LATTICE POINTS IN MULTIDIMENSIONAL BODIES

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Abstract. Asymptotic expansions for the number of lattice points in bounded regions of the d-dimensional Euclidean space are obtained. We assume that the regions are level sets of polynomials of degree $p \geq 2$. The expansions are valid provided that some 'local' bounds on trigonometric sums related to these polynomials are satisfied. Using this approach we obtain as an initial application optimal bounds for the lattice point rest in large bodies generated by polynomials such as $\mathbb{Q}(x) = \mathbb{Q}_p(x) + \mathbb{P}(x)$, $\deg \mathbb{P} < p$, $x \in \mathbb{R}^d$, where $\mathbb{Q}_p$ has the form $\mathbb{Q}_p(x) = \lambda_1 x_1^p + \cdots + \lambda_d x_d^p$ with some $\lambda_j > 0$, for $1 \leq j \leq d$, provided that the dimension $d \geq c_p$ is sufficiently large. The proofs are based on an extension of techniques developed for quadratic forms by the authors. Similar to the quadratic case the results allow to prove variants of the quantitative Oppenheim conjecture and the Davenport and Lewis conjecture for classes of higher order polynomials assuming that the dimension is sufficiently large.

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1. Introduction and results

Let $\mathbb{R}^d$, $1 \le d < \infty$, denote a real d-dimensional Euclidean space with scalar product $\langle \cdot, \cdot \rangle$ and norm

$$|x|^2 = \langle x, x \rangle = x_1^2 + \cdots + x_d^2, \qquad \text{for} \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

Let $\mathbb{Z}^d$ be the standard lattice of points with integer coordinates in $\mathbb{R}^d$.

For a (measurable) set $B \subset \mathbb{R}^d$ let $\operatorname{vol} B$ denote the Lebesgue measure of B, and let $\operatorname{vol}_{\mathbb{Z}} B$ denote the lattice volume of B, that is the number of points in $B \cap \mathbb{Z}^d$.

For $p \in \mathbb{N}$, $p \ge 2$, consider a polynomial

$$\mathbb{Q}(x) = \sum_{j=1}^{p} \mathbb{Q}_j\,(x), \qquad \deg \mathbb{Q}_j = j, \qquad x \in \mathbb{R}^d, \tag{1.1}$$

where $\mathbb{Q}_j$ denotes the j-th homogeneous part of $\mathbb{Q}$. We shall write $\|Q_j\| = \sup\limits_{|x| \le 1} \bigl| Q_j(x) \bigr|$.

For $\lambda_j \ne 0$, $\lambda_j \in \mathbb{R}$, and $1 \le d_0 \le d$, write

$$\lambda_* = |\lambda_1| \wedge \cdots \wedge |\lambda_{d_0}|, \qquad \lambda^* = |\lambda_1| \vee \cdots \vee |\lambda_{d_0}|, \tag{1.2}$$

where $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. Let

$$\mathbb{B}_p\,(x;\, d_0) = \mathbb{B}_p\,[x, \dots, x] = \sum_{j=1}^{d_0} \lambda_j\, x_j^p, \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d, \tag{1.3}$$

and

$$\mathbb{Q}_p\,(x) = \mathbb{B}_p\,(x;\, d_0) + \mathbb{P}_p\,(x), \tag{1.4}$$

where $\mathbb{P}_p : \mathbb{R}^d \to \mathbb{R}$, $\deg \mathbb{P}_p = p$, is a homogeneous polynomial such that its degree in the variables $x_1, \dots, x_{d_0}$ is strictly smaller than p. For example, the polynomial

$$\sum_{j=1}^{d_0} x_j^p + \sum_{j=1}^{d_0} x_j^2 \sum_{j=d_0+1}^{d} x_j^{p-2}, \qquad p > 2,$$

has the form (1.4).

We call a polynomial $\mathbb{Q}_p$ positive definite if $\mathbb{Q}_p\,(x) > 0$, for all $x \ne 0$. Otherwise $\mathbb{Q}_p$ is called indefinite. In this Introduction we shall restrict ourselves to problems related to positive definite polynomials $\mathbb{Q}$ with the leading part of the form (1.4). The reason is that in order to apply our main result—Theorem 2.2, we have to verify a condition (see (2.8)) for trigonometric sums. This is done in Section 8 for polynomials with the leading part (1.4). In more general situations one has to develop a technique more refined than that used in Section 8. We assume positive definiteness in this Introduction only in order to avoid additional technicalities associated with the indefinite case, cf. proofs in both cases for quadratic forms in Bentkus and Götze (1997c) (henceforth briefly BG).

For positive definite $\mathbb{Q}_p$ we have

$$q_* \, |x|^p \ll \mathbb{Q}_p(x) \ll \|\mathbb{Q}_p\| \, |x|^p \tag{1.5}$$

with some $q_* > 0$.

Define the region

$$E_s = \{ x \in \mathbb{R}^d : \ \mathbb{Q}(x) \le s \}, \qquad \text{for} \quad s \in \mathbb{R}. \tag{1.6}$$

For positive definite $\mathbb{Q}_p$ the region E_s is a compact subset of $\mathbb{R}^d$, and $E_s \ne \varnothing$, for $s \ge 0$. Hence, there exists a minimal number $r = r_s \ge 1$ such that the cube

$$B(r) = \{ x \in \mathbb{R}^d : \ |x|_\infty \le r \}, \qquad |x|_\infty = |x_1| \vee \cdots \vee |x_d|, \tag{1.7}$$

contains the set E_s. Clearly,

$$r_s = 1 \vee \sup_{x \in E_s} |x|_\infty.$$

Write as well

$$r_* = \sup_{x \in E_1} |x|_\infty. \tag{1.8}$$

For $\mathbb{Q}$ with a positive definite leading part the number r_* satisfies $0 < r_* < \infty$. Furthermore, for such $\mathbb{Q}$ we have $\lim_{s \to \infty} r_s / s^{1/p} = r_*$, which means that asymptotically the quantity r_s is proportional to $s^{1/p}$, as $s \to \infty$ (see Proposition 3.5).

There are two natural interrelated lattice points problems associated to a a polynomial. The first one is to investigate the asymptotic as $s \to \infty$ of

$$\operatorname{vol}_{\mathbb{Z}} E_s = \operatorname{vol}_{\mathbb{Z}} \{ x \in \mathbb{Z}^d : \ \mathbb{Q}(x) \le s \}, \tag{1.9}$$

for given polynomial $\mathbb{Q}$. Another way is to blow up a body, that is, to study the asymptotic of

$$\operatorname{vol}_{\mathbb{Z}}(r\, E_1) = \operatorname{vol}_{\mathbb{Z}} \{ r\, x \in \mathbb{Z}^d : \ \mathbb{Q}(x) \le 1 \} \tag{1.10}$$

as $r \to \infty$. For both bodies (1.9) and (1.10), we shall obtain bounds for the lattice point remainders

$$\delta(s, \mathbb{Q}) \overset{\text{def}}{=} |\operatorname{vol}_{\mathbb{Z}} \, E_s - \operatorname{vol} \, E_s| \quad \text{and} \quad \Delta(r, \mathbb{Q}) \overset{\text{def}}{=} |\operatorname{vol}_{\mathbb{Z}} \, rE_1 - \operatorname{vol} \, rE_1|. \tag{1.11}$$

It is clear that

$$rE_1 = \{ x \in \mathbb{Z}^d : \ \mathbb{Q}_p(x) + r\, \mathbb{Q}_{p-1}(x) + \cdots + r^{p-1}\, \mathbb{Q}_1(x) \le r^p \}. \tag{1.12}$$

The relation (1.12) shows that in order to derive bounds in both cases, it is desirable that in the case (1.9) they depend on the polynomial $\mathbb{Q}$ via the quantities

$$\|\mathbb{Q}_p\|, \ \|r^{-1}\,\mathbb{Q}_{p-1}\|, \dots, \|r^{1-p}\,\mathbb{Q}_1\|.$$

Define

$$q=\sum_{j=1}^{p} r^{j-p}\,\|Q_j\|, \qquad q_0=\sum_{j=1}^{p}\|Q_j\|. \tag{1.13}$$

Our bounds shall depend on $1+q$ in the case (1.9), and on $1+q_0$ in the case (1.10). In the case (1.10) it is convenient to replace r_s by the parameter

$$\rho_r = 1\vee (r_* r), \qquad\qquad r>0, \tag{1.14}$$

which for $r\geq 1/r_*$ is proportional to r. Notice that in cases when the lower order terms of the polynomial $\mathbb{Q}$ vanish (that is, when $\mathbb{Q}=\mathbb{Q}_p$), the problems related to (1.9) and (1.10) coincide.

Our main result is a general Theorem 2.2 given in Section 2 below. All results of the Introduction are corollaries of this theorem, for their proofs see Section 3. In this paper we consider the case of positive definite polynomials only, although Theorem 2.2 is applicable in the case of indefinite polynomials as well. Results for indefinite polynomials will be considered in a forthcoming paper.

The formulations of the results will depend on a sequence of *positive* exponents α_p, for example, we can choose

$$\alpha_p=\exp\bigl\{-3\,p\,\log p\bigr\}. \tag{1.15}$$

The results hold for sufficiently large dimensions $d\geq c(p)$ and $d_0\geq c_0(p)$, where the constants $c(p)$ and $c_0(p)$ depend on α_p.

Theorem 1.1. *Assume that a polynomial $\mathbb{Q}$ has a positive definite leading part of the form* (1.4) *with some $1\leq d_0\leq d$ and positive $\lambda_j>0$, for $1\leq j\leq d_0$. Let α_p satisfy* (1.15)*. If the dimension d_0 is sufficiently large, that is, if*

$$2^{1-2p}\,\alpha_p\,d_0>p\vee 4, \tag{1.16}$$

then the quantities $\delta(s,\mathbb{Q})$ and $\Delta(r,\mathbb{Q})$ defined by (1.11) *can be bounded as follows*

$$\delta(s,\mathbb{Q})\ll_{d,p} r_s^{d-p}\,C\,(1+q)^p, \qquad \Delta(r,\mathbb{Q})\ll_{d,p} \rho_r^{d-p}\,C\,(1+q_0)^p \tag{1.17}$$

with some constant $C=C(\lambda_1,\dots,\lambda_{d_0})$. In the case $d=d_0$ the condition (1.16) *can be weakened to*

$$2^{1-p}\,\alpha_p\,d>p\vee 4. \tag{1.18}$$

The constant C in (1.17) allows the following bounds

$$C \le \big((1 \vee \lambda^*)\,(1 \vee \lambda_*^{-1})\big)^{d_0\, 2^{2-p}}, \qquad \text{if } d_0 < d, \tag{1.19}$$

and

$$C \le \big((1 \vee \lambda^*)\,(1 \vee \lambda_*^{-1})\big)^{d\, 4^{1-p}}, \qquad \text{if } d_0 = d, \tag{1.20}$$

with λ_* and λ^* defined by (1.2).

Below we shall characterize cases when $\mathcal{O}(r_s^{d-p})$ and $\mathcal{O}(\rho_s^{d-p})$ in Theorem 1.1 can be improved to $o(r_s^{d-p})$ and $o(\rho_s^{d-p})$. These results can be used to describe the structure of the set $\mathbb{Q}(\mathbb{Z}^d) \subset \mathbb{R}$ extending the Davenport and Lewis (1972) conjecture to higher order polynomials, cf. the quadratic case BG (1997c). The bounds (1.17) extend as well to the case of shifted sets $E_s + a$, $a \in \mathbb{R}^d$.

Remark 1.2. Our requirements on the dimension in Theorem 1.1 and 1.3 may be improved to

$$\alpha_2 = 1, \ \ \alpha_4 = 4/189, \ \ \alpha_6 = 2/154275, \ \ \dots \tag{1.21}$$

An algorithm which allows to obtain further values of α_p is described in the proof of Lemma 8.6. In general the sequence (1.21) is not optimal as well since it is possible to prove Theorem 1.1 with $\alpha_p = \exp\big\{-c\,p\big\}$ with some absolute constant $c > 0$. However, the question of finding optimal α_p is a well-known difficult problem. In particular, it is not clear whether bounds of type (1.17) can be proved with some α_p decreasing as a power of p or with dimensions d_0 growing as a power of p. To conclude the remark, let us provide values of the dimensions d for which (1.17) holds assuming that the condition (1.4) is fulfilled with $d_0 = d$:

$$p = 2, \ \ d > 8, \qquad p = 4, \ \ d > 1512, \qquad p = 6, \ \ d > 14810400, \qquad \dots. \tag{1.22}$$

To obtain (1.22), it suffices to combine (1.18) and (1.21). The improvements described in this Remark will appear in a separate paper.

Let us compare Theorem 1.1 with known results and discuss the methods of proof. We prove Theorem 1.1 (and the main Theorem 2.2) using an extension of methods introduced in Bentkus and Götze (1994b, 1995a) to prove the bound $\mathcal{O}(r_s^{d-2})$ for *general* ellipsoids (that is, for $p = 2$). In the case of ellipsoids Theorem 1.1 holds without the condition (1.4). For references related to the bounds of lattice rest for ellipsoids, see the monographs by Walfisz (1957), Fricker (1982) and Krätzel (1988) and the papers cited above. For sufficiently large dimensions and $p > 2$, the bound of Theorem 1.1 improves a result $\mathcal{O}\Big(r_s^{d-2d/(d+1)} \log r_s\Big)$, $d \ge p-1$, of Randol (1966) obtained in the case $\mathbb{Q}(x) = x_1^p + \cdots + x_d^p$. Krätzel (1973) extended this result to functions $\mathbb{Q}(x) = |x_1|^p + \cdots + |x_d|^p$. They noticed as well that in low dimensions $d < p - 1$ bounds as in Theorem 1.1 are not available.

Since the lower bound $\delta(s,\mathbb{Q}) = \Omega(r_s^{d-p})$ holds in the case $\mathbb{Q}(x) = x_1^p + \cdots + x_d^p$ and $d \ge p$ (see, Krätzel (1988), Th. 4.4), Theorem 1.1 solves the problem of error bounds for regions E_s provided that the dimension d is sufficiently large. The bound of Theorem 1.1 combined with (1.19) and (1.20) shows that for large dimensions the number of lattice points in E_s depends asymptotically only on the size of E_s, i.e., only on radii of the largest inscribed and the smallest circumscribed spheres. It can be shown that gaps between large consecutive values $\mathbb{Q}(x)$, $x \in \mathbb{Z}^d$, say $v = \mathbb{Q}(x)$, converge to zero as $v \to \infty$, provided that remainders are $o(r_s^{d-p})$ or $o(\rho_s^{d-p})$. For polynomials with *integer* coefficients these gaps are bounded from below, which implies for such polynomials $\delta(s,\mathbb{Q}) = \Omega(r_s^{d-p})$ and $\Delta(r,\mathbb{Q}) = \Omega(\rho_s^{d-p})$.

The condition (1.4) on the leading part of polynomial $\mathbb{Q}$ seems to be rather restrictive. However, bounds of type (1.17) do not hold for arbitrary positive definite $\mathbb{Q}_p$: a simple example is provided by the polynomial $\mathbb{Q}(x) = (x_1^2 + \cdots + x_d^2)^{p/2}$, where $\delta(s,\mathbb{Q}) = \Omega(r_s^{d-2})$, by a reduction to the quadratic case $p = 2$. This suggests to consider representations of $\mathbb{Q}_p$ as a sum

$$\mathbb{Q}_p = \mathbb{P}_1\,\mathbb{S}_1 + \cdots + \mathbb{P}_m\,\mathbb{S}_m, \tag{1.23}$$

with polynomials $\mathbb{P}_j$ and $\mathbb{S}_j$ of positive degree, for all j (cf. Schmidt 1984). Let m_0 denote the minimal m in (1.23). One might expect that bounds of type (1.17) hold for $\mathbb{Q}$ such that the number m_0 is sufficiently large.

Bounds of type (1.17) are related to estimates of the accuracy of normal approximations in the Central Limit Theorem in multidimensional spaces and to limit theorems for U-statistics, see BG (1996, 1997a,b), where results corresponding to the quadratic case $p = 2$ were obtained. Proofs of corresponding results in probability theory are technically much more involved. They are as well more general in the sense that there is an equivalence between number theoretical problems and some special cases in probability. For $p > 2$ this correspondence to probability theory remains open.

Irrational polynomials. A polynomial $\mathbb{Q}_p$ of the form (1.4) is called rational if there exists an $M \ne 0$, $M \in \mathbb{R}$, such that the numbers $M\,\lambda_j$ are integers, for all j. Otherwise $\mathbb{Q}_p$ is called irrational.

Let us pass to a description of situations when remainders in Theorem 1.1 may be improved to $o(r_s^{d-p})$ and $o(\rho_r^{d-p})$. In order to describe these rates we shall use functions ω, ω_0, σ_0, σ which we have to define. Any of these functions, say ω, has the following essential property: $\lim_{s\to\infty} \omega(s) = 0$ if and only if the polynomial $\mathbb{Q}_p$ of the form (1.4) is irrational.

Let $\mathbb{P}$ denote a polynomial such that $\deg \mathbb{P} < p$. For $r > 0$, introduce the function (see (2.10) for a more natural and compact expression of this function)

$$\varphi_{\mathbb{P}}(t) = (2\,[r] + 1)^{-dp-d}\,\Big|\sum_{x_1,\dots,x_{p+1}} \mathrm{e}\big\{\,t\,\mathbb{Q}(x) + t\mathbb{P}(x)\,\big\}\Big|, \tag{1.24}$$

where we write $x = x_1 + \cdots + x_{p+1}$, and where the summation in (1.24) extends over all $x_j \in \mathbb{Z}^d$ such that $|x_j|_\infty \le r$, for all $1 \le j \le p+1$. For a number $T \ge 1$, define

$$\gamma\big(r^p, T\big) \stackrel{\text{def}}{=} \sup_{\deg \mathbb{P} < p} \; \sup_{r^{-p/2} \le t \le T} \varphi_{\mathbb{P}}(t). \tag{1.25}$$

Write (cf. the bounds (3.2) and (3.3) below):

$$\omega_0(s) \stackrel{\text{def}}{=} \inf_{T \ge 1} \big(r_s^{p-n} + T^{-1} + T^{\varepsilon}\, \gamma^{1-\varepsilon-4/(\alpha\varkappa_0)}\,(r_s^p, T)\big), \tag{1.26}$$

$$\sigma_0(r) \stackrel{\text{def}}{=} \inf_{T \ge 1} \big(\rho_r^{p-n} + T^{-1} + T^{\varepsilon}\, \gamma^{1-\varepsilon-4/(\alpha\varkappa_0)}\,(\rho_r^p, T)\big), \tag{1.27}$$

and

$$\omega(\tau) \stackrel{\text{def}}{=} \sup_{s \ge \tau} \omega_0(s), \qquad\qquad \sigma(\tau) \stackrel{\text{def}}{=} \sup_{r \ge \tau} \sigma_0(r) \tag{1.28}$$

with parameters $n > p$ and $\varepsilon > 0$. In Section 9 we show (see (9.5) and (9.9)) that the irrationality of $\mathbb{Q}_p$ of the form (1.4) guarantees the existence of $T_s \to \infty$, $s \to \infty$, and $T_r \to \infty$, $r \to \infty$, such that $\gamma(r_s^p, T_s) \to 0$ and $\gamma(\rho_r^p, T_r) \to 0$. As a consequence, it follows that $\omega_0(s), \omega(s) \to 0$ and $\sigma_0(r), \sigma(r) \to 0$, for irrational $\mathbb{Q}_p$ of the form (1.4).

Theorem 1.3. *Let $n > p$ and $\varepsilon > 0$. Let α_p satisfy* (1.15). *Assume that $\mathbb{Q}$ has a positive definite leading part of the form* (1.4) *with some $1 \le d_0 \le d$ and positive $\lambda_j > 0$, for $1 \le j \le d_0$. If the dimension d_0 is sufficiently large, that is, if*

$$2^{1-2p}\, \alpha_p\, d_0 > n \vee 4, \tag{1.29}$$

then the quantities $\delta(s, \mathbb{Q})$ and $\Delta(r, \mathbb{Q})$ defined by (1.11) *allow the following bounds*

$$\delta(s, \mathbb{Q}) \ll_{d,p,n,ve} r_s^{d-p}\, C\,(1+q)^p\, \omega(s), \qquad \Delta(r, \mathbb{Q}) \ll_{d,p,n,ve} \rho_r^{d-p}\, C\,(1+q_0)^p\, \sigma(r) \tag{1.30}$$

with a constant C as in Theorem 1.1. *In the case $d = d_0$ the condition* (1.29) *can be weakened to*

$$2^{1-p}\, \alpha_p\, d > n \vee 4. \tag{1.31}$$

If $\mathbb{Q}_p$ is irrational then

$$r_s^{p-d}\, \delta(s, \mathbb{Q}) \to 0, \text{ as } s \to \infty, \qquad \text{and} \qquad \rho_r^{p-d}\, \Delta(r, \mathbb{Q}) \to 0, \text{ as } r \to \infty. \tag{1.32}$$

If the polynomials $\mathbb{Q}_j$, for $1 \le j \le p-1$, and $\mathbb{P}_p$ in (1.4) *have integer coefficients then* (1.32) *holds if and only if $\mathbb{Q}_p$ is irrational* (*that is, if the vector $(\lambda_1, \dots, \lambda_{d_0})$ is not proportional to an integer vector*).

Theorem 1.3 extends a similar result of BG (1997c) in the quadratic case $p = 2$, which holds for arbitrary quadratic forms (that is, without the assumption (1.4) on $\mathbb{Q}_2$). See as well BG (1997c) for further references related to the question. In the non-quadratic case we are not aware on results similar to Theorem 1.3.

NOTATION. By c with or without indices we shall denote generic absolute constants. We shall write $B \ll D$ instead of $B \le cD$, with $c > 0$. If a constant depends on a parameter, say p, then we write c_p or $c(p)$ and use $A \ll_p B$ instead of $A \le c_p B$. The notation $A \asymp B$ (or $A \asymp_p B$ means that $A \ll B$ and $B \ll A$ (resp. $A \ll_p B$ and $B \ll_p A$).

By $[B]$ we denote the integer part of a real number B.

We write $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

We shall write $\overline{r} = [r] + 1/2$, for $r \ge 0$. Thus $r \ll \overline{r}$, and for $r \ge 1$ the reverse inequality holds, $\overline{r} \ll r$.

The set of natural numbers is denoted as $\mathbb{N} = \{1, 2, \dots\}$, the set of integer numbers as $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$, and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

We write $B(r) = \big\{ x \in \mathbb{R}^d : \ |x|_\infty \le r \big\}$ and $|x|_\infty = |x_1| \vee \dots \vee |x_d|$, $|x|_1 = \sum_{1 \le j \le d} |x_j|$.

The region of integration is specified only in cases when it differs from the whole space. Hence, $\int_{\mathbb{R}} = \int$ and $\int_{\mathbb{R}^d} = \int$.

We use the notation

$$\mathrm{e}\big\{ t \big\} = \exp\big\{ i\, t \big\}, \qquad i = \sqrt{-1}, \tag{1.33}$$

which differs from $\mathrm{e}\big\{ t \big\} = \exp\big\{ 2\,\pi\, i\, t \big\}$ by an unessential factor 2π. Since we consider polynomials with arbitrary real coefficients, the normalization as in (1.33) is more natural.

The Fourier–Stieltjes transforms of functions, say $F : \mathbb{R} \to \mathbb{R}$, of bounded variation are denoted as

$$\widehat{F}(t) = \int \mathrm{e}\big\{ t\, x \big\}\, dF(x).$$

Throughout $\mathbf{I}\big\{ A \big\}$ denotes the indicator function of event A, that is, $\mathbf{I}\big\{ A \big\} = 1$ if A occurs, and $\mathbf{I}\big\{ A \big\} = 0$ otherwise.

For $s > 0$, define the function

$$\mathcal{M}(t; s) = \big(|t|\, s \big)^{-1}\, \mathbf{I}\big\{ |t| \le s^{-1/2} \big\} + |t|\, \mathbf{I}\big\{ |t| > s^{-1/2} \big\}. \tag{1.34}$$

Obviously $|t\, s|^{-1} + |t| \le 2\, \mathcal{M}(t; s)$ and $\mathcal{M}(t; s) \le |t\, s|^{-1} + |t|$. Therefore, for positive absolute constants $c_1, c_2 > 0$ and arbitrary $\varepsilon > 0$, we have

$$\mathcal{M}(t; s) \ll \mathcal{M}(c_1\, t;\, c_2\, s) \ll \mathcal{M}(t; s), \qquad \mathcal{M}(\varepsilon t; s) \ll (1 \vee \varepsilon^{-1})\, \mathcal{M}(t; s). \tag{1.35}$$

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$, we write $\alpha! = \alpha_1! \dots \alpha_d!$. Partial derivatives of functions $f : \mathbb{R}^d \to \mathbb{C}$ we denote by

$$\partial^\alpha f(x) = \partial_x^\alpha f(x) = \frac{\partial^{\alpha_1}}{(\partial x_1)^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{(\partial x_d)^{\alpha_d}}\, f(x).$$

Sometimes we shall use notation related to Fréchet derivatives: for $\alpha = (\alpha_1, \dots, \alpha_n)$, we write

$$f^{(|\alpha|_1)}(x)h_1^{\alpha_1}\dots h_n^{\alpha_n} = \partial_{t_1}^{\alpha_1}\dots\partial_{t_n}^{\alpha_n} f(x+t_1h_1+\dots+t_nh_n)\Big|_{t_1=\dots=t_n=0}. \tag{1.36}$$

For a real-valued polynomial

$$\mathbb{Q}(x) = \sum_{j=0}^{p} \mathbb{Q}_j(x), \qquad \mathbb{Q}_j(x) = \sum_{|\alpha|=j} a_\alpha x^\alpha, \qquad \text{for } \ x\in\mathbb{R}^d, \tag{1.38}$$

by $\mathbb{Q}_j$ we denote its j-th homogeneous part. With $\mathbb{Q}_j$ we associate a j-linear symmetric form $\mathbb{Q}_j[x_1,\dots,x_j]$ such that $\mathbb{Q}_j(x) = \mathbb{Q}_j[x,\dots,x]$, for $x\in\mathbb{R}^d$. Using Fréchet derivatives we have

$$j!\,\mathbb{Q}_j[x,\dots,x] = \mathbb{Q}^{(j)}(0)x\dots x. \tag{1.38}$$

We write

$$\|\mathbb{Q}\| = \sum_{j=0}^{p}\|\mathbb{Q}_j\|, \quad \|\mathbb{Q}_j\| = \sup_{|x|\le 1}\big|\mathbb{Q}_j(x)\big|, \quad \|\mathbb{Q}_j\| \asymp_d \sup_{|x_1|\le 1,\dots,|x_j|\le 1}\big|\mathbb{Q}_j[x_1,\dots,x_j]\big|. \tag{1.39}$$

By $\deg\mathbb{Q} = p$ we denote the degree of the polynomial (1.38). We write $\deg_x \mathbb{Q} = p$ for the degree in the variable x in cases when a polynomial $\mathbb{Q}$ depends on several variables.

Define the symmetrization $\widetilde{\mu}$ of a measure μ on $\mathbb{R}^d$ by $\widetilde{\mu}(C) = \int\mu(C+x)\,\mu(dx)$, for $C\in\mathcal{B}^d$. In particular, $\widetilde{\mu}$ satisfies

$$\int f(x)\,\widetilde{\mu}(dx) = \iint f(x-y)\,\mu(dx)\,\mu(dy), \tag{1.40}$$

for any integrable function $f:\mathbb{R}^d\to\mathbb{C}$.

