A_1^{-1}$;
(c) $\tau A_j \tau = \tau\eta W^{j(k-1)}\eta_j\tau = \eta_1\tau(\tau\sigma)^{j(k-1)}\tau\eta_{1-j} = \eta_1(\sigma\tau)^{j(k-1)}\eta_{1-j} = \eta_1 W^{-j(k-1)}\eta_{1-j} = \eta_1 W^{-(k-1)}\eta\eta W^{(1-j)(k-1)}\eta_{1-j} = A_1^{-1}A_{1-j}$, for $j = 2, ..., p$;
(d) $\sigma A_j \sigma = \sigma\eta W^{j(k-1)}\eta_j\sigma = \eta\sigma(\tau\sigma)^{j(k-1)}\sigma\eta_{p+1-j} = \eta(\sigma\tau)^{j(k-1)}\eta_{p+1-j} = \eta W^{-j(k-1)}\eta_{p+1-j} = A_{p+1-j}$, for $j = 1, ..., p$.

Since $\tau\sigma A_j\sigma\tau = A_1^{-1}A_{j+1}$, A_1 belongs to J_k and J_k is normal subgroup in K, we get inductively that A_2,..., A_p also belong to J_k and, in particular, $\widehat{J}_k < J_k$.

Part (a) says that $W^{j(k^2-1)}$ belongs to J_k. In particular, we have that J_k has torsion if k^2 is not equivalent to 1 mod $(p+1)$ and, in particular, J_k cannot be a Schottky group.

On the other hand, if $k^2 \equiv 1 \bmod (p+1)$, then (a)–(d) asserts that $\widehat{J}_k$ is a normal subgroup of K containing $T_k = A_1$ and, in consequence, we must have that $\widehat{J}_k = J_k$. But, $\widehat{J}_k$ turns out to be a Schottky group by construction (see lemma below) if we have $k^2 \equiv 1 \bmod (p+1)$ (see figures 2 and 3 for $p=2$). $\square$

Lemma. *If $k^2 \equiv 1$ mod $(p+1)$, then $\widehat{J}_k$ is a Schottky group of genus p.*

Proof. We only need to check that $A_r(C_r) \neq A_R(C_R)$ for $r \neq R$, where $C_j = W^j(L_3)$ and $r, R \in \{1, ..., p\}$.

If we assume $r > R$ and equality $A_r(C_r) = A_R(C_R)$, then we have that $A_R^{-1}A_r\eta_r(A_R^{-1}A_r)^{-1} = \eta_R$. From this we obtain that $\eta_R = \eta_{r+(r-R)(k-1)}$ and, in particular, $(r-R)k \equiv 0 \bmod (p+1)$. Since we are assuming that $k^2 \equiv 1 \bmod (p+1)$, we have that the prime factors of $(p+1)$ are either prime factors of $(k-1)$ or $(k+1)$. In particular, they cannot be prime factors of k. From this we must have that $r - R \equiv 0 \bmod (p+1)$, giving a contradiction to the fact that $0 < r - R < p - 1$. $\square$

As a consequence of the above, for $k \in \{1, ..., p\}$ so that $k^2 \equiv 1 \bmod (p+1)$, we have that J_k is a Schottky group. We have that $S_k = \overline{\Delta}/J_k$ is a compact bordered Riemann surface of algebraic genus p, with double cover the closed Riemann surface of genus p uniformized by the Schottky group J_k, that is, Ω/J_k. Moreover, the surface S_k admits the group $G_k = K/J_k$ as group of automorphisms so that $S_k/G_k = \Omega/\widehat{K} = \overline{\Delta}/K$ is a closed disc with four branched values in the border, one of order $(p+1)$ and three of order 2. The condition on k also asserts that G_k has order $4(p+1)$ (see Bujalance's theorem).

In this way, one gets Schottky uniformizations of a closed Riemann surface S^d of genus p admitting an anticonformal involution $\theta : S^d \to S^d$ so that S^d/θ is a bordered compact Riemann surface of algebraic genus p with a group of automorphisms of order $4(p+1)$ as in Bujalance's theorem.

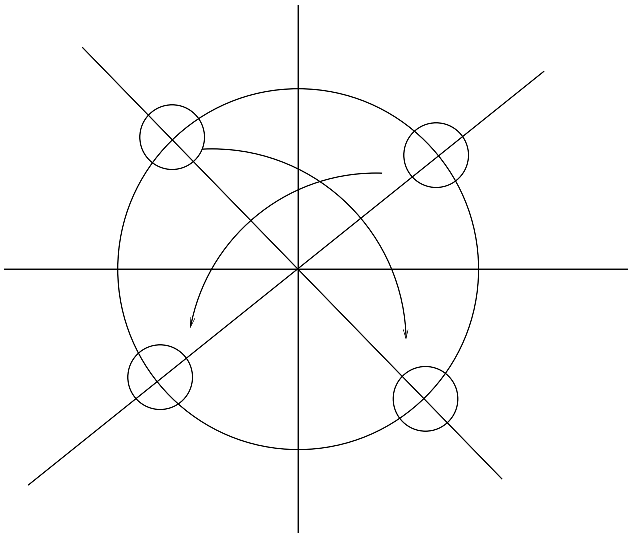

Figure 1

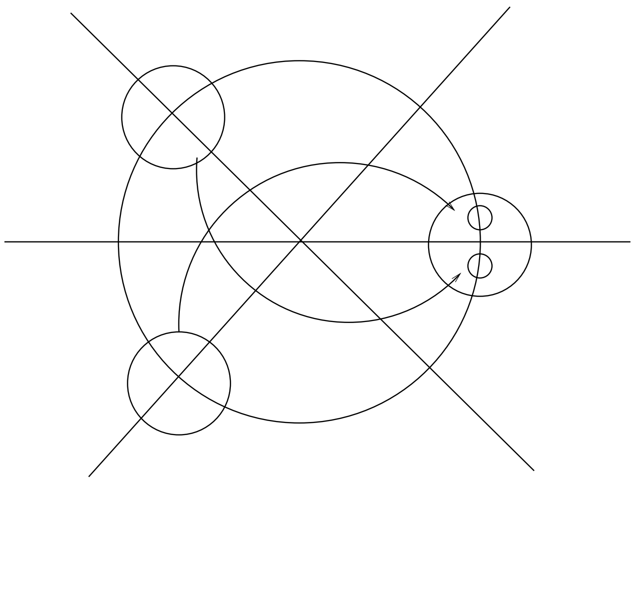

Figure 2: p=2, k=2, r=1

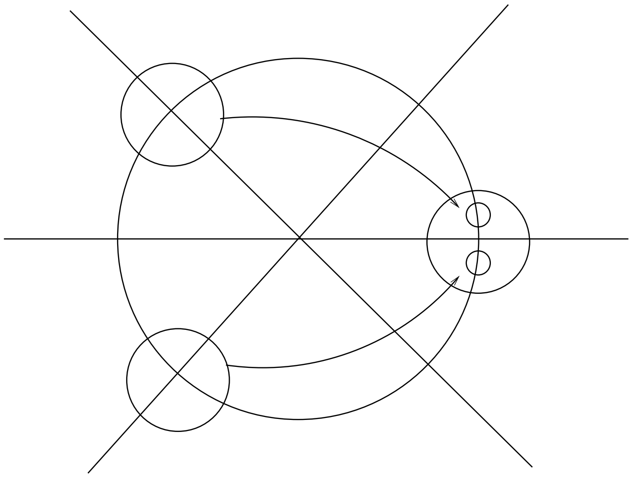

Figure 3: p=2, k=1, r=2

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