2. The Main Result

For the formulation of the main result—Theorem 2.1 we need some simple notions related to measures on $\mathbb{R}^d$. We shall consider *signed* measures, that is, σ-additive set functions $\mu:\mathcal{B}^d\to\mathbb{R}$, where $\mathcal{B}^d$ denotes the σ-algebra of Borel subsets of $\mathbb{R}^d$. If a signed measure μ is non-negative and normalized, that is, $\mu(\mathbb{R}^d)=1$ and $\mu(C)\ge 0$, for $C\in\mathcal{B}^d$, then we shall call μ measure. We shall write $\int f(x)\,\mu(dx)$ for the (Lebesgue) integral over $\mathbb{R}^d$ of a measurable function $f:\mathbb{R}^d\to\mathbb{C}$ with respect to a signed measure μ, and denote as usual by $\mu*\nu(C) = \int\mu(C-x)\,\nu(dx)$, for $C\in\mathcal{B}^d$, the convolution of the signed measures μ and ν. Equivalently, $\mu*\nu$ is defined as the signed measure such that

$$\int f(x)\,\mu*\nu(dx) = \iint f(x+y)\,\mu(dx)\,\nu(dy), \tag{2.1}$$

for any integrable function f. We say that a set A supports a measure μ if $\mu(A)=1$. The support $\operatorname{supp}\mu$ of μ is defined as the smallest closed set A such that $\mu(A)=1$.

Let $w_x \in \mathbb{R}$, $x \in \mathbb{Z}^d$, be a system of weights. Using signed measures, weighted trigonometric sums, say,

$$\sum_{x \in \mathbb{Z}^d} \mathrm{e}\big\{ t\, \mathbb{Q}(x) \big\}\, w_x = \int \mathrm{e}\big\{ t\, \mathbb{Q}(x) \big\}\, \theta(dx), \qquad \mathrm{e}\{v\} = \exp\{iv\},$$

can be represented as an integral with respect to the signed measure θ supported by the lattice $\mathbb{Z}^d$ such that $\theta\big(\{x\}\big) = w_x$, for $x \in \mathbb{Z}^d$.

We define the *uniform lattice measure* $\mu(\,\cdot\,; r)$ supported by the lattice points in the cube $B(r) = \big\{ x \in \mathbb{R}^d : \ |x|_\infty \le r \big\}$ by

$$\mu(C; r) = \frac{\mathrm{vol}_{\mathbb{Z}}\big(C \cap B(r)\big)}{\mathrm{vol}_{\mathbb{Z}}\, B(r)}, \qquad \text{for} \quad C \in \mathcal{B}^d. \tag{2.2}$$

In other words, the measure $\mu(\,\cdot\,; r)$ assigns equal weights $\mu(\{x\}; r) = (2\,\overline{r})^{-d}$ to lattice points in the cube $B(r)$, where $\overline{r} = [r] + 1/2$. Notice as well that $\mu(\,\cdot\,; r) = \mu(\,\cdot\,; \overline{r})$.

We define the *uniform measure* $\nu(\,\cdot\,; r)$ in $B(r)$ by

$$\nu(C; r) = \frac{\mathrm{vol}\big(C \cap B(r)\big)}{\mathrm{vol}\, B(r)}, \qquad \text{for} \quad C \in \mathcal{B}^d. \tag{2.3}$$

The measure $\nu(\cdot; r)$ is just a normalized (that, is, $\nu(\mathbb{R}^d; r) = 1$) restriction of the Lebesgue measure to the cube $B(r)$.

For a number $R > 0$ write $\Phi = \mu(\,\cdot\,; R)$ and $\Psi = \nu(\,\cdot\,; \overline{R})$, and introduce the measures

$$\mu = \Phi * \mu^{*k}(\,\cdot\,; r), \qquad \nu = \Psi * \nu^{*k}(\,\cdot\,; \overline{r}), \qquad k \in \mathbb{N}. \tag{2.4}$$

The distribution function, say G, of a mapping $\mathbb{P} : \mathbb{R}^d \to \mathbb{R}$ with respect to a signed measure, say λ, on $\mathbb{R}^d$ is defined as

$$G(s) = \lambda\big\{ x \in \mathbb{R}^d : \ \mathbb{P}(x) \le s \big\} = \int \mathbf{I}\big\{ \mathbb{P}(x) \le s \big\}\, \lambda(dx), \tag{2.5}$$

where $\mathbf{I}\big\{ A \big\}$ denotes the indicator function of event A. The function $G : \mathbb{R} \to \mathbb{R}$ is right continuous and satisfies $G(-\infty) = 0$, $G(\infty) = \lambda(\mathbb{R}^d)$. If λ is a measure (i.e., non-negative and normalized) then we have in addition: $G : \mathbb{R} \to [0, 1]$ is non-decreasing and $G(\infty) = 1$.

We shall obtain an asymptotic expansion of the distribution function, say F, of $x \mapsto \mathbb{Q}(x)$ with respect to the measure μ defined by (2.4). The first term of this expansion will be the distribution function, say F_0, of $x \mapsto \mathbb{Q}(x)$ with respect to the measure ν defined by (2.4). Other terms of this expansion will be distribution functions F_j, $j \in 2\,\mathbb{N}$, of certain signed measures related to the measure ν (or, in other words, to certain Lebesgue type volumes). A description of F_j will be given after Theorem 2.2.

We shall impose the following conditions related to trigonometric sums: the sums

$$\varphi(t;\mu,r)=\int\cdots\int \mathrm{e}\big\{\,t\,\mathbb{Q}_p[x_1,\dots,x_p]\big\}\,\widetilde{\mu}(dx_1;r)\dots\widetilde{\mu}(dx_p;r), \tag{2.6}$$

$$\varphi(t;\nu,r)=\int\cdots\int \mathrm{e}\big\{\,t\,\mathbb{Q}_p[x_1,\dots,x_p]\big\}\,\widetilde{\nu}(dx_1;\overline{r})\dots\widetilde{\nu}(dx_p;\overline{r}), \tag{2.7}$$

satisfy

$$\varphi(t;\mu,r)\le\Lambda\,\mathcal{M}^{\varkappa_0}\big(t;r^p\big), \tag{2.8}$$

$$\varphi(t;\nu,r)\le\Lambda\,\big|\,t\,r^p\big|^{-\varkappa_0}, \tag{2.9}$$

for $t\in\mathbb{R}$, with some $\varkappa_0>0$ and $\Lambda\ge 1$ (symmetrization $\widetilde{\mu}$ of μ is defined by ((1.40); the function $\mathcal{M}$ is defined by (1.34)). In the quadratic case $p=2$ the condition (2.8) is satisfied with a sufficiently large $\varkappa_0$ such that Theorem 2.1 yields optimal results concerning the estimation of lattice points remainders for ellipsoids and Oppenheim's (1929, 1931) conjecture, for $d\ge 9$, assuming that the matrix of the quadratic form $\mathbb{Q}_2[x,x]$ is invertible. In this case (2.8) and (2.9) are implied (see BG 1997c) by the well-known double large sieve bounds. We shall prove a satisfactory variant of (2.8) in the case $p>2$ for polynomials with the leading part (1.4) assuming that the dimension d is sufficiently large. However, in general the problem of finding optimal bounds of type (2.8) remains open. The condition (2.9) is a continuous counterpart of (2.8) and usually is fulfilled whenever (2.8) holds.

Let $\mathbb{P}$ be a polynomial such that $\deg\mathbb{P}<p$. Introduce the function (cf. (1.24))

$$\varphi_{\mathbb{P}}(t)=\Big|\int \mathrm{e}\big\{\,t\mathbb{Q}(x)+t\mathbb{P}(x)\,\big\}\,\mu^{*(p+1)}(dx;r)\Big|. \tag{2.10}$$

Recall that

$$\gamma\big(r^p,T\big)\overset{\mathrm{def}}{=}\sup_{\deg\mathbb{P}<p}\ \sup_{r^{-p/2}\le t\le T}\varphi_{\mathbb{P}}(t) \tag{2.11}$$

(see (1.25)), where $T\ge 1$. For proving remainders of order $\mathcal{O}(r^{-p})$ only, it will suffice to choose $T=1$ and to use the obvious estimate $\gamma\le 1$.

Write

$$\Lambda_1\overset{\mathrm{def}}{=}\frac{1+q}{r}\Big(\frac{R}{r}\Big)^{p-1},\qquad q\overset{\mathrm{def}}{=}\sum_{j=1}^{p}\frac{\|Q_j\|}{r^{p-j}}. \tag{2.12}$$

Our main result is the following theorem.

Theorem 2.1. *Assume that $\mathbb{Q}=\mathbb{Q}_p+\cdots+\mathbb{Q}_0$ is a polynomial of degree $p\ge 2$. Let $n\ge 0$ denote a non-negative number and let $n_0\in\mathbb{N}_0$ be the largest even integer such that $n_0<n$. Consider the following asymptotic expansion*

$$F(s)=F_0(s)+F_2(s)+\cdots+F_{n_0-2}(s)+F_{n_0}(s)+\mathcal{R}_n, \tag{2.13}$$

where F and F_0 are the distribution functions of measures μ and ν defined by (2.4),

$$F(s)=\mu\big\{x\in\mathbb{R}^d:\,\mathbb{Q}(x)\le s\big\},\qquad F_0(s)=\nu\big\{x\in\mathbb{R}^d:\,\mathbb{Q}(x)\le s\big\}, \tag{2.14}$$

and expansion terms $F_j(s)$ are defined by (2.23) below.

Assume that the conditions (2.8) and (2.9) hold with some sufficiently large $\varkappa_0$, that is, such that

$$2^{-p+1}\,\varkappa_0 > 4, \qquad 2^{-p+1}\,\varkappa_0 > n. \tag{2.15}$$

Suppose that the number k of the copies of the measure $\mu(\,\cdot\,;r)$ in the definition of the measure $\mu = \Phi * \mu^{*k}(\cdot\,;r)$ is sufficiently large, that is,

$$k \geq n+2, \qquad \big[k/(pn-n+1)\big] \geq p+1. \tag{2.16}$$

Furthermore, assume that the sizes r and R of the measures $\mu(\,\cdot\,;r)$ and Φ satisfy $R \geq r \geq 1$, and let $T \geq 1$. Then the remainder term $\mathcal{R}_n$ in the expansion (2.13) allows the following bound

$$|\mathcal{R}_n| \ll_{d,k,p,n,\varkappa_0,\varepsilon} \Lambda^\alpha \left(\Lambda_1^n + r^{-p}\,T^{-1} + r^{-p}\,T^{\varepsilon}\,\gamma^{1-\varepsilon-4/(\alpha\varkappa_0)}\,(r^p, T)\right), \tag{2.17}$$

for any $\varepsilon > 0$, where $\alpha = 2^{-p+1}$.

Notice, that the estimate (2.17) is uniform in s. For example, in the special case $n = p$ and $R = 2r$ the bound (2.17) yields $|\mathcal{R}_p| = \mathcal{O}(r^{-p})$ with an implied constant which can depend on $\mathbb{Q}, k, d, \varkappa_0$, provided that the number $\varkappa_0$ in conditions (2.8) and (2.9) satisfies $\varkappa_0 > 2^{p-1}\,(4 \vee p)$ and that the number k of copies of the measure $\mu(\,\cdot\,;r)$ is sufficiently large, that is, $k \geq p^3 + 1$. To see this, it suffices to choose $T = 1$ and to estimate $\gamma(r^{-p}, T) \leq 1$. In order to obtain better bounds of type $o(r^{-p})$ one has to prove that $\gamma(r^{-p}, T) \to 0$ with some $T = T(r) \to \infty$, as $r \to \infty$, see Section 9 for results in this direction.

The measure Φ (or its support $B(R) \cap \mathbb{Z}^d$) represents the main box of size R from which lattice points are taken. The convolution of Φ with $\mu^{*k}(\cdot\,;r)$ is a somewhat "smoother" lattice measure than Φ. Note though that the weights assigned by μ to the lattice points near the boundary of the box $B(R)$ become smaller when the points approach the boundary of the box $B(R + kr)$. The weights assigned to lattice points in $B(R - kr)$ remain unchanged. Later on we shall choose the size r of the smoothing measure $\mu(\cdot\,;r)$ smaller in comparison with R, that is, we shall assume that $R \geq ckr$ with a sufficiently large constant c. This smoothing near the boundary simplifies the derivation of approximations and helps to avoid extra logarithmic factors in the estimates of errors. The corresponding measure ν is the continuous counterpart of μ with the dominating counting measure on $\mathbb{Z}^d$ replaced by the Lebesgue measure on $\mathbb{R}^d$. Theorem 2.1 allows a generalization. The measure Φ can be replaced by an arbitrary uniform lattice measure with support in a cube of size R, see Theorem 2.2 below. Theorem 2.1 is a partial case of Theorem 2.2. In order to formulate that result, we extend our notation.

We shall denote by $\pi = \nu(\cdot\,;1/2)$ the restriction to the cube $B(1/2)$ of the Lebesgue measure in $\mathbb{R}^d$. The measure π has the density $d\pi/dx = \mathbf{I}\big\{\,|x|_\infty \leq 1/2\,\big\}$ with respect to the Lebesgue measure in $\mathbb{R}^d$, so that $\pi(dx) = \mathbf{I}\big\{ |x|_\infty \le 1/2 \big\}\, dx$. The measure $\nu(\,\cdot\,;\overline{r})$ has the density $(2\overline{r})^{-d}\,\mathbf{I}\big\{ |x|_\infty \le \overline{r} \big\}$. Notice as well that $\nu(\,\cdot\,;\overline{r}) = \pi * \mu(\,\cdot\,;\overline{r})$.

Henceforth Φ will denote a measure on $\mathbb{R}^d$ such that $\Phi(A) = 1$, for some subset $A \subset B(R) \cap \mathbb{Z}^d$ and

$$\Phi\big(\{x\}\big) = 1/\operatorname{card} A, \qquad \text{for all } x \in A. \tag{2.18}$$

We do not impose restrictions on the structure of the support $A = \operatorname{supp}\Phi$ of Φ except that $A \subset B(R) \cap \mathbb{Z}^d$ and $A \ne \varnothing$. Write $\Psi = \Phi * \pi$. It is easy to see that Ψ has the density

$$\frac{d\Psi}{dx} = \frac{1}{\operatorname{card} A} \sum_{y\in A} \mathbf{I}\big\{ |x-y|_\infty \le 1/2 \big\}. \tag{2.19}$$

We define measures μ and ν as in (2.4), and denote distribution functions of the mapping $x \mapsto \mathbb{Q}(x)$ with respect to μ and ν again as F and F_0 respectively. Notice that the measure ν has the density

$$D(x) \overset{\text{def}}{=} \frac{d\nu}{dx} = \frac{d\Psi}{dx} * \Big(\frac{d\nu(\,\cdot\,;\overline{r})}{dx} \Big)^{*k}, \tag{2.20}$$

where $f * g$ denotes the convolution of functions f and g,

$$f * g(x) = \int f(x-y)\, g(y)\, dy.$$

Using Fourier transform, it is easy to verify that the density D admits continuous bounded partial derivatives $|\partial^\alpha D(x)| \ll_{d,k} \overline{r}^{\,-d-|\alpha|_1}$, for $|\alpha|_\infty \le k-2$ (see Lemma 7.1 in BG 1997c).

Theorem 2.2. *Theorem* 2.1 *holds with* Φ *and* Ψ *defined by* (2.18) *and* (2.19) *respectively.*

Let us now define the functions F_j, for $j \in 2\mathbb{N}$. Let $\eta = (\eta_1, \dots, \eta_m)$ denote a multi-index with entries $\eta_1, \dots, \eta_m \in \mathbb{N}$. Write $\sum\limits_{\eta:\ |\eta|_1 = j}^{**}$ for the sum which extends over all possible representations of the *even* number j as a sum $j = \eta_1 + \dots + \eta_m$ of *even* $\eta_1, \dots, \eta_m \ge 2$, for all possible $m \ge 1$. For example, for $j = 6$, we have

$$6 = 6, \quad 6 = 4+2, \quad 6 = 2+4, \quad 6 = 2+2+2.$$

Introduce the functions

$$D_j(x) = \sum_{\eta:\ |\eta|_1 = j}^{**} D_{j\eta}(x) \tag{2.21}$$

with

$$D_{j\eta}(x) = \frac{(-1)^m}{\eta!} \int \cdots \int D^{(j)}(x) u_1^{\eta_1} \dots u_m^{\eta_m} \prod_{l=1}^m \pi^{*(k+1)}(du_l), \tag{2.22}$$

where the density D is defined by (2.20), and where we use the notation (1.36) for the

Fréchet derivatives. For example, we have

$$D_2(x) = -\frac{1}{2} \int D''(x)u^2\, \pi^{*(k+1)}(du),$$

and $D_4(x) = D_{44}(x) + D_{422}(x)$ with

$$\begin{aligned} D_{44}(x) &= -\frac{1}{24} \int D^{(4)}(x)u^4\, \pi^{*(k+1)}(du), \\ D_{422}(x) &= \frac{1}{4} \iint D^{(4)}(x)u_1^2u_2^2\, \pi^{*(k+1)}(du_1)\, \pi^{*(k+1)}(du_2). \end{aligned}$$

Let ν_j denote the signed measure on $\mathbb{R}^d$ with density D_j. We define the function F_j, for $j \in 2\,\mathbb{N}$, as the distribution function of $x \mapsto \mathbb{Q}(x)$ with respect to the signed measure ν_j, that is,

$$F_j(s) = \nu_j\big\{ x \in \mathbb{R}^d : \ \mathbb{Q}(x) \le s \big\} = \int \mathbf{I}\big\{ \mathbb{Q}(x) \le s \big\}\, D_j(x)\, dx. \tag{2.23}$$

The function $F_j : \mathbb{R} \to \mathbb{R}$ is a function of bounded variation. Its properties are described in Lemma 2.3.

In the remaining part of the section we shall describe some properties of functions F_j and prove Theorem 2.2. We shall use the following approximate (see, e.g., relation (8.4) in BG 1997) and precise (see, e.g., Chung 1974) formulas for the Fourier–Stieltjes inversion. For any $T > 0$ and any distribution function F of a (normalized non-negative) measure with the Fourier–Stieltjes transform

$$\widehat{F}(t) = \int \mathrm{e}\big\{ t x \big\}\, dF(x), \qquad t \in \mathbb{R},$$

we have

$$F(x) = \frac{1}{2} + \frac{i}{2\pi}\, \mathrm{V.P.} \int_{-T}^{T} \mathrm{e}\big\{ -x t \big\}\, \widehat{F}(t)\, \frac{dt}{t} + \widetilde{\mathcal{R}} \tag{2.24}$$

with remainder term $\widetilde{\mathcal{R}}$ such that

$$|\widetilde{\mathcal{R}}| \le \frac{1}{T} \int_{-T}^{T} |\widehat{F}(t)|\, dt.$$

Here $\mathrm{V.P.} \int f(t)\, dt = \lim\limits_{\tau \to 0} \int_{|t| > \tau} f(t)\, dt$ denotes the Principal Value of the integral. Furthermore, for any function $F : \mathbb{R} \to \mathbb{R}$ of bounded variation such that $F(-\infty) = 0$ and $2\,F(x) = F(x+) + F(x-)$, for all $x \in \mathbb{R}$, we have

$$F(x) = \frac{1}{2} F(\infty) + \frac{i}{2\pi} \lim_{M \to \infty} \mathrm{V.P.} \int_{|t| \le M} \mathrm{e}\{-x t\}\, \widehat{F}(t)\, \frac{dt}{t}. \tag{2.25}$$

The formula is well-known for distribution functions. To functions of bounded variation it extends by linearity arguments.

The next Lemma 2.3 establishes properties of the expansion terms F_j used in the proof of Theorems 2.1 and 2.2.

Lemma 2.3. *Assume that the condition* (2.9) *is fulfilled and that* $R \geq r \geq 1$,

$$n \geq 0, \quad k \geq n+2, \quad \left[k/(pn-n+1)\right] \geq p+1, \quad n < \varkappa_0\, \alpha, \quad \alpha \stackrel{\text{def}}{=} 2^{-p+1}. \tag{2.26}$$

Then the functions $s \mapsto F_j(s)$, $0 \leq j < n$, *are continuous functions of bounded variation and*

$$\sup_s \left| F_j(s) \right| \ll_{d,k,p,n,\varkappa_0} \Lambda^\alpha \Lambda_1^j, \tag{2.27}$$

$$\int\limits_{|t| \geq r^{-p/2}} \left| \widehat{F}_j(t) \right| \frac{dt}{|t|} \ll_{d,k,p,n,\varkappa_0} \Lambda^\alpha \Lambda_1^n, \tag{2.28}$$

with Λ_1 *defined by* (2.12). *Furthermore,*

$$F_0(-\infty) = 0, \quad F_0(\infty) = 1, \quad F_j(-\infty) = F_j(\infty) = 0, \quad j \geq 1. \tag{2.29}$$

Proof. The functions F_j are continuous functions of bounded variation since they are distribution functions of the signed measures ν_j (see (2.23)) and the measures ν_j have densities $D_j : \mathbb{R}^d \to \mathbb{R}$ (see (2.21)) which by Lemma 7.1 in BG (1997c) are continuous functions with compact supports, for $j < n \leq k-2$. It is clear as well that (2.29) holds. Thus it remains to establish the bounds (2.27) and (2.28).

Let us prove (2.27). In the case $j = 0$ the bound (2.27) is implied by the obvious inequality $0 \leq F_0(s) \leq 1$. Therefore let us assume that $j > 0$. Estimating $\Lambda_0^j \Lambda_{1,j} \ll_n \Lambda_1^j$ in (4.15) of Lemma 4.1, we have

$$\left| \widehat{F}_j(t) \right| \ll_{d,k,p,j} \psi_0(t)\, \Lambda_1^j\, \Lambda_{2,j} \quad \text{with } \Lambda_{2,j} = |t\, r^p|^{\theta(j)} + |t\, r^p|^j \tag{2.30}$$

with $\theta(j) > 0$ defined by (4.10). We can estimate $\psi_0(t)$ (defined by (4.13)) applying the symmetrization Lemma 7.1 replacing in that lemma the measures μ_j by $\nu(\cdot;\overline{r})$, for $1 \leq j \leq p+1$, and the measure ν by $\Psi * \nu^{*(\kappa-p-1)}(\cdot;\overline{r})$. We obtain

$$\left(\psi_0(t)\right)^{1/\alpha} \leq \int \cdots \int \mathrm{e}\left\{ p!\, t\, \mathbb{Q}_p[x_1, \dots, x_p] \right\} \prod_{j=1}^p \widetilde{\nu}(dx_j;\overline{r}) = \varphi(p!\, t;\, \nu, r). \tag{2.31}$$

Combining the obvious bound $\varphi(p!\, t;\, \nu, r) \leq 1$ with the condition (2.9) and using $\Lambda \geq 1$, we have

$$\varphi(t;\nu,r) \ll_{p,\kappa_0} \Lambda\, \left(1 \wedge \left| t\, r^p \right|^{-\varkappa_0}\right). \tag{2.32}$$

Applying now the inversion formula (2.25), using (2.29), (2.30), (2.31) and (2.32), we get

$$\left| F_j(s) \right| \ll \int \left| \widehat{F}_j(t) \right| \frac{dt}{|t|} \ll_{d,k,p,n} \Lambda^\alpha \Lambda_1^j\, J \tag{2.33}$$

with

$$J = \int \Big(|t\, r^p|^{\theta(j)} + |t\, r^p|^j \Big) \, \Big(1 \wedge \big| t\, r^p \big|^{-\varkappa_0\, \alpha} \Big) \, \frac{dt}{|t|}$$
$$= 2 \int_0^\infty (\tau^{\theta(j)} + \tau^j) \, \Big(1 \wedge \tau^{-\varkappa_0\, \alpha} \Big) \, \frac{d\tau}{\tau}. \tag{2.34}$$

The bound (2.34) proves (2.27) since $\theta(j) \le j \le n$, the number $\theta(j)$ is positive for positive j, and the integral J converges due to the assumption $j < n < \varkappa_0\, \alpha$.

The proof of (2.28) is similar to that of (2.27). Arguing as in (2.30)–(2.33) we have

$$\int_{|t| \ge r^{-p/2}} \big| \widehat{F}_j(t) \big| \, \frac{dt}{|t|} \ll_{d,k,p,n,\varkappa_0} \Lambda^\alpha \, \Lambda_1^j \, J \ll_n \Lambda^\alpha \, \Lambda_1^j \, r^{p(j - \alpha \varkappa_0)/2}, \tag{2.35}$$

where now

$$J = \int_{r^{p/2}}^\infty (\tau^{\theta(j)} + \tau^j) \, \Big(1 \wedge \tau^{-\varkappa_0\, \alpha} \Big) \, \frac{d\tau}{\tau}$$
$$\ll \int_{r^{p/2}}^\infty \tau^{j - \varkappa_0\, \alpha} \, \frac{d\tau}{\tau} \ll r^{p(j - \varkappa_0\, \alpha)/2} \tag{2.36}$$

since $r \ge 1$. Writing $\Lambda_2 = 1 + q \left(\frac{R}{r} \right)^{p-1}$, we have $\Lambda_1 = r^{-1} \Lambda_2$. Using $\Lambda_2 \ge 1$, we obtain

$$\Lambda_1^j \, r^{p(j - \alpha \varkappa_0)/2} = r^{-j + p(j - \alpha \varkappa_0)/2} \, \Lambda_2^j \ll r^{-n} \, \Lambda_2^n = \Lambda_1^n \tag{2.37}$$

since $r \ge 1$, $\alpha\, \varkappa_0 > n$, $j < n$, $p \ge 2$ and therefore

$$-j + p\,(j - \alpha\, \varkappa_0)/2 = j\,(p/2 - 1) - p\, \alpha\, \varkappa_0/2 \le n\,(p/2 - 1) - p\, n/2 = -n.$$

The bounds (2.35) and (2.37) imply (2.28). $\square$

Let us return to the proof of Theorems 2.1 and 2.2. Theorem 2.1 is implied by Theorem 2.2 and we have to prove Theorem 2.2 only.

Proof of Theorem 2.2. The expansion (2.13) yields

$$|\mathcal{R}_n| = \Big| F - \sum_{j \in 2\,\mathbb{N}_0,\ j < p} F_j \Big|. \tag{2.38}$$

Using (2.38), representing F by the approximate Fourier–Stieltjes inversion (2.24) and the functions F_j by the inversion formula (2.25) and splitting the intervals of integration, we obtain

$$|\mathcal{R}_n| \ll \frac{1}{T} \int_{-T}^{T} \big| \widehat{F}(t) \big| \, dt + \int_{-T}^{T} \big| \widehat{\mathcal{R}}_n(t) \big| \, \frac{dt}{|t|} + \sum_{j \in 2\,\mathbb{N}_0,\ j < n} \ \int_{|t| \ge T} \big| \widehat{F}_j(t) \big| \, \frac{dt}{|t|} \tag{2.39}$$

with

$$\widehat{\mathcal{R}}_n(t) = \widehat{F}(t) - \sum_{j\in 2\,\mathbb{N}_0,\ j<p} \widehat{F}_j(t).$$

Splitting the integrals in (2.39) again and using the obvious estimate

$$\frac{1}{T}\int\limits_{r^{-p/2}\le|t|\le T} \big|\,\widehat{F}(t)\,\big|\,dt \le \int\limits_{r^{-p/2}\le|t|\le T} \big|\,\widehat{F}(t)\,\big|\,\frac{dt}{|t|}\,,$$

we have

$$|\mathcal{R}_n| \ll I_1 + I_2 + I_3 + \sum_{j\in 2\,\mathbb{N}_0,\ j<p} I_{j+4} \tag{2.40}$$

with

$$\begin{aligned}
I_1 &= \int\limits_{|t|\le r^{-p/2}} \big|\,\widehat{\mathcal{R}}_n(t)\,\big|\,\frac{dt}{|t|}\,,\\
I_2 &= \frac{1}{T}\int\limits_{|t|\le r^{-p/2}} \big|\,\widehat{F}(t)\,\big|\,dt,\\
I_3 &= \int\limits_{r^{-p/2}\le|t|\le T} \big|\,\widehat{F}(t)\,\big|\,\frac{dt}{|t|}\,,\\
I_{j+4} &= \int\limits_{|t|\ge r^{-p/2}} \big|\,\widehat{F}_j(t)\,\big|\,\frac{dt}{|t|}\,, \qquad\qquad j\in 2\,\mathbb{N}_0,\ j<n.
\end{aligned}$$

The estimate (2.40) shows that in order to prove the theorem (i.e., the bound (2.17)) it suffices to prove that

$$I_1,\, I_{j+4} \ll_{d,k,p,n,\varkappa_0} \Lambda^{\alpha}\,\Lambda_1^{n}, \qquad\qquad j\in 2\,\mathbb{N}_0,\ j<n, \tag{2.41}$$

$$I_2 \ll_{d,p,\varkappa_0} \Lambda^{\alpha}\, r^{-p}\, T^{-1}, \tag{2.42}$$

$$I_3 \ll_{d,p,\varkappa_0,\varepsilon} \Lambda^{\alpha}\, r^{-p}\, T^{\varepsilon}\, \gamma^{1-\varepsilon-4/(\alpha\varkappa_0)}\,(r^p, T), \qquad \text{for any } \varepsilon>0, \tag{2.43}$$

Let us estimate I_3. Changing variables, we have

$$\widehat{F}(t) = \int \mathrm{e}\big\{\,t\,\mathbb{Q}(x)\,\big\}\;\mu(dx).$$

Let us split

$$\mu = \mu^{*(p+1)}(\,\cdot\,;r) * \chi \quad \text{with} \quad \chi = \mu^{*(k-p-1)}(\,\cdot\,;r) * \Phi$$

and notice that $\mathbb{Q}(x+z) = \mathbb{Q}(x) + \mathbb{P}(x)$ with a polynomial $\mathbb{P}$ such that $\deg\mathbb{P} < p$. Naturally, coefficients of $\mathbb{P}$ can depend on z. For any $A\le B$ we have

$$\begin{aligned}
\int\limits_A^B \big|\,\widehat{F}(t)\,\big|\,\frac{dt}{|t|} &\le \int\int\limits_A^B \Big|\int \mathrm{e}\big\{\,t\,\mathbb{Q}(x+y)\,\big\}\;\mu^{*(p+1)}(dx;r)\Big|\,\frac{dt}{|t|}\,\chi(dy)\\
&\le \sup_{\deg\mathbb{P}<p}\int\limits_A^B \big|\,\varphi_{\mathbb{P}}(t)\,\big|\,\frac{dt}{|t|}\,,
\end{aligned} \tag{2.44}$$

where $\varphi_{\mathbb{P}}$ is given by (2.10),

$$\varphi_{\mathbb{P}}(t) = \Big| \int \mathrm{e}\big\{ t\,\mathbb{Q}(x) + t\,\mathbb{P}(x) \big\}\, \mu^{*(p+1)}(dx;r) \Big|, \qquad \deg \mathbb{P} < p. \tag{2.45}$$

Using (2.44), we obtain

$$I_3 \le I_3', \qquad\qquad I_3' \overset{\text{def}}{=} \sup_{\deg \mathbb{P}<p} \int_{r^{-p/2} \le |t| \le T} \varphi_{\mathbb{P}}(t)\, \frac{dt}{|t|}\,. \tag{2.46}$$

By Theorem 5.1 the function $\varphi_{\mathbb{P}}$ satisfies the following inequalities

$$\varphi_{\mathbb{P}}(t)\, \varphi_{\mathbb{P}}(t+\tau) \ll_{d,p} \varphi^{\alpha}(p!\, 2^{-p}\, \tau; \mu, 2\,[r]), \tag{2.47}$$

$$\varphi_{\mathbb{P}}(t) \ll_{d,p} \varphi^{\alpha}(p!\, 2^{-p}\, t; \mu, 2\,[r]), \qquad t,\ \tau \in \mathbb{R}. \tag{2.48}$$

We can estimate the function $\varphi(t;\mu,r)$ (defined by (2.6)) using the condition (2.8). Then (2.47) and (2.48) imply

$$\varphi_{\mathbb{P}}(t)\, \varphi_{\mathbb{P}}(t+\tau) \ll_{d,p} \Lambda^{\alpha}\, \mathcal{M}^{\varkappa_0 \alpha}(\tau; r^p), \tag{2.49}$$

$$\varphi_{\mathbb{P}}(t) \ll_{d,p} \Lambda^{\alpha}\, \mathcal{M}^{\varkappa_0 \alpha}(t; r^p), \qquad t,\ \tau \in \mathbb{R}. \tag{2.50}$$

with the function $\mathcal{M}(t;s)$ introduced by (1.34). The inequality (2.49) allows to apply Corollary 6.2 of Theorem 6.1. Choosing in this corollary

$$\varphi = \varphi_{\mathbb{P}}, \quad \Lambda = c(p,d)\, \Lambda^{\alpha}, \quad \varkappa = \varkappa_0\, \alpha, \quad s = r^p,$$

we obtain

$$I_3' \ll_{d,p,\varkappa_0,\varepsilon} \Lambda^{\alpha}\, r^{-p}\, T^{\varepsilon}\, \gamma^{1-\varepsilon-4/(\alpha\varkappa_0)}\big(r^p, T\big), \tag{2.51}$$

and (2.46) yields the desired bound (2.43) for I_3.

Let us estimate I_2. Similarly to the estimation of I_3 we obtain

$$I_2 \le T^{-1}\, I_2', \qquad I_2' \overset{\text{def}}{=} \sup_{\deg \mathbb{P}<p} \int_{|t| \le r^{-p/2}} \varphi_{\mathbb{P}}(t)\, dt. \tag{2.52}$$

The bound (2.50) for $\varphi_{\mathbb{P}}$ and the definition (1.34) of the function $\mathcal{M}(t;s)$ together with the inequality $\varphi_{\mathbb{P}} \le 1$ yield

$$\varphi_{\mathbb{P}}(t) \ll_{d,p} \Lambda^{\alpha}\big(1 \wedge |r^p\, t|^{-\alpha\varkappa_0}\big), \qquad \text{for } |t| \le r^{-p/2}. \tag{2.53}$$

Hence, due to the assumption $\alpha\,\varkappa_0 > 4$, we have

$$I_2' \ll_{d,p} \Lambda^{\alpha} \int_{\mathbb{R}} \big(1 \wedge |r^p\, t|^{-\alpha\varkappa_0}\big)\, dt = \Lambda^{\alpha}\, r^{-p} \int \big(1 \wedge |t|^{-\alpha\varkappa_0}\big)\, dt \ll \Lambda^{\alpha}\, r^{-p},$$

which together with (2.52) yields the bound (2.42) for I_2.

Let us prove (2.41) for the integral I_1. Estimating $|\widehat{\mathcal{R}}_n(t)|$ by (4.14) of Lemma 4.1, using $\Lambda_0\,\Lambda_{1,n} \ll_n \Lambda_1^n$ and taking into account Corollary 4.2, we have

$$I_1 \ll_{d,k,p,n} \Lambda_1^n\, I_1' \tag{2.54}$$

with

$$I_1' = \int\limits_{|t|\le r^{-p/2}} \psi(t)\,\Lambda_{2,n}\,\frac{dt}{|t|}, \qquad \Lambda_{2,n} = |t\,r^p|^{\theta(n)} + |t\,r^p|^n. \tag{2.55}$$

The function $\psi(t)$ defined by (4.12) allows the estimate $\psi(t) \le \sup\limits_{\deg\mathbb{P}<p} \varphi_{\mathbb{P}}(t)$ (cf. the proof of (2.46)). Using (2.53) in order to bound $\varphi_{\mathbb{P}}(t)$, we obtain

$$I_1' \ll_{d,p} \int \big(|t\,r^p|^{\theta(n)} + |t\,r^p|^n\big)\,\big(1\wedge|t\,r^p|^{-\alpha\varkappa_0}\big)\,\frac{dt}{|t|} \ll 1. \tag{2.56}$$

similarly to the estimation of the integral J in (2.33). The bounds (2.54) and (2.56) yield (2.41) for I_1.

We conclude the proof of the Theorem 2.2 noting that the bound (2.41) for I_{j+4} is proved as the estimate (2.28) of Lemma 2.3. □

3. An application of the main result: proofs of theorems of the introduction

In cases when the polynomial $\mathbb{Q}$ has a positive definite leading part $\mathbb{Q}_p$, the choice of the measure Φ (in other words, of the "main box" from which lattice points are taken) in the definition of $\mu = \Phi * \mu^{*k}(\,\cdot\,;r)$ (see (2.4)) is immaterial as long as $\operatorname{supp}\Phi$ contains a sufficiently massive box of lattice points. Therefore throughout the section for $\mathbb{Q}$ with a positive definite leading part we assume that $\Phi = \mu(\,\cdot\,;R)$.

Recall that $r_s \ge 1$ denotes a minimal number such that the cube $B(r_s)$ contains the set E_s. The parameter ρ_r and constants q and q_0 are defined by (1.14) and (1.13) respectively.

Lemma 3.1. *Assume that $\mathbb{Q}$ has a positive definite leading part. Let $n \ge 0$. Suppose that the conditions* (2.8) *and* (2.9) *are fulfilled with some $\varkappa_0$ such that*

$$2^{-p+1}\,\varkappa_0 > 4 \vee n. \tag{3.1}$$

Then the quantities $\delta(s,\mathbb{Q})$ and $\Delta(r,\mathbb{Q})$ defined by (1.11) *satisfy*

$$\delta(s,\mathbb{Q}) \ll_{d,p,n,\varkappa_0,\varepsilon} r_s^d\,\Lambda^\alpha\,\big((1+q)^n\,r_s^{-n} + r_s^{-p}\,T^{-1} + r_s^{-p}\,T^\varepsilon\,\gamma^{1-\varepsilon-4/(\alpha\varkappa_0)}\,(r_s^p,T)\big) \tag{3.2}$$

and

$$\Delta(r,\mathbb{Q}) \ll_{d,p,n,\varkappa_0,\varepsilon} \rho_r^d\,\Lambda^\alpha\,\big((1+q_0)^n\,\rho_r^{-n} + \rho_r^{-p}\,T^{-1} + \rho_r^{-p}\,T^\varepsilon\,\gamma^{1-\varepsilon-4/(\alpha\varkappa_0)}\,(\rho_r^p,T)\big), \tag{3.3}$$

for $T \ge 1$ and any $\varepsilon > 0$, where $\alpha = 2^{-p+1}$. The constant Λ is from the conditions (2.8) *and* (2.9), *and $\gamma(\,\cdot\,,T)$ is defined by* (2.11).

Corollary 3.2. *Assume that $\mathbb{Q}$ has a positive definite leading part. Suppose that the conditions* (2.8) *and* (2.9) *are fulfilled with some $\varkappa_0$ such that $2^{-p+1}\,\varkappa_0 > 4 \vee p$. Then we have*

$$\delta(s,\mathbb{Q}) \ll_{d,p,\varkappa_0} r_s^{d-p}\,\Lambda^\alpha\,(1+q)^p, \qquad \Delta(r,\mathbb{Q}) \ll_{d,p,\varkappa_0} \rho_r^{d-p}\,\Lambda^\alpha\,(1+q_0)^p \tag{3.4}$$

with Λ and α as in Lemma 3.1.

Proof of Corollary 3.2. It suffices to substitute $T = 1$ and $n = p$ in bounds (3.2) and (3.3) and to estimate $\gamma(\,\cdot\,,T) \le 1$. $\square$

Proof of Lemma 3.1. We shall apply Theorem 2.1. Choose

$$r = r_s, \quad k = pn+3, \quad R = (pn+5)\,r_s. \tag{3.5}$$

Then the condition (2.16) is fulfilled. The measure μ restricted to the cube $B(2r_s)$ is proportional to the uniform lattice measure in this cube. Indeed, using (3.5) it is easy to see that

$$E_s - x_1 - \cdots - x_k \subset B(R) \qquad \text{provided that } |x_j|_\infty \le r.$$

Therefore, using the *shift invariance* of the measure μ *inside* the cube $B(R)$, we have

$$\begin{aligned} F(s) = \mu(E_s) &= \textstyle\int\cdots\int \mu\big(E_s - x_1 - \cdots - x_k; R\big) \displaystyle\prod_{j=1}^k \mu(dx_j; r) \\ &= \mu\big(E_s; R\big) = (2\overline{R})^{-d}\,\mathrm{vol}_{\mathbb{Z}}\, E_s. \end{aligned} \tag{3.6}$$

Similarly, for the measure ν we have

$$F_0(s) = \nu(E_s) = \nu\big(E_s; R\big) = (2\overline{R})^{-d}\,\mathrm{vol}\, E_s. \tag{3.7}$$

In particular, the density $D(x)$ of ν with respect to the Lebesgue measure inside the cube $B(2r_s)$ is a constant function. Hence, the terms $F_j(s)$, for $j \ge 1$, of the expansion (2.13) vanish since they are integrals of the derivatives of the constant function $D(x) \equiv$constant inside the cube $B(2r_s)$, see (2.21–(2.23).

As a consequence, using (3.5)–(3.7) and estimating $(2\overline{R})^d \ll_{p,n,d} r_s^d$, in our specific case we can rewrite Theorem 2.1 as the bound (3.2).

Let us prove (3.3). Write

$$\mathbb{Q}(x;r) = \mathbb{Q}_p(x) + r\,\mathbb{Q}_{p-1}(x) + \cdots + r^{p-1}\,\mathbb{Q}_1(x). \tag{3.8}$$

Let us apply (3.2) replacing $\mathbb{Q}$ by the polynomial $\mathbb{Q}(x;r)$. Since the polynomial $\mathbb{Q}(x;r)$ depends on the parameter r, the set $E_s = E_s(r)$ and the number $r_s = r_s(r)$ defined using this polynomial shall depend on r as well. It is clear however that for $s = r^p$ we have

$$r_s(r) = 1 \vee (r_*\,r) = \rho_r, \qquad r > 0. \tag{3.9}$$

Hence, the bound (3.2) yields (3.3). $\square$

In the next Lemma we show that conditions (2.8) and (2.9) are fulfilled for polynomials with the leading part (1.4).

Lemma 3.3. *Assume that the leading part $\mathbb{Q}_p$ of polynomial $\mathbb{Q}$ satisfies condition* (1.4). *Suppose that $\alpha_p = \exp\{-3p \log p\}$ or that α_p satisfy a weaker condition* (8.23). *Then we have:*

i) if the number d_0 in (1.4) *satisfies $d_0 = d$ then both conditions* (2.8) *and* (2.9) *are fulfilled with*

$$\Lambda \ll_{p,d} \big((1 \vee \lambda^*)\,(1 \vee \lambda_*^{-1})\big)^{2d}, \qquad \varkappa_0 = d\alpha_p, \tag{3.10}$$

where λ^* and λ_* are defined by (1.2)*;*

ii) if d_0 in (1.4) *is such that $d_0 < d$ then both conditions* (2.8) *and* (2.9) *are fulfilled with*

$$\Lambda \ll_{p,d} \big((1 \vee \lambda^*)\,(1 \vee \lambda_*^{-1})\big)^{d_0\, 2^{1-p}}, \qquad \varkappa_0 = 2^{-p}\, d_0\, \alpha_p. \tag{3.11}$$

Proof. Let us start with the proof of i). Since $d_0 = d$, we can apply Corollary 8.8. We obtain

$$\varphi(t; \mu, r) \ll (1 \vee \lambda_*^{-d\alpha_p})\, \mathcal{M}^{-d\alpha_p}(t;\, r^p). \tag{3.12}$$

Hence, the condition (2.8) is fulfilled with

$$\Lambda \ll_{p,d} 1 \vee \lambda_*^{-d\alpha_p}, \qquad \varkappa_0 = d\alpha_p. \tag{3.13}$$

Similarly, using Corollary 8.5 and choosing operators $\mathbb{A}_j$ in Lemma 8.1 and the number s in (8.1) in an appropriate way (we omit related simple considerations), we see that (2.9) is fulfilled with

$$\Lambda \ll_{p,d} (1 \vee \lambda^*)^{2d}\, \lambda_*^{-2d}, \qquad \varkappa_0 = d/(p-1). \tag{3.14}$$

Combining (3.13) with (3.14) shows that the choice of Λ and $\varkappa_0$ as in (3.10) guarantees fulfillment of conditions (2.8) and (2.9).

Let us prove ii) using a reduction to the case i). Suppose that a polynomial $\mathbb{Q}_p = \mathbb{B}_p(\cdot; d_0) + \mathbb{P}_p$ has the form (1.4). Writing $\mathbb{B}_p[x, \dots, x; d_0]$ and $\mathbb{P}_p[x, \dots, x]$ for the associated symmetric p-linear forms and representing

$$x = (x_1, \dots, x_d) = u + v, \quad u = (x_1, \dots, x_{d_0}, 0, \dots, 0), \quad v = (0, \dots, 0, x_{d_0+1}, \dots, x_d),$$

we have

$$\mathbb{Q}_p(x) = \mathbb{B}_p[u, \dots, u; d_0] + \sum_{j=1}^{p} c_{pj}\, \mathbb{P}_p[u^{p-j}, v^j], \tag{3.15}$$

where c_{pj} denote binomial coefficients and $v^j = v, \dots, v$ (j times). The term $\mathbb{P}_p[u, \dots, u]$ in (3.15) vanishes since by our assumption (1.4) the order $\deg_u \mathbb{P}_p < p$ of $\mathbb{P}_p$ in variable u is strictly smaller than p.

The measure $\mu(dx; r) = \mu(du; r)\, \mu(dv; r)$ is a product measure, where uniform lattice measures $\mu(du; r)$ and $\mu(dv; r)$ are defined on corresponding subspaces of $\mathbb{R}^d$ of dimensions d_0 and $d - d_0$ respectively. Hence, using $\widetilde{\mu}(dx; r) = \widetilde{\mu}(du; r)\, \widetilde{\mu}(dv; r)$, the definition (2.6) of $\varphi(t; \mu, r)$ and (3.15) we get

$$\begin{aligned}\varphi(t; \mu, r) &= \int \cdots \int \mathrm{e}\big\{ t\, \mathbb{Q}_p[x_1, \dots, x_p] \big\}\, \widetilde{\mu}(dx_1; r) \dots \widetilde{\mu}(dx_p; r) \\ &\le \int \cdots \int \big| \varphi_0(t; \mu, r) \big| \prod_{j=1}^{p} \widetilde{\mu}(dv_j; r) \end{aligned} \tag{3.16}$$

with

$$\varphi_0(t;\mu,r) \stackrel{\text{def}}{=} \int\cdots\int \mathrm{e}\{\,t\mathbb{B}_p[u_1,\dots,u_p;d_0]+f_1+\cdots+f_p\,\}\prod_{j=1}^{p}\widetilde{\mu}(du_j;r) \tag{3.17}$$

and some functions

$$f_j = f_j(u_1,\dots,u_{j-1},u_{j+1},\dots,u_p) \tag{3.18}$$

independent of the variable u_j. Naturally, f_j can depend on $v_1,\dots,v_p$ and t. Using the definition of symmetrization of measures, we have

$$\bigl|\,\varphi_0(t;\mu,r)\,\bigr| \le \int\cdots\int \bigl|\,\varphi_1(t;\mu,r)\,\bigr|\prod_{j=1}^{p}\mu(dw_j;r) \tag{3.19}$$

with

$$\varphi_1(t;\mu,r) \stackrel{\text{def}}{=} \int\cdots\int \mathrm{e}\{\,t\mathbb{B}_p[u_1,\dots,u_p;d_0]+g_1+\cdots+g_p\,\}\prod_{j=1}^{p}\mu(du_j;r) \tag{3.20}$$

and some functions g_j which have the structure of functions f_j described in (3.18). Naturally, the functions g_j can depend in addition on variables $w_1,\dots,w_p$. In order to estimate the function $\bigl|\,\varphi_1(t;\mu,r)\,\bigr|$ we shall apply the symmetrization Lemma 7.2 replacing in that lemma the measures $\mu_j(dx_j)$ by $\mu(du_j;r)$, the symmetric form $\mathbb{Q}_p[x_1,\dots,x_p]$ by $\mathbb{B}_p[u_1,\dots,u_p;d_0]$, the functions f_j by g_j, and the space $\mathbb{R}^d$ by $\mathbb{R}^{d_0}$. We obtain

$$\bigl|\,\varphi_1(t;\mu,r)\,\bigr|^{2^p} \le \varphi_2(t;\mu,r)$$

with

$$\varphi_2(t;\mu,r) \stackrel{\text{def}}{=} \int\cdots\int \mathrm{e}\{\,t\mathbb{B}_p[u_1,\dots,u_p;d_0]\,\}\prod_{j=1}^{p}\widetilde{\mu}(du_j;r) \tag{3.21}$$

The trigonometric sum $\varphi_2(t;\mu,r)$ has the form of the sum $\varphi(t;\mu,r)$ in the case $i)$ with dimension d replaced by d_0. Hence, using $i)$ and applying in addition Hölder's inequality, we get

$$\varphi^{2^p}(t;\mu,r) \ll_{p,d} \bigl((1\vee\lambda^*)\,(1\vee\lambda_*^{-1})\bigr)^{2d_0}\,\mathcal{M}^{d_0\alpha_p}(t;r^p),$$

which is equivalent to (3.11). □

Proof of Theorem 1.1 *and of* (1.19) *and* (1.20). It suffices to combine bounds of Corollary 3.2 and Lemma 3.3, where the conditions (2.8) and (2.9) are verified with Λ and $\varkappa_0$ specified in (3.10) and (3.11). □

Proof of Theorem 1.3. This proof is similar to that of Theorem 1.1 replacing Corollary 3.2 by Lemma 3.1. The relation (1.32) follows from results of Section 9. Finally, (1.32) cannot hold for $\mathbb{Q}$ with integer coefficients since this would contradict to the fact that gaps between successive values $\mathbb{Q}(x)$, $x\in\mathbb{Z}^d$, shrink when values grow and (1.32) holds. □

The following Propositions 3.4–3.6 are useful in descriptions of the structure of the set $\mathbb{Q}(Z^d)$.

Proposition 3.4. *Let $f(u) = u^p + \varepsilon_1 u^{p-1} + \dots + \varepsilon_{p-1} u$, $u \geq 0$, denote a polynomial of degree $p \geq 2$ such that coefficients $\varepsilon_j \in \mathbb{R}$ satisfy*

$$|\varepsilon|_1 \overset{\text{def}}{=} |\varepsilon_1| + \dots + |\varepsilon_{p-1}| < c_p$$

with some sufficiently small positive constant $c_p > 0$. Then

$$u^p/2 < f(u) < 2u^p, \qquad p\, u^{p-1}/2 < f'(u) < 2\, p\, u^{p-1}, \qquad \text{for } u \geq 1/2,$$

and $\bigl| f(u) \bigr| < 1/2$, for $u \leq 1/2$.

Let $g(x) = f^{-1}(x)$ denote the inverse of f, for $x \geq 1$. Then

$$2^{-1/p}\, x^{1/p} \leq g(x) \leq 2^{1/p}\, x^{1/p}, \qquad g'(x) \asymp_p x^{-1+1/p}. \tag{3.22}$$

The set

$$\bigl\{ u : \ 1 \leq f(u) \leq 1 + \delta \bigr\} = \bigl\{ u : \ g(1) \leq u \leq g(1+\delta) \bigr\}, \qquad \text{for } \delta \geq 0, \tag{3.23}$$

is an interval of length $g(1+\delta) - g(1) \asymp_p \delta\, (1 \wedge \delta^{-1+1/p})$.

Proof. Obvious. □

Proposition 3.5. *Assume that $\mathbb{Q}_p$ is a positive definite polynomial. Let $s > 0$ satisfy*

$$(1 \vee q_*^{-1}) \sum_{j=1}^{p-1} s^{-j/p}\, \|\mathbb{Q}_{p-j}\| \leq c_p \tag{3.24}$$

with a sufficiently small positive constant $c_p > 0$. Then

$$\operatorname{vol}(E_{s+\delta} \setminus \operatorname{vol} E_s) \gg_{d,p} s^{d/p}\, q_*^{-d/p}\, \frac{\delta}{s} \Bigl(1 \wedge \Bigl(\frac{\delta}{s} \Bigr)^{-1+1/p} \Bigr), \qquad \text{for } \delta \geq 0. \tag{3.25}$$

Furthermore, for s satisfying (3.24) we have

$$1 \vee \Bigl(\frac{s^{1/p}}{2\, d\, \|Q_p\|^{1/p}} \Bigr) \leq r_s \leq 1 \vee \Bigl(r_s \leq \frac{2 s^{1/p}}{q_*^{1/p}} \Bigr). \tag{3.26}$$

Proof. We shall use polar coordinates $dx = r^{d-1}\, dr\, d\eta$ in $\mathbb{R}^d$, where $r = |x|$ and $d\eta$ is area element of the $(d-1)$-dimensional unit sphere S^{d-1} in $\mathbb{R}^d$. We obtain

$$\operatorname{vol}(E_{s+\delta} \setminus \operatorname{vol} E_s) = \int_{\mathbb{R}^d} \mathbf{I}\bigl\{ s \leq \mathbb{Q}(x) \leq s + \delta \bigr\}\, dx = \int_{S^{d-1}} J(\eta)\, d\eta$$

with

$$J(\eta) \overset{\text{def}}{=} \int_0^\infty \mathbf{I}\bigl\{ s \leq \mathbb{Q}_p(\eta)\, r^p + \dots + \mathbb{Q}_1(\eta)\, r \leq s + \delta \bigr\}\, r^{d-1} dr.$$

Changing the variable $r = \big(s/\mathbb{Q}_p(\eta)\big)^{1/p}\, u$, we get

$$J(\eta) = \big(s/\mathbb{Q}_p(\eta)\big)^{d/p} \int\limits_0^\infty \mathbf{I}\big\{ 1 \le f(u) \le 1+\delta/s \big\}\, u^{d-1} dr$$

with

$$f(u) \overset{\text{def}}{=} u^p + \varepsilon_1\, u^{p-1} + \cdots + \varepsilon_{p-1}\, u, \quad \varepsilon_j \overset{\text{def}}{=} s^{-j/p} \mathbb{Q}_{p-j}(\eta)\, \big(\mathbb{Q}_p(\eta)\big)^{(j-p)/p}. \tag{3.27}$$

Writing $g(x) = f^{-1}(x)$ for the inverse function of f, estimating $\mathbb{Q}_p(\eta) \ge q_*$, using (3.23) of the Proposition 3.4 with δ replaced by δ/s and (3.22) with $\delta = 0$, we obtain

$$\begin{aligned} J(\eta) &= \big(s/\mathbb{Q}_p(\eta)\big)^{d/p} \int\limits_{g(1)}^{g(1+\delta/s)} u^{d-1} du \gg_p s^{d/p}\, q_*^{-d/p} g^{d-1}(1)\, \big(g(1+\delta/s) - g(1)\big) \\ &\gg_p s^{d/p}\, q_*^{-d/p}\, \frac{\delta}{s}\, \Big(1 \wedge \Big(\frac{\delta}{s}\Big)^{-1+1/p}\Big) \end{aligned} \tag{3.28}$$

provided that

$$\sum_{j=1}^{p-1} |\varepsilon_j| \ll_p 1. \tag{3.29}$$

Estimating $\mathbb{Q}_p(\eta) \ge q_*$ and $\mathbb{Q}_j(\eta) \le \|\mathbb{Q}_j\|$, we see that (3.29) is implied by the stronger condition (3.24). Integrating the bound (3.28) over S^{d-1} we get (3.25).

It remains to prove (3.26). Since $r_s = 1 \vee \sup_{x \in E_s} |x|_\infty$, it suffices to estimate the last supremum. Writing $x = r\,\eta$ with $r = |x|$ and changing the variable $r = (s/\mathbb{Q}_p(\eta))^{1/p}\, u$, we have

$$\begin{aligned} \sup_{x\in E_s} |x|_\infty &= \sup_{|\eta|=1} |\eta|_\infty \sup_{r \ge 0 \colon \mathbb{Q}_p(\eta)\, r^p + \cdots + \mathbb{Q}_1(\eta)\, r \le s} r \\ &= s^{1/p} \sup_{|\eta|=1} |\eta|_\infty\, (Q_p(\eta))^{-1/p} \sup_{u \ge 0 \colon f(u) \le 1} u \end{aligned}$$

with f as in (3.27). Using the properties of f described in Proposition 3.4, we have

$$\sup_{u \ge 0 \colon f(u) \le 1} u = \sup_{u \ge 0 \colon u \le g(1)} u = g(1), \qquad 1/2 \le g(1) \le 2. \tag{3.30}$$

Combining (3.30) with $|\eta|/d \le |\eta|_\infty \le |\eta|$ and $q_* \le \mathbb{Q}_p(\eta) \le \|\mathbb{Q}\|$, we obtain (3.26). $\square$

Proposition 3.6. *Assume that $\mathbb{Q}_p$ is a positive definite polynomial. Let $\delta \ge 0$ and $r > 0$ satisfy $\delta \le r$. Let*

$$\sum_{j=1}^{p-1} q_*^{(j-p)/p} \, \|\mathbb{Q}_{p-j}\| \le c_p \tag{3.31}$$

with a sufficiently small positive constant $c_p > 0$. Then

$$\operatorname{vol}\big((r+\delta)\, E_1 \setminus r\, E_1\big) \gg_{d,p} \delta\, r^{d-1}\, q_*^{-d/p}. \tag{3.32}$$

Proof. The proof is similar to that of Proposition 3.5. Using polar coordinates and writing $\mathbb{Q}(x,r) = \mathbb{Q}_p(x) + r\,\mathbb{Q}_{p-1}(x) + \cdots + r^{p-1}\,\mathbb{Q}_1(x)$ (cf. (1.12)), we obtain

$$\operatorname{vol}\big((r+\delta)\, E_1 \setminus r\, E_1\big) = \int_{\mathbb{R}^d} \mathbf{I}\big\{\,\mathbb{Q}(x,r) \ge r^p,\ \mathbb{Q}(x, r+\delta) \le (r+\delta)^p\,\big\}\, dx = \int_{S^{d-1}} J(\eta)\, d\eta$$

with

$$J(\eta) \overset{\text{def}}{=} \int_0^\infty \mathbf{I}\big\{\,\mathbb{Q}(\rho\,\eta, r) \ge r^p,\ \mathbb{Q}(\rho\,\eta, r+\delta) \le (r+\delta)^p\,\big\}\, \rho^{d-1} d\rho.$$

Changing the variable $\rho = r\, u \mathbb{Q}_p^{-1/p}(\eta)$ and using the inequality $(1+t)^p \ge 1 + p\,t$, for $0 \le t \le 1$, we get

$$J(\eta) \ge r^d\, \mathbb{Q}_p^{-d/p}(\eta) \int_0^\infty \mathbf{I}\big\{\, f(u) \ge 1,\ f_0(u) \le 1 + p\,\delta/r\,\big\}\, u^{d-1} du \tag{3.33}$$

with

$$f(u) \overset{\text{def}}{=} u^p + \varepsilon_1\, u^{p-1} + \cdots + \varepsilon_{p-1}\, u, \quad \varepsilon_j \overset{\text{def}}{=} \mathbb{Q}_{p-j}(\eta)\, \big(\mathbb{Q}_p(\eta)\big)^{(j-p)/p}.$$

and

$$f_0(u) \overset{\text{def}}{=} u^p + \tau_1\, u^{p-1} + \cdots + \tau_{p-1}\, u, \quad \tau_j \overset{\text{def}}{=} (1+\delta/r)^j\, \varepsilon_j.$$

Due to the condition (3.31) and $\delta \le r$, both functions f and f_0 satisfy the conditions of Proposition 3.4. Using this proposition, it is easy to see that there exists a constant $c(p)$ such that

$$f(u) \le 1 + \delta/r \le 2 \implies u \le c(p), \qquad f_0(u) \le 1 + p\,\delta/r \le 1 + p \implies u \le c(p).$$

Applying the elementary inequality $\big|(1+t)^j - 1\big| \le j\, 2^{j-1}\, t$, for $0 \le t \le 1$, estimating $\mathbb{Q}_p(\eta) \ge q_*$ and $\big|\mathbb{Q}_j(\eta)\big| \le \|\mathbb{Q}_j\|$, and using the condition (3.31) and $\delta/r \le 1$, we have

$$\big|\, f_0(u) - f(u)\big| \le \sum_{j=1}^{p-1} |\varepsilon_j|\, u^j\, \big|(1+\delta/r)^j - 1\big| \le c_0(p)\, c_p\, \delta/r, \quad \text{for } u \le c(p), \tag{3.34}$$

with some constant $c_0(p)$, and c_p from the condition (3.31). Using (3.34), we have

$$\mathbf{I}\Big\{ f_0(u) \le 1 + \tfrac{p\,\delta}{r}\Big\} \ge \mathbf{I}\Big\{ f(u) \le 1 + \tfrac{p\,\delta}{r} - \big|\, f_0(u) - f(u)\big|\Big\} \ge \mathbf{I}\Big\{ f(u) \le 1 + \tfrac{\delta}{r}\Big\} \tag{3.35}$$

since the constant c_p in the condition (3.31) is sufficiently small and therefore we can estimate $p - c_0(p)\, c_p \geq 1$, for $p \geq 2$. Combining (3.33) and (3.35), we obtain

$$J(\eta) \geq r^d\, \mathbb{Q}_p^{-d/p}(\eta) \int_0^\infty \mathbf{I}\big\{ 1 \leq f(u) \leq 1 + \delta/r \big\}\, u^{d-1} du. \tag{3.36}$$

Let g denote the inverse function of f. Due to the assumptions (3.31), the function f satisfies the conditions of Proposition 3.4. In particular, the inequality $1 \leq f(u) \leq 1 + \delta/r$ is equivalent to $g(1) \leq u \leq g(1+\delta/r)$, and we can estimate the right hand side of (3.36) as in the final stage of the proof of (3.25), starting from (3.28), which leads to (3.32). □

4. An expansion for the Fourier–Stieltjes transform $\widehat{F}$

In this section we shall obtain an expansion (see Lemma 4.1 below) of the Fourier-Stieltjes transform $\widehat{F}$ of the distribution function F extending an approach developed in BG (1997c) for $p = 2$. Let us explain the ideas and methods of proof using the following simplified model. The Fourier-Stieltjes transform $\widehat{F}(t)$ is an integral with respect to the discrete measure μ defined by (2.4). Hence, we may interpret $\widehat{F}(t)$ as a sum which approximates $\widehat{F}_0(t)$, interpreting $\widehat{F}_0(t)$ as an integral with respect to the absolutely continuous measure ν defined by (2.4). Therefore let us consider the following simple problem: to obtain an expansion of the sum $\int f(x)\, \Phi(dx)$, where $f : \mathbb{R}^d \to \mathbb{C}$ is a sufficiently smooth function, using integrals with respect to the absolutely continuous measure Ψ, see (2.4) or (2.18), (2.19) for the definitions of Φ and Ψ. Using the identity $\Phi * \pi = \Psi$, where π denotes the Lebesgue measure restricted to the cube $B(1/2)$, and expanding into the Taylor series in powers of y, we have

$$\int f(x)\, \Psi(dx) = \iint f(x+y)\, \Phi(dx)\, \pi(dy) = \int f(x)\, \Phi(dx) + \sum_{j=1}^{n-1} \frac{a_j}{j!} + \mathcal{R} \tag{4.1}$$

with the expansion terms $a_j = \iint f^{(j)}(x) y^j\, \Phi(dx)\, \pi(dy)$ and the remainder term

$$\mathcal{R} = \frac{1}{(n-1)!} \int_0^1 (1-\tau)^{n-1} \iint f^{(n)}(x + \tau\, y) y^n\, \Phi(dx)\, \pi(dy)\, d\tau,$$

see (1.36) for the notation related to the Fréchet derivatives. Using (4.1) and ignoring the remainder term $\mathcal{R}$, we may express the sum $\int f(x)\, \Phi(dx)$ via the integral $\int f(x)\, \Psi(dx)$ and a weighted sum of expansion terms a_j. We may apply to the sums a_j expansions similar to (4.1) including in the remainder term the integrals containing derivatives

$f^{(n)}$. After a finite number of steps of such kind we shall obtain a representation of $\int f(x)\,\Phi(dx)$ through a weighted sum of integrals of type

$$\int \cdots \int f^{(j)}(x) y_1^{\eta_1} \dots y_m^{\eta_m}\, \Psi(dx) \prod_{s=1}^{m} \pi(dy_s), \qquad 0 \le j < n, \tag{4.2}$$

where $\eta = (\eta_1, \dots, \eta_m)$ denotes a non-negative integer multi-index such that $|\eta|_1 = j$. Concerning the remainder terms of expansions, a simple their estimation is not sufficient for our purposes since we need to have certain trigonometric sums (such as in (4.12) and (4.13)) as factors in our bounds in order to ensure the convergence of integrals which appear in Section 3 by the inversion of the Fourier-Stieltjes transforms. This is one reason why we need sufficiently many copies of measures $\mu(\cdot\,; r)$ in the definition (2.4) of the measure μ. It turns out that the remainders in expansions of type (4.1) for $\widehat{F}(t)$ have the following structure

$$\mathcal{R}_s \overset{\text{def}}{=} \int \mathbb{S}(x)\, \mathrm{e}\big\{ t\, \mathbb{Q}(x) \big\}\, \rho^{*(s+1)}(dx),$$

where $\mathbb{S}(x) = \mathbb{S}[x, \dots, x]$ is an s-linear form, $s \in \mathbb{N}$, and ρ is a convolution of measures $\mu(\cdot\,; r)$ such that the size of its support is $\ll r$. The desired factors of are obtained as follows. Using the properties of convolutions and the s-linearity of $\mathbb{S}$ and writing for a while $x = x_1 + \dots + x_{s+1}$, we can proceed as follows:

$$|\mathcal{R}_s| = \Big| \int \cdots \int \mathbb{S}(x)\, \mathrm{e}\big\{ t\, \mathbb{Q}(x) \big\} \prod_{j=1}^{s+1} \rho(dx_j) \Big| \le \sum_{1 \le j_1, \dots, j_s \le s+1} \big| \mathcal{R}_s(j) \big|$$

$$\mathcal{R}_s(j) \overset{\text{def}}{=} \int \cdots \int \mathbb{S}[x_{j_1}, \dots, x_{j_s}]\, \mathrm{e}\big\{ t\, \mathbb{Q}(x) \big\} \prod_{j=1}^{s+1} \rho(dx_j).$$

In order to estimate $\mathcal{R}_s(j)$ notice that among the variables $x_{j_1}, \dots, x_{j_s}$ at least one of $x_1, \dots, x_{s+1}$ is absent. To be definite, assume that x_1 is absent. Then

$$\big| \mathcal{R}_s(j) \big| \le \int \cdots \int \big| \mathbb{S}[x_{j_1}, \dots, x_{j_s}] \big| \; \Big| \int \mathrm{e}\big\{ t\, \mathbb{Q}(x) \big\}\, \rho(dx_1) \Big| \prod_{j=2}^{s+1} \rho(dx_j) \le C\, \|\mathbb{S}\|\, J \tag{4.3}$$

with

$$J = \sup_{\deg \mathbb{P} < p} \Big| \int \mathrm{e}\big\{ t\, \mathbb{Q}_p(y) + t\, \mathbb{P}(y) \big\}\, \rho(dy) \Big|$$

and

$$C = \int \cdots \int \big| x_{j_1} \big| \dots \big| x_{j_s} \big| \prod_{j=2}^{s+1} \rho(dx_j) \ll_s 1 + r^s$$

since the measure ρ has the support of size $\ll r$. While proving (4.3) we used the decompositions $x = x_1 + \dots + x_{s+1}$, $\mathbb{Q} = \mathbb{Q}_0 + \dots + \mathbb{Q}_p$ and the following estimate

$$\Big| \int \mathrm{e}\big\{ t\, \mathbb{Q}(x) \big\}\, \rho(dx_1) \Big| \le J.$$

The integral J in (4.3) serves as the factor (cf. (4.12) and (4.13)) which ensures the convergence of the aforementioned integrals.

Let us return to the exposition of the results of the section. By change of variables we can write the Fourier–Stieltjes transforms of the distribution functions F and F_0 as

$$\widehat{F}(t) = \int \mathrm{e}\big\{ t\,\mathbb{Q}(x) \big\}\, \mu(dx), \qquad \widehat{F_0}(t) = \int \mathrm{e}\big\{ t\,\mathbb{Q}(x) \big\}\, \nu(dx). \tag{4.4}$$

Similarly, for the Fourier–Stieltjes transforms of the functions of bounded variation F_j (see (2.21)–(2.23)) we have

$$\widehat{F}_j(t) = \sum_{\eta:\ |\eta|_1 = j}^{**} \widehat{F}_{j\eta}(t), \tag{4.5}$$

where the sum $\sum\limits_{\eta:\ |\eta|_1=j}^{**}$ is defined as in (2.21), and

$$\widehat{F}_{j\eta}(t) = \frac{(-1)^m}{\eta!} \int \cdots \int \mathrm{e}\big\{ t\,\mathbb{Q}(x) \big\}\, D^{(j)}(x) u_1^{\eta_1} \dots u_m^{\eta_m}\, dx \prod_{l=1}^m \pi^{*(k+1)}(du_l). \tag{4.6}$$

Introducing the notation

$$g(x) = \exp\big\{ h(x) \big\}, \qquad h(x) = i\,t\,\mathbb{Q}(x), \tag{4.7}$$

integrating by parts and using $D(x)\,dx = \nu(dx)$, we have

$$\widehat{F}(t) = \int g(x)\, \mu(dx), \qquad \widehat{F_0}(t) = \int g(x)\, \nu(dx) \tag{4.8}$$

and

$$\widehat{F}_{j\eta}(t) = \frac{(-1)^m}{\eta!} \int \cdots \int g^{(j)}(x) u_1^{\eta_1} \dots u_m^{\eta_m}\, \nu(dx) \prod_{l=1}^m \pi^{*(k+1)}(du_l). \tag{4.9}$$

In the proof of (4.9) we used that the function D has compact support and the derivatives $\partial^\alpha D$ are continuous functions, for $|\alpha|_\infty \le k-2$.

Write

$$\theta(s) = [s/p] + \mathbf{I}\big\{ s/p - [s/p] > 0 \big\}. \tag{4.10}$$

Notice that $\theta(s) \ge 1$, for $s > 0$. Write as well

$$\kappa = \big[k/(pn - n + 1) \big] \tag{4.11}$$

and define

$$\psi(t) = \sup_{\deg \mathbb{P} < p} \Big| \int \mathrm{e}\big\{ t\,\mathbb{Q}(x) + t\,\mathbb{P}(x) \big\}\, \mu^{*\kappa}(dx; r) \Big|, \tag{4.12}$$

$$\psi_0(t) = \sup_{\deg \mathbb{P} < p} \Big| \int \mathrm{e}\big\{ t\,\mathbb{Q}(x) + t\,\mathbb{P}(x) \big\}\, \nu^{*\kappa}(dx; \overline{r}) \Big|, \tag{4.13}$$

where $\sup\limits_{\deg \mathbb{P} < p}$ is taken over all polynomials $\mathbb{P} : \mathbb{R}^d \to \mathbb{R}$ such that $\deg \mathbb{P} < p$. In the case $\kappa = 0$ (that is, if $k < pn - n + 1$) we define $\psi(t) = \psi_0(t) \equiv 1$. Notice that $\psi(t)$ and $\psi_0(t)$ are independent of $\mathbb{Q}_0, \dots, \mathbb{Q}_{p-1}$.

Lemma 4.1. *Assume that $\mathbb{Q} = \mathbb{Q}_p + \dots + \mathbb{Q}_0$ is a polynomial of degree $p \ge 2$. Write*

$$q = \sum_{j=1}^{p} \frac{\|Q_j\|}{r^{p-j}}.$$

Let $n \in \mathbb{N}$ denote a natural number and let $n_0 \in \mathbb{N}_0$ be the largest even integer such that $n_0 < n$. Then the remainder term $\widehat{\mathcal{R}}_n(t)$ in the asymptotic expansion

$$\widehat{F}(t) = \widehat{F}_0(t) + \widehat{F}_2(t) + \dots + \widehat{F}_{n_0-2}(t) + \widehat{F}_{n_0}(t) + \widehat{\mathcal{R}}_n(t)$$

satisfies

$$|\widehat{\mathcal{R}}_n(t)| \ll_{d,k,p,n} \psi(t)\, \Lambda_0^n\, \Lambda_{1,n}\, \Lambda_{2,n} , \tag{4.14}$$

where

$$\Lambda_0 \overset{\text{def}}{=} \frac{1}{r}\left(\frac{R}{r}\right)^{p-1}, \qquad \Lambda_{1,n} \overset{\text{def}}{=} q^{\theta(n)} + q^n, \qquad \Lambda_{2,n} \overset{\text{def}}{=} |t r^p|^{\theta(n)} + |t r^p|^n,$$

the number $\theta(n)$ is defined by (4.10)*, and where k denotes the number of the copies of the measure $\mu(\,\cdot\,; r)$ in the definition* (2.4) *of measures μ and ν, and $R \ge r \ge 1$ is the size of the measure Φ in* (2.4)*. Furthermore, the expansion terms $\widehat{F}_j(t)$ satisfy*

$$\big|\,\widehat{F}_j(t)\,\big| \ll_{d,k,p,n} \psi_0(t)\, \Lambda_0^j\, \Lambda_{1,j}\, \Lambda_{2,j}, \qquad \textit{for } j \in 2\,\mathbb{N}_0,\ \ j < n. \tag{4.15}$$

Notice that we can represent the remainder term in the asymptotic expansion of Lemma 4.1 as

$$\widehat{\mathcal{R}}_n(t) = \widehat{F}(t) - \widehat{F}_0(t) - \sum_{j \in 2\,\mathbb{N},\ j<n} \widehat{F}_j(t). \tag{4.16}$$

Corollary 4.2. *The bound* (4.14) *of Lemma* 4.1 *remains true for all non-negative $n \in \mathbb{R}$ provided that $\theta(n)$ is replaced by $\theta_0(n)$ defined below by* (4.17)*.*

Proof of Corollary 4.2. In this proof we shall write R_n instead of $|\widehat{\mathcal{R}}_n(t)|$. Assume that $n \notin \mathbb{N}$, $n \ge 0$ and consider the following three cases: *a*) $n = 0$; *b*) the number $[n]$ is odd and $n > 0$; *c*) the number $[n]$ is even and $n > 0$.

In the case *a*) the bound (4.14) reduces to the inequality $R_0 \le 2\psi(t)$, which is fulfilled since $\big|\,\widehat{F}(t)\,\big| \le \psi(t)$ and $\big|\,\widehat{F}_0(t)\,\big| \le \psi_0(t) \le \psi(t)$.

In the case *b*) we shall estimate R_n interpolating (4.14). We have $R_n = R_{[n]} = R_{[n]+1}$ since $[n]$ is odd. Writing $\alpha = 1 - n + [n]$ we have $R_n = R_{[n]}^{\alpha}\, R_{[n]+1}^{1-\alpha}$. Estimating now $R_{[n]}$ and $R_{[n]+1}$ by (4.14), we see that R_n satisfies (4.14) with $\theta(n)$ replaced by

$$\theta_0(n) = \big(1 - n + [n]\big)\,\theta([n]) + \big(n - [n]\big)\,\theta([n]+1) \tag{4.17}$$

Notice that $\theta_0(n) \le n$ and $\theta_0(n)$ is positive for $n > 0$.

In the case *c*) we interpolate (4.14) again. We have $R_n = R_{[n]+1}$ and $R_n \le R_{[n]} + |\widehat{F}_{[n]}(t)|$ since $[n]$ is even. Estimating again $R_{[n]}$ and $R_{[n]+1}$ by (4.14), applying the bound (4.15) to $|\widehat{F}_{[n]}(t)|$, using $\psi_0(t) \le \psi(t)$ and repeating the argumentation of the case *b*), we obtain (4.14) with $\theta_0(n)$ instead of $\theta(n)$.

Proof of Lemma 4.1. In this proof we shall write $\ll$ instead of $\ll_{d,k,p,n}$.

Let $g : \mathbb{R}^d \to \mathbb{C}$ denote a sufficiently smooth complex valued function. Assume that $x, u_1, \dots, u_n \in \mathbb{R}^d$. We shall use the following decomposition

$$g(x) = g(x+u_1) + g_1 + \dots + g_n, \tag{4.18}$$

where

$$g_j = \sum_{\eta:\ |\eta|_1 = j}^{*} c(\eta)\, g^{(j)}(x+u_{m+1}) u_1^{\eta_1} \dots u_m^{\eta_m}, \quad \text{for} \quad 1 \le j < n, \tag{4.19}$$

$$g_n = \sum_{\eta:\ |\eta|_1 = n}^{*} c_n(\eta) \int_0^1 (1-\tau)^{\eta_m}\, g^{(n)}(x+\tau u_m) u_1^{\eta_1} \dots u_m^{\eta_m}\, d\tau, \tag{4.20}$$

and

$$c(\eta) = \frac{(-1)^m}{\eta_1! \dots \eta_m!}, \qquad c_n(\eta) = \frac{(-1)^m}{\eta_1! \dots \eta_{m-1}!\,(\eta_m - 1)!}. \tag{4.21}$$

Here the sum $\sum_{\eta:\ |\eta|_1=j}^{*}$ is taken over all representations of j as a sum $j = \eta_1 + \dots + \eta_m$, $m \le j$, of integers $\eta_1, \dots, \eta_m \in \mathbb{N}$. For example, in the case $j = 3$ we have the following four representations: $3 = 3$, $3 = 2+1$, $3 = 1+2$, $3 = 1+1+1$. In particular, if $n = 1$ then $g(x) = g(x+u_1) + g_1$ with

$$g_1 = -\int_0^1 (1-\tau)\, g'(x+\tau u_1) u_1\, d\tau,$$

and if $n = 2$ then $g(x) = g(x+u_1) + g_1 + g_2$ with

$$g_1 = -g'(x+u_2)u_1, \quad g_2 = \int_0^1 (1-\tau)\, g''(x+\tau u_2) u_1 u_2\, d\tau - \int_0^1 (1-\tau)^2\, g''(x+\tau u_1) u_1^2\, d\tau.$$

In order to obtain the decomposition (4.18) it suffices to iteratively apply Taylor expansions. In the first step we use the Taylor expansion of the function g:

$$g(x) = g(x+u_1) - \sum_{\eta_1=1}^{n-1} \frac{1}{\eta_1!}\, g^{(\eta_1)}(x) u_1^{\eta_1} - \frac{1}{(n-1)!} \int_0^1 (1-\tau)^n\, g^{(n)}(x+\tau u_1) u_1^n\, d\tau. \tag{4.22}$$

In the second step we apply expansions of type (4.22) to the functions $x \mapsto g^{(\eta_1)}(x) u_1^{\eta_1}$, for $1 \le \eta_1 < n$, using u_2 instead of u_1. After n such steps we arrive at (4.18).

For the derivatives of the composite function $g(x) = \exp\{\, h(x)\,\}$, we shall use the following formula, which follows from a general formula (see Theorem 2.5 in Averbuh and Smoljanov 1967). Let $v_1, \dots, v_s \in \mathbb{R}^d$. Let $\mathcal{A} = \{A_1, \dots, A_\alpha\}$ denote a partition of the set $\{v_1,\dots,v_s\} = A_1\cup\dots\cup A_\alpha$ into α disjoint subsets A_j such that $1\le \operatorname{card} A_j\le p$, for all j. Notice that $\theta(s)\le\alpha\le s$, where $\theta(s)$ is defined by (4.10). For a subset $A\subset\{v_1,\dots,v_s\}$, say $A=\{v_{i_1},\dots,v_{i_l}\}$, introduce the derivative $\partial_A h(x)\overset{\text{def}}{=} h^{(l)}(x)v_{i_1}\dots v_{i_l}$. Then

$$g^{(s)}(x)v_1\dots v_s = g(x)\sum_{\mathcal A}\partial_{A_1}h(x)\dots\partial_{A_\alpha}h(x), \tag{4.23}$$

where the sum $\sum\limits_{\mathcal A}$ extends over all partitions $\mathcal A$ with properties specified above. One can easily prove (4.23) using $h^{(p+1)}(x)\equiv 0$ and induction in s.

The following identities

$$\nu(\,\cdot\,;\overline r)=\mu(\,\cdot\,;r)*\pi, \qquad \nu=\Psi*\nu^{*k}(\cdot;\overline r)=\mu*\pi^{*(k+1)}, \tag{4.24}$$

are obvious (see (2.2), (2.3) and (2.4) for the definitions of measures which appear in (4.24)). For example, for any integrable function $u:\mathbb R^d\to\mathbb R$, we have

$$\int u(x)\,\nu(dx;\overline r) = \sum_{y\in\mathbb Z^d\cap B(r)}(2\overline r)^{-d}\int u(x+y)\,\pi(dx)=\iint u(x+y)\,\pi(dx)\,\mu(dy;r),$$

proving the first identity in (4.24).

Choosing in (4.18) $g(x)=\mathrm e\{t\,\mathbb Q(x)\}$ and integrating both sides of the identity (4.18) with respect to the measures

$$\mu(dx),\ \ \pi^{*(k+1)}(du_1),\ \ \dots,\ \ \pi^{*(k+1)}(du_n)$$

and using (4.8), (4.9), (4.19), (4.20), (4.24), we obtain

$$\widehat F(t)=\widehat F_0(t)+f_1(t)+\dots+f_{n-1}(t)+f_n(t), \tag{4.25}$$

with $\widehat F$ and $\widehat F_0$ defined by (4.8), and

$$f_j(t)=\sum_{\eta:\ |\eta|_1=j}{}^{*}\, c(\eta)\int\cdots\int g^{(j)}(x)u_1^{\eta_1}\dots u_m^{\eta_m}\,\nu(dx)\prod_{s=1}^{m}\pi^{*(k+1)}(du_s), \tag{4.26}$$

for $1\le j<n$, and

$$f_n(t)=\sum_{\eta:\ |\eta|_1=n}{}^{*}\, c_n(\eta)\int_0^1(1-\tau)^{\eta_m}f_n(t;\eta)\,d\tau, \tag{4.27}$$

$$f_n(t;\eta)\overset{\text{def}}{=}\int\cdots\int g^{(n)}(x+\tau\,u_m)u_1^{\eta_1}\dots u_m^{\eta_m}\,\mu(dx)\prod_{l=1}^{m}\pi^{*(k+1)}(du_l).$$

The sums in (4.26) and (4.27) are the same as in (4.19) and (4.20) respectively. However, notice that $f_j = 0$, for odd $j < n$ since the measure π is symmetric. Moreover, $f_j = \widehat{F}_j$, for even $j < n$, since all terms in the sum (4.26) vanish unless all $\eta_1, \dots, \eta_m$ are even, due to the symmetry of π. Hence, (4.16) and (4.25) together yield

$$\big|\widehat{\mathcal{R}}_n(t)\big| = \Big|\widehat{F}(t) - \widehat{F}_0(t) - \sum_{j \in 2\mathbb{N},\ j<n} \widehat{F}_j(t)\Big| = \big|f_n(t)\big|. \tag{4.28}$$

Now we can return to the proof of (4.14). Using (4.28) and (4.27), we have

$$\big|\widehat{\mathcal{R}}_n(t)\big| \ll \max_{\eta} \sup_{0\le\tau\le 1} \big|f_n(t;\eta)\big|, \tag{4.29}$$

where $\max\limits_{\eta}$ is taken over all multi-indices $\eta = (\eta_1, \dots, \eta_s)$ which are allowed in (4.27). Define $v_1, \dots, v_n$ repeating u_1 η_1 times, followed by u_2 η_2 times, etc. For example, $v_j = u_1$, for all $1 \le j \le \eta_1$. Using the formula (4.23) with $s = n$ for derivatives of the composite function $g(x) = \exp\big\{h(x)\big\}$ and applying (4.27), we obtain

$$\sup_{0\le\tau\le1} \big|f_n(t;\eta)\big| \ll \sup_{0\le\tau\le1} \max_{\mathcal{A}} \int\cdots\int \sup_{|z|_\infty\le1} I(\mathcal{A}) \prod_{l=1}^{m} \pi^{*(k+1)}(du_l), \tag{4.30}$$

where $\max\limits_{\mathcal{A}}$ is taken over all possible partitions $\mathcal{A}$ used in (4.23),

$$I(\mathcal{A}) = \Big|\int g(x+z)\,\partial_{A_1} h(x+z)\dots\partial_{A_\alpha} h(x+z)\,\mu(dx)\Big|, \qquad z \overset{\text{def}}{=} \tau\, u_m, \tag{4.31}$$

and $\theta(n) \le \alpha \le n$. Let us fix in (4.30) a partition $\mathcal{A} = \big\{A_1, \dots, A_\alpha\big\}$. Recall that

$$h(x) = i\,t\,\mathbb{Q}(x), \qquad \mathbb{Q} = \sum_{j=0}^{p} \mathbb{Q}_j, \tag{4.32}$$

where $\mathbb{Q}_j(x) = \mathbb{Q}_j[x, \dots, x]$ denotes the j-linear part of the polynomial $\mathbb{Q}$. Using (4.32), we get

$$I(\mathcal{A}) \ll |t|^\alpha \max_{0\le j_1,\dots,j_\alpha\le p} I(\mathcal{A}, j) \tag{4.33}$$

with

$$I(\mathcal{A}, j) = \Big|\int g(x+z)\,\partial_{A_1}\mathbb{Q}_{j_1}(x+z)\dots\partial_{A_\alpha}\mathbb{Q}_{j_\alpha}(x+z)\,\mu(dx)\Big|. \tag{4.34}$$

Let us denote the elements of the set A_l by $w_1, \dots, w_{\beta_l}$, where $\beta_l = \operatorname{card} A_l$. The elements w_j may depend on the set A_l, which is not reflected in the notation. This will not lead to confusion since we shall use the sole property of w_j, namely that $|w_j| \ll 1$. The inequality $|w_j| \ll 1$ holds since w_j is just one of the variables $u_1, \dots, u_m$ of integration in (4.30) and $\operatorname{supp}\pi = B(1/2)$. Using the multi-linearity of $\mathbb{Q}_{j_l}(u) = \mathbb{Q}_{j_l}[u, \dots, u]$, we can write

$$\begin{aligned} &\partial_{A_1}\mathbb{Q}_{j_l}(x+z) = c\,\mathbb{Q}_{j_l}[x+z, \dots, x+z, w_1, \dots, w_{\beta_l}], && \text{if } \beta_l \le j_l,\\ &\partial_{A_1}\mathbb{Q}_{j_l}(x+z) = 0, && \text{if } \beta_l > j_l \end{aligned} \tag{4.35}$$

with $c = j_l(j_l-1)\dots(j_l-\beta_l)$.

The relations (4.34)–(4.35) yield

$$I(\mathcal{A},j) \ll \max_{\mathbb{S}} I(\mathcal{A},j,\mathbb{S}), \tag{4.36}$$

with

$$I(\mathcal{A},j,\mathbb{S}) \stackrel{\text{def}}{=} \Big| \int g(x+z)\,\mathbb{S}(x)\,\mu(dx) \Big| \tag{4.37}$$

and

$$\mathbb{S}(x) \stackrel{\text{def}}{=} \mathbb{S}[x,\dots,x] = \prod_{l=1}^{\alpha} \mathbb{Q}_{j_l}[x,\dots,x,z,\dots,z,w_1,\dots,w_{\beta_l}]. \tag{4.38}$$

The $\max_{\mathbb{S}}$ in (4.36) is taken over all possible $\mathbb{S}$ of the form (4.38); deriving (4.36) we used the fact that the number of such $\mathbb{S}$ is bounded from above by a constant which depends on p and n only. The number of variables x in $\mathbb{Q}_{j_l}[x,\dots]$ in (4.38), say u_l, depends on the aforementioned term $I(\mathcal{A},j,\mathbb{S})$. However, it is clear that $u_l+\beta_l \le j_l$. Hence, writing $u = u_1+\dots+u_\alpha$ and $|j|_1 = j_1+\dots+j_\alpha$, and using $\beta_1+\dots+\beta_\alpha = n$, $j_l \le p$, we have

$$u \le |j|_1 - n \le pn - n. \tag{4.39}$$

Using the multi-linearity of $\mathbb{Q}_j$ and estimating $|z| \ll 1$ and $|w_s| \ll 1$, we have

$$\big|\mathbb{S}[x_1,\dots,x_u]\big| \ll \prod_{l=1}^{\alpha} \|\mathbb{Q}_{j_l}\|\,|x_1|\cdots|x_u| \ll \Lambda_{1,n}\, r^{p\alpha-|j|_1}\,|x_1|\cdots|x_u|, \tag{4.40}$$

for any $x_1,\dots,x_u \in \mathbb{R}^d$, since $|z| \ll 1$, $|w_j| \ll 1$, $\theta(n) \le \alpha \le n$, $\Lambda_{1,n} = q^{\theta(n)} + q^n$, and

$$\prod_{l=1}^{\alpha} \|\mathbb{Q}_{j_l}\| = r^{p\alpha-|j|_1} \prod_{l=1}^{\alpha} \|\mathbb{Q}_{j_l}\|\, r^{j_l-p} \ll r^{p\alpha-|j|_1}\, q^{\alpha} \ll r^{p\alpha-|j|_1}\,(q^{\theta(n)}+q^n).$$

Henceforth the estimation depends on whether the number κ defined by (4.11) equals zero or not.

Let us consider the simpler case $\kappa = 0$. Using (4.40) with $x_1 = \dots = x_u = x$ and estimating $\big|g(x+z)\big| \le 1$, we obtain

$$I(\mathcal{A},j,\mathbb{S}) \ll \Lambda_{1,n}\, r^{p\alpha-|j|_1} \int |x|^u\,\mu(dx) \ll \Lambda_{1,n}\, r^{p\alpha-|j|_1} R^u \tag{4.41}$$

since the measure μ has a support of size $\ll R$ and we can estimate $|x| \ll R$.

Combining (4.41), (4.36) and (4.33) and using (4.39) we get

$$\begin{aligned} I(\mathcal{A}) &\ll \Lambda_{1,n}\; |t|^{\alpha}\, r^{p\alpha-|j|_1} R^u = \Lambda_{1,n}\, |t\,r^p|^{\alpha}\, r^{u-|j|_1} (R/r)^u \\ &\ll \Lambda_{1,n}\, \big(|t\,r^p|^{\theta(n)} + |t\,r^p|^n\big)\, r^{-n} (R/r)^{pn-n} = \Lambda_0^n\,\Lambda_{1,n}\,\Lambda_{2,n}. \end{aligned} \tag{4.42}$$

Proving (4.42) we used the inequalities

$$\theta(n) \le \alpha \le n, \qquad R \ge r \ge 1. \tag{4.43}$$

The bound (4.42) combined with (4.30) and (4.29) yields the desired bound (4.14) for $\widehat{\mathcal{R}}_n(t)$ in the case $\kappa = 0$ since in this case $\psi(t) \equiv 1$.

Let us consider the case $\kappa > 0$. Compared to the case $\kappa = 0$ we have to improve our bounds by the extra factor $\psi(t)$. Using the definition (4.11) of the number κ, we may write $k = \kappa\, s + k_0$ with $s \overset{\text{def}}{=} pn - n + 1$ and some $0 \le k_0 < s$, and split the measure $\mu = \Phi * \mu^{*k}(\,\cdot\,; r)$ as follows

$$\mu = \chi_0 * \chi^{*s}, \qquad \chi_0 \overset{\text{def}}{=} \Phi * \mu^{*k_0}(\,\cdot\,; r), \quad \chi \overset{\text{def}}{=} \mu^{*\kappa}(\,\cdot\,; r). \tag{4.44}$$

Then we have

$$\int U(x)\, \mu(dx) = \int U(x_0 + \cdots + x_s)\, \chi_0(dx_0)\, \chi(dx_1) \dots \chi(dx_s), \tag{4.45}$$

for any integrable function U. Writing $x = x_0 + \cdots + x_s$ and using the u-linearity of $\mathbb{S}(x) = \mathbb{S}[x, \dots, x]$ defined by (4.38), we have

$$\mathbb{S}(x) = \sum_{i_1=0}^{s} \cdots \sum_{i_u=0}^{s} \mathbb{S}[x_{i_1}, \dots, x_{i_u}]. \tag{4.46}$$

Applying (4.45) to (4.37) and using (4.46), we obtain

$$I(\mathcal{A}, j, \mathbb{S}) \ll \max_{0 \le i_1, \dots, i_u \le s} I(\mathcal{A}, j, \mathbb{S}, i) \tag{4.47}$$

with

$$I(\mathcal{A}, j, \mathbb{S}, i) = \int \Big| \int \cdots \int g(x+z)\, \mathbb{S}[x_{i_1}, \dots, x_{i_u}] \prod_{l=1}^{s} \chi(dx_l) \Big|\, \chi_0(dx_0). \tag{4.48}$$

Recall that $u \le pn - n = s - 1$. Hence, given the variables $x_{i_1}, \dots, x_{i_u}$, we find among $x_1, \dots, x_s$ at least one variable, say x_l such that $l \notin \big\{ i_1, \dots, i_u \big\}$. Without loss of generality we can assume that $l = 1$. Then (4.48) yields

$$I(\mathcal{A}, j, \mathbb{S}, i) \ll \int \cdots \int \big| \mathbb{S}[x_{i_1}, \dots, x_{i_u}] \big|\; J\; \chi_0(dx_0) \prod_{l=2}^{s} \chi(dx_l) \tag{4.49}$$

with

$$J = \Big| \int g(x+z)\, \chi(dx_1) \Big|, \qquad x = x_0 + \cdots + x_s.$$

Recall (see (4.7)) that $g(x) = \mathrm{e}\big\{ t\, \mathbb{Q}(x) \big\}$. Therefore

$$J \le \sup_{y \in \mathbb{R}^d} \Big| \int \mathrm{e}\big\{ t\, \mathbb{Q}(x+y) \big\}\, \mu^{*\kappa}(dx; r) \Big| \le \psi(t) \tag{4.50}$$

with ψ defined by (4.12). Hence, the bound (4.49) combined with (4.40) and (4.50) yields

$$I(\mathcal{A}, j, \mathbb{S}, i) \ll \psi(t)\, \Lambda_{1,n}\, r^{p\alpha-|j|_1} \int \cdots \int |x_{i_1}| \cdots |x_{i_u}|\, \chi_0(dx_0) \prod_{l=2}^{s} \chi(dx_l). \tag{4.51}$$

The supports of the measures χ_0 and χ defined by (4.44) have sizes $\ll R$. Hence, the bound (4.51) combined with (4.47) yields

$$I(\mathcal{A}, j, \mathbb{S}) \ll \psi(t)\, \Lambda_{1,n}\, r^{p\alpha-|j|_1}. \tag{4.52}$$

The bound (4.52) compared to (4.41) contains the extra factor $\psi(t)$. Hence, we can conclude the proof of (4.14) for $\widehat{\mathcal{R}}_n(t)$ similarly as in the case $\kappa = 0$, cf. (4.42) and (4.43).

The proof of (4.15) repeats the proof of (4.14) starting from (4.28) since now we have to estimate $|\widehat{F}_j| = |f_j|$ (see (4.26)) instead of $|f_n|$. In this proof we have to use (4.26) instead of (4.27) and to replace everywhere n by j. Furthermore, instead of (4.44) we have to use a similar splitting of ν of the form $\nu = \chi_0 * \chi^{*s}$ with

$$\chi_0 = \Psi * \nu^{*k_0}(\,\cdot\,; \overline{r}) \quad \text{and} \quad \chi = \nu^{*\kappa}(\,\cdot\,; \overline{r}). \tag{4.53}$$

In particular, the splitting (4.53) yields that the integral corresponding to J in (4.50) now satisfies

$$J \le \sup_{y\in\mathbb{R}^d} \Big| \int \mathrm{e}\big\{ t\,\mathbb{Q}(x+y) \big\}\, \nu^{*\kappa}(dx; \overline{r}) \Big| \le \psi_0(t),$$

which leads to the factor ψ_0 in (4.15). □

5. Multiplicative inequalities for trigonometric sums

We shall consider polynomials $\mathbb{Q}$ defined by (1.38). Let $\mathbb{P}$ denote a polynomial of degree $\deg \mathbb{P} < p$. For $t \in \mathbb{R}$, introduce the trigonometric sum

$$\varphi(t) = \varphi(t; \mathbb{P}) = \Big| \int \mathrm{e}\big\{ t\,\mathbb{Q}(x) + t\,\mathbb{P}(x) \big\}\, \mu^{*(p+1)}(dx; r) \Big|. \tag{5.1}$$

In order to illustrate the basic argument in the following inequalities, assume for a while that $p = 2$ and $\mathbb{Q} = \mathbb{Q}_2$ is a quadratic form. For a system of weights q_x, $x \in \mathbb{R}$, such that only a finite number of q_x is non zero, define the trigonometric sum

$$S(t) = \Big| \sum_x p_x\, \mathrm{e}\big\{ t\,\mathbb{Q}_2(x) \big\} \Big|.$$

Notice that $\mathbb{Q}_2(x-u) = \mathbb{Q}_2(x) - 2\,\mathbb{Q}_2[x,u] + \mathbb{Q}_2(u)$, where $\mathbb{Q}_2[x,u]$ denotes the bilinear symmetric form associated with $\mathbb{Q}_2$. Hence, changing the variable $y = x - u$ of summation and using $\big|\,\mathrm{e}\big\{\,t\,\mathbb{Q}_2(u)\,\big\}\big| = 1$, we obtain

$$
\begin{aligned}
S(t-\varepsilon)\,S(t+\varepsilon) &= \Big|\sum_{x,y} p_x\,p_y\,\mathrm{e}\big\{\,(t+\varepsilon)\,\mathbb{Q}_2(x) - (t-\varepsilon)\,\mathbb{Q}_2(y)\,\big\}\Big| \\
&\le \sum_u \Big|\sum_x p_x\,p_{x-u}\,\mathrm{e}\Big\{\,t\,\big(\mathbb{Q}_2(x) - \mathbb{Q}_2(x-u)\big) + \varepsilon\big(\mathbb{Q}_2(x) + \mathbb{Q}_2(x-u)\big)\,\Big\}\Big| \\
&= \sum_u \Big|\sum_x p_x\,p_{x-u}\,\mathrm{e}\big\{\,2\,(t-\varepsilon)\,\mathbb{Q}_2[x,u] + 2\,\varepsilon\,\mathbb{Q}_2(x)\,\big\}\Big|,
\end{aligned}
\tag{5.2}
$$

for $t, \varepsilon \in \mathbb{R}$. The further bounds of the right hand side of (5.2) use symmetrization arguments and double large sieve type bounds They will depend on the coefficient (here $2\,\varepsilon$) of the leading part (here $2\,\varepsilon\,\mathbb{Q}_2(x)$) in the last exponent in (5.2), which is *independent* of t.

Write

$$\varphi(t;\mu,r) = \int\cdots\int \mathrm{e}\big\{\,t\,\mathbb{Q}_p[x_1,\dots,x_p]\,\big\}\,\widetilde{\mu}\big(dx_1;r\big)\dots\widetilde{\mu}\big(dx_p;r\big), \tag{5.3}$$

where $\widetilde{\mu}$ denotes symmetrization of the measure μ (see (2.2)), and $\mathbb{Q}_p[\dots]$ is the p-linear symmetric form associated with the polynomial $\mathbb{Q}_p(x)$, see (1.38).

Theorem 5.1. *Let $p = 2, 3, \dots$ and $\alpha = 2^{-p+1}$. Then the function φ defined by* (5.1) *satisfies the following multiplicative type inequality*

$$\varphi(t)\,\varphi(t+\varepsilon) \ll_{d,p} \varphi^{\alpha}(p!\,2^{-p}\,\varepsilon;\mu,2\,[r]), \qquad \text{for } t,\varepsilon \in \mathbb{R}. \tag{5.4}$$

In particular,

$$\sup_{\deg \mathbb{P} < p} \varphi(\varepsilon) \ll_{d,p} \varphi^{\alpha}(p!\,2^{-p}\,\varepsilon;\mu,2\,[r]). \tag{5.5}$$

Notice that the right hand side of (5.4) is independent of t and $\mathbb{P}$. In Theorem 5.2 below we show that the function $t \mapsto \varphi(t;\mu,r)$ satisfies a multiplicative inequality of type (5.4) as well.

Remark. For the case $p = 1$ the arguments used to prove (5.4) fail in the final stage of the proof, where we use the positivity of certain trigonometric sums (see the inequalities following (5.33)).

Proof of Theorem 5.1. The estimate (5.4) implies (5.5) in the case $t = 0$ since $\varphi(0) = 1$.
Let us prove (5.4). In order to prove (5.4) it suffices to show that

$$\varphi^{1/\alpha}(t-\varepsilon)\,\varphi^{1/\alpha}(t+\varepsilon) \ll_{d,p} \varphi(p!\,\alpha\varepsilon;\mu,2\,[r]), \qquad \text{for } \varepsilon,\ t \in \mathbb{R}. \tag{5.6}$$

Furthermore, without loss of generality we can assume that $\mathbb{Q} = \mathbb{Q}_p$, i.e., that

$$\varphi(t) = \Big| \int \mathrm{e}\big\{ t\,\mathbb{Q}_p(x) + t\,\mathbb{P}(x) \big\}\, \mu^{*(p+1)}(dx; r) \Big|, \tag{5.7}$$

with some polynomial $\mathbb{P}$ such that $\deg \mathbb{P} < p$.

Let us start with some simple algebraic identities:

$$\begin{aligned} &\mathbb{Q}_p(x) + \mathbb{P}(x) + \mathbb{Q}_p(y) + \mathbb{P}(y) = \alpha\, \mathbb{Q}_p(x+y) + \mathbb{S}_1, \\ &\mathbb{Q}_p(x) + \mathbb{P}(x) - \mathbb{Q}_p(y) - \mathbb{P}(y) = \mathbb{S}_2, \end{aligned} \tag{5.8}$$

where $\alpha = 2^{-p+1}$, and $\mathbb{S}_j = \mathbb{S}_j(u,v)$ are polynomials in $u = x - y$ and $v = x + y$ such that $\deg_v \mathbb{S}_j < p$, for $j = 1, 2$. To prove (5.8), consider a symmetric s-linear form $\mathbb{W}(x) = \mathbb{W}[x, \dots, x]$, for $x \in \mathbb{R}^d$. Writing $c_j = \binom{s}{j}$, we have

$$\mathbb{W}(u+v) = \sum_{j=0}^{s} c_j\, \mathbb{W}[u^{s-j}, v^j], \qquad \mathbb{W}(u-v) = \sum_{j=0}^{s} (-1)^j\, c_j\, \mathbb{W}[u^{s-j}, v^j], \tag{5.9}$$

where for brevity $\mathbb{W}[u^{s-j}, v^j] = \mathbb{W}[u, \dots, u, v \dots, v]$ (the variable v is repeated j times). Substituting in (5.9) $u = x + y$ and $v = x - y$, we get

$$\begin{aligned} 2^s\, \mathbb{W}(x) &= \sum_{j=0}^{s} c_j\, \mathbb{W}\big[(x+y)^{s-j}, (x-y)^j \big], \\ 2^s\, \mathbb{W}(y) &= \sum_{j=0}^{s} (-1)^j\, c_j\, \mathbb{W}\big[(x+y)^{s-j}, (x-y)^j \big], \end{aligned}$$

whence

$$\mathbb{W}(x) + \mathbb{W}(y) = 2^{-s+1}\, \mathbb{W}(x+y) + 2^{-s+1} \sum_{j \in 2\mathbb{N},\ 2 \le j \le s} c_j\, \mathbb{W}\big[(x+y)^{s-j}, (x-y)^j \big], \tag{5.10}$$

and

$$\mathbb{W}(x) - \mathbb{W}(y) = 2^{-s+1} \sum_{j \notin 2\mathbb{N},\ 1 \le j \le s} c_j\, \mathbb{W}\big[(x+y)^{s-j}, (x-y)^j \big]. \tag{5.11}$$

Writing $\mathbb{Q}_p(x) = \mathbb{Q}_p[x, \dots, x]$, decomposing $\mathbb{P}$ into a sum of multi-linear symmetric forms and applying (5.10) and (5.11), we easily derive (5.8).

Let us return to the proof of (5.6). Using (5.7), we have

$$\varphi(t-\varepsilon)\, \varphi(t+\varepsilon) = \Big| \iint \mathrm{e}\big\{ I \big\}\, \mu^{*(p+1)}(dx; r)\, \mu^{*(p+1)}(dy; r) \Big| \tag{5.12}$$

with

$$\begin{aligned} I &= (t+\varepsilon)\big(\mathbb{Q}_p(x) + \mathbb{P}(x)\big) - (t-\varepsilon)\big(\mathbb{Q}_p(y) + \mathbb{P}(y)\big) \\ &= t\Big(\mathbb{Q}_p(x) + \mathbb{P}(x) - \mathbb{Q}_p(y) - \mathbb{P}(y)\Big) + \varepsilon\Big(\mathbb{Q}_p(x) + \mathbb{P}(x) + \mathbb{Q}_p(y) + \mathbb{P}(y)\Big). \end{aligned}$$

Using (5.8), we can write

$$I = \alpha\varepsilon\,\mathbb{Q}_p(x+y) + \mathbb{S}(x+y) \tag{5.13}$$

with some polynomial $v \mapsto \mathbb{S}(v)$ such that $\deg_v \mathbb{S} < p$. Coefficients of $\mathbb{S}$ can depend on t, ε and $x - y$. Using the definition of a convolution of measures and (5.13), we can rewrite (5.12) as

$$\varphi(t-\varepsilon)\varphi(t+\varepsilon) = \Big| \int \cdots \int \mathrm{e}\big\{ I \big\} \prod_{j=1}^{p+1} \big(\mu(dx_j; r)\, \mu(dy_j; r) \big) \Big| \tag{5.14}$$

where

$$I = \alpha\varepsilon\,\mathbb{Q}_p(z) + \mathbb{S}(z), \qquad z \overset{\mathrm{def}}{=} \sum_{j=1}^{p+1} (x_j + y_j),$$

with a polynomial $z \mapsto \mathbb{S}(z)$ such that $\deg_z \mathbb{S} < p$. Coefficients of $\mathbb{S}$ can depend on t, ε and $x_j - y_j$, for $1 \le j \le p+1$.

We are going to change variables in the integrals in (5.14). Let us describe this change which unfortunately is somewhat cumbersome. The integrand $\mathrm{e}\big\{ I \big\}$ in (5.14) is a function of $u_j = x_j - y_j$ and $v_j = x_j + y_j$. Therefore it will be convenient to use the following formula

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(x-y, x+y)\, \mu(dx; r)\, \mu(dy; r) = \int_{\mathbb{R}^d} \Big(\gamma(u) \int_{\mathbb{R}^d} h(u, v)\, \eta(dv; u) \Big)\, \mu\big(du; 2\,[r]\big), \tag{5.15}$$

which holds for arbitrary function $h : \mathbb{R}^d \to \mathbb{C}$, with some weight function $\gamma(u)$ and measure $\eta(\,\cdot\,; u)$ which depends on the parameter u. Let us start the description of the construction of $\gamma(u)$ and $\eta(\,\cdot\,; u)$ by the the following simple identity

$$\sum_{n\in\mathbb{Z}: |n|\le r} \ \sum_{m\in\mathbb{Z}: |m|\le r} h(n-m, n+m) = \sum_{j\in\mathbb{Z}: |j|\le 2\,[r]} \ \sum_{k\in\mathbb{Z}: |k|\le 2\,[r]-|j|,\, j-k\in 2\mathbb{Z}} h(j,k). \tag{5.16}$$

Notice that the last sum in (5.16) is taken over *even* k if j is even, and over *odd* k if j is odd, such that $|k| \le 2\,[r] - |j|$. To prove (5.16) it suffices to reorder the summands in the first double sum in (5.16).

Recalling that $\mu(\,\cdot\,; r)$ is the uniform lattice measure supported by lattice points in the cube $B(r) \subset \mathbb{R}^d$, let $\mu_0(\,\cdot\,; r)$ denote the uniform lattice measure supported by integer points in the interval $[-r, r] \subset \mathbb{R}$. Using the measure $\mu_0(\,\cdot\,; r)$, we can rewrite the left hand side of (5.16) as

$$(2\overline{r})^2 \int_{\mathbb{R}} \int_{\mathbb{R}} h(x-y, x+y)\, \mu_0(dx; r)\, \mu_0(dy; r), \qquad 2\overline{r} = 2\,[r] + 1. \tag{5.17}$$

In order to rewrite the right hand side of (5.16) in a form similar to (5.17), let us define the measure $\theta(\cdot\,; j)$ on $\mathbb{R}$. This measure depends on the parameter $j \in \mathbb{Z}$ such that $|j| \le 2\,[r]$ and assigns equal weights

$$a_0(j) = (2\,\overline{r} - |j|)^{-1} \tag{5.18}$$

to *even* integer points in the interval $[-2[r]+|j|-1, 2[r]-|j|+1]$, if j is even, and respectively to *odd* points, if j is odd. Using the measure $\theta(\cdot;j)$ we can rewrite the right hand side of (5.16) as

$$(4[r]+1)\int_{\mathbb{R}}\Big(\frac{1}{a_0(j)}\int_{\mathbb{R}} h(u,v)\,\theta(dv;u)\Big)\,\mu_0\big(du;2[r]\big). \tag{5.19}$$

Hence, the identity (5.16) combined with (5.17) and (5.19) yields

$$\int_{\mathbb{R}}\int_{\mathbb{R}} h(x-y,x+y)\,\mu_0(dx;r)\,\mu_0(dy;r) = \int_{\mathbb{R}}\Big(\gamma_0(u)\int_{\mathbb{R}} h(u,v)\,\theta(dv;u)\Big)\,\mu_0\big(du;2[r]\big), \tag{5.20}$$

where

$$\gamma_0(j) = (4[r]+1)\,(2\overline{r})^{-2}/a_0(j), \qquad \text{for } j\in\mathbb{Z},\ |j|\le 2[r], \tag{5.21}$$

with $a_0(j)$ defined by (5.18). For $|j|\ge 2[r]$ or $j\notin\mathbb{Z}$ we define $\gamma_0(j)=0$.

The formula (5.20) obviously extends to the multidimensional case and provides the identity (5.15) replacing weights (respectively measures) by products of weights (respectively measures). Notice that the measure $\mu(\,\cdot\,;r)$ is a product measure on $\mathbb{R}^d$, that is, $\mu(dx;r)=\prod_{k=1}^{d}\mu_0(dx_{(k)};r)$, where a vector $x\in\mathbb{R}^d$ in coordinates is represented as $x=(x_{(1)},\dots,x_{(d)})$. Write

$$\gamma(u)=\prod_{k=1}^{d}\gamma_0(u_{(k)}), \qquad \text{for } u=(u_{(1)},\dots,u_{(d)})\in\mathbb{R}^d, \tag{5.22}$$

and define the product measure

$$\eta(dv;u)=\prod_{k=1}^{d}\theta(dv_{(k)};u_{(k)}).$$

Introducing the weights

$$a(u)=\prod_{k=1}^{d}a_0(u_{(k)}), \qquad |u|_\infty\le 2[r], \quad u=(u_{(1)},\dots,u_{(d)})\in\mathbb{R}^d, \tag{5.23}$$

with $a_0(j)$ defined by (5.18), we can describe the measure $\eta(\cdot;u)$ as a measure supported by a sub-lattice of $\mathbb{Z}^d$ (in particular, we have $\eta(\mathbb{Z}^d;u)=1$) and such that $\eta\big(\{z\};u\big)$ is equal either to $a(u)$ or to 0, for any $z\in\mathbb{Z}$. Moreover, $\eta(\cdot;u)$ is supported by the cube

$$\prod_{j=1}^{d}\big[-2[r]+|u_j|,\,2[r]-|u_j|\big]\subset B\big(2[r]\big).$$

Let us return to the proof of (5.6). Writing

$$x_j-y_j=u_j, \quad x_j+u_j=v_j, \quad 1\le j\le p+1, \qquad \gamma=\prod_{j=1}^{p+1}\gamma(u_j) \tag{5.24}$$

with $\gamma(u)$ defined by (5.22), and applying the identity (5.15) $p+1$ times, we can rewrite (5.14) as

$$\varphi(t-\varepsilon)\,\varphi(t+\varepsilon) = \Big|\int\cdots\int\gamma\,\Big(\int\cdots\int \mathrm{e}\big\{I\big\}\prod_{j=1}^{p+1}\eta(dv_j;u_j)\Big)\prod_{j=1}^{p+1}\mu\big(du_j;2[r]\big)\Big| \tag{5.25}$$

where

$$I=\alpha\varepsilon\,\mathbb{Q}_p(v_1+\cdots+v_{p+1})+\mathbb{S}(v_1+\cdots+v_{p+1}) \tag{5.26}$$

with a polynomial $z\mapsto\mathbb{S}(z)$ such that $\deg\mathbb{S}<p$. Coefficients of $\mathbb{S}$ can depend on t,ε and u_j, for $1\le j\le p+1$.

Hölder's inequality and (5.25) yield

$$\varphi^{1/\alpha}(t-\varepsilon)\varphi^{1/\alpha}(t+\varepsilon)\le\int\cdots\int\gamma^{1/\alpha}\,J\prod_{j=1}^{p+1}\mu\big(du_j;2[r]\big) \tag{5.27}$$

with

$$J=\Big|\int\cdots\int\mathrm{e}\big\{I\big\}\prod_{j=1}^{p+1}\eta(dv_j;u_j)\Big|^{1/\alpha}.$$

In order to estimate J, let us apply the symmetrization Lemma 7.1 replacing the polynomial $\mathbb{Q}$ by I (see (5.26), the measures μ_j by $\eta(\cdot;u_j)$, for $1\le j\le p+1$, and choosing the measure ν such that $\nu\big(\{0\}\big)=1$. We obtain

$$2\,J\le J_1+J_2,\qquad\qquad J_1\overset{\text{def}}{=}\int\cdots\int\mathrm{e}\big\{\alpha p!\varepsilon\,\mathbb{Q}[v_1,\dots,v_p]\big\}\prod_{j=1}^{p}\widetilde{\eta}(dv_j;u_j), \tag{5.28}$$

where the integral J_2 is defined as J_1 but replacing v_p and $\widetilde{\eta}(\cdot;u_p)$ by v_{p+1} and $\widetilde{\eta}(\cdot;u_{p+1})$ respectively. Substituting $J\le(J_1+J_2)/2$ in (5.27) and noting that the integrals obtained are equal (just the enumerations of variables of integration in these integrals are different), we get

$$\varphi^{1/\alpha}(t-\varepsilon)\varphi^{1/\alpha}(t+\varepsilon)\le\int\cdots\int\gamma^{1/\alpha}\,J_1\prod_{j=1}^{p+1}\mu\big(du_j;2[r]\big). \tag{5.29}$$

The estimate (5.29) implies (5.6) provided that we show that

$$J_1\ll_{d,p}\varphi(p!\,\alpha\varepsilon;\mu,2[r])\;\prod_{j=1}^{p}\big(\overline{r}^{\,2d}\,a^2(u_j)\big), \tag{5.30}$$

and

$$\gamma^{1/\alpha}\;\prod_{j=1}^{p}\big(\overline{r}^{\,2d}\,a^2(u_j)\big)\ll_{d,p}1,\qquad\text{for }\;|u_j|_\infty\le 2[r]. \tag{5.31}$$

Let us prove (5.31). The quantity γ is a product of $\gamma(u_j)$, see (5.24). Both $\gamma(u_j)$ and $a(j)$ are products as well, see (5.22) and (5.23). Hence, the proof of (5.31) reduces to showing that

$$\gamma_0(k) \ll 1, \qquad \gamma_0^{1/\alpha}(k)\,\overline{r}^{\,2}\,a_0^2(k) \ll 1, \qquad \text{for } |k| \le 2\,[r]. \tag{5.32}$$

Using the definitions (5.21) and (5.18) of $\gamma_0(k)$ and $a_0(k)$, the first inequality in (5.32) obviously holds, and the second one reduces to $\big((2\overline{r}-k)/\overline{r}\big)^{-2+1/\alpha} \ll 1$, for $|k| \le 2\,[r]$. But this is fulfilled since we assume that $p \ge 2$ and therefore $1/\alpha \ge 2$.

To conclude the proof of the lemma we have to prove (5.30). The proof consists of step-wise replacements of the measures $\widetilde{\eta}(\cdot\,; u_j)$, which are present in the integral J_1 defined by (5.28), by the measures $\widetilde{\mu}(\cdot\,; 2\,[r])$, for $1 \le j \le p$. The resulting integral is just

$$\varphi(p!\,\alpha\,\varepsilon; \mu, 2[r])\,\prod_{j=1}^{p}\big(\overline{r}^{\,2d}\,a^2(u_j)\big)$$

with the trigonometric sum $\varphi(p!\,\alpha\,\varepsilon; \mu, 2[r])$ defined by (5.3). We shall provide here all details related to such a replacement in the case of the measure $\widetilde{\eta}(\cdot\,; u_1)$ only; the replacements of $\widetilde{\eta}(\cdot\,; u_j)$, for $j = 2, \dots, p$ are similar. By the definition of symmetrization of measures, we can represent J_1 as follows

$$J_1 = \int f(v_1)\widetilde{\eta}(dv_1; u_1) = \iint f(x_1 - x_2)\,\eta(dx_1; u_1)\,\eta(dx_2; u_1) \tag{5.33}$$

with

$$\begin{aligned} f(v_1) &= \int \cdots \int \mathrm{e}\big\{\,\alpha\,p!\,\varepsilon\,\mathbb{Q}[v_1, \dots, v_p]\,\big\}\,\prod_{j=2}^{p}\widetilde{\eta}(dv_j; u_j) \\ &= \int \cdots \int \Big|\int \mathrm{e}\big\{\,\alpha\,p!\,\varepsilon\,\mathbb{Q}[v_1, \dots, v_p]\,\big\}\,\eta(dv_2; u_2)\Big|^2\,\prod_{j=3}^{p}\widetilde{\eta}(dv_j; u_j) \\ &\ge 0. \end{aligned}$$

Using the *positivity* of f, the representation (5.33) and properties of the measure $\eta(\cdot\,; u)$ described in (5.15)–(5.23), we obtain

$$\begin{aligned} J_1 &\le a^2(u_1) \sum_{x_1, x_2 \in \mathbb{Z}^d,\ |x_1|_\infty \le 2\,[r],\ |x_2|_\infty \le 2\,[r]} f(x_1 - x_2) \\ &\ll \overline{r}^{\,2\,d}\,a^2(u_1)\,\iint f(x_1 - x_2)\,\mu(dx_1; 2\,[r])\,\mu(dx_2; 2\,[r]) \\ &= \overline{r}^{\,2\,d}\,a^2(u_1)\,\int f(x_1)\,\widetilde{\mu}(dx_1; 2\,[r]) \\ &= \overline{r}^{\,2\,d}\,a^2(u_1)\,\int \cdots \int \mathrm{e}\big\{\,\alpha p!\,\varepsilon\,\mathbb{Q}[x_1, v_2, \dots, v_p]\,\big\}\,\widetilde{\mu}(dx_1; 2\,[r])\,\prod_{j=2}^{p}\widetilde{\eta}(dv_j; u_j), \end{aligned}$$

thus replacing $\widetilde{\eta}(\cdot\,; u_1)$ by $\widetilde{\mu}(\cdot\,; 2\,[r])$. $\square$

Theorem 5.2. *Let $p = 2, 3, \dots$ and $\beta = 2^{-p}$. Then the function $t \to \varphi(t;\mu,[r])$ defined by* (5.3) *satisfies the following multiplicative inequality*

$$\varphi(t;\mu,[r])\,\varphi(t+\varepsilon;\mu,[r]) \ll_{d,p} \varphi^{\beta}(\beta\,\varepsilon;\mu,2\,[r]), \qquad \text{for } t,\varepsilon \in \mathbb{R}. \tag{5.34}$$

In particular,

$$\varphi(\varepsilon;\mu,[r]) \ll_{d,p} \varphi^{\beta}(\beta\,\varepsilon;\mu,2\,[r]). \tag{5.35}$$

Notice that the right hand side of (5.34) is independent of t.

Proof of Theorem 5.2. It suffices to prove (5.34) since (5.35) follows from (5.34) substituting $t = 0$ and using $\varphi(0;\mu,[r]) = 1$.

The proof of (5.34) is similar to that of Theorem 5.1. The main changes are that now we use the algebraic identities (5.39) instead of (5.8), and that instead of Lemma 7.1 we apply the symmetrization Lemma 7.2. In order to prove (5.34) it suffices to show that

$$\varphi^{1/\beta}(t-\varepsilon;\mu,[r])\,\varphi^{1/\beta}(t+\varepsilon;\mu,[r]) \ll_{d,p} \varphi(2^{-p+1}\,\varepsilon;\mu,2\,[r]), \qquad \text{for } \varepsilon,\, t \in \mathbb{R}. \tag{5.36}$$

Let us start with some simple algebraic identities:

$$\begin{aligned} \mathbb{Q}_p[x_1,\dots,x_p] + \mathbb{Q}_p[y_1,\dots,y_p] &= 2^{-p+1}\,\mathbb{Q}_p[x_1+y_1,\dots,x_p+y_p] + \sum_{j=1}^{p} h_j, \\ \mathbb{Q}_p[x_1,\dots,x_p] - \mathbb{Q}_p[y_1,\dots,y_p] &= \sum_{j=1}^{p} g_j, \end{aligned} \tag{5.37}$$

where, for given $x_j - y_j = u_j$, $1 \le j \le p$, the functions

$$h_j = h_j(v_1,\dots,v_{j-1},v_{j+1},\dots,v_p) \quad \text{and} \quad g_j = g_j(v_1,\dots,v_{j-1},v_{j+1},\dots,v_p) \tag{5.38}$$

are independent of the variable $v_j = x_j + y_j$, for all $1 \le j \le p$. Both h_j and g_j may depend on $u_1,\dots,u_p$.

The identities (5.37) easily follow from the obvious relations

$$\begin{aligned} \mathbb{Q}_p[z_1+w_1,\dots,z_p+w_p] &= \mathbb{Q}_p[z_1,\dots,z_p] + \sum_{j=1}^{p} h_j^*, \\ \mathbb{Q}_p[z_1-w_1,\dots,z_p-w_p] &= \mathbb{Q}_p[z_1,\dots,z_p] + \sum_{j=1}^{p} g_j^*, \end{aligned} \tag{5.39}$$

where the functions

$$h_j^* = h_j^*(z_1,\dots,z_{j-1},z_{j+1},\dots,z_p) \quad \text{and} \quad g_j^* = g_j^*(z_1,\dots,z_{j-1},z_{j+1},\dots,z_p)$$

are independent of the variable z_j, for all $1 \le j \le p$. Both h_j^* and g_j^* can depend on $w_1,\dots,w_p$. Indeed, substituting $z_j = x_j + y_j$ and $w_j = x_j - y_j$ in (5.39), we easily derive (5.37).

Let us return to the proof of (5.36). Obviously,

$$\varphi(t-\varepsilon;\mu,[r])\,\varphi(t+\varepsilon;\mu,[r]) = \Big|\int\cdots\int \mathrm{e}\big\{\,I\,\big\}\ \prod_{j=1}^{p}\widetilde{\mu}(dx_j;[r])\ \prod_{j=1}^{p}\widetilde{\mu}(dy_j;[r])\Big| \tag{5.40}$$

with

$$\begin{aligned} I &= (t+\varepsilon)\,\mathbb{Q}_p[x_1,\dots,x_p]-(t-\varepsilon)\,\mathbb{Q}_p[y_1,\dots,y_p] \\ &= t\Big(\mathbb{Q}_p[x_1,\dots,x_p]-\mathbb{Q}_p[y_1,\dots,y_p]\Big)+\varepsilon\Big(\mathbb{Q}_p[x_1,\dots,x_p]+\mathbb{Q}_p[y_1,\dots,y_p]\Big). \end{aligned}$$

Using (5.37), we can write

$$I = 2^{-p+1}\,\varepsilon\,\mathbb{Q}_p[x_1+y_1,\dots,x_p+y_p]+\sum_{j=1}^{p} f_j \tag{5.41}$$

with some f_j as in (5.38) independent of $v_j = x_j + y_j$. The functions f_j can depend on t, ε and $u_j = x_j - y_j$, $1 \le j \le p$.

By the definition of symmetrizations of measures we have,

$$\int h(z_j)\,\widetilde{\mu}(dz_j;[r]) = \iint h(z_j-\bar z_j)\,\mu(dz_j;[r])\mu(d\bar z_j;[r]), \tag{5.42}$$

for any function h. Using (5.42) with $z_j = x_j$ and $z_j = y_j$, for $1 \le j \le p$, and applying Hölder's inequality, we derive from (5.40) that

$$\varphi^{1/\beta}(t-\varepsilon;\mu,[r])\,\varphi^{1/\beta}(t+\varepsilon;\mu,[r]) \le \int\cdots\int \Gamma\ \prod_{j=1}^{p}\mu(d\bar x_j;[r])\ \prod_{j=1}^{p}\mu(d\bar y_j;[r]) \tag{5.43}$$

with

$$\Gamma = \Big|\int\cdots\int \mathrm{e}\big\{\,I\,\big\}\ \prod_{j=1}^{p}\mu(dx_j;[r])\ \prod_{j=1}^{p}\mu(dy_j;[r])\Big|^{1/\beta} \tag{5.44}$$

and I as in (5.41); however now the functions f_j can depend on $\bar x_j$ and $\bar y_j$ as well. To estimate Γ, we can apply the change of variables described in (5.15)–(5.23). Applying in addition Hölder's inequality (cf. the proof of (5.27)), we obtain

$$\Gamma \le \int\cdots\int \gamma^{1/\beta}\,J\ \prod_{j=1}^{p}\mu(du_j;2\,[r]) \tag{5.45}$$

with

$$J = \Big|\int\cdots\int \mathrm{e}\big\{\,I\,\big\}\ \prod_{j=1}^{p}\eta(dv_j;u_j)\Big|^{1/\beta}$$

and $\gamma = \prod_{j=1}^{p}\gamma(u_j)$ (see (5.22) for the definition of $\gamma(u_j)$). In order to estimate J, let us apply the symmetrization Lemma 7.2 with measures μ_j replaced by $\eta(\cdot;u_j)$, for $1 \le j \le p$. We obtain

$$2\,J \le J_1 \overset{\mathrm{def}}{=} \int\cdots\int \mathrm{e}\big\{\,2^{1-p}\,\varepsilon\,\mathbb{Q}[v_1,\dots,v_p]\,\big\}\ \prod_{j=1}^{p}\widetilde{\eta}(dv_j;u_j). \tag{5.46}$$

Proceeding similarly to the proof of (5.30) and (5.31) we have

$$\gamma^{1/\beta}\, J_1 \ll_{d,p} \varphi(2^{1-p}\,\varepsilon;\mu, 2\,[r]).$$

Therefore collecting of (5.43)–(5.46) yields (5.36). $\square$

6. The integration procedure for large $|t|$

Recall that

$$\mathcal{M}(t;s) = \big(|t|\,s\big)^{-1}\,\mathbf{I}\big\{\,|t| \le s^{-1/2}\big\} + |t|\,\mathbf{I}\big\{\,|t| > s^{-1/2}\big\}. \tag{6.1}$$

where $s \ge 1$ will be a positive large parameter.

For a number $T \ge 1$ and a family of functions $\varphi(\,\cdot\,) = \varphi(\,\cdot\,;s) : \mathbb{R} \to \mathbb{R}$ introduce

$$\gamma = \gamma\big(s, T\big) \overset{\text{def}}{=} \sup\Big\{\,\big|\,\varphi(t)\,\big| :\ s^{-1/2} \le t \le T\Big\}. \tag{6.2}$$

Theorem 6.1 (see BG (1997c), Theorem 5.1)**.** *Let $\varphi(t)$, $t \ge 0$ denote a continuous function such that $\varphi(0) = 1$ and $0 \le \varphi \le 1$. Assume that, for some $\varkappa > 4$ and $\Lambda \ge 1$,*

$$\varphi(t)\,\varphi(t+\varepsilon) \le \Lambda\,\mathcal{M}^{\varkappa}(\varepsilon;s), \qquad \text{for all} \quad t \ge 0 \quad \text{and} \quad \varepsilon \ge 0. \tag{6.3}$$

Let $T \ge 1$. Assume that the number γ defined by (6.2) satisfies

$$\begin{aligned} &\gamma > 4^{\varkappa/(\varkappa-4)}\, s^{-\varkappa/4}, && \text{if} \ \ -1 < \alpha \le 0,\\ &\gamma\,\Big(1 + \ln\frac{1}{\gamma}\Big)^{\varkappa/(\varkappa-4)} > 4^{\varkappa/(\varkappa-4)}\, s^{-\varkappa/4}\,(1+\ln s)^{\varkappa/(\varkappa-4)}, && \text{if} \ \ \alpha = -1. \end{aligned} \tag{6.4}$$

Then the integral

$$J = \int_{s^{-1/2}}^{T} \varphi(t)\, t^{\alpha}\, dt$$

can be bounded as follows

$$J \ll_{\alpha,\varkappa} \Big(\frac{\gamma}{\Lambda}\Big)^{1-4/\varkappa}\, T^{\alpha+1}\,\frac{\Lambda}{s}, \qquad \text{for} \ \ -1 < \alpha \le 0, \tag{6.5}$$

and

$$J \ll_{\alpha,\varkappa} \Big(\frac{\gamma}{\Lambda}\Big)^{1-4/\varkappa}\Big(1 + \ln\frac{\Lambda}{\gamma}\Big)\,(1+\ln T)\,\frac{\Lambda}{s}, \qquad \text{for} \ \ \alpha = -1. \tag{6.6}$$

If (6.4) is not fulfilled then the following trivial bounds hold

$$J \ll_{\alpha} \gamma T^{\alpha+1}, \quad \text{for} \ \ \alpha > -1, \qquad J \ll \gamma\,(1+\ln s)\,(1+\ln T), \quad \text{for} \ \ \alpha = -1. \tag{6.7}$$

Corollary 6.2. *Under the conditions of Theorem* 6.1 *we have*

$$\int_{s^{-1/2}}^{T} \varphi(t)\, \frac{dt}{t} \ll_{\varkappa,\varepsilon} \frac{\Lambda}{s}\, T^{\varepsilon}\, \gamma^{1-\varepsilon-4/\varkappa}, \qquad \textit{for } s \geq 1, \;\; \varepsilon > 0. \tag{6.8}$$

Proof. The bound (6.8) is implied by Theorem 6.1 with $\alpha = -1$. The proof depends on whether the condition (6.4) holds or not. If (6.4) is fulfilled then (6.8) follows from (6.6) using the inequalities $\gamma \leq 1$, $T \geq 1$, $\Lambda \geq 1$ and applying $\ln u \ll_{\varepsilon} u^{\varepsilon}$ with $u = T$ and $u = \Lambda/\gamma$. If (6.4) does not hold then, in particular, we have

$$\gamma \ll_{\varkappa,\varepsilon_1} s^{-\varkappa/4+\varepsilon_1} \qquad \text{for any } \varepsilon_1 > 0. \tag{6.9}$$

Splitting $\gamma = \gamma^{1-\varepsilon-4/\varkappa}\gamma^{\varepsilon+4/\varkappa}$, using the first bound in (6.7) and estimating $\gamma^{\varepsilon+4/\varkappa}$ by (6.9), we derive (6.8). $\square$

Corollary 6.3. *Under the conditions of Theorem* 6.1 *we have*

$$\int_{s^{-1/2}}^{T} \varphi(t)\, dt \ll_{\varkappa} \Lambda\, T/s, \qquad \textit{for } s \geq 1. \tag{6.10}$$

Proof. Using Theorem 6.1 with $\alpha = 0$, the proof is similar to that of Corollary 6.2 provided that we estimate $\gamma \leq 1$. $\square$

7. Symmetrization inequalities

The following symmetrization inequality is a generalization of a well-known classical inequality due to Weyl (1915/1916). In the case $p = 2$ the inequality as Lemma 7.1 was used in BG (1997c). In probability similar inequalities were extensively used to estimate the accuracy of normal approximations in the CLT on balls in Euclidean multidimensional spaces starting with a paper of Götze (1979) and followed by other authors, e.g., Yurinskii (1982), Bentkus (1984), Nagaev and Chebotarev (1986), Zalesskii, Sazonov and Ulyanov (1988), BG (1987a,b).

Recall that $\mathbb{Q} = \sum_{j=0}^{p} \mathbb{Q}_j$ denotes a real-valued polynomial on $\mathbb{R}^d$ (see (1.38)). With $\mathbb{Q}_j$ we associate a j-linear symmetric form $\mathbb{Q}_j[x_1, \dots, x_j]$ such that $\mathbb{Q}_j(x) = \mathbb{Q}_j[x, \dots, x]$, for $x \in \mathbb{R}^d$ (see (1.38)). The symmetrization $\widetilde{\mu}$ of a measure μ is defined by (1.40).

Lemma 7.1. *Let $\mathbb{Q}$ be a polynomial of order $p \geq 1$. Let ν and μ_j, $1 \leq j \leq p+1$, denote measures on $\mathbb{R}^d$. Then the integral*

$$J = \Big| \int \cdots \int \mathrm{e}\big\{ \mathbb{Q}(x_1 + \cdots + x_{p+1} + w) \big\} \nu(dw) \prod_{j=1}^{p+1} \mu_j(dx_j) \Big|^{2^{p-1}}$$

satisfies $2\,J \leq J_1 + J_2$, where

$$J_1 = \int \cdots \int \mathrm{e}\big\{ p!\,\mathbb{Q}_p[x_1, \dots, x_{p-1}, x_p] \big\} \prod_{j=1}^{p} \widetilde{\mu}_j(dx_j),$$

$$J_2 = \int \cdots \int \mathrm{e}\big\{ p!\,\mathbb{Q}_p[x_1, \dots, x_{p-1}, x_{p+1}] \big\} \,\widetilde{\mu}_{p+1}(dx_{p+1}) \prod_{j=1}^{p-1} \widetilde{\mu}_j(dx_j).$$

In particular, if $\mu_p = \mu_{p+1}$ then $J \leq J_1$.

Proof. We shall prove the lemma using induction in p.

Let us prove the lemma for $p = 1$. We have $\mathbb{Q}(x) = a_0 + \langle a, x \rangle$, for some $a_0 \in \mathbb{R}$ and $a \in \mathbb{R}^d$. Therefore

$$\begin{aligned} 2\,J &= 2 \Big| \iiint \mathrm{e}\big\{ a_0 + \langle a, x_1 + x_2 + w \rangle \big\} \,\mu_1(dx_1)\, \mu_2(dx_2)\, \nu(dw) \Big| \\ &\leq 2 \int \Big| \iint \mathrm{e}\big\{ \langle a, x_1 + x_2 \rangle \big\} \,\mu_1(dx_1)\, \mu_2(dx_2) \Big| \,\nu(dw) \\ &= 2 \Big| \int \mathrm{e}\big\{ \langle a, x \rangle \big\} \,\mu_1(dx) \Big| \, \Big| \int \mathrm{e}\big\{ \langle a, x \rangle \big\} \,\mu_2(dx) \Big| \leq J_1 + J_2 \end{aligned}$$

provided that we apply $2\,a\,b \leq a^2 + b^2$ and notice that $\Big| \int \mathrm{e}\big\{ \langle a, x \rangle \big\} \,\mu_1(dx) \Big|^2 = J_1$.

Assuming that we have proved the lemma for $1, \dots, p-1$, let us prove it for p. Introduce the measure $\theta = \mu_2 * \cdots * \mu_{p+1} * \nu$. Using Fubini's Theorem and Hölder's inequality, we obtain

$$J \leq \Big(\int \Big| \int \mathrm{e}\big\{ \mathbb{Q}(x + y) \big\} \,\mu_1(dx) \Big|^2 \theta(dy) \Big)^{2^{p-2}}. \tag{7.1}$$

We can write

$$\Big| \int \mathrm{e}\big\{ \mathbb{Q}(x + y) \big\} \,\mu_1(dx) \Big|^2 = \iint \mathrm{e}\big\{ \mathbb{Q}(x_1 + y) - \mathbb{Q}(\bar{x}_1 + y) \big\} \,\mu_1(dx_1)\, \mu_1(d\bar{x}_1). \tag{7.2}$$

Using the decomposition $\mathbb{Q}(x+y) = \mathbb{Q}_p[x+y, \dots, x+y] + \mathbb{Q}_{p-1}[x+y, \dots, x+y] + \mathbb{S}_1$, where $\mathbb{S}_1 = \mathbb{S}_1(x, y)$ is a polynomial in the variables x and y such that its degree $\deg_y \mathbb{S}_1$ in y is at most $p-2$, it is easy to verify that

$$\mathbb{Q}(x_1 + y) - \mathbb{Q}(\bar{x}_1 + y) = p\,\mathbb{Q}_p[x_1 - \bar{x}_1, y, \dots, y] + \mathbb{S},$$

where $\mathbb{S} = \mathbb{S}(x_1, \bar{x}_1, y)$ is a polynomial in $x_1, \bar{x}_1, y$ such that $\deg_y \mathbb{S} \le p-2$. Thus (7.1) and (7.2) combined with Fubini's Theorem and Hölder's inequality imply

$$\begin{aligned} J &\le \Big(\iint \Big| \int \mathrm{e}\big\{ p\,\mathbb{Q}_p[x_1 - \bar{x}_1, y, \dots, y] + \mathbb{S} \big\}\, \theta(dy) \Big|\, \mu_1(dx_1)\, \mu_1(d\bar{x}_1) \Big)^{2^{p-2}} \\ &\le \iint \Big| \int \mathrm{e}\big\{ p\,\mathbb{Q}_p[x_1 - \bar{x}_1, y, \dots, y] + \mathbb{S} \big\}\, \theta(dy) \Big|^{2^{p-2}} \mu_1(dx_1)\, \mu_1(d\bar{x}_1). \end{aligned} \tag{7.3}$$

To the inner integral in (7.3) we can apply the induction assumption for $p-1$. Indeed, the leading part of the polynomial

$$p\,\mathbb{Q}_p[x_1 - \bar{x}_1, y, \dots, y] + \mathbb{S}$$

in y equals to $p\,\mathbb{Q}_p[x_1 - \bar{x}_1, y, \dots, y]$, which is a polynomial of order $p-1$ in the variable $y = x_2 + \dots + x_{p+1} + w$. Thus, using the induction assumption, we have

$$\begin{aligned} 2 \Big| \int & \mathrm{e}\big\{ p\,\mathbb{Q}_p[x_1 - \bar{x}_1, y, \dots, y] + \mathbb{S} \big\}\, \theta(dy) \Big|^{2^{p-2}} \\ &\le \int \cdots \int \mathrm{e}\big\{ p!\,\mathbb{Q}_p[x_1 - \bar{x}_1, x_2, \dots, x_p] \big\} \prod_{j=2}^{p} \widetilde{\mu}_j(dx_j) \\ &\quad + \int \cdots \int \mathrm{e}\big\{ p!\,\mathbb{Q}_p[x_1 - \bar{x}_1, x_2, \dots, x_{p-1}, x_{p+1}] \big\} \widetilde{\mu}_{p+1}(dx_{p+1}) \prod_{j=1}^{p-1} \widetilde{\mu}_j(dx_j). \end{aligned} \tag{7.4}$$

Integrating (7.4) with respect to the measure $\mu_1(dx_1)\, \mu_1(d\bar{x}_1)$, using (7.3) and noticing that $\iint f(x - \bar{x})\, \mu(dx)\, \mu(d\bar{x}) = \int f(x)\, \widetilde{\mu}(dx)$ holds for arbitrary functions f and measures μ, we obtain $2\,J \le J_1 + J_2$. $\square$

Lemma 7.2. *Let $\mathbb{Q}_p$ be a symmetric p-linear form of order $p \ge 1$. Let f_j, $1 \le j \le p$, denote real-valued functions such that the function*

$$f_j = f_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_p)$$

is independent of the variable x_j. Let μ_j, $1 \le j \le p$, denote measures on $\mathbb{R}^d$. Then the integral

$$J = \Big| \int \cdots \int \mathrm{e}\big\{ \mathbb{Q}_p[x_1, \dots, x_p] + f_1 + \dots + f_p \big\} \prod_{j=1}^{p} \mu_j(dx_j) \Big|^{2^p}$$

satisfies $J \le J_1$, where

$$J_1 = \int \cdots \int \mathrm{e}\big\{ \mathbb{Q}_p[x_1, \dots, x_p] \big\} \prod_{j=1}^{p} \widetilde{\mu}_j(dx_j).$$

Proof. We shall prove the lemma using induction in p.

Let us prove the lemma for $p=1$. In this case the function f_1 is a constant, and we have

$$J=\Big|\int \mathrm{e}\big\{\,\mathbb{Q}_1[x_1]+f_1\,\big\}\,\mu_1(dx_1)\Big|^2=\iint \mathrm{e}\big\{\,\mathbb{Q}_1[x_1-y_1]\,\big\}\,\mu_1(dx_1)\,\mu_1(dy_1)=J_1.$$

Assuming that we have proved the lemma for $1,\dots,p-1$, let us prove it for p. Using Hölder's inequality, we obtain

$$J\le\Big(\int\cdots\int I\prod_{j=1}^{p-1}\mu_j(dx_j)\Big)^{\alpha},\qquad \alpha\overset{\text{def}}{=}2^{p-1},\tag{7.5}$$

with

$$I\overset{\text{def}}{=}\Big|\int \mathrm{e}\big\{\,\mathbb{Q}_p[x_1,\dots,x_p]+\sum_{j=1}^{p}f_j\,\big\}\,\mu_p(dx_p)\Big|^2=\iint I_0\,\mu_p(dx_p)\mu_p(dy_p),$$

where we write

$$I_0=\mathrm{e}\big\{\,\mathbb{Q}_p[x_1,\dots,x_{p-1},x_p-y_p]+\sum_{j=1}^{p-1}(f_j-\overline{f}_j)\,\big\}$$

and

$$\overline{f}_j=f_j(x_1,\dots,x_{j-1},x_{j+1},\dots,x_{p-1},y_p).$$

Hence, using (7.5) and applying Fubini's Theorem and Hölder's inequality, we have

$$J\le\Big(\int\cdots\int I_0\prod_{j=1}^{p-1}\mu_j(dx_j)\,\mu_p(dx_p)\mu_p(dy_p)\Big)^{\alpha}\le\iint I_1\,\mu_p(dx_p)\mu_p(dy_p)\tag{7.6}$$

with

$$I_1\overset{\text{def}}{=}\Big|\int\cdots\int \mathrm{e}\big\{\,\mathbb{Q}_p[x_1,\dots,x_{p-1},x_p-y_p]+\sum_{j=1}^{p-1}(f_j-\overline{f}_j)\,\big\}\prod_{j=1}^{p-1}\mu_j(dx_j)\Big|^{\alpha}.$$

Using the induction assumption we get

$$I_1\le\int\cdots\int \mathrm{e}\big\{\,\mathbb{Q}_p[x_1,\dots,x_{p-1},x_p-y_p]\,\big\}\prod_{j=1}^{p-1}\widetilde{\mu}_j(dx_j),$$

and (7.6) yields

$$J\le\int\cdots\int \mathrm{e}\big\{\,\mathbb{Q}_p[x_1,\dots,x_{p-1},x_p-y_p]\,\big\}\prod_{j=1}^{p-1}\widetilde{\mu}_j(dx_j)\,\mu_p(dx_p)\mu_p(dy_p)=J_1.\qquad\square$$

8. Upper bounds for trigonometric sums

In this section for a class of polynomials we shall verify the conditions (2.8) and (2.9). We shall assume that $p \geq 2$. We shall start with the condition (2.9) since it is simpler to verify.

Write

$$\mathbb{B}_p(x) = \mathbb{B}_p[x, \dots, x] = x_1^p + \dots + x_s^p - x_{s+1}^p - \dots - x_d^p, \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d. \tag{8.1}$$

We shall consider the following p-linear form

$$\mathbb{Q}_p(x) = \mathbb{B}_p[\mathbb{A}_1 x, \dots, \mathbb{A}_1 x], \tag{8.2}$$

where $\mathbb{A}_j : \mathbb{R}^d \to \mathbb{R}^d$ denote linear invertible operators. We extend the notation (2.6) and (2.7) to

$$\begin{aligned} \varphi(t; \mu, r, \mathbb{Q}_p) &= \int \cdots \int \mathrm{e}\big\{ t\, \mathbb{Q}_p[x_1, \dots, x_p] \big\}\, \widetilde{\mu}(dx_1; r) \dots \widetilde{\mu}(dx_p; r), \\ \varphi(t; \nu, r, \mathbb{Q}_p) &= \int \cdots \int \mathrm{e}\big\{ t\, \mathbb{Q}_p[x_1, \dots, x_p] \big\}\, \widetilde{\nu}(dx_1; \overline{r}) \dots \widetilde{\nu}(dx_p; \overline{r}). \end{aligned} \tag{8.3}$$

For a linear operator $\mathbb{A} : \mathbb{R}^d \to \mathbb{R}^d$ introduce the norm $\|A\| = \sup\limits_{|x|=1} |\mathbb{A}\, x|$. It is clear that for invertible $\mathbb{A}$ we have

$$\|\mathbb{A}^{-1}\|^{-1}\, |x|_\infty \ll_d |\mathbb{A}\, x|_\infty \ll_d \|\mathbb{A}\|\, |x|_\infty, \quad \|\mathbb{A}\|^{-1}\, |x|_\infty \ll_d |\mathbb{A}^{-1}\, x|_\infty \ll_d \|\mathbb{A}^{-1}\|\, |x|_\infty, \tag{8.4}$$

and

$$\|\mathbb{A}^{-1}\|^{-d} \ll_d |\det \mathbb{A}| \ll_d \|\mathbb{A}\|^d. \tag{8.5}$$

Lemma 8.1. *Assume that $\mathbb{Q}_p$ is defined by* (8.2). *Then*

$$\varphi(t; \nu, r, \mathbb{Q}_p) \ll_{d,p} C_1^{2d}\, C_2^{2d}\, \varphi(c_{d,p}\, C_1\, t; \nu, r, \mathbb{B}_p) \tag{8.6}$$

with

$$C_1 = \prod_{j=1}^{p} \|\mathbb{A}_j\|, \qquad C_2 = \prod_{j=1}^{p} \|\mathbb{A}_j^{-1}\|. \tag{8.7}$$

Proof. The proof consists in step-wise replacements of the variables $\mathbb{A}_j\, x_j$ in the integral defining $\varphi(t; \mu, r, \mathbb{Q}_p)$ by the variables x_j, for all $1 \leq j \leq p$. Let us demonstrate the details in the case $j = 1$ only. Introduce the function

$$f(x) = \int \cdots \int \mathrm{e}\big\{ t\, \mathbb{B}_p[x, \mathbb{A}_2\, x_2, \dots, \mathbb{A}_p\, x_p] \big\}\, \widetilde{\nu}(dx_2; \overline{r}) \dots \widetilde{\nu}(dx_p; \overline{r}). \tag{8.8}$$

Notice that the function $f \geq 0$ is non-negative (cf. the proof that a similar function f in (5.33) is non-negative). We shall prove that

$$J \overset{\text{def}}{=} \int f(\mathbb{A}_1\, x)\, \widetilde{\nu}(dx; \overline{r}) \ll_d \|\mathbb{A}_1^{-1}\|^{2d}\, \|\mathbb{A}_1\|^{2d} \int f(c_d\, \|A_1\|\, x)\, \widetilde{\nu}(dx; \overline{r}). \tag{8.9}$$

Application of inequalities similar to (8.9) with $\mathbb{A}_j$ instead of $\mathbb{A}_1$ proves the lemma.

Let us prove (8.9). Using $\nu(dx;\overline{r}) = \mathbf{I}\big\{ |x|_\infty \le \overline{r} \big\}\, dx$, we have

$$J = \iint f\big(\mathbb{A}_1\,(x-\overline{x})\big)\,\mathbf{I}\big\{ |x|_\infty \le \overline{r} \big\}\,\mathbf{I}\big\{ |\overline{x}|_\infty \le \overline{r} \big\}\, dx\, d\overline{x}. \tag{8.10}$$

Changing the variables in (8.10), using (8.5) to estimate $|\det \mathbb{A}_1|$, applying (8.4) to derive $\mathbf{I}\big\{ |\mathbb{A}_1^{-1}\, y|_\infty \le \overline{r} \big\} \le \mathbf{I}\big\{ \|\mathbb{A}_1^{-1}\|\, |y|_\infty \le \overline{r} \big\}$, using $f \ge 0$ and changing the variables again, we obtain (8.9).

Lemma 8.1 reduces the estimation of $\varphi(t;\nu,r,\mathbb{Q}_p)$ to the estimation of $\varphi(t;\nu,r,\mathbb{B}_p)$. Since the measures $\nu(\cdot;\overline{r})$ are symmetric product measures and the form $\mathbb{B}_p$ is diagonal, we have

$$\varphi(t;\nu,r,\mathbb{B}_p) = \varphi_p^d(t) \tag{8.11}$$

with

$$\varphi_p(t) = \varphi_p(t;\nu,r) = \int\cdots\int \mathrm{e}\big\{ t\,x_1\cdots x_p \big\}\,\widetilde{\nu}_1(dx_1;\overline{r})\ldots\widetilde{\nu}_1(dx_p;\overline{r}),$$

where $\nu_1(\cdot;\overline{r})$ denotes the measure on $\mathbb{R}$ with density $(2\,\overline{r})^{-1}\mathbf{I}\big\{ |x| \le \overline{r} \big\}$.

We shall use the following double large sieve type bound (Lemma 8.2 below) which is a corollary of Lemma 2.4 in Bombieri and Iwaniec (1986). It follows from Corollary 5.3 in BG (1997).

Lemma 8.2. *Assume that functions $g, h : \mathbb{R}^d \to \mathbb{C}$ satisfy $\big|g(x)\big| \le 1$ and $\big|h(x)\big| \le 1$. Let μ and ν denote arbitrary measures on $\mathbb{R}^d$ such that*

$$\mu\Big(\big\{ x \in \mathbb{R}^d :\ |x|_\infty \le T \big\}\Big) = 1 \quad \text{and} \quad \nu\Big(\big\{ x \in \mathbb{R}^d :\ |x|_\infty \le S \big\}\Big) = 1,$$

for some $T > 0$ and $S > 0$. Write

$$J = \Big| \int \Big(\int g(x)\,h(y)\,\mathrm{e}\big\{ t\,\langle \mathbb{A}x, y\rangle \big\}\ \mu(dx)\Big)\,\nu(dy)\Big|^2, \qquad t \in \mathbb{R},$$

where $\mathbb{A} : \mathbb{R}^d \to \mathbb{R}^d$ is an invertible symmetric operator. Then there exists a positive constant c_d depending on the dimension d only such that

$$J \ll_d C_3^d\,\big(1 + S\,T\,|t|\big)^d \sup_{x\in\mathbb{R}^d} \mu\Big(x + \frac{c_d\,B}{|t|\,S}\Big) \sup_{x\in\mathbb{R}^d} \nu\Big(x + \frac{c_d\,B}{|t|\,T}\Big),$$

where $B = \big\{ x \in \mathbb{R}^d :\ |x|_\infty \le 1 \big\}$ and $C_3 = \|\mathbb{A}\|\,\|\mathbb{A}^{-1}\|$.

The following bound (8.12) for concentration of measures can be easily derived from Lemma 6.1 in Esseen (1968).

Lemma 8.3. *Let λ be a measure on $\mathbb{R}^d$. Then, for $r > 0$, we have*

$$\sup_{x\in\mathbb{R}^d} \lambda\big\{ B(r) + x \big\} \ll_d r^d \int\limits_{B(r^{-1})} \big|\widehat{\lambda}(t)\big|\, dt, \tag{8.12}$$

where $\widehat{\lambda}(t) = \int \mathrm{e}\{ \langle t, u\rangle \} \, \lambda(du)$, *and* $B(r) = \{ |x|_\infty \le r \}$.

Let us recall the notion of an image of a measure. Consider a measurable mapping $f : \mathbb{R}^d \to \mathbb{R}^m$ and a measure λ on $\mathbb{R}^d$. The image, say θ, of λ under the mapping f is defined as a measure on $\mathbb{R}^m$ such that $\theta(A) = \lambda\big(f^{-1}(A)\big)$, for all measurable sets $A \subset \mathbb{R}^m$. We shall use the following formula

$$\int_{\mathbb{R}^m} g(x)\, \theta(dx) = \int_{\mathbb{R}^d} g\big(f(y)\big)\, \lambda(dy)$$

to change the variables, which is valid for any measurable function $g : \mathbb{R}^m \to \mathbb{C}$.

Lemma 8.4. *For* $p \ge 2$ *we have*

$$\varphi_p(t) \ll_p |t\, \overline{r}|^{-1/(p-1)}. \tag{8.13}$$

Corollary 8.5. *Assume that* $\mathbb{Q}_p$, $p \ge 2$, *is defined by* (8.2)*. Then we have*

$$\varphi(t; \nu, r, \mathbb{Q}_p) \ll_{d,p} C_1^{2d - d/(p-1)}\, C_2^{2d}\, |t\, \overline{r}|^{-d/(p-1)}.$$

with C_1 *and* C_2 *defined by* (8.7)*.*

Proof of Corollary 8.5. It suffices to combine the bound (8.6) of Lemma 8.1, to use (8.11) and to apply Lemma 8.4. □

Proof of Lemma 8.4. We shall use induction in p.

Let us prove the following bound

$$\varphi_p^s(t) \ll_{p,s} \big(1 + |t\, \overline{r}^p|\big)\, (t\, \overline{r}^p)^{-2} \int_{|y| \le c_1 |t\, \overline{r}^p|} \varphi_{p-1}^s(y\, \overline{r}^{\,1-p})\, dy, \qquad s \in \mathbb{N}. \tag{8.14}$$

Let θ denote the image of the product measure $\lambda(dx_2, \dots, dx_p) = \prod_{j=2}^{p} \nu_1(dx_j; \overline{r})$ on $\mathbb{R}^{p-1}$ under the mapping $(x_2, \dots, x_p) \mapsto x_2 \cdots x_p : \mathbb{R}^{p-1} \to \mathbb{R}$. It is clear that θ is a symmetric measure on $\mathbb{R}$ such that $\int \mathrm{e}\{ t x v \}\, \theta(dv) \ge 0$. Applying Hölder's inequality, we have

$$\begin{aligned} \varphi_p^s(t) &= \Big(\iint \mathrm{e}\{ t x v \}\, \nu_1(dx; \overline{r})\, \theta(dv) \Big)^s \le \int \Big(\int \mathrm{e}\{ t x v \}\, \theta(dv) \Big)^s \nu_1(dx; \overline{r}) \\ &= \int \int \mathrm{e}\{ t x v \}\, \theta_s(dv)\, \nu_1(dx; \overline{r}), \end{aligned} \tag{8.15}$$

where θ_s denote the image of the product measure $\prod_{j=1}^{s} \theta(du_j)$ on $\mathbb{R}^s$ under the mapping $(u_1, \dots, u_s) \mapsto u_1 + \cdots + u_s : \mathbb{R}^s \to \mathbb{R}$. To the last double integral in (8.15) we can apply the double large sieve bound of Lemma 8.2. Taking in that lemma $d = 1$, $\mathbb{A} = 1$, $\mu = \nu_1(\cdot; \overline{r})$, $\nu = \theta_s$, noting that the supports of the measures $\nu_1(\cdot; \overline{r})$ and θ_s are of sizes $\ll \overline{r}$ and $\ll_{p,s} \overline{r}^{\,p-1}$ respectively, and using (8.15), we obtain

$$\varphi_p^s(t) \ll_{p,s} \big(1 + |t\, r^p|\big)\, J_1\, J_2, \tag{8.16}$$

where $B = [-1, 1]$, and

$$J_1 = \sup_{x \in \mathbb{R}} \nu_1\Big(x + \frac{c_{p,s}\, B}{|t\, r^{p-1}|}\,;\ \overline{r}\Big) \qquad\qquad J_2 = \sup_{x \in \mathbb{R}} \theta_s\Big(x + \frac{c\, B}{|t\, \overline{r}|}\Big).$$

The measure $\nu_1(\cdot; \overline{r})$ has a density bounded from above by $1/\overline{r}$. Hence

$$J_1 \ll_{p,s} |t\, r^p|^{-1}. \tag{8.17}$$

In order to estimate the concentration of θ_s, we apply Lemma 8.3. Using $\widehat{\theta}_s = \widehat{\theta}^{\,s}$ and $\widehat{\theta} = \varphi_{p-1}$, we obtain

$$\begin{aligned} J_2 &\ll |t\, \overline{r}|^{-1} \int\limits_{|\tau| \le c_1\, |t\, \overline{r}|} \widehat{\theta}_s(\tau)\ d\tau = |t\, \overline{r}|^{-1} \int\limits_{|\tau| \le c_1\, |t\, \overline{r}|} \varphi_{p-1}^s(\tau)\ d\tau \\ &= |t\, \overline{r}^p|^{-1} \int\limits_{|y| \le c_1\, |t\, \overline{r}^p|} \varphi_{p-1}^s(y\, r^{1-p})\ dy. \end{aligned} \tag{8.18}$$

Collecting (8.16)-(8.18), we obtain (8.14).

Let us return to the proof of (8.13). We shall assume that $|t\, r^p| \ge 1$ since otherwise (8.13) is implied by the obvious estimate $\varphi_p \le 1$.

Let us prove (8.13) for $p = 2$. Choosing in (8.16) $s = 1$, noting that $J_1 = J_2$, for $p = 2$ and $s = 1$, and using (8.17), we obtain $\varphi_2(t) \ll |t\, \overline{r}^p|^{-1}$.

Let us prove (8.13) for p, assuming that (8.13) is fulfilled for $2, \dots, p-1$. The bound (8.14) with $s = p - 1$ and the induction assumption imply

$$\varphi_p^{p-1}(t) \ll_p |t\, \overline{r}^p|^{-1} \int\limits_{|y| \le c_1\, |t\, \overline{r}^p|} \big(1 \wedge |y|^{-(p-1)/(p-2)}\big)\, dy \ll_p |t\, \overline{r}^p|^{-1}, \tag{8.19}$$

which yields (8.13). $\square$

Since the form $\mathbb{B}_p$ is diagonal, we have

$$\varphi(t; \mu, r, \mathbb{B}_p) = \psi_p^d(t) \tag{8.20}$$

with

$$\psi_p(t) = \psi_p(t; \mu, r) = \int \cdots \int \mathrm{e}\big\{\, t\, x_1 \cdots x_p \big\}\, \widetilde{\mu}_1(dx_1; r) \dots \widetilde{\mu}_1(dx_p; r),$$

where $\mu_1(\cdot; r)$ denotes the measure $\mu(\cdot; r)$ on $\mathbb{R}$; we use an index 1 in order to emphasize this fact.

Lemma 8.6. *For $p \ge 2$ we have*

$$\psi_p(t) \ll_p \mathcal{M}^{\alpha_p}(t;\, r^p) \tag{8.21}$$

with some $\alpha_p > 0$. The exponents α_p satisfy

$$\alpha_2 = 1, \quad \alpha_3 = 2/51, \quad \alpha_4 = 4/189, \quad \alpha_5 = 2/4083, \quad \alpha_6 = 2/154275, \dots, \tag{8.22}$$

$$\alpha_p \ge \exp\big\{ -3\, p\, \log p \big\}. \tag{8.23}$$

Corollary 8.7. *For the polynomial* $\mathbb{B}_p$ *defined by* (8.1), *we have*

$$\varphi(t; \mu, r, \mathbb{B}_p) \ll_{d,p} \mathcal{M}^{d\alpha_p}(t;\, r^p)$$

with α_p *from Lemma* 8.6.

Proof of Corollary 8.7. It suffices to use (8.20) and to apply Lemma 8.6. □

Corollary 8.8. *For the polynomial* $\mathbb{B}_p(x) = \sum\limits_{s=1}^{d} \lambda_s\, x_s^p$ *we have*

$$\varphi(t; \mu, r, \mathbb{B}_p) \ll_{d,p} (1 \vee \lambda_*^{-d\alpha_p})\, \mathcal{M}^{d\alpha_p}(t;\, r^p), \qquad \lambda_* \stackrel{\text{def}}{=} |\lambda_1| \wedge \cdots \wedge |\lambda_d|,$$

provided that $\lambda_j \neq 0$, *for all* j, *where* α_p *is from Lemma* 8.6.

Proof of Corollary 8.8. The measure $\widehat{\mu}(\cdot; r)$ is a product measure of the measures $\widehat{\mu}_1(\cdot; r)$. Hence, applying the arithmetic-geometric mean inequality, we have

$$\varphi(t; \mu, r, \mathbb{B}_p) = \textstyle\prod_{j=1}^{d} \psi_p(\lambda_j\, t) \ll_d \sum_{j=1}^{d} \psi_p^d(\lambda_j\, t).$$

An application of Lemma 8.6 and of the bound (1.35) with $\varepsilon = |\lambda_j|$ concludes the proof. □

Proof of Lemma 8.6. The proof is based on induction in p. In this proof we shall use the following three statements a), b) and c):

a) For $p = 2$ we have

$$\psi_2(t) \ll \mathcal{M}(t;\, r^2); \tag{8.24}$$

b) For $p \geq 2$ and $s \in \mathbb{N}$, we have

$$\psi_p(t) \ll_p \mathcal{M}^{2/(ps)}(t;\, r^p),$$

provided that

$$\psi_{p-1}(t) \ll_{p,\alpha} \mathcal{M}^{\alpha}(t;\, r^{p-1})$$

with some $\alpha > 0$ such that

$$s > 2^{p+1}/\alpha;$$

c) For $p \geq 3$ and $n, q, s \in \mathbb{N}$, $1 \leq n \leq p/2$, we have

$$\psi_p(t) \ll_{p,q,s} \mathcal{M}^{2n/(pqs)}(t;\, r^p), \tag{8.25}$$

provided that

$$\psi_n(t) \ll_{p,\delta} \mathcal{M}^{\delta}(t;\, r^n), \qquad \psi_{p-n}(t) \ll_{p,\alpha} \mathcal{M}^{\alpha}(t;\, r^{p-n}) \tag{8.26}$$

with some $\delta, \alpha > 0$ such that

$$q > 2^{n+2}/\delta, \qquad s > 2^{p-n+2}/\alpha. \tag{8.27}$$

In the proof of $a)$, $b)$ and $c)$ without loss of generality we shall assume that

$$r \geq 1 \quad \text{and} \quad r^{-p} \leq |t| \leq 1 \tag{8.28}$$

since otherwise these statements are implied by the obvious bounds $\psi(t) \leq 1$, for all t, and $\mathcal{M}(t;\, r) \geq 1$, for $r \geq 1$.

Let us prove the statements $a)$, $b)$ and $c)$. Assume that $1 \leq n \leq p/2$, $n \in \mathbb{N}$. Let λ and θ denote the images of the product measures

$$\prod_{j=1}^{n} \widetilde{\mu}_1(dx_j; r) \qquad \text{and} \qquad \prod_{j=n+1}^{p} \widetilde{\mu}_1(dx_j; r)$$

on $\mathbb{R}^n$ and $\mathbb{R}^{p-n}$ under the mappings

$$(x_1, \ldots, x_n) \mapsto x_1 \cdots x_n : \mathbb{R}^n \to \mathbb{R} \quad \text{and} \quad (x_{n+1}, \ldots, x_p) \mapsto x_{n+1} \cdots x_p : \mathbb{R}^{p-n} \to \mathbb{R} \tag{8.29}$$

respectively. It is clear that λ and θ are symmetric measures on $\mathbb{R}$ such that

$$\int \mathrm{e}\{\, t\,u\,v \,\}\, \lambda(du) \geq 0 \qquad \text{and} \qquad \int \mathrm{e}\{\, t\,u\,v \,\}\, \theta(dv) \geq 0.$$

Applying Hölder's inequality, we have

$$\begin{aligned} \psi_p^{qs}(t) &= \Big(\iint \mathrm{e}\{\, t\,u\,v \,\}\, \lambda(du)\, \theta(dv) \Big)^{qs} \leq \Big(\int \Big(\int \mathrm{e}\{\, t\,u\,v \,\}\, \lambda(dv) \Big)^{q} \theta(du) \Big)^{s} \\ &= \Big(\int\int \mathrm{e}\{\, t\,u\,v \,\}\, \lambda_q(du)\, \theta(dv) \Big)^{s} \leq \int\int \mathrm{e}\{\, t\,u\,v \,\}\, \lambda_q(du)\, \theta_s(dv), \end{aligned} \tag{8.30}$$

where λ_q and θ_s denote measures on $\mathbb{R}$ which are images of the product measures

$$\prod_{j=1}^{q} \lambda(du_j) \qquad \text{and} \qquad \prod_{j=1}^{s} \theta(dv_j)$$

on $\mathbb{R}^q$ and $\mathbb{R}^s$ under the mappings

$$(u_1, \ldots, u_q) \mapsto u_1 + \cdots + u_q : \mathbb{R}^q \to \mathbb{R} \text{ and } (u_1, \ldots, u_s) \mapsto u_1 + \cdots + u_s : \mathbb{R}^s \to \mathbb{R} \tag{8.31}$$

respectively. To the last double integral in (8.30) we can apply the double large sieve bound of Lemma 8.2. Taking in that lemma $d = 1$, $\mathbb{A} = 1$, $\mu = \lambda_q$, $\nu = \theta_s$, noting that the supports of the measures λ_q and θ_s are of sizes $\ll_{p,q} r^{\,n}$ and $\ll_{p,s} r^{\,p-n}$ respectively, and using (8.30), we obtain

$$\psi_p^{qs}(t) \ll_{p,q,s} \big(1 + |t\, r^{\,p}| \big)\, J_1\, J_2, \tag{8.32}$$

where $B = [-1,1]$, and

$$J_1 = \sup_{x\in\mathbb{R}} \lambda_q\Big(x + \frac{c\,B}{|t\,r^{\,p-n}|}\Big), \qquad J_2 = \sup_{x\in\mathbb{R}} \theta_s\Big(x + \frac{c\,B}{|t\,r^{\,n}|}\Big)$$

with a constant $c = c(p,q,s)$.

For estimation of J_1 and J_2 we shall use the following obvious inequality: for any measure μ on $\mathbb{R}$ which is supported by the set $\mathbb{Z}$, and for any interval $[a,b] \subset \mathbb{R}$, we have

$$\sup_{x\in\mathbb{R}} \mu\big(x + [a,b]\big) \ll A\,\mathbf{I}\big\{\,b-a \le 1\,\big\} + (b-a)\,A\,\mathbf{I}\big\{\,b-a \ge 1\,\big\}, \tag{8.33}$$

where $A = \sup\limits_{x\in\mathbb{R}} \mu\big(\{x\}\big)$ denotes the maximal atom of the measure μ.

Let us estimate J_1 in the case $n = 1$. In this case we choose $q = 1$. Hence, we have $\lambda_q = \widetilde{\mu}_1(\cdot\,; r)$. The measure $\widetilde{\mu}_1(\cdot\,; r)$ is a measure on $\mathbb{R}$ supported by the set $\mathbb{Z}$ and its maximal atom is $\ll r^{\,-1}$. Hence, using (8.33), we obtain

$$J_1 \ll_{p,s} r^{\,-1}\,\mathbf{I}\big\{\,|t\,r^{\,p-1}| \ge 1\,\big\} + |t\,r^{\,p}|^{-1}\,\mathbf{I}\big\{\,|t\,r^{\,p-1}| < 1\,\big\}. \tag{8.34}$$

Let us prove the statement $a)$. In this case $p = 2$. We choose $n = q = s = 1$. Then we have $J_1 = J_2$ and (8.34) yields

$$J_1\,J_2 \ll r^{\,-2}\,\mathbf{I}\big\{\,|t\,r^{\,p-1}| \ge 1\,\big\} + |t\,r^{\,p}|^{-2}\,\mathbf{I}\big\{\,|t\,r^{\,p-1}| < 1\,\big\}. \tag{8.35}$$

Using (8.28) we can estimate $1 \le |t\,r^p|$, and (8.32) together with (8.35) implies (8.24), proving $a)$.

In the case $n \ge 2$ the estimation of J_1 is more involved. We shall prove that

$$J_1 \ll_{p,q,\delta} r^{-n}\,\mathbf{I}\big\{\,|t\,r^{p-n}| \ge 1\,\big\} + |t\,r^{p}|^{-1}\,\mathbf{I}\big\{\,|t\,r^{p-n}| \le 1\,\big\}, \tag{8.36}$$

assuming that the first inequalities in (8.26) and (8.27) are fulfilled. Let us start with estimation of the concentration of the measure λ_q. This measure is supported by the set $\mathbb{Z}$ since it is an image under the mappings (8.29) and (8.31) of measures which are supported by the standard lattices. Let us estimate the maximal atom $A = \sup\limits_{x\in\mathbb{R}} \lambda_q\big(\{x\}\big)$ of λ_q. Using Lemma 8.3 and and $\widehat{\lambda}_q = \widehat{\lambda}^{\,q}$, $\widehat{\lambda} = \psi_n$, we obtain

$$A \ll \int\limits_{|\tau|\le 1} \widehat{\lambda}_q(\tau)\;d\tau = \int\limits_{|\tau|\le 1} \psi_n^q(\tau)\;d\tau. \tag{8.37}$$

To estimate the last integral in (8.37) we shall apply Corollary 6.3 of Theorem 6.1 replacing in that corollary φ by ψ_n and choosing $T = 1$. We have to verify a multiplicative inequality of type (6.3). Choosing in Theorem 5.2 $d = 1$, $p = n$ and replacing the function $\varphi(t;\mu,[r])$ by $\psi_n(t)$, we obtain

$$\psi_n(t)\,\psi_n(t+\varepsilon) \ll_n \psi_n^{\beta}(\beta\varepsilon; 2\,[r]), \qquad \psi_n(\varepsilon) \ll_n \psi_n^{\beta}(\beta\varepsilon; 2\,[r]), \quad \beta = 2^{-n}, \tag{8.38}$$

for $t, \varepsilon \in \mathbb{R}$, where the function $\psi_n(t; 2[r])$ is defined as $\psi_n(t)$ just replacing the underlying measure $\mu_1(\cdot; r)$ by $\mu_1(\cdot; 2[r])$. We assume that the first inequality in (8.26) holds. Hence, (8.38) yields

$$\psi_n^q(t)\,\psi_n^q(t+\varepsilon) \ll_n \mathcal{M}^{q\delta\beta}(\varepsilon; r^n), \qquad \psi_n^q(\varepsilon) \ll_n \mathcal{M}^{q\delta\beta}(\varepsilon; r^n), \quad \beta = 2^{-n}, \tag{8.39}$$

The inequalities (8.39) show that ψ_n^q satisfies the multiplicative type inequality (6.3) with $\varkappa = q\,\delta\,2^{-n}$ and $\Lambda = c(n)$. Hence, applying Corollary 6.3 to the last integral in (8.37), we obtain

$$A \ll_{p,q,\delta} r^{-n}, \quad \text{provided that} \;\; q\,\delta\,2^{-n} > 4. \tag{8.40}$$

Using (8.40) and applying (8.33) to the measure λ_q, we get (8.36).

We have

$$J_2 \ll_{p,s,\alpha} r^{-p+n}\, \mathbf{I}\big\{ |t\,r^n| \ge 1 \big\} + |t\,r^p|^{-1}\, \mathbf{I}\big\{ |t\,r^n| \le 1 \big\}, \tag{8.41}$$

provided that the last two inequalities in (8.26) and (8.38) are fulfilled. Indeed, to prove (8.41) it suffices to replace in (8.36) n by $p-n$.

Let us return to the estimation of ψ_p^{qs}. We can assume that (8.36) holds for $n=1$ as well, since for $n=1$ the bound (8.36) reduces to (8.34). Since $r \ge 1$ (see (8.28)) and $1 \le n \le p/2$, we obviously have

$$r^{-p} \le r^{n-p} \le r^{-p/2} \le r^{-n} \le 1. \tag{8.42}$$

Combining (8.32) with (8.36) and (8.41) and using (8.28), (8.42), we obtain

$$\psi_p^{qs}(t) \ll_p \frac{1}{|t\,r^p|}\, \mathbf{I}\big\{ r^{-p} \le |t| \le r^{n-p} \big\} + \frac{1}{r^n}\, \mathbf{I}\big\{ r^{n-p} \le |t| \le r^{-n} \big\} + |t|\, \mathbf{I}\big\{ r^{-n} \le |t| \le 1 \big\}. \tag{8.43}$$

Using $r^{-p/2} \le \mathcal{M}(t;\, r^p)$ and $\mathcal{M}(t;\, r^p) \le 1$, for $r^{-p} \le |t| \le 1$, the bound (8.43) yields

$$\psi_p(t) \ll_p \mathcal{M}^{2n/(pqs)}(t;\, r^p),$$

which proves the statements *b*) and *c*1).

The statement *a*) proves the lemma for $p=2$. It is clear as well that using induction in p the statement *b*) (or/and *c*)) implies the bound (8.21) with some positive α_p. The values of α_p given by (8.22) one can find via simple calculation and optimal choices of the numbers q and s, Moreover, one has

$p=3:\quad n=1,\ \alpha_3 = 2/51.$

$p=4:\quad n=1,\ \alpha_4 = 1/1634;\qquad n=2,\ \alpha_4 = 4/189.$

$p=5:\quad n=1,\ \alpha_5 = 2/4083;\qquad n=2,\ \alpha_5 = 4/59445.$

$p=6:\quad n=1,\ \alpha_6 = 1/391971;\qquad n=2,\ \alpha_6 = 2/154275;\qquad n=3,\ \alpha_6 = 1/667489.$

Here n is the value used by application of the statements b) or c); if $n = 1$ then b) is applied, and if $n > 1$ then c) is applied.

To conclude the proof of the lemma we have to prove the lower bound (8.23). We shall prove it by induction in p. For $p \le 6$ the bound (8.23) is implied by (8.22). Hence, assuming that $p \ge 7$ and that (8.23) holds for $2, \dots, p$, we have to prove (8.23) for p. Let us apply (8.25) choosing $n = [p/2]$. Choosing the numbers $q = 2^{n+3}/\alpha_n$ and $s = 2^{p-n+3}/\alpha_{p-n}$, we see that the conditions (8.27) are satisfied. Hence, using (8.25) and the induction assumption, we get

$$\alpha_p \ge 2^{-p-5}\, n\, p^{-1}\, \alpha_{p-n}\, \alpha_{p-n} \ge \exp\bigl\{\, a \,\bigr\} \, \exp\bigl\{ -3\, p\, \log p \bigr\}$$

with

$$a = -(p+5)\, \log 2 + \log(n/p) + 3\, p\, \log p - 3\, n\, \log n - 3\, (p-n)\, \log(p-n).$$

Consequently, it remains to prove that the number a in non-negative.

If $p = 2\, k$, $k \ge 4$, is even, then $n = k$ and $a = (4\, k - 6)\, \log 2 \ge 0$. Somewhat more complicated calculations show that $a \ge 0$, for odd $p = 2\, k + 1$ with $k \ge 3$. $\square$

9. Trigonometric sums of irrational polynomials

In this section we introduce a criterion for the rationality of a polynomial of type (1.4)

$$\mathbb{Q}_p\,(x) = \mathbb{B}_p\,(x;\, d_0) + \mathbb{P}_p\,(x), \tag{9.1}$$

in terms of the behavior of an associated sequence of trigonometric sums. Recall that (see (1.3))

$$\mathbb{B}_p\,(x;\, d_0) = \mathbb{B}_p\,[x, \dots, x;\, d_0] = \sum_{j=1}^{d_0} \lambda_j\, x_j^p, \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d, \tag{9.2}$$

where $1 \le d_0 \le d$ and $\lambda_j \in \mathbb{R}$, $\lambda_j \ne 0$, and that we assume that either $\mathbb{P}_p = 0$ or $\mathbb{P}_p : \mathbb{R}^d \to \mathbb{R}$, $\deg \mathbb{P}_p = p$, is a homogeneous polynomial such that its degree in variables $x_1, \dots, x_{d_0}$ is strictly smaller than p. A polynomial of the form (9.1) (or $\mathbb{Q}_p$, $\mathbb{B}_p$) we call rational if there exists an $M \ne 0$, $M \in \mathbb{R}$ such that the numbers $M\, \lambda_j$ are rational, for all j. Otherwise $\mathbb{Q}_p$ is called irrational.

By change of variables we can write the Fourier–Stieltjes transforms of the distribution function $F(s) = \mu\bigl(\mathbb{Q}(x) \le s\bigr)$ as the following integral (=trigonometric sum)

$$\widehat{F}(t) = \int \mathrm{e}\bigl\{\, t\, \mathbb{Q}(x-a) \,\bigr\}\, \mu(dx), \tag{9.3}$$

where the polynomial $\mathbb{Q} = \sum\limits_{j=1}^{p} \mathbb{Q}_j$ (see (1.1)), and the measure $\mu = \Phi * \mu^{*k}(\cdot; r)$ is defined by (2.4). Splitting $x = y + z \in \mathbb{R}^d$ with $x = (x_1, \dots, x_{d_0}, 0, \dots, 0) \in \mathbb{R}^{d_0}$ and $z = x - y \in \mathbb{R}^{d-d_0}$, using the fact that $\mu(\cdot; r)$ is a product measure of $\mu(\cdot; r, d_0)$ and $\mu(\cdot; r, d - d_0)$ on $\mathbb{R}^{d_0}$ and $\mathbb{R}^{d-d_0}$ respectively, we obtain

$$\bigl|\widehat{F}(t)\bigr| \le \sup_{\deg \mathbb{P} < p} \Bigl| \int \cdots \int \mathrm{e}\bigl\{ \mathbb{B}_p(y) + \mathbb{P}(y) \bigr\} \prod_{j=1}^{k} \mu(dy_j; r, d_0) \Bigr|, \tag{9.4}$$

where we write $y = y_1 + \cdots + y_k$. Applying to the integral in (9.4) the symmetrization Lemma 7.1 with $\mu_j = \mu(\cdot; r, d_0)$ and choosing the measure ν such that $\nu(\{0\}) = 1$, we obtain

$$\bigl|\widehat{F}(t)\bigr|^{2^{p-1}} \le \varphi(t; \mu, r, d_0), \qquad \text{for} \quad k \ge p + 1, \tag{9.5}$$

with

$$\varphi(t; \mu, r, d_0) = \int \cdots \int \mathrm{e}\bigl\{ t \mathbb{B}_p[x_1, \dots, x_p; d_0] \bigr\} \prod_{j=1}^{d_0} \widetilde{\mu}(dx_j; r, d_0).$$

In this section we shall prove that the function $\varphi(t; \mu, r, d_0)$ converges uniformly to 0, as $r \to \infty$, on any bounded interval separated from zero if and and only if $\mathbb{B}_p$ is irrational. Due to the upper bound (9.5), we have $\widehat{F}(t) \to 0$ for irrational $\mathbb{B}_p$. We shall prove these facts by a reduction to the case $p = 2$, for which we can apply a proof used in BG (1997c).

For a family of functions $g = g(\cdot; r) : \mathbb{R} \to \mathbb{C}$ parameterized by $r \ge 0$, consider the following condition:

$$\text{for any } \ 0 < \delta_0 \le \delta < \infty, \qquad \lim_{r \to \infty} \sup_{\delta_0 \le |t| \le \delta} \bigl| g(t; r) \bigr| = 0. \tag{9.6}$$

We shall apply condition (9.6) to the trigonometric sums $\widehat{F}(t)$ and $\varphi(t; \mu, r, d_0)$. See (9.8) and (9.9) for equivalent formulations of (9.6).

Theorem 9.1. *The polynomial $\mathbb{B}_p$ is irrational if and only if $\varphi(t; \mu, r, d_0)$ satisfies condition* (9.6)*. If $k \ge p + 1$ and $\mathbb{B}_p$ is irrational then $\widehat{F}(t)$ satisfies* (9.6)*. If $\mathbb{B}_p$, $\mathbb{P}_p$, a and $\mathbb{Q}_j$, $1 \le j \le p - 1$, are rational, that is, there exists $M \ne 0$ such that*

$$M\, \mathbb{B}_p(\mathbb{Z}^d - a) \subset \mathbb{Z}, \quad M\, \mathbb{P}_p(\mathbb{Z}^d - a) \subset \mathbb{Z}, \quad M\, \mathbb{Q}_j(\mathbb{Z}^d - a) \subset \mathbb{Z}, \tag{9.7}$$

then $\widehat{F}(t)$ does not satisfy (9.6)*.*

The proof of Theorem 9.1 will be given later.

For a given family of functions, the condition (9.6) obviously allows the following equivalent reformulation: there exist functions $\delta_0(r) \downarrow 0$ and $\delta(r) \uparrow \infty$ such that

$$\lim_{r \to \infty} \sup_{\delta_0(r) \le |t| \le \delta(r)} \bigl| g(t; r) \bigr| = 0. \tag{9.8}$$

The irrationality of $\mathbb{B}_p$ is equivalent to the following condition: there exists a function $\delta(r) \uparrow \infty$ such that

$$\lim_{r \to \infty} \sup_{\delta_1(r) \le |t| \le \delta(r)} \varphi(t; \mu, r, d_0) = 0, \tag{9.9}$$

for any function δ_1 such that $\delta_1(r) \to 0$ and $r^p\, \delta_1(r) \to \infty$, as $r \to \infty$. In order to prove (9.9) we use (9.8) with $\varphi(t;\mu,r,d_0)$ instead of $g(t;r)$, Theorem 9.1 and Corollary 8.8. By this corollary we have

$$\varphi(t;\mu,r,d_0) \ll_{d_0,p,\lambda_1,\dots,\lambda_d} \mathcal{M}^{d\alpha_p}(t;\, r^p) \tag{9.10}$$

since we assume that $\lambda_j \neq 0$, for all j, where α_p is a positive number. The definition of the function $\mathcal{M}$ and $r^p\,\delta_1(r) \to \infty$ yield

$$\lim_{r\to\infty} \sup_{\delta_1(r)\le|t|\le\delta_0(r)} \mathcal{M}(t;\, r^p) = 0. \tag{9.11}$$

Combining (9.10), (9.11) and (9.8), we get (9.9).

Due to the upper bound (9.5), the relation (9.9) holds as well for $\widehat{F}$ uniformly in a, $\mathbb{Q}_j$, $0 \le j \le p-1$, and $\mathbb{P}_p$.

Remark 9.2. An inspection of the proof shows that the condition (9.6) for the trigonometric sums $\widehat{F}(t)$ and $\varphi(t;\mu,r,d_0)$ holds uniformly over compact sets, say $\mathcal{C}$, of irrational vectors $\lambda \stackrel{\text{def}}{=} (\lambda_1,\dots,\lambda_{d_0})$, that is,

$$\lim_{r\to\infty} \sup_{\lambda\in\mathcal{C}} \sup_{\delta_0\le|t|\le\delta} \varphi(t;\mu,r,d_0) = 0.$$

We shall use some well-known known bounds for trigonometric sums which go back to Weyl (1915/1916). For our purposes we formulate a variation of such bounds as the following Lemma 9.3. For a proof see BG (1997c), Lemma 6.4.

Lemma 9.3. *For $k\in\mathbb{N}$ and $z=(z_1,\dots,z_d)\in\mathbb{R}^d$, we have*

$$g(z) \stackrel{\text{def}}{=} \int \mathrm{e}\big\{\langle z,y\rangle\big\}\, \widetilde{\mu}^{*k}(dy;\, r) \ll_{d,k} \sum_{m\in\mathbb{Z}^d} h(m),$$

where

$$h(m) = \prod_{j=1}^{d} h_0(z_j - 2\pi m_j), \qquad h_0(s) = \big(1 + r^2 s^2\big)^{-k}, \qquad s\in\mathbb{Z}. \tag{9.12}$$

Proof of Theorem 9.1. Let us show that for rational polynomials $\mathbb{B}_p$ the trigonometric sum $\varphi(t;\mu,r,d_0)$ does not satisfy (9.6). Assume for simplicity that λ_j are integers. Then $\mathbb{B}_p[\mathbb{Z}^{d_0},\dots,\mathbb{Z}^{d_0};d_0] \subset \mathbb{Z}$. The measure $\widetilde{\mu}(\cdot;r,d_0)$ is supported by $\mathbb{Z}^{d_0}$, that is, $\widetilde{\mu}(\mathbb{Z}^{d_0};r,d_0)=1$. Hence, the function

$$\begin{aligned}\varphi(t+2\,\pi;\mu,r,d_0) &= \int_{\mathbb{Z}^{d_0}}\cdots\int_{\mathbb{Z}^{d_0}} \mathrm{e}\big\{(t+2\,\pi)\,\mathbb{B}_p[x_1,\dots,x_p;d_0]\big\} \prod_{j=1}^{d_0} \widetilde{\mu}(dx_j;r,d_0)\\ &= \varphi(t;\mu,r,d_0)\end{aligned} \tag{9.13}$$

is a 2π-periodic function such that $\varphi(0;\mu,r,d_0)=1$, and the condition (9.6) is not fulfilled. Similar arguments show that under the condition (9.7) the trigonometric sum $\widehat{F}(t)$ does not satisfy (9.6).

To complete the proof of the theorem, it remains to prove that if $\widehat{F}(t)$ or $\varphi(t;\mu,r,d_0)$ does not satisfy condition (9.6) then $\mathbb{B}_p$ is rational. The upper bound (9.5) shows that we have to consider the case of $\varphi(t;\mu,r,d_0)$ only. Since (9.6) is not fulfilled, we have

$$\text{for some } 0<\delta_0\le\delta<\infty, \qquad \limsup_{r\to\infty}\ \sup_{\delta_0\le|t|\le\delta} \varphi(2\pi t;\mu,r,d_0)>0.$$

Hence, there exist $\varepsilon>0$ and sequences $t_n\to t_0$, $\delta_0\le|t_n|\le\delta$, and $r_n\uparrow\infty$ such that

$$\varphi(2\pi t_n;\mu,r_n,d_0)\ge\varepsilon>0, \qquad \text{for all } n\in\mathbb{N}. \tag{9.14}$$

We shall prove that (9.14) implies the rationality of $\mathbb{B}_p$. We shall write t and r instead of t_n and r_n.

Introduce the vector $z=(z_1,\dots,z_{d_0})\in\mathbb{R}^d$ with the coordinates

$$z_j=x_{2j}\cdots x_{pj}, \qquad \text{for } x_l=(x_{l1},\dots,x_{ld_0})\in\mathbb{R}^{d_0}. \tag{9.15}$$

Then we can write $\mathbb{B}_p[x_1,\dots,x_p;d_0]=\langle\Lambda x_1,z\rangle$, where Λ denotes a linear operator on $\mathbb{R}^{d_0}$ such that $\Lambda x=\sum_{j=1}^{d_0}\lambda_j x_j$. Let χ denote the image of the product measure $\prod_{j=2}^{d_0}\widetilde{\mu}(dx_j;r,d_0)$ (which is a measure on $\mathbb{R}^{(p-1)d_0}$) under the mapping

$$(x_2,\dots,x_{p-1})\mapsto z:\mathbb{R}^{(p-1)d_0}\to\mathbb{R}^{d_0},$$

where the vector z is defined by (9.15). The measure χ is a measure on $\mathbb{R}^{d_0}$ supported by the set $B\big((2r)^{p-1}\big)\cap\mathbb{Z}^{d_0}$, and we can write

$$\varphi(2\pi t;\mu,r,d_0)=\iint \mathrm{e}\big\{\langle x_1,2\pi t\Lambda z\rangle\big\}\,\widetilde{\mu}(dx_1;r,d_0)\ \chi(dz). \tag{9.16}$$

Let us prove that, for $k\ge3$, $k\in\mathbb{N}$, and sufficiently large $l\ge c(p)$, $l\in\mathbb{N}$, the following bound holds

$$\varepsilon^{kl}\ll_{d_0,p,k,l}P^{-d_0}\sum_{z\in B(2P)\cap\mathbb{Z}^d}\ \sum_{m\in\mathbb{Z}^d}h(m), \tag{9.17}$$

where a number $P=P(r)$ satisfies $P\to\infty$, as $r\to\infty$, and

$$h(m)\overset{\text{def}}{=}\prod_{j=1}^{d}h_0(u_j-m_j), \qquad h_0(s)=\big(1+r^2s^2\big)^{-k}, \qquad u=u(z)=t\Lambda z. \tag{9.18}$$

Using (9.14) and applying Hölder's inequality, we obtain

$$\begin{aligned}\varepsilon^l&\le\varphi^l(t;\mu,r,d_0)\le\int\Big(\int\mathrm{e}\big\{\langle x_1,t\Lambda z\rangle\big\}\ \chi(dz)\Big)^l\,\widetilde{\mu}(dx_1;r,d_0)\\&=\iint\mathrm{e}\big\{\langle x_1,t\Lambda z\rangle\big\}\,\widetilde{\mu}(dx_1;r,d_0)\ \chi^{*l}(dz).\end{aligned} \tag{9.19}$$

Using (9.19) and arguing similarly to (9.19), we obtain

$$\varepsilon^{kl} \le \iint \mathrm{e}\big\{ \langle x_1, t\,\Lambda\, z\rangle \big\}\, \widetilde{\mu}^{*k}(dx_1; r, d_0)\; \chi^{*l}(dz). \tag{9.20}$$

The measure χ^{*l} is a measure supported by $\mathbb{Z}^{d_0}$, its support is of size $\ll_l r^{p-1}$, and by ??? its maximal atom satisfies

$$\sup_{x\in\mathbb{Z}^{d_0}} \chi^{*m}\big(\{x\}\big) \ll_{p,d_0} r^{(p-1)d_0} \tag{9.21}$$

provided that the number $l \ge c(p)$ is sufficiently large. Hence, for any non-negative summable function G we have

$$\int G(x)\,\chi^{*m}(dx) \ll_{p,d_0} r^{(p-1)d_0} \sum_{|x|_\infty \ll r^{p-1},\ x\in\mathbb{Z}^d} G(x). \tag{9.22}$$

Now (9.20) combined with (9.22) and Lemma 9.3 yield (9.17) with $P = P(r)$ such that $0 < \liminf_{r\to\infty} r^{p-1}/P$ and $\limsup_{r\to\infty} r^{p-1}/P < \infty$.

The bound (9.17) allows to apply a result of BG (1997c). In this paper it is proved that the bound (9.17) implies the rationality of $\lambda = (\lambda_1, \dots, \lambda_{d_0})$, see the proof of Lemma 6.1 in BG (1997c) starting from (9.15) (in the bound (9.15) in BG (1997c) one has to replace ε by ε^{kl}, the number r by P, the dimension d by d_0, and the operator $\mathbb{Q}$ by the diagonal operator Λ).

